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Equivalence of Measures and Classification of Groups of Non-singular Transformations*

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Contents

§1. Introduction

By a non-singular transformation of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$ we mean a bi-measurable 1-1 transformation of Ω onto itself that preserves null sets. For a long time, it had been an open problem to find an example of a non-singular transformation without an invariant measure. The first example was affirmatively given by D. S. Ornstein [13] in 1960. This was followed by other examples of R. V. Chacon [4], A. Brunel [3], L. K. Arnold [2], C. C. Moore [12], O. Takenouchi [15] and D. Hill [6]. It is necessary for us not only to give such an exampl but also to investigate the structure of such transformations. Also it is useful for the classification problem of factors of type III in the theory of von Neumann algebras.

This thesis concerns with the classification of groups of non-singular transformations without an invariant measure under the weak equivalence relation. Two groups G and G' of nonsingular transformations of $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}' P')$ respectively, are said to be weakly equivalent with each other if there exists an isomorphism φ from Ω onto Ω' such that $\varphi[G]\varphi^{-1}=[G']$ where $[G]$ ($[G']$) is the group of all G (G')-orbits preserving transformations.

* This is the doctoral thesis at Kyushu University.

W. Krieger [9] [10] introduced an invariant $r(G)$ for the classification. We shall obtain a more detailed classification by introducing a new invariant $T(G)$, the definition of which is based on the homological equation appearing in the invariant measure problem. By this it becomes possible to classify groups of nonsingular transformations of type III_0, to which the set $r(G)$ of Krieger is not applicable.

We also discuss the equivalence of measures on an infinite product space. We shall obtain a nice criterion for the equivalence of a σ-finite measure with product property and an infinite direct product probability measure in terms of some convergence of independent random variables.

We now want to emphasize that this criterion for the equivalence of such measures and the invariant $T(G)$ are closely related with each other and make important roles in classifying certain class of non-singular transformations without an invariant measure, which includes examples of D. S. Ornstein and others mentioned before.

Before stating our results we note that the algebraic invariant $T(M)$ of a factor M of A. Connes [5] which is based on the Tomita and Takesaki's theory of generalized Hilbert algebra, corresponds to $T(G)$ in the sense that M is a crossed product factor $W^*(G, L^\infty(\Omega, P))$ associated with a group G of non-singular transformations.

Our main results are the followings. In section 3 we define the set $T(G)$ for every group G of non-singular transformations, which is invariant under the weak equivalence relation and is an additive subgroup of R. Given any countable subgroup Γ of R, we construct in section 4 an ergodic group G of non-singular transformations of type III_0 such as $T(G)=\Gamma$ (Theorem 4.2). In section 5 a criterion for the equivalence of a quasi-product measure and an infinite direct product probability measure is established in terms of almost sure convergence of infinite direct products of independent, positively valued random variables (Theorem 5.1). In sections 6 and 7 we concern with a class of groups of non-singular transformations of infinite product type (hereafter referred to as IPT transformation groups). Such groups of non-singular transformations correspond to ITPFI factors of Araki-Woods [1]. Using a useful criterion for the equivalence of measures of section 5 we obtain a necessary and sufficient condition for the existence of an invariant measure of an IPT transformation group in terms of almost sure convergence of infinite products of independent random variables (Theorem 6.1) and apply it to show that examples of D. S. Ornstein and others are of type III. In the last section we characterize the set $T(G)$ of an ITP transformation group G in terms of almost sure convergence of infinite products of independent random variables (Theorem 7.1).

Using this criterion we construct a new example of an IPT transformation group G of type III_0, the set $T(G)$ of which is the countable additive subgroup generated by dyadic points (Example of section 7).

§ 2. Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a Lebesgue measure space with $P(\Omega)=1([14])$. Finite or σ-finite measures μ and ν defined on $\mathscr{F}$ are said to be equivalent with each other, $\mu\sim\nu$, if $\mu(A)=0$ if and only if $\nu(A)=0$ $A\in\mathscr{F}$. A bi-measurable 1-1 transformation g of Ω onto itself, that is $g^{-1}\mathscr{F}=g\mathscr{F}=\mathscr{F}$, is said to be P-non-singular (or simply non-singular) if $Pg\sim P$, where $Pg(A)=P(gA)$ $A\in\mathscr{F}$. For a countable group G of non-singular transformations of $(\Omega, \mathscr{F}, P)$ we denote by $\frac{dPg}{dP}(\omega)$ $g\in G$ the Radon-Nikodym density of a measure Pg with respect to a measure P. A function $\frac{dPg}{dP}(\omega)$ is positively valued and satisfiies

$$P((gh)B) = P(g(hB)) = \int_{hB} \frac{dPg}{dP}(\omega)\ dP(\omega) = \int_{B} \frac{dPg}{dP}(h\omega)\frac{dPh}{dP}(\omega)dP(\omega) \quad B\in\mathscr{F}.$$

So we have

$$\frac{dPgh}{dP}(\omega) = \frac{dPg}{dP}(h\omega)\frac{dPh}{dP}(\omega) \quad a.s.\ \omega,\ g,\ h\in G.$$

A measure μ defined on $(\Omega, \mathscr{F})$ is said to be invariant under G if $\mu g(A)=\mu(A)$ $A\in\mathscr{F}$, $g\in G$. A measrable function $f(\omega)$ on $(\Omega, \mathscr{F}, P)$ is said to be invariant under G if $f(g\omega)=f(\omega)$ $a.s.\ \omega$, $g\in G$. G is said to be ergordic if every invariant measurable functions is constant $a.s.\ \omega$. We denote by $[G]$ the group of all non-singular transformations $g\in G$ of $(\Omega, \mathscr{F}, P)$ satisfying that there exist measurable sets A_n $n=1,2,$ and nonsingular transformations $g_n\in G$ $n=1,2,\cdots$ such that $\Omega=\bigcup_{n=1}^{\infty} A_n$ (disjoint) and $g(\omega)=g_n\omega$ $a.s.\ \omega\in A_n$ $n=1,2,\cdots$. $[G]$ is said to be the full group of G. Two countable groups G and G' of non-singular transformations of $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$ respectively, are said to be weakly equivalent if there exists a bimeasurable 1-1 mapping φ from Ω onto Ω' such that $\varphi[G]\varphi^{-1}=[G']$ and $P\sim P'\varphi$.

From now on our arguments are concerned with the case when G is countable.

§ 3. An invariant $T(G)$.

In this section we shall introduce a new invariant "$T(G)$" for the weak equivalence.

DEFINITION 3.1. *Let G be a countable group of non-singular transformations of $(\Omega, \mathscr{F}, P)$. We define the set $T(P, G)$ as the set of all $t \in \mathrm{R}$ with the following property: There exists a $\mathscr{F}$-measurable function* $\exp i\,\xi_t(\omega)$ *such that*

$$\exp(i(\xi_t(g\omega)-\xi_t(\omega)) = \exp\left(it \log \frac{dPg}{dP}(\omega)\right), \quad a.\,s.\ \omega, \quad g \in G.$$

We say "$T(P, G)$" a T-set of G.

THEOREM 3.1. *$T(P, G)$ is an additive subgroup of* R, *and is invariant for the weak equivalence.*

Hence we may denote $T(P, G)$ by $T(G)$.

PROOF. Let t and s be in $T(P, G)$. Then

$$\begin{aligned}
&\exp\,(i((\xi_t(g\omega)-\xi_s(g\omega))-(\xi_t(\omega)-\xi_s(\omega))))\\
&\quad = \exp(i(\xi_t(g\omega)-\xi_t(\omega)))\exp\,(-i(\xi_s(g\omega)-\xi_s(\omega)))\\
&\quad = \exp\left(it\log\frac{dPg}{dP}(\omega)\right)\exp\left(-is\log\frac{dPg}{dP}(\omega)\right)\\
&\quad = \exp\left(i(t-s)\mathrm{iog}\frac{dPg}{dP}(\omega)\right).
\end{aligned}$$

Therefore, $t-s \in T\ (P, G)$. It is shown that if $P \sim Q$ then $T(P, G)=T(Q, G)$. Indeed, let $f(\omega)$ be the Radon-Nikodym density $\frac{dQ}{dP}(\omega)$. Then,

$$\begin{aligned}
&\exp(i((\xi_t(g\omega)+t\log f(g\omega))-(\xi_t(\omega)+t\log f(\omega))))\\
&\quad = \exp(i(\xi_t(g\omega)-\xi_t(\omega)))\exp(it\log\ (f(g\omega)f^{-1}(\omega)))\\
&\quad = \exp\left(it\log\frac{dPg}{dP}(\omega)\right)\mathrm{exq}(it\log\ (f(g\omega)f^{-1}(\omega)))\\
&\quad = \exp\left(it\log\frac{dQg}{dQ}(\omega)\right).
\end{aligned}$$

Notice that $T(P, G)=T(P, [G])$. If G and G' of $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$ respectively, are weakly equivalent under an isomorphism φ then

$$T(P, G) = T(P, [G]) = T(P'\varphi, \varphi^{-1}[G']\varphi) = T(P', [G']) = T(P', G').$$

PROPOSITION 3.1. *For the product transformation group $G \times G'$: $G \times G' = \{g \otimes g';\ g \in G,\ g' \in G'\}$ $g \otimes g'(\omega, \omega') = (g\omega, g'\omega')$ $(\omega, \omega') \in \Omega \times \Omega'$ we obtain $T(G \times G') = T(G) \cap T(G')$.*

