On the existence of the continuous solution of the problem u_t＝(u^m)_<xx>＋f(u)u

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On the existence of the continuous solution of the problem $u_t=(u^m)_{xx}+f(u)u$

By

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Introduction

With respect to the Cauchy Problem for the equation of the parabolic type:

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|u|^{m-1}\frac{\partial u}{\partial x_i}\right)+f(x,t,u) \qquad (m>1) \tag{1}$$

J. A. Dubinskii [2] proved the existence of weak solutions. In the case of $n=1$, especially, O. A. Oleĭnik, A. S. Kalashnikov and Chzhou Yui-lin' [4] proved the existence of the continuous weak solution for the equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u^m}{\partial x^2} \qquad (m>1) \tag{2}$$

Moreover, it was established that the solution of the above is Hölder continuous, by D. G. Aronson [1] and S. N. Krujkov [3]. In this paper, we will show the existence of the ***continuous*** weak solution of the Cauchy Problem:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u^m}{\partial x^2}+f(u)u, \quad -\infty<x<+\infty, \qquad 0<t<T, \ (m>1)\ , \tag{3}$$

$$u(x,0)=\varphi(x)\ . \tag{4}$$

It is assumed that functions f and φ satisfy the conditions described in the following § 1.

§ 1. The Theorem

In the Cauchy Problem (3), (4), $\varphi(x)$ is a given function bounded ($\sup \varphi(x)=L_1$) and nonnegative in $(-\infty,+\infty)$, for which $\varphi(x)^m$ is Lipshitz continuous, i.e.,

$$|\varphi^m(x)-\varphi^m(y)|\leq K|x-y|, \tag{1.1}$$

and

$f=f(\lambda)$ is a given function belonging to the class C^5 in $[0,\infty)$ which satisfies the following conditions:

$$0 \leq f(\lambda) \leq 1,\ |f'(\lambda)| \leq 1 \ \text{in}[0,\infty)\ , \tag{1.2}$$

$$f(L) = 0,\ \text{ here, }\ L = (2K+L_1^m)^{\frac{1}{m}}, \tag{1.3}$$

(for example, $f(\lambda)=(1/2)\sin^2(\lambda-L)$ and $f(\lambda)=(\lambda-L)^2/(1+(\lambda-L)^2)$.

Then, there exists a continuous weak solution $u(x,t)$ in $S=(-\infty,+\infty)\times[0,T]$. Namely, $u(x,t)$ is a nonnegative and bounded continuous function, for which the generalized derivative $(u^m)_x$ exists and is bounded, and which satisfies the identity

$$\iint_S \left\{ u\frac{\partial\rho}{\partial t} - \frac{\partial u^m}{\partial x}\frac{\partial\rho}{\partial x} + f(u)u\rho \right\} dxdt + \int_{-\infty}^{+\infty} \varphi(x)\rho(x,0)dx = 0 \tag{1.4}$$

for all $\rho(x,t)\in C^\infty(S)$ that vanish for sufficiently large $|x|$ and at $t=T$.

§ 2. The Lemma and its Proof

If we set $v=u^m$ in (3), it follows that

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{m} v^{\frac{1-m}{m}} \frac{\partial v}{\partial t} - f(v^{\frac{1}{m}})\, v^{\frac{1}{m}}\ . \tag{2.1}$$

Now, in $R=[a,b]\times[0,T]$ we consider the problem (2.1) under the conditions:

$$v(x,0) = \psi_0(x),\ \ v(a,t) = \psi_1(t),\ \ v(b,t) = \psi_2(t) \tag{2.2}$$

LEMMA. In the initial-boundary value problem (2.1) (2.2), if $\psi_0\in C^4$, $\psi_i\in C^3$ $(i=1,2)$ and the compatibility conditions are satisfied, i.e.,