PROOF. It follows from

$$\exp\left(it\log\frac{dP\times P'\,g\otimes g'}{dP\times P'}(\omega, \omega')\right) = \exp\left(it\,\mathrm{iog}\frac{dPg}{dP}(\omega)\right)\exp\left(it\log\frac{dP'g'}{dP'}(\omega')\right).$$

PROPOSITION 3.2. *A real number $t \in T(G)$ if and only if there exists a σ-finite measure μ_t of $(\Omega, \mathscr{F}, P)$ such that $\mu_t \sim P$ and*

$$\frac{d\mu_t g}{d\mu_t}(\omega) \in \left\{\exp\frac{2n\pi}{t}\,;\ n = 0\,, \pm 1\,, \cdots\right\} \quad a.\,s.\,\omega, \quad g \in G\,.$$

PROOF. "If" part. Let $t \in T(G)$ and let $\xi_t(\omega)$ be a measurable soluton of

$$\exp(i(\xi_t(g\omega) - \xi_t(\omega))) = \exp\Bigl(it \log\frac{dPg}{dP}(\omega)\Bigr) a.\,s.\,\omega\,,\ g \in G\,.$$

Putting $d\mu_t(\omega) = \exp\Bigl(-\frac{\xi_t(\omega)}{t}\Bigr)\,dP(\omega)$, we have

$$\begin{aligned}\exp\Bigl(it\log\frac{d\mu_t g}{d\mu_t}(\omega)\Bigr) &= \exp\Bigl(it\Bigl(\log\frac{d\mu_t}{dP}(g\omega) + \log\frac{dPg}{dP}(\omega) + \log\frac{dP}{d\mu_t}(\omega)\Bigr)\Bigr)\\ &= \exp\Bigl(it\Bigl(-\frac{\xi_t(g\omega)}{t} + \log\frac{dPg}{dP}(\omega) + \frac{\xi_t(\omega)}{t}\Bigr)\Bigr)\\ &= 1\,.\end{aligned}$$

The proof of "only if" part is evident.

REMARK. By the virtue of Theorem 1.4.8 of [5] and Proposition 3.2 we have $T(G) = T(W^*(G, L^\infty(\Omega, P)))$, where $W^*(G, L^\infty(\Omega, P))$ is the crossed product von Neumann algebra associated with a group G of non-singular transformations of $(\Omega, \mathscr{F}, P)$ and so $T(G)$ is an algebraic invariant of $W^*(G, L^\infty(\Omega, P))$.

§ 4. III_λ $0 < \lambda < 1$, III_1 and III_0.

4.1. Definitions of type III_λ $0 < \lambda < 1$, III_1 and III_0.

Let G be a countable group of non-singular transformations of $(\Omega, \mathscr{F}, P)$. G is said to be of semi-finite type if it admits an equivalent σ-finite invariant measure and is said to be of type III if otherwise. It is easy to see that if G is ergodic and of semi-finite type and if μ and ν are two finite or σ-finite measures on $\mathscr{F}$, both equivalent with P and invariant under G then there exists a positive constant α such that $\mu = \alpha\nu$.

Let G be an ergodic countable group of non-singular transformations of $(\Omega, \mathscr{F}, P)$, μ be an equivalent measure on $\mathscr{F}$ and H be a subgroup of $[G]$. A pair (μ, H) is said to be an admissible pair of G if H is an ergodic subgroup of μ preserving transformations of $[G]$.

For an ergodic countable group G of non-singular transformations of type III of $(\Omega, \mathscr{F}, P)$ we consider the following cases:

(III_λ) *There exists an admissible pair (μ, H) and the smallest number $0 < \lambda < 1$ such that*

$$\frac{d\mu g}{d\mu}(\omega)\in\{\lambda^n:\ n=0,\pm1,\pm2,\cdots\}\quad a.s.\ \omega,\quad g\in G.$$

(III$_1$) *There exists an admissiblep air* (μ, H) *without satisfying the cases* (III$_\lambda$) $0<\lambda<1$.

(III$_0$) *There is no such an admissible pair* (μ, H) *as in* III$_\lambda$ $0<\lambda<1$ *or* III$_1$.

It is clear that these cases are exclusive and exaustive. It is shown that a parameter λ of the case (III$_\lambda$) is independent of the chice of an admissible pair (μ, H). For the proof it is enough to show the following lemma.

LEMM 4.1. *Let* (μ, H) *and* (μ', H') *be two admissible pairs of an ergodic countable group* G *of non-singular transformations of* $(\Omega, \mathscr{F}, P)$. *If for some* $0<\lambda\leqq 1$ $\frac{d\mu g}{d\mu}(\omega)\in\{\lambda^n;\ n=0,\pm1,\cdots\}$ $a.s.\ \omega$, $g\in G$, *then* $\frac{d\mu' g}{d\mu'}(\omega)\in\{\lambda^n;\ n=0,\pm1,\cdots\}$ $a.s.\ \omega$, $g\in G$.

PROOF. Let us denote by $f(\omega)$ the Radon-Nikodym density $\frac{d\mu'}{d\mu}(\omega)$ and by $n_g(\omega)$ the integer valued $\mathscr{F}$-measurable function such as $\frac{d\mu g}{d\mu}(\omega)=\lambda^{n_g(\omega)}$ $g\in G$. Then we have

$$\begin{aligned} f(h'\omega) &= \frac{d\mu' h'}{d\mu'}(\omega)\frac{d\mu'}{d\mu}(\omega)\frac{d\mu}{d\mu h'}(\omega)\\ &= f(\omega)\lambda^{-n_{h'}(\omega)}\quad a.s.\ \omega,\quad h'\in H' \end{aligned}$$

because $\frac{d\mu' h'}{d\mu'}(\omega)=1$ $a.s.\ \omega$, $h'\in H'$. If $\lambda=1$ then $f(\omega)$ is invariant. Since H' is ergodic, $f(\omega)$ is constant $a.s.\ \omega$. If $\lambda<1$ then choose any numbers $\lambda<c<d<1$. The set $\{\omega;\ \lambda^m c\leqq f(\omega)\leqq\lambda^m d$ for some $m=0,\pm1,\cdots\}$ is H'-invariant and then it has measre 0 or 1 since H' is ergodic Therefore for an integer valued measurable function $m(\omega)$ and a constant c, $f(\omega)=c\lambda^{m(\omega)}$ and

$$\begin{aligned} \frac{d\mu' g}{d\mu'}(\omega) &= f(g\omega)\cdot\frac{d\mu g}{d\mu}(\omega)\cdot f^{-1}(\omega)\\ &= \lambda^{m(g\omega)-m(\omega)+n_g(\omega)}\quad a.s.\omega,\quad g\in G. \end{aligned}$$

REMARK. The proof of this lemma means that if for an ergodic non-singular transformation group G and a random variable $Y(\omega)$, $Y(g\omega)-Y(\omega)$ is lattice distributed with span $k_g c$ where c is a constant and k_g an integer, $g\in G$, then there exists a constant a such that $Y(\omega)-a$ is lattice distributed with span c.

DEFINITION 4.1. *Let G be an ergodic countable group of non-singular transformation of $(\Omega, \mathscr{F}, P)$. We say that G is of type* III_λ $0<\lambda<1$, III_1, *or* III_0 *accordingly as the case* (III_λ), (III_1) *or* (III_0) *happens.*

It is obvious that the type of G is an invariant under the weak equivalence relation.

4.2. A characterization of type of groups of non-singular transformations by $T(\mathrm{G})$.

THEOREM 4.1. *Let G be a countable group of non-singular transformations of $(\Omega, \mathscr{F}, P)$. Then*

(1) $T(G)=\mathrm{R}$ *if G is of semi-finite type,*

(2) $T(G)=\frac{2\pi}{\log\lambda}\mathrm{Z}$ *if G is of type* III_λ $0<\lambda<1$ *and*

(3) $T(G)=\{0\}$ *if G is of type* III_1.

LEMMA 4.2. *G is of semi-finite type if and only if there exists a positively valued measurable function $f(\omega)$ such that*

$$\frac{f(\omega)}{f(g\omega)}=\frac{dPg}{dP}(\omega)\ \ a.s.\ \omega,\ g\in G.$$

PROOF. A positively valued measurable function $f(\omega)$ satisfies the above equation if and only if $d\mu(\omega)=f(\omega)\,dP(\omega)$ is an equivalent σ-finite measure and is invariant under G.

PROOF *of Theorem* 4.1. (1). By Lemma 4.2, a measurable function $\xi(t,\omega)=-t\log f(\omega)$ satisfies

$$\exp(i(\xi(t,g\omega)-\xi(t,\omega)))=\exp\left(it\log\frac{dPg}{dP}(\omega)\right)\ \ a.s.\ \omega,\ \ g\in G \text{ and } t\in\mathrm{R}.$$

(2), (3). Let (μ, H) be an admissible pair of G. If $\exp(i(\xi_t(g\omega)-\xi_t(\omega)))=\exp\left(it\log\frac{d\mu g}{d\mu}(\omega)\right)$ $a.s.\ \omega,\ g\in G$, then $\exp(i(\xi_t(h\omega)-\xi_t(\omega)))=1$ $a.s.\ \omega,\ h\in H$. Since H is ergodic, a measurable function $\exp(i\xi_t(\omega))$ is constant $a.s.\ \omega$. Thus, $t\in T(G)$ if and only if $\exp\left(it\log\frac{d\mu g}{d\mu}(\omega)\right)=1$ $a.s.\ \omega,\ g\in G$. If G is of type III_λ $0<\lambda<1$, then for an integer valued measurable function $n_g(\omega)$ with $\frac{d\mu g}{d\mu}(\omega)=\lambda^{n_g(\omega)}$ $a.s.\ \omega$, $g\in G$ it follows that $\exp(itn_g(\omega)\log\lambda)=1$ $a.s.\ \omega,\ g\in G$. Since $\{n_g(\omega);\ a.s.\ \omega \in\Omega,\ g\in G\}$ generates the additive group Z, $\exp(it\log\lambda)=1$. Therefore, $t\in \frac{2\pi}{\log\lambda}Z$. Let t be a nonzero number of $T(G)$. Then from $\exp\Big(it\log\frac{d\mu g}{d\mu}$

$(\omega)\Big)=1$ $a.s.\ \omega$, $g\in G$, is follows that $\frac{d\mu g}{d\mu}(\omega)\in\left\{\exp\frac{2n\pi}{t};\ n=0, \pm1,\cdots\right\}$ $a.s.\omega$, $g\in G$. This also means that $T(G)=\{0\}$ if G is of type III_1.

4.3. Skew product transformations and type III_0.

We are going to construct a group of non-singular transformations of type III_0 whose T-set is a given countable subgroup of R. For this we shall introduce a skew product transformation group due to W. Krieger ([11]).

Let G be an ergodic countable group G of type III_λ $0<\lambda<1$ of non-singular transformations of $(\Omega, \mathscr{F}, P)$ with an admissible pair (μ, H) and U be an ergodic measure preserving transformation of a σ-finite measure space $(X, \mathscr{B},\nu)$. We define the G_U as the set of all non-singular transformations g_U:

$$g_U(\omega, x) = (g\omega, U^{\log_\lambda \frac{d\mu g}{d\mu}(\omega)}x) \quad g\in G$$

of the product measure space $(\Omega\times X, \mathscr{F}\times\mathscr{B}, P\times\nu)$. It is shown that the group G_U is ergodic. Indeed, let $f(\omega, x)$ be a G_U-invariant $\mathscr{F}\times\mathscr{B}$-measurable function, that is $f(g\omega, U^{\log_\lambda \frac{d\mu g}{d\mu}(\omega)}x)=f(\omega, x)$ $a.s.$ (ω, x), $g\in G$. Then we have $f(h\omega, x)=f(\omega, x)$ $a.s.$ (ω, x), $h\in H$. Since H is ergodic there exists a $\mathscr{B}$-measurable function $f(x)$ such that $f(\omega, x)=f(x)$ $a.s.$ (ω, x). Thus we have $f(U^{\log_\lambda \frac{d\mu g}{d\mu}(\omega)}x)=f(x)$ $a.s.$ (ω, x), $g\in G$. Since the set $\left\{\log_\lambda\frac{d\mu g}{d\mu}(\omega);\ a.s.\ \omega\in\Omega,\ g\in G\right\}$ generates the additive group Z, $f(Ux)=f(x)$ $a.s.\,x$. Since U is ergodic, $f(\omega, x)=f(x)$ is constant $a.s.$ (ω, x). Therefore, G_U is ergodic.

PROPOSITION 4.1. *Let G be an ergodic group of non-singular transformations of type III_λ $0<\lambda<1$ on $(\Omega, \mathscr{F}, P)$ with an admissible pair (μ, H). Then for sets $A\in\mathscr{F}$ and $B\in\mathscr{F}$, and for each integer k there exists $g\in[G]$ such that*

$$\mu(gA\cap B)>0 \text{ and } \frac{d\mu g}{d\mu}(\omega)=\lambda^k \quad \omega\in A\cap g^{-1}B.$$

PROOF. Let U be an ergodic measure preserving transformation of (Z, m) defined by $Ui=i+1$ $i\in Z$, $m(i)=1$. Then from the ergodicity of G_U there exists $g_U\in[G_U]$ such that

$$P\times m(g_U(A\times\{0\})\cap B\times\{k\})>0.$$

Hence, $P(gA\cap B)>0$ and $\frac{d\mu g}{d\mu}(\omega)=\lambda^k \quad \omega\in A\cap g^{-1}$B.

PROPOSITION 4.2. *Let G be an ergodic countable group of non-singular trans-*

formations of type III_λ $0<\lambda<1$ *on* $(\Omega, \mathscr{F}, P)$ *and* ρ *be an eqivalent measure on* $\mathscr{F}$ *such that* $\frac{d\rho g}{d\rho}(\omega) \in \{\lambda^k;\ k=0, \pm 1, \cdots\}$ *a. s.* ω, $g \in G$. *Then the non-singular transformation group* K:

$$K=\left\{g \in [G]\ ;\ \frac{d\rho g}{d\rho}(\omega)=1 \quad a.s.\ \omega\right\}$$

is weakly equivalent with an induced ergodic non-singular transformation group of $[G_U]$, *where* $Ui=i+1$ $i \in \mathrm{Z}$, *and hence* $\{\rho, K\}$ *is an admissible pair of* G.

Proof. Let $\{\mu, H\}$ be an admissible pair of G and let $\frac{d\mu g}{d\mu}(\omega)=\lambda^{n_g(\omega)}$ and $\frac{d\rho g}{d\rho}(\omega)=\lambda^{m_g(\omega)}$. Since $\frac{d\rho}{d\mu}(g\omega)=\lambda^{m_g(\omega)-n_g(\omega)} \frac{d\rho}{d\mu}(\omega)$ and G is ergodic, there exists a constant c and an integer valued function $l(\omega)$ such that $\frac{d\rho}{d\mu}(\omega)=c\lambda^{l(\omega)}$ from Lemma 4.1. Let $E=\bigcup_{k=-\infty}^{\infty}\{(\omega, k) \in \Omega \times \mathrm{Z};\ l(\omega)=-k\}$ and let π be the projection $\pi(\omega, k)=\omega$ from $\Omega \times \mathrm{Z}$ onto Ω. The restriction $\pi|_E$ of π on the measurable subset E is a 1-1, measure preserving mapping from $(E, \mu\times\nu|_E)$ onto (Ω, μ). The ergodic non-singular transformation group G_U has the equivalent invariant masure $\lambda^{-j}d\mu(\omega)\,d\nu(j)$, where $\nu(j)=1$. The induced transformation group $[G_U]|_E=\{g_U \in [G_U]\ ;\ g_U E=E\}$ of the measure space $(E, \mu\times\nu|_E)$ is ergodic and has an equivalent invariant measure and satisfies

$$\begin{aligned}\pi|_E[G_U]|_E\pi|_E^{-1} &= \{g \in [G]\ ;\ l(g\omega)=l(\omega)-n_g(\omega)\ \ a.s.\ \omega.\}\\ &= \left[\begin{aligned} g \in [G]\ ;\ \frac{d\rho g}{d\rho}(\omega) &= \frac{d\rho}{d\mu}(g\omega)\frac{d\mu g}{d\mu}(\omega)\frac{d\mu}{d\rho}(\omega)\\ &= \lambda^{l(g\omega)+n_g(\omega)-l(\omega)}\\ &= 1\end{aligned}\right]\\ &= K\end{aligned}$$

Thus, K is weakly equivalent with the induced ergodic non-singular transformation group $[G_U]|_E$ which admits an invariant measure and hence (ρ, K) is an admissible pair of G.

Remark. Let G be aner godic countable group of non-singular transformations. If $T(G)=\frac{2\pi}{\log\lambda}\mathrm{Z}$ $0<\lambda<1$ then theie exists an equivalent measure μ with $\frac{d\mu g}{d\mu}(\omega) \in \{\lambda^k;\ k=0, \pm 1, \cdots\}$ by Proposition 3.2. For such a measure μ G is of type III_λ if $\left\{g \in [G]\ ;\ \frac{d\mu g}{d\mu}(\omega)=1\ \ a.s.\ \omega\right\}$ is ergodic and G is of type III_0 if otherwise from Proposition 4.2.

THEOREM 4.2. *Let G be an ergodic countable group of nonsingular transformations of type* III_λ *of* $(\Omega, \mathscr{F}, P)$ *and U be an ergodic finite measure preserving transformation of* $(X, \mathscr{B}, \nu)$. *Then*

(1) $T(G_U)=\dfrac{2\pi}{\log \lambda}\sigma(U)$, *where $\sigma(U)$ is the set of point spectra s of U: There existe* $h(x)\in L^2(X, \nu)$ *such that*

$$h(Ux) = \exp(2\pi is)\ h(x) \quad a.s.x.$$

(2) G_U *is of type* III_0 *if ν is non-atomic, and*
G_U *is of type* III_{λ^k} *if ν has k atomic points.*

PROOF. (1) Let (μ, H) bean admissible pair of G and let $\exp\left(i\left(\xi_t\left(g\omega, U^{\log_\lambda \frac{d\mu g}{d\mu}(\omega)}x\right)-\xi_t\ (\omega, x)\right)\right)=\exp\left(it\log\frac{d\mu g}{d\mu}(\omega)\right)$ $a.s.(\omega, x)$, $g\in$ G. Then $\exp(i(\xi_t(h\omega, x)-\xi_t(\omega, x)))=1$ $a.s.(\omega, x)$, $h\in H$. Since H is ergodic there exists a $\mathscr{B}$-measurable function $\xi_t(x)$ such that $\exp i\xi_t(\omega, x)=\exp i\xi_t(x)$ $a.s.(\omega, x)$. Then we have

$$\exp\left(i\left(\xi_t\left(U^{\log_\lambda \frac{d\mu g}{d\mu}(\omega)}x\right)-\xi_t(x)\right)\right)$$
$$=\exp\left(it\log\frac{d\mu g}{d\mu}(\omega)\right)a.s.\ (\omega. x) \quad g\in G,$$

and $$\exp(i(\xi_t(Ux)-\xi_t(x))) = \exp it \quad a.s.x.$$

Therefore $\dfrac{t}{2\pi}\in\sigma(U)$. Conversely, let s be a point spectrum of U with $f(Ux)=\exp(2\pi is)f(x)$ $f(x)\in L^2(\mathfrak{x}, \nu)$.
Since U *is* ergodic, then for a constant c $f(x)=c\exp(i\xi(x))$. So we have

$$\exp\left(i\left(\xi\left(U^{\log_\lambda \frac{d\mu g}{d\mu}(\omega)}x\right)-\xi(x)\right)\right)=\exp\left(2\pi\ is\log\frac{d\mu g}{d\mu}(\omega)\right)$$

$a.s.(\omega, x)$, $g\in G$.