$$\left\{\begin{aligned} &\psi_0(a) = \psi_1(0),\ \ \psi_0(b) = \psi_2(0) \\ &\frac{d^2\psi_0(a)}{dx^2} = \frac{1}{m}\psi_1(0)^{\frac{1-m}{m}}\frac{d\psi_1(0)}{dt} - f(\psi_1(0)^{\frac{1}{m}})\,\psi_1(0)^{\frac{1}{m}} \\ &\frac{d^2\psi_0(b)}{dx^2} = \frac{1}{m}\psi_2(0)^{\frac{1-m}{m}}\frac{d\psi_2(0)}{dt} - f(\psi_2(0)^{\frac{1}{m}})\,\psi_2(0)^{\frac{1}{m}} \end{aligned}\right. \tag{2.3}$$

and if $f=f(\lambda)\in C^5$ with the condition (1.2), and moreover

$$0 < \tilde{l} \leq \psi_i \leq \tilde{L}\ \ (i=0,1,2)\ , \tag{2.4}$$

then there exists a classical solution $v(x,t)$ of the problem and it holds the estimate $\tilde{l}\leq v(x,t)\leq \tilde{L}e^{mT}$ for any $(x,t)\in R$.

Proof. Here we use the method of O. A. Oleĭnik, A. S. Kalashnikov and Chzhou Yui-lin' [4]. Considering the "truncating function" we can construct the functions $A(\lambda)$ and $F(\lambda)$ belonging to the class C^5 in $(-\infty, +\infty)$ which satisfy the following relations.

$$A(\lambda)=\begin{cases}\dfrac{1}{m}\left(\dfrac{\tilde{l}}{4}\right)^{\frac{1-m}{m}} & \left(\lambda\leqq\dfrac{\tilde{l}}{2},\ 2\tilde{L}e^{mT}\leqq\lambda\right)\\ \dfrac{1}{m}\lambda^{\frac{1-m}{m}} & (\tilde{l}\leqq\lambda\leqq\tilde{L}e^{mT})\end{cases}$$

$$\frac{1}{m}\lambda^{\frac{1-m}{m}}\leqq A(\lambda)\leqq\frac{1}{m}\left(\frac{\tilde{l}}{4}\right)^{\frac{1-m}{m}}\quad\left(\frac{\tilde{l}}{2}\leqq\lambda\leqq\tilde{l},\ \tilde{L}e^{mT}\leqq\lambda\leqq2\tilde{L}e^{mT}\right),$$

$$F(\lambda)=\begin{cases}\left(\dfrac{\tilde{l}}{4}\right)^{\frac{1-m}{m}} & \left(\lambda\leqq\dfrac{\tilde{l}}{4},\ 2\tilde{L}e^{mT}+1\leqq\lambda\right)\\ mA(\lambda)f\left(\lambda^{\frac{1}{m}}\right) & \left(\dfrac{\tilde{l}}{2}\leqq\lambda\leqq2\tilde{L}e^{mT}\right)\end{cases}$$

$$mA(\lambda)f\left(\lambda^{\frac{1}{m}}\right)\leqq F(\lambda)\leqq\left(\frac{\tilde{l}}{4}\right)^{\frac{1-m}{m}}\quad\left(\frac{\tilde{l}}{4}\leqq\lambda\leqq\frac{\tilde{l}}{2},\ 2\tilde{L}e^{mT}\leqq\lambda\leqq2\tilde{L}e^{mT}+1\right).$$

Now, we consider the problem

$$\frac{\partial^2 v}{\partial x^2}=A(v)\frac{\partial v}{\partial t}-F(v)v\quad\text{in } R. \tag{2.5}$$

Since it follows that $A(\lambda)\geqq\text{const.}>0$ and $(-F(\lambda)\lambda)'\geqq\text{const.}$, we can apply the main theorem of O. A. Oleĭnik and T. D. Venttsel' [5] under the assumption of the lemma.

Thus, we have a classical solution $v(x,t)$ of the problem (2.5), (2.2). Setting $v=we^{\alpha t}$ $(\alpha>m)$ it follows that

$$\frac{\partial^2 w}{\partial x^2}=A(v)\frac{\partial w}{\partial t}+\{A(v)\alpha-F(v)\}w\quad\text{in } R. \tag{2.6}$$

Here, it holds that $w\leqq\tilde{L}$ on the boundary $\Gamma(t=0, x=a, x=b)$.