(2) Assume that ν is a non-atomic measure. Since $\sigma(U)$ is a countable subgroup, then by Theorem 4.1 we may assume $T(G_U)=\dfrac{2\pi}{\log\lambda}Z$ $k=1, 2, \cdots$. If G_U has an admissible pair, G_U is of type III_{λ^k}. Since $\sigma(U)=\dfrac{1}{k}Z$, there exists a partition $\{A_i\}_{i=0,1,\cdots k-1}$ of X such that $UA_i=A_{i+1}$ $0\leq i\leq k-1$. Let $d\mu\times\nu^*\ (\omega, x)=\lambda^{-j}d\mu\times\nu(\omega, x)$, $x\in A_j$, then

$$\frac{d\mu\times\nu^* g_U}{d\mu\times\nu^*}(\omega, x)\in\{\lambda^{kn};\ n=0, \pm 1, \cdots\} \quad a.s.(\omega, x).$$

By Remark of Proposition 4.2

$$\left\{g_U \in [G_U]\ ;\ \frac{d\mu\times\nu^* g_U}{d\mu\times\nu^*}(\omega, x)=1 \quad a.s.(\omega, x)\right\}$$

$$= \{g_U \in [G_U]\ ;\ g_U(\omega, x) = (h\omega, x) \text{ for some } h \in H\ a.s.(\omega, x)\}$$

must be ergodic. But this is a contradiction.
Let ν be supported by k atomic points and let $\mathfrak{x}=\{0,1,\cdots k-1\}$ and $\nu(j)=1$. It is easy to show that for a measure $\nu^*(j)=\frac{1}{\lambda^j}$, $\left\{g_U \in [G_U]\ ;\ \frac{d\mu\times\nu^* g_U}{d\mu\times\nu^*}(\omega, j)=1\ a.s.(\omega, j)\right\}$ is ergodic.

REMARK. W. Krieger ([11]) gave another proof of (2) of Theorem 4.2 using the theorem of L. K. Arnold ([2]).

THEOREM 4.3. *Let G be of type* III_λ $0<\lambda<1$ *and U_1 and U_2 be ergodic measure preserving transformations of measure spaces $(X_1, \mathscr{B}_1, \nu_1)$ and $(X_2, \mathscr{B}_2, \nu_2)$ respectively. Then a product measure preserving transformations $U_1\times U_2^{-1}$ is ergodic if and only if $G_{U_1}\times G_{U_2}$ is of type* III_λ.

PROOF Let (μ, H) be an admissible pair of G, then

$$\frac{d\mu\times\nu_1\times\mu\times\nu_2 g_{U_1}\otimes g'_{U_2}}{d\mu\times\nu_1\times\mu\times\nu_2}(\omega, x_1, \omega', x_2)=\frac{d\mu g}{d\mu}(\omega)\frac{d\mu g'}{d\mu}(\omega') = \lambda^{n_g(\omega)+n_{g'}(\omega')}.$$

For each measurable function $f(\omega, x_1, \omega', x_2)$ which is invariant under

$$\left\{g_{U1}\otimes g'_{U2}\ ;\ \frac{d\mu\times\nu_1\times\mu\times\nu_2 g_{U_1}\otimes g'_{U2}}{d\mu\times\nu_1\times\mu\times\nu_2}(\omega, x_1, \omega', x_2) = 1 \quad a.s.(\omega, x_1, \omega', x_2)\right\}$$

it follows that $f(h\omega, x_1, h'\omega', x_2)=f(\omega, x_1, \omega', x_2)$ $a.s.(\omega,x_1, \omega', x_2)$ $h\otimes h' \in H\times H$. Since $H\times H$ is ergodic, there exists a measurable function $f(x_1, x_2)$ on $X_1\times X_2$ such that $f(\omega, x_1, \omega', x_2)=f(x_1, x_2)$ $a.s.$ $(\omega, x_1, \omega', x_2)$. Therefore $f(U_1^{n_g(\omega)} x_1, U_2^{-n_{g'}(\omega')} x_2)=f(x_1, x_2)$ if $n_g(\omega)=n_{g'}(\omega')$. By Proposition 4.1 we have $f(U, x_1, U_2^{-1}x_2)=f(x_1, x_2)$ $a.s.$ (x_1, x_2). If $U_1\times U_2^{-1}$ is ergodic then $f(x_1, x_2)$ is a constant and $G_{U_1}\times G_{U_2}$ is of type III_λ. Conversely if $G_{U_1}\times G_{U_2}$ is of type III_λ then

$$\left\{g_{U_1}\otimes g'_{U_2}\ ;\ \frac{d\mu\times\nu_1\times\mu\times\nu_2 g_{U_1}\otimes g'_{U2}}{d\mu\times\nu_1\times\mu\times\nu_2}(\omega, x_1, \omega', x_2) = 1\right\}$$

must be ergodic by Proposition 4.2. Therefore $U_1\times U_2^{-1}$ is ergodic.

§ 5. Equivalence of puasi-product measures.

We now want to discuss a class of equivalent measures on an infinite direct product probability measure space. A useful condition for the equivalence of a quasi-product measure and an infinit direct product probability measure is obtained and is applied to show that the groups of non-singular transformations of D. Ornstein ([13]) and L. Arnold ([2]) are of type III.

5.1. Quasi-product measure. Let $(\Omega_n, \mathscr{F}_n)$ be a measurable space and μ_n be a σ-finite measure on it for each $n \geqq 1$, and let $(\Omega, \mathscr{F}) = \prod_{n=1}^{\infty} (\Omega_n, \mathscr{F}_n)$ be the infinite direct product measurable space.

DEFINITION 5.1. *Let μ_n be a σ-finite measure on $(\Omega_n, \mathscr{F}_n)$ $n=1,2,\cdots$. A σ-finite measure μ on $(\Omega, \mathscr{F})$ is said to be a quasi-product measure of $(\mu_n)_{n\geq 1}$ if there exists for each $n \geqq 1$ a σ-finite measure μ^*_{n+1} on $\prod_{i=n+1}^{\infty} (\Omega_i, \mathscr{F}_i)$ such that*

$$\mu = \prod_{i=1}^{n} \mu_i \times \mu^*_{n+1}\,.$$

An example of a quasi-product measure is given as follows. For each $n \geqq 1$, let A_n be chosen in $\mathscr{F}_n$ such that $0 < \mu_n(A_n) < \infty$. We define the normalized measures $\bar{\mu}_n = \frac{\mu_n}{\mu_n(A_n)}$ and the restriction of this to A_n, λ_n. $\left(i.e.\text{ for all } B_n \in \mathscr{F}_n\ \lambda_n(B_n) = \frac{\mu_n(B_n \cap A_n)}{\mu_n(A_n)}\right)$. Let $\mu^{(n)} = \prod_{i=1}^{n} \bar{\mu}_i \times \prod_{j=n+1}^{\infty} \lambda_j$ and μ be the inductive limit measure of $\mu^{(n)}$. Then μ is a quasi-product measure of $(\mu_n)_{n\geq 1}$.

If $\mu_n(\Omega_n) = 1$ $n \geqq 1$ then the infinite direct product probability measure $\mu = \prod_{n=1}^{\infty} \mu_n$ is quasi-product.

Oui interest is that; let μ_n be a σ-finite measure on $\mathscr{F}_n$ and let P_n be another probability measure on $\mathscr{F}_n$ and assume $\mu_n \sim P_n$ for each $n \geqq 1$. Under what condition does there exist a quasi-product measre μ of $(\mu_n)_{n\geq 1}$ equivalent with the infinite direct product probability measure $P = \prod_{n=1}^{\infty} P_n$?

This problem was first discussed by C. C. Moore ([1]) and 0. Takenouchi ([15]) when each Ω_n is a finite or countable set. The general case was discussed by D. G. Hill ([6]) and a necessary and sufficient condition for the existence was obtained: *There exists a quasi-product measure μ of $(\mu_n)_{n\geq 1}$ equivalent with $P = \prod_{n=1}^{\infty} P_n$ if and only if there exists, for each n, a measurable subset $A_n \in \mathscr{F}_n$ with the following properties*:

(1) $0 < \mu_n(A_n) < \infty$

(2) $\prod_{n=1}^{\infty} P_n(A_n) > 0$ *or equivalently*, $\sum_{n=1}^{\infty} (1 - P_n(A_n)) < \infty$

(3) $P' = \prod_{n=1}^{\infty} P'_n$ *and* $\mu' = \prod_{n=1}^{\infty} \mu'_n$

are equivalent, where P'_n and μ'_n are measures defined on $\mathscr{F}_n$ by $P'_n(B_n) = \Gamma_n$

$(B_n \cap A_n)/P_n(A_n)$, $\mu'_n(B_n)=\mu_n(B_n \cap A_n)/\mu_n(A_n)$.

Here we shall give another simple criterion for the existence of an equivalent quasi-product measure. Of course the general Hill's condition is easily obtained from our condition.

We denote by $X_n(\omega_n)$ the Radon-Nikodym density $\frac{d\mu_n}{dP_n}(\omega_n)$. We may consider that $\mathscr{F}_n$ and $X_n(\omega_n)$ are defined on $(\Omega, \mathscr{F}, P)=\prod_{n=1}^{\infty}(\Omega_n, \mathscr{F}_n, P_n)$. Then $X_1(\omega)$, $X_2(\omega),\cdots$ are mutually independent random variables and each $X_n(\omega)$ is $\mathscr{F}_n$-measurable and positively valued.

THEOREM 5.1. *The following conditions are equivalent.*

(1) *There exists a quasi-product σ-finite measure μ of $(\mu_n)_{n\geq 1}$ equivalent with the infinite direct product probability measure $P=\prod_{n=1}^{\infty} P_n$.*

(2) *There are positive constants b_1, $b_2,\cdots$. such that*

$$\prod_{n=1}^{\infty}\frac{X_n(\omega)}{b_n}$$

converges almost surely.

PROOF. First we remark the condition (1) is equivalent with the following condition (1)′ through the relation $\frac{d\mu}{dP}(\omega)=X(\omega)$.

(1)′ There exists a positively valued, $\mathscr{F}$-measurable random variable $X(\omega)$ such that for each $n\geqq 1$

$$X(\omega)=\prod_{i=1}^{n} X_1(\omega)\times X_{n+1}^*(\omega)$$

where $X_{n+1}^*(\omega)$ is a $\bigvee_{i=n+1}^{\infty}\mathscr{F}_i$-measurable random variable. (The sub σ-algebra $\bigvee_{i=1}^{n+1}\mathscr{F}_i$ is the σ-algebra generated by $\mathscr{F}_{n+1}$, $\mathscr{F}_{n+2},\cdots$.) So it is enough to show the equivalence of (1)′ and (2).

LEMMA 5.1. *If random variables $U_1(\omega)$, $U_2(\omega),\cdots$ satisfy*

$$|\mathrm{E}\{\exp(-it\,U_n(\omega))\}|\xrightarrow[n\to\infty]{} 1 \qquad t\in\mathrm{R}$$

then there are constants a_1, $a_2,\cdots$ such that

$$U_n(\omega)-a_n \xrightarrow[n\to\infty]{} 0 \quad \textit{in probability}.$$

PROOF. Let $\tilde{U}_n(\omega)$ be an independent random variable of $U_n(\omega)$ with the same distribution. Then

$$\int_{-1}^{1} |-|\mathrm{E}\{\exp(-it\ U_n(\omega))\}|^2 dt = 2\mathrm{E}\{1-\psi(U_n(\omega)-\tilde{U}_n(\omega))\}$$

where

$$\psi(u) = \begin{cases} \dfrac{\sin u}{u} & \text{if} \quad u \neq 0 \\ 1 & \text{if} \quad u = 0 . \end{cases}$$

Choosing small $\varepsilon>0$,

$$\begin{aligned} \mathrm{E}\{1-\psi(U_n(\omega)-\tilde{U}_n(\omega))\} &\geqq \{1-\psi(\varepsilon)\}P(|U_n(\omega)-\tilde{U}_n(\omega)|>\varepsilon) \\ &\geqq \{1-\psi(\varepsilon)\}P(U_n(\omega)-a_n>\varepsilon,\ \tilde{U}_n(\omega)\leqq a_n) \\ &+ \{1-\psi(\varepsilon)\}P(U_n(\omega)-a_n<-\varepsilon,\ \tilde{U}_n(\omega)\geqq a_n) \\ &\geqq \frac{1}{2}P(|U_n(\omega)-a_n|>\varepsilon), \end{aligned}$$

where a_n is a median of $U_n(\omega)$;

$$P(U_n(\omega) \leqq a_n) \geqq \frac{1}{2}$$

$$P(U_n(\omega) \geqq a_n) \geqq \frac{1}{2} .$$

By the bounded convergence theorem

$$\int_{-1}^{1} 1-|\mathrm{E}\{\exp(-it\ U_n(\omega))\}|^2 dt \xrightarrow[n\to\infty]{} 0 ,$$

and so $U_n(\omega)-a_n \xrightarrow[n\to\infty]{} 0$ in probability.

We now continue to give a proof of Theorem 5.1. $(1)' \longrightarrow (2)$.
Let $Y_n(\omega)=\log X_n(\omega)$, $Y^*_{n+1}(\omega)=\log X^*_{n+1}(\omega)$ and $Y(\omega)=\log X(\omega)$.

$$\mathrm{E}\left\{\exp(-it\ Y(\omega))\middle|\bigvee_{j=1}^{n}\mathscr{F}_j\right\} = \prod_{j=1}^{n}\exp(-itY_j(\omega))\times\mathrm{E}\{\exp(-itY^*_{n+1}(\omega))\} .$$

By the martingale covergence theorem

$$\mathrm{E}\left\{\exp(-it\ Y(\omega))\middle|\bigvee_{j=1}^{n}\mathscr{F}_j\right\} \xrightarrow[n\to\infty]{} \exp(-itY(\omega)) \quad a.s.\ \omega ,$$

and so

$$|\mathrm{E}\{\exp(-itY^*_{n+1}(\omega))\}| \xrightarrow[n\to\infty]{} 1 \qquad t\in\mathrm{R} .$$

By Lemma 5.1 there are constants $a_1, a_2, \cdots$ such that

$$Y^*_{n+1}(\omega)-a_n \xrightarrow[n\to\infty]{} 0 \quad \text{in probability.}$$

Then for a sequence $\{c_n\}_{n\geq1}$ defined by $\sum_{i=1}^{n} c_i = -a_n$ $n \geqq 1$

$$\sum_{i=1}^{n} Y_i(\omega) - c_i \longrightarrow Y(\omega) \quad \text{in probability.}$$

Since $Y_1(\omega)$, $Y_2(\omega), \cdots$ are independent, $\sum_{i=1}^{n} Y_i(\omega) - c_i$ converges to $Y(\omega)$ almost surely as $n \longrightarrow \infty$.
Therefore, it follows that for $b_n = \exp c_n$

$$\prod_{n=1}^{\infty} \frac{X_n(\omega)}{b_n}$$

converges almost surely. Proof of (2) $\longrightarrow$ (1)$'$ is evident.

REMARK 1. A quasi-product measure μ equivalent with the infinite direct product probability measure P, if it exists, is unique up to constants by the 0-1 law.

REMARK 2. If $X_1(\omega)$, $X_2(\omega), \cdots$ are independent identically distributed (non-deterministic) then, there are no equivalent quasi-product measures of $(\mu_n)_{n \geq 1}$.

5.2. Hill's condition.

LEMMA 5.2. *Let $Z_1(\omega)$, $Z_2(\omega), \cdots$ be a sequence of positively valued independent random variables of $L^2(\Omega, \mathscr{F}, P)$. Then $\prod_{n=1}^{\infty} Z_n(\omega)$ converges in $L^2(\Omega, P)$-sense if and only if* (1) $0 < \prod_{n=1}^{\infty} \mathrm{E}(Z_n(\omega)^2) < \infty$ *and* (2) $0 < \prod_{n=1}^{\infty} \mathrm{E}(Z_n(\omega)) < \infty$.

PROOF. "If" part. (1) follows from the definition. Since $\left\{ \prod_{k=1}^{n} \frac{Z_k(\omega)}{\mathrm{E}(Z_k^2(\omega))} \,;\, n=1, 2, \cdots \right\}$ is a Cauchy-sequence of $\mathrm{L}^2(\Omega, P)$,

$$\left\| \prod_{k=1}^{n} \frac{Z_k(\omega)}{\mathrm{E}(Z_k^2(\omega))} - \prod_{k=1}^{m} \frac{Z_k(\omega)}{\mathrm{E}(Z_k(\omega)^2)} \right\|_{L^2(\Omega, P)} = \left\| 1 - \prod_{k=n+1}^{m} \frac{Z_k(\omega)}{\mathrm{E}(Z_k^2)} \right\|_{L^2(\Omega, P)}$$

$$= \sqrt{2} \left\{ 1 - \prod_{k=n+1}^{m} \frac{\mathrm{E}(Z_k)}{\mathrm{E}(Z_k^2)} \right\}$$

tends to 0 as $n, m \longrightarrow \infty$. The proof of "only if" part is the same.

Lemma 5.3. (Kakutani [8]). *Two infinite direct product probability measures $P = \prod_{n=1}^{\infty} P_n$ and $Q = \prod_{n=1}^{\infty} Q_n$, $P_n(\Omega_n) = Q_n(\Omega_n) = 1$, are equivalent with each other if and only if*

$$0 < \prod_{n=1}^{\infty} \mathrm{E}_p(\sqrt{Z_n(\omega)}) < \infty$$

where $Z_n(\omega) = \dfrac{dQ_n}{dP_n}(\omega)$.