If the function w had a positive maximal value inside the $R-\Gamma$, it would follow that $\frac{\partial w}{\partial t}\geqq0$ and $\frac{\partial^2 w}{\partial x^2}\leqq0$ at the point. Since $A(v)\alpha-F(v)>0$, the left-hand side of (2.6) would be nonpositive and the right-hand side of (2.6) would be positive. It is inconsistent.
Hence we have

$$w(x,t)\leqq\tilde{L},\ \text{or}\ v(x,t)\leqq\tilde{L}e^{mT}\quad\text{in } R.$$

Setting $v=\tilde{l}+we^{\beta t}(\beta>m)$, it follows that

(2.7) $$\frac{\partial^2 w}{\partial x^2}=A(v)\frac{\partial w}{\partial t}+(A(v)\beta-F(v))w-F(v)\tilde{l}e^{-\beta t}.$$

Here, it holds that $0\leqq w$ on the boundary Γ.

If the function w had a negative minimal value inside the R-Γ, relations $\frac{\partial w}{\partial t}\leqq 0$ and $\frac{\partial^2 w}{\partial x^2}\geqq 0$ would be satisfied at the point. In (2.7), the left-hand side would be nonnegative and the right-hand side would be negative. It is inconsistent. Hence we have

$$w(x,t)\geqq 0, \text{ or } v(x,t)\geqq \tilde{l} \text{ in } R.$$

Thus, with respect to the problem (2.5), (2.2), we have a classical solution satisfying the estimate

(2.8) $$\tilde{l}\leqq v\leqq \tilde{L}e^{mT} \text{ in } R.$$

Next, we return to the equation (2.1). Since the function $v(x,t)$ of the above satisfies the relation (2.8), it holds

$$A(v)=\frac{1}{m}v^{\frac{1-m}{m}},\quad F(v)=v^{\frac{1-m}{m}}f(v^{\frac{1}{m}}).$$

Consequently, the function v satisfies the equation (2.1), and the proof of the lemma is complete.

§ 3. The Proof of the Theorem

If we set $v_0(x)=\varphi(x)^m$ and

$$v_{0n}(x)=\int_{|x-\xi|<2^{-n}}\rho_{2^{-n}}(x-\xi)\varphi(\xi)^m d\xi+2^{-n+2}K,\quad (-\infty<x<+\infty).$$

(here, $\rho_\varepsilon(x)$ is a Friedrichs mollifier), we have the following relations.

$$v_{0n+1}(x)\leqq v_{0n}(x),\quad v_{0n}(x)\in C^\infty,\quad 0<v_{0n}(x)\leqq L^m,$$

$$\left|\frac{dv_{0n}(x)}{dx}\right|\leqq K \quad \text{for any } x\in(-\infty,+\infty),\quad v_{0n}(x)\rightrightarrows v_0(x)\,(n\to\infty)$$

(uniformly convergence on any compact set).

Further, we can construct the function $\tilde{v}_n(x)\in C^\infty$ which satisfies the following relations.

$$\tilde{v}_n(x)=v_{0n}(x),\quad |x|\leqq n-2,\quad \tilde{v}_n(x)=L^m,\quad |x|\geqq n-1, \text{ and}$$

$$v_{0n+1}(x)\leqq \tilde{v}_{n+1}(x)\leqq \tilde{v}_n(x)\leqq L^m,\quad \left|\frac{d\tilde{v}_n(x)}{dx}\right|\leqq K \text{ in } (-\infty,+\infty).$$

Now in $G_n=[-n,n]\times[0,T]$ we consider the problem

(3.1) $$\frac{\partial^2 v}{\partial x^2}=\frac{1}{m}v^{\frac{1-m}{m}}\frac{\partial v}{\partial t}-f(v^{\frac{1}{m}})v^{\frac{1}{m}},$$

(3.2) $$v(x,0)=\tilde{v}_n(x)\,,\quad v(\pm n,t)=L^m\,.$$

Taking notice of the assumption $f(L)=0$, we find that the compatibility conditions are satisfied. Moreover it holds $0<\inf_x \tilde{v}_n(x)\leqq \tilde{v}_n(x)\leqq L^m$, and so we have the classical solution $v_n(x,t)$ of the problem (3.1), (3.2) satisfying the estimate

$$\inf_x \tilde{v}_n(x)\leqq v_n(x,t)\leqq L^m e^{mT}\,.$$

This is established by the lemma. Next, setting

$$w_n=v_n(x,t)-v_{n+1}(x,t)\,,$$

we have the equality

$$\frac{1}{B(v_n)}\frac{\partial^2 w_n}{\partial x^2}=\frac{\partial w_n}{\partial t}+\frac{1}{B(v_n)}\left\{B'(\theta_n)\frac{\partial v_{n+1}}{\partial t}-C'(\tilde{\theta}_n)\right\}w_n\,,$$

where

$$B(\lambda)=\frac{1}{m}\lambda^{\frac{1-m}{m}},\quad C(\lambda)=f(\lambda^{\frac{1}{m}})\lambda^{\frac{1}{m}},\quad 0<\inf\tilde{v}_{n+1}(x)\leqq\theta_n(x,t)\,,\quad \tilde{\theta}_n(x,t)\leqq L^m e^{mT}\,.$$

In consideration of the boundedness

$$\frac{1}{B(v_n)}\left|B'(\theta_n)\frac{\partial v_{n+1}}{\partial t}-C'(\tilde{\theta}_n)\right|\leqq M_n \text{ in } G_n\,,$$

and the boundary values $w_n(x,0)\geqq 0$, $w_n(\pm n,t)=0$, we have $w_n(x,t)\geqq 0$ in G_n, or $v_n(x,t)\geqq v_{n+1}(x,t)$.

Setting $P_n=\frac{\partial v_n}{\partial x}$, it gives the equality

$$\frac{\partial^2 P_n}{\partial x^2}=B(v_n)\frac{\partial P_n}{\partial t}+\frac{B'(v_n)}{B(v_n)}P_n\frac{\partial P_n}{\partial x}+\left(\frac{B'(v_n)}{B(v_n)}C(v_n)-C'(v_n)\right)P_n$$

in G_n,

in which it holds $\alpha_0 B(v_n)+\left(\frac{B'(v_n)}{B(v_n)}C(v_n)-C'(v_n)\right)\geqq C_0>0$, where $\alpha_0=2(m+Le^T)$. Hence, the function $\tilde{P}_n=P_n e^{-\alpha_0 T}$ gives the equality

$$\frac{\partial^2\tilde{P}_n}{\partial x^2}=B(v_n)\frac{\partial\tilde{P}_n}{\partial t}+\left\{\alpha_0 B(v_n)+\frac{B'(v_n)}{B(v_n)}C(v_n)-C'(v_n)\right)\tilde{P}_n+\frac{B'(v_n)}{B(v_n)}\tilde{P}_n\frac{\partial\tilde{P}_n}{\partial x}e^{\alpha_0 t}\,.$$

Applying the maximum principle, we have the estimate

(3.3) $$|P_n(x,t)|\leqq \operatorname*{Max}_{\Gamma_n}|P_n(x,t)|e^{\alpha_0 T},\quad \Gamma_n=(x=\pm n,\ t=0).$$

To investigate the property of $v_n(x,t)$ in $S_n=[n-1,n]\times[0,T]$, put $w_n(x,t)=v_n(x,t)+e^{K_1(x-n+1)}-L^m$, then it gives the equality

$$\frac{\partial^2 w_n}{\partial x^2} - \frac{1}{m} v_n^{\frac{1-m}{m}} \frac{\partial w_n}{\partial t} = -f(v_n^{\frac{1}{m}}) v_n^{\frac{1}{m}} + K_1^2 e^{K_1(x-n+1)},$$

where, K_1 is a sufficiently large number to hold inequalities

$$K_1^2 - Le^T > 0, \text{ and } L^m e^{mT} + 1 \leq e^{K_1}$$

Thereby it gives that $w_n(x, t)$ has a maximal value e^{K_1} on the boundary $x=n$, and so $\left.\frac{\partial w_n}{\partial x}\right|_{x=n} \geq 0$, equivalently, $P_n(n, t) \geq -K_1 e^{K_1}$. By