LEMMA 5.4. (Kolmogorov [7]). *Let $Y_1(\omega)$, $Y_2(\omega)$,$\cdots$ be a sequence of real independent random variables and put*

$$Y'_n(\omega) = \begin{cases} Y_n(\omega) & \text{if} \quad |Y_n(\omega)| \leqq 1 \\ 0 & \text{if} \quad \text{othewise} \end{cases} .$$

Then $\sum_{n=1}^{\infty} Y_n(\omega)$ converges almost surely if and only if the three series $\sum_{n=1}^{\infty} P(|Y_n(\omega)|>1)$, $\sum_{n=1}^{\infty} \mathrm{E}(Y_n'(\omega))$ and $\sum_{n=1}^{\infty} \mathrm{Var}\ (Y_n')$ converge.

PROPOSITION 5.5. *Let $X_1(\omega)$, $X_2(\omega)$,$\cdots$ be a sequence of positively valued independent random variables. Then $\prod_{n=1}^{\infty} X_n(\omega)$ converges almost surely if and only if there are $X_n(\omega)$-measurable sets A_n $n\geqq 1$ such that the three infinite products $\prod_{n=1}^{\infty} P(A_n)$, $\prod_{u=1}^{\infty} \mathrm{E}(x_n: A_n)$ and $\prod_{n=1}^{\infty} \mathrm{E}(\sqrt{x_n}: A_n)$ converge.*

PROOF. "Only if" part. Let $\Omega^* = \bigcap_{n=1}^{\infty} A_n$ and $(\Omega^*, \mathscr{F} \cap \Omega^*, P^*)$ be the restriction of $(\Omega, \mathscr{F}, P)$ onto Ω^*: $P^*(A) = \frac{P(A)}{P(\Omega^*)}$ $A \in \mathscr{F} \cap \Omega^*$. We define a sequence of independent random variables $X_n^*(\omega)$ on $(\Omega^*, \mathscr{F} \cap \Omega^*, P^*)$:

$$X_n^*(\omega) = \frac{X_n(\omega) P(A_n)}{\mathrm{E}(X_n: A_n)} \quad \omega \in \Omega^* .$$

By the condition, $\prod_{n=1}^{\infty} \mathrm{E}_{p^*}(\sqrt{X_n^*})^2 = \prod_{n=1}^{\infty} \frac{\mathrm{E}(\sqrt{X_n(\omega)}\,;\,A_n)^2}{P(A_n)\mathrm{E}(X_n(\omega)\,;\,A_n)}$ coverges and $\mathrm{E}_{p^*}(X_n^*(\omega))=1$. By Lemma 5.2, $\prod_{n=1}^{\infty} \sqrt{X_n^*(\omega)}$ converges in $L^2(\Omega^*, P^*)$-sence and converges almost surely, because of independence. Since $\prod_{n=1}^{\infty} \frac{P(A_n)}{\mathrm{E}(X_n(\omega)\,;\,A_n)}$ converges, $\prod_{n=1}^{\infty} X_n(\omega)$ converges $a.s.\ \omega \in \Omega^*$, from 0-1 law, $\prod_{n=1}^{\infty} X_n(\omega)$ converges almost surely on Ω.

"If" part. Let $Y_n(\omega) = \log X_n(\omega)$ and $A_n = \{\omega \mid |Y_n(\omega)| \leqq 1\}$. Then we have

$$\begin{aligned}
\mathrm{Var}(Y_n') &\geqq \mathrm{E}\left\{\left(\exp \frac{Y_n(\omega) - \mathrm{E}(Y_n(\omega)\,;\,A_n)}{2} - 1\right)^2 ;\ A_n\right\} \\
&= \mathrm{E}\left\{\left(\exp\left(-\frac{Y_n(\omega)}{2}\right) - \exp\left(-\frac{\mathrm{E}(Y_n:\ A_n)}{2}\right)\right)^2 \exp Y_n(\omega)\,;\ A_n\right\} \\
&\geqq \mathrm{E}\left\{\left(\exp\left(-\frac{Y_n(\omega)}{2}\right) - \frac{\mathrm{E}\left(\exp\frac{Y_n}{2}\,;\ A_n\right)}{\mathrm{E}(\exp Y_n:\ A_n)}\right)^2 \exp Y_n(\omega)\,;\ A_n\right\} \\
&= P(A_n) - \frac{\mathrm{E}\left\{\exp \frac{Y_n(\omega)}{2}\,;\ A_n\right\}^2}{\mathrm{E}\{\exp Y_n(\omega)\,;\ A_n\}}
\end{aligned}$$

$$=P(A_n)-\frac{\mathrm{E}\{\sqrt{X_n(\omega)}\,;\,A_n\}^2}{\mathrm{E}\{X_n(\omega)\,;\,A_n\}}$$

, because $u^2\geqq\left(\exp\frac{u}{2}-1\right)^2$ $u\in\mathrm{R}$. Therefore by Lemma 5.4

$$0<\prod_{n=1}^{\infty}\frac{\mathrm{E}\{\sqrt{X_n}:A_n\}^2}{\mathrm{E}\{X_n:A_n\}}<\infty .$$

Since for a large number n $|\mathrm{E}(Y_n;A_n)|<\frac{1}{2}$, it holds that

$$\begin{aligned}&|\mathrm{E}\{1-(\exp(-\mathrm{E}(Y_n;A_n)))X_n(\omega)\,;\,A_n\}|\\&\quad=|\mathrm{E}\{1-\exp(Y_n(\omega)-\mathrm{E}(Y_n;A_n))\,;\,A_n\}|\\&\quad\leqq 2\mathrm{Var}(Y_n')\end{aligned}$$

, because $|\exp u-1-u|<2|u|^2$ if $|u|\leqq\frac{3}{2}$. Therefore by Lemma 5.4

$$0<\prod_{n=1}^{\infty}\exp(-\mathrm{E}(Y_n;A_n))\times\mathrm{E}(Y_n;A_n)<\infty .$$

By Lemma 5.4, the ratio

$$\frac{\prod_{k=1}^{n}\mathrm{E}(X_k;A_k)}{\prod_{k=1}^{n}\exp(-\mathrm{E}(Y_k;A_k))\times\mathrm{E}(X_k:A_k)}=\exp\left(\sum_{k=1}^{n}\mathrm{E}(Y_k;A_k)\right)$$

coverges as $n\longrightarrow\infty$. Thus we have

$$0<\prod_{n=1}^{\infty}\mathrm{E}(X_n:A_n)<\infty \quad\text{and}\quad 0<\prod_{n=1}^{\infty}\mathrm{E}(\sqrt{X_n};A_n)<\infty .$$

THEOREM 5.2 (Hill [6]). *There exists an equivalent quasi-product measure if and only if there exists, for each* n, $A_n\in\mathscr{F}_n$ *sucht hat* (1) $0<\mu_n(A_n)<\infty$ (2) $\prod_{n=1}^{\infty}P_n(A_n)>0$ and (3) $P'=\prod_{n=1}^{\infty}P_n'$ and $\mu'=\prod_{n=1}^{\infty}\mu_n'$ *are equivalent with each other, where* P_n' *are* μ'_n *are the restriction of* P_n *and* μ_n *onto* A_n *respectively.*

PROOF. It follows from Theorem 5.1, Proposition 5.5 and Lemma 5.3.

§ 6. Classification of IPT transformation groups I.

In this section and the following section we are going to discuss a class of ergodic groups of non-singular transformations for which interesting criterions are obtained.

6.1. IPT transformation groups.

Let $\{(\Omega_n,\mathscr{F}_n,P_n)\,|\,n=1,2,\cdots\}$ be a sepuence of Lebesgue measure spaces with $P_n(\Omega_n)=1$ $n=1,2,\cdots$. For each n, let G_n be an ergodic countable group of non-singular transformations of $(\Omega_n,\mathscr{F}_n,P_n)$. Assume that, for each n, there exists a finite or σ-finite measure μ_n defined on $\mathscr{F}_n$, which is equivalent with P_n and invariant under G_n. We denote by $X_n(\omega)$ the Radon-Nikodym density $\frac{d\mu_n}{dP_n}(\omega_n)$.

Let $(\Omega, \mathscr{F}, P)$ be the infinite direct product probabilty measure space $\prod_{n=1}^{\infty}(\Omega_n, \mathscr{F}_n, P_n)$. We may consider that G_n, $\mathscr{F}_n$ and X_n are defined on the infinite direct product probability measure space $(\Omega, \mathscr{F}, P)$. The group $G=\sum_{n=1}^{\infty}\oplus G_n$ generated by $\bigcup_{n=1}^{\infty} G_n$ is said to be an ITP transformation group of $(\Omega, \mathscr{F}, P)$.

PROPOSITION 6.1. *An ITP transformation group* $G=\sum_{n=1}^{\infty}\oplus G_n$ *is ergodic and non-singular.*

PROOF. For $g=g_1g_2\cdots g_n\in G$ with $g_i\in G_i$

$$\frac{dPg}{dP}(\omega)=\frac{dP_1g_1\times P_2g_2\times\cdots\times P_ng_n}{dP_1\times P_2\times\cdots\times P_n}(\omega)$$
$$=\prod_{i=1}^{n}\frac{X_i(\omega)}{X_i(g_i\omega)}$$

If a measurable function $f(\omega)$ is invariant under G, then for each $n\geq 1$ $f(\omega)$ is $G_1\times\cdots\times G_n$-invariant. Since $G_1\times\cdots\times G_n$ is ergodic, $f(\omega)$ is $\bigvee_{i=n+1}^{\infty}\mathscr{F}_i$-measurable. Since n is any, $f(\omega)$ is tail σ-field measurable and is constant *a.s.* ω, because of the 0-1 law.

PROPOSITION 6.2. *Let* $G=\sum_{n=1}^{\infty}\oplus G_n$ *be an IPT transformation group of* $(\Omega, \mathscr{F}, P)$ *and let* μ *be a* σ*-finite measure equivalent with* P. *Then* μ *is invariant under* G *if and only if* μ *is a quasi-product measure of* $(\mu_n)_{n\geq 1}$.

PROOF. Let $X(\omega)=\frac{d\mu}{dP}(\omega)$. From Lemma 4.2 μ is invariant under G if and only if for $g=g_1g_2\cdots g_n\in G$ with $g_i\in G_i$

$$\frac{\prod_{i=1}^{n}X_i(g_i\omega)}{X(g\omega)}=\frac{\prod_{i=1}^{n}X_i(\omega)}{X(\omega)}\quad a.s.\ \omega.$$

Since $G_1\times G_2\times\cdots G_n$ is ergodic, $\frac{\prod_{i=1}^{n}X_i(\omega)}{X(\omega)}$ is $\bigvee_{j=n+1}^{\infty}\mathscr{F}_j$-measurable.

THEOREM 6.1. *Let* $G=\sum_{n=1}^{\infty}\oplus G_n$ *be an IPT transformation group of* $(\Omega, \mathscr{F}, P)$. *Then* G *is of semi-finite type if and only if there are positive constants* b_1,

$b_2, \cdots$ *such that*

$$\prod_{n=1}^{\infty} \frac{X_n(\omega)}{b_n}$$

converges almost surely.

PROOF is from Proposition 6.2 and Theorem 5.1.

6.2. Examples of groups of non-singular transformations of type III.

Let $\Omega_n=\{0,1,\cdots\cdots, k_n-1\}$, $g_n i=i+1 \pmod{k_n}$, $G_n=\{g_n^j;\ j=0,1,\cdots\cdots, k_n-1\}$, P_n be a probability measure with $P_n(i)>0$ and μ_n be a uniform measure on Ω_n. We note that in this case an IPT transformation group $G=\sum_{n=1}^{\infty}\oplus G_n$ is weakly equivalent with a cyclic group generated by a single ergodic non-singular transformation of the usual Lebesgue measure space $([0,1], \mathscr{B}[0,1], \lambda)$ ([6]).

EXAMPLE 1 (*Ornstein* [13]) $\Omega_n=\{0,1,\cdots, k_n-1\}$ $k_n\geqq 3$ $P_n(i)=\frac{1}{2}$ if $i=0$, $=\frac{1}{2(k_n-1)}$ if $i\geqq 1$.

EXAMPLE 2 (*Brunel* [3]). $\Omega_n=\{0,1,2\}$ $P_n(i)=\frac{1}{4}$ if $i=0$ or $2,=\frac{1}{2}$ if $i=1$.

EXAMPLE 3 (*Arnold* [2]). $\Omega_n=\{0,1\}$, $P_n(0)=\frac{1}{1+\lambda}$ $P_n(1)=\frac{\lambda}{1+\lambda}$ $0<\lambda<1$.

Examples 2, 3 are cases of identical distribution and then they are of type III by Theorem 6.1 and Remark 2 of Theorem 5.1. Next suppose that Example 1 has an equivalent invariant measure. Then by Theorem 6.1 for some positive constants $b_1, b_2, \cdots\cdots$, $\frac{X_n(\omega)}{b_n}$ must converge to 1 in probability. For any $\varepsilon>0$

$$P\left(\left|\frac{X_n(\omega)}{b_n}-1\right|>\varepsilon\right)=\frac{1}{2}\chi_{(-\infty,1-\varepsilon)\cup(1+\varepsilon,\infty)}\left(\frac{2}{b_n}\right)$$
$$+\frac{1}{2}\chi_{(-\infty,1-\varepsilon)\cup(1+\varepsilon,\infty)}\left(\frac{2(k-1)}{b_n}\right)\xrightarrow[n\to\infty]{}0$$

, where χ denotes an indicator function of a set. Then both $\frac{2}{b_n}$ and $\frac{2(k_n-1)}{b_n}$ converge to 1, which contradicts to the condition $k_n\geqq 3$. Therefore Example 1 is also of type III.

§7. Classification of IPT transformation groups II.

7.1. $T(G)$ of IPT transformation groups $G=\sum_{n=1}^{\infty}\oplus G_n$.

THEOREM 7.1. *Let* $G=\sum_{n=1}^{\infty}\oplus G_n$ *be an IPT transformation group of* $(\Omega,$

$\mathscr{F}$, P). Then a real number $t \in T(G)$ if and only if there are real constants $a_{t,1}$, $a_{t,2}$, ……, such that

$$\exp\left(i \sum_{j=1}^{n} (t \log X_j(\omega) - a_{t,j}\right)$$

converges almost surely as $n \longrightarrow \infty$.

PROOF. Let a non-singular transformation $g = g_1 g_2 \cdots\cdots g_n \in G (g_i \in G_i)$ satisfy $\exp\left(i(\xi_t(g\omega) - \xi_t(\omega))\right) = \exp\left(it \log \frac{dPg}{dP}(\omega)\right)$.
Then

$$\begin{aligned} &\exp(i(\xi_t(g_1\omega_1, \cdots\cdots, g_n\omega_n, \omega_{n+1}, \cdots) + t \log X_1(g_1\omega_1) \cdots\cdots X_n(g_n\omega_n))) \\ &\quad = \exp(i(\xi_t(\omega_1, \cdots\cdots, \omega_n, \omega_{n+1}, \cdots\cdots) + t \log X_1(\omega_1) \cdots X_n(\omega_n))) \end{aligned}$$

Since $G_1 \times \cdots \times G_n$ is ergodic there exists a $\bigvee_{j=n+1}^{\infty} \mathscr{F}_j$-measurable function $\xi_{t,n+1}^*(\omega)$ such that

$$\exp(i(\xi_t(\omega) + t \log X_1(\omega) \cdots\cdots X_n(\omega))) = \exp(i\xi_{t,n+1}^*(\omega)) .$$

Hence we have that $t \in T(G)$ if and only if there exist a $\mathscr{F}$-measurable function $\xi_t(\omega)$ and a $\bigvee_{j=n+1}^{\infty} \mathscr{F}_j$-measurable function $\xi_{t,n+1}^*(\omega)$ for each $n=1, 2, ,\cdots$ such that

$$\exp(i\xi_t(\omega)) = \exp\left(-it \sum_{j=1}^{n} \log X_j(\omega)\right) \exp\left(i\xi_{t,n+1}^*(\omega)\right) .$$

By this

$$\mathrm{E}\left\{\exp(i\xi_t(\omega)) \,\middle|\, \bigvee_{j=1}^{n} \mathscr{F}_j\right\} = \exp\left(-it \sum_{j=1}^{n} \log X_j(\omega)\right) \times \mathrm{E}\{\ \exp i\xi_{t,n+1}^*\} .$$

Let $c_{t,n+1}$ be the angle in polar coordinate of $\mathrm{E}\{\exp i\xi_{t,n+1}\}$. We have

$$\begin{aligned} &\lim_{n\to\infty} \exp\left(i\left(-t \sum_{j=1}^{n} \log X_j(\omega) + c_{t,n+1}\right)\right) \\ &\qquad = \lim_{n\to\infty} \frac{\mathrm{E}\left\{\exp i\xi_t(\omega) \,\middle|\, \bigvee_{j=1}^{n} \mathscr{F}_j\right\}}{\left|\mathrm{E}\left\{\exp i\xi_t(\omega) \,\middle|\, \bigvee_{j=1}^{n} \mathscr{F}_j\right\}\right|} \\ &\qquad = \exp i\xi_t(\omega) \quad a.\ s.\ \omega, \end{aligned}$$

by the martingale convergence theorem, and

$$\lim_{n\to\infty} \exp\left(-i \sum_{j=1}^{n} (t \log X_j(\omega) - a_{t,j})\right) = \exp(i\xi_t(\omega)) \quad a.\ s.\ \omega ,$$

where $a_{t,j} = c_{t,j+1} - c_{t,j}$ $j=1, 2, \cdots\cdots$ and $c_{t,1}=0$.

Conversely the last equation implies

$$\exp(i\xi_t(\omega)) = \exp\Big(-it\sum_{j=1}^{n}\log X_j(\omega)\Big)\times\exp(i\xi^*_{t,n+1}(\omega))$$

where

$$\exp(i\xi^*_{t,n+1}(\omega)) = \exp\Big(i\sum_{j=1}^{n}a_{t,j}\Big)\lim_{m\to\infty}\exp\Big(-i\sum_{j=n+1}^{m}(t\log X_j(\omega)-a_{t,j})\Big)$$

$$a.\,s.\,\omega\,,\quad n=1,2,\cdots\cdots.$$

REMARK. Let $G=\sum_{n=1}^{\infty}\oplus G_n$ be an IPT transformation group. If a real number $t\in T(G)$ then there exist real constants $a_{t,1}$, $a_{t,2},\cdots\cdots$ such that

$$\lim_{n\to\infty}\exp(i(t\log X_n(\omega)+a_{t,n})) = 1\quad a.\,s.\,\omega\,.$$

7.2. A simple proof of the converse of (1) of Theorem 4.1.

PROPOSITION 7.1. *Let $G=\sum_{n=1}^{\infty}\oplus G_n$ be an IPT transformation group of $(\Omega,\mathscr{F}, P)$. If $T(G)=R$ then G is of semifinite type.*

PROOF. Since by Theorem 7.1 there are constants $a_{t,1}$, $a_{t,2},\cdots$ such that

$$\lim_{n,m\to\infty}\exp\Big(-i\sum_{j=n+1}^{m}(t\log X_j(\omega)-a_{t,j})\Big)=1\quad a.\,s.\,\omega\,,$$

we have

$$\lim_{n,m\to\infty}\Big|\mathrm{E}\Big\{\exp\Big(-it\sum_{j=n+1}^{m}\log X_j(\omega)\Big)\Big\}\Big|=1\quad t\in\mathrm{R}\,.$$

By Lemma 5.1 there are real constants $c_{n,m}$ $n,m=1,2,\cdots\cdots$ such that

$$\sum_{j=n+1}^{m}(\log X_j(\omega)-c_{n,m})$$

converges to 0 in probability as $n,m\longrightarrow\infty$. And then

$$\lim_{n,m\to\infty}\exp\Big(i\Big(\sum_{j=n+1}^{m}a_{t,j}-tc_{n,m}\Big)\Big)=1\quad t\in\mathrm{R}\,.$$

Therefore

$$\lim_{n,m\to\infty}\exp\Big(-it\sum_{j=n+1}^{m}(\log X_j(\omega)-c_{n,m})\Big)=1\quad a.\,s.\,\omega\,,$$

and

$$\lim_{n,m\to\infty}\sum_{j=n+1}^{m}(\log X_j(\omega))-c_{n,m}=0\quad a.\,s.\,\omega\,,$$

which means that putting $b_j=X_j(\omega_0)$ for $\omega_0\in\Omega j=1,2,\cdots\cdots$, the infinite product

$$\prod_{j=1}^{\infty} \frac{X_j(\omega)}{b_j}$$

converges almost surely. By Theorem 6.1 G is of semi-finite type.

7.3. Examples.

Each non-singular transformation group of type III_λ $0<\lambda<1$, III_1 and III_0 is given by the infinite product method as follows,

III_λ and III_1. Let $\Omega_n=\{0,1,\cdots\cdots, d\text{-}1\}$ $g_n j=j+1 \pmod d$ $G_n=\{g_n^k;\ 0\leqq k<d\}$, $P_n(j)=P_m(j)>0$ and μ_n be a uniform measure of Ω_n. Then IPT transformation group $G=\sum_{n=1}^{\infty}\oplus G_n$ is of type III_λ if $\log X_n(\omega)$ is lattice distributed with span $-\log\lambda$ $0<\lambda<1$ and G is of type III_1 if it is non-lattice distributed.

III_0. Let $\Omega_n=\{0,1\}$, $g_n(j)=j+1 \pmod 2$, $G_n=\{g_n^k;\ k=0,1,\}$ and μ_n be a uniform measure of Ω_n. For $0<\lambda<1$, let N_0, $N_1,\cdots\cdots$ be an increasing sequence of positive integers such that

$$\left(\frac{1}{1+\lambda^{2^k}}\right)^{N_k-N_{k-1}}<c<1 \quad k=0,1,\cdots\cdots.$$

Put $P_n(0)=\dfrac{1}{1+\lambda^{2^k}}$ and $P_n(1)=\dfrac{\lambda^{2^k}}{1+\lambda^{2^k}}$ if $N_{k-1}<n\leqq N_k$ $k=0,1,\cdots\cdots$. Then

$$X_n(0)=\frac{d\mu_n}{dP_n}(0)=1+\lambda^{2^k}$$

with probability $\dfrac{1}{1+\lambda^{2^k}}$, and

$$X_n(1)=\frac{1+\lambda^{2^k}}{\lambda^{2^k}}$$

with probability $\dfrac{\lambda^{2^k}}{1+\lambda^{2^k}}$ if $N_{k-1}<n\leqq N_k$ $k=0,1,\cdots\cdots$. We show

$$T(G)=\frac{2\pi}{\log\lambda}\Gamma$$

, where $\Gamma=\left\{\dfrac{l}{2^m};\ m=0,1,\cdots\cdots,\ l=0,\pm1,\pm2,\cdots\cdots\right\}$.

Putting

$$t=\frac{2\pi}{\log\lambda}\frac{l}{2^m}$$

and

$$a_n=\frac{2\pi}{\log\lambda}\frac{l}{2^m}\log(1+\lambda^{2^k}) \quad N_{k-1}<n\leqq N_k\ ,$$

we have

$$\exp\Big(-i\sum_{j=1}^{n}(t\log X_j(\omega)-a_j)\Big)$$

$$=\exp\Big(-i\Big(\sum_{k=1}^{k_0-1}\sum_{\substack{N_{k-1}<j\le N_k\\ \omega_j=1}}\frac{2\pi}{\log\lambda}\frac{l}{2^m}\log\lambda^{2^k}+\sum_{\substack{N_{k_0-1}<j\le n\\ \omega_j=1}}\frac{2\pi}{\log\lambda}\frac{l}{2^m}\log\lambda^{2^k}\Big)\Big)$$

$$=\exp\Big(-i\Big(\sum_{k=1}^{m-1}\sum_{\substack{N_{k-1}<j\le N_k\\ \omega_j=1}}2\pi\cdot\frac{l}{2^{m-k}}\Big)$$

, $N_{k_0-1}<n\leqq N_{k_0}$ $m\leqq k_0$. By Theorem 7.1 $t\in T(G)$.

Assume that a real number t is not in $\dfrac{2\pi}{\log\lambda}\Gamma$.

For the 2-adic expansion

$$\frac{t\log\lambda}{2\pi}=\sum_{m=0}^{\infty}\frac{p_m}{2_m}\quad p_m\in\{0,1\}\quad m=1,2,\cdots\cdots,$$

let

$$\gamma_k=\sum_{m=k+1}^{\infty}\frac{p_m}{2^{m-k}}\quad k=1,2,\cdots\cdots.$$

Then $\exp(2\pi i\gamma_k)$ does not converge as $k\longrightarrow\infty$. Hence there exists a positive number ε and an increasing sub-sequence k_n $n=1,2,\cdots\cdots$ such that

$$|\exp(2\pi i\gamma_{k_n})-1|>\varepsilon\quad n=1,2,\cdots\cdots.$$

Putting

$$b_j=t\log(1+\lambda^{2^{k_n}})\quad N_{k-1}<j\leqq N_k,$$

we have

$$\exp(-i(t\log X_j(\omega_j)-b_j))=\begin{cases}1 & \text{if}\quad \omega_j=0\\ \exp(2\pi i\gamma_k) & \text{if}\quad \omega_j=1\end{cases}$$

, $N_{k-1}<j\leqq N_k$ $k=1,2,\cdots\cdots$. Since

$$P\Big(\bigcap_{n=1}^{\infty}\bigcap_{j=N_{k_{n-1}}+1}^{N_{k_n}}\{\omega;\ \omega_j=1\}\Big)$$

$$=\prod_{n=1}^{\infty}\Big(\frac{1}{1+\lambda^{2^{k_n}}}\Big)^{N_{k_n}-N_{k_{n-1}}}$$

$$=0,$$

there exist for almost sure ω infinitely often numbers n with $\omega_j=1$ for some $N_{k_{n-1}}<j\leqq N_{k_n}$, because of the Borel-Cantelli lemma. Therefore $\exp(-i(t\log X_j(\omega)-b_j))$ does not converge as $j\longrightarrow\infty$ almost surely. This implies that t is not in $T(G)$.

References

[1] Araki, H. and Woods, E. J., A classification of factors, Publ. RIMS Kyoto Univ. 4 (1968), 51-130.

[2] Arnold, L. K. On σ-finite invariant measures, Z. Wahrscheinlichkeitstheorie verw. Geb. 9 (1968), 85-97.

[3] Brunel, A., Sur les measures invariantes, Z. Wahrscheinlichkeitstheorie verw. Gebt. 5 (1966), 300-303.

[4] Chacon, R. V., A class of linear transformations, Proc. Amer. Math. Soc., 15 (1964), 560-564.

[5] Connes, A., Une classification de facteurs de type III. Ann. Scient. Ec. Norm. Sup., 6 (1973), 133-252.

[6] Hill, D., σ-finite invariant measures on infinite product spaces, Trans. Amer. Math. Soc., 153 (1971), 347-370.

[7] Ito, K., The theory of probability, Iwanami-shoten (1963). (In Japanese).

[8] Kakutani, S., On equivalence of infinite product measures, Ann. of Math., 49 (1948), 214-224.

[9] Krieger, W., On non-singular transformations of a measure space I, II, Z. Wahrscheinlichkeitstheorie verw. Geb. 11 (1968) 83-97, 98-119.

[10] ——, On the Araki-Woods' asymptotic ratio set and non-singular transformations of a measure space, Lecture note in Math. 160 (1970), 158-177.

[11] ——, On a class of hyperfinite factors that arise from null-recurrent Markov chains, J. Funct. Anal. 7 (1971), 27-42.

[12] Moore, C. C., Invariant measures on product spaces, Fifth Berkeley Symposium on Math. Stat. and Prob. (1967), 447-459.

[13] Ornstein, D. S., On invariant measure, Bull. Amer. Math. Soc., 66 (1960), 297-300.

[14] Rohlin, V. A., On the fundamental ideas of a measure theory, Math. Sb. 25 (67) (1949), 107-150 = Amer. Math. Soc. Transl. 71 (1952).

[15] Takenouchi, O., On type classification of factors constructed as infinite tensor products, Publ. RIMS Kyoto Univ., 4 (1968), 467-482.

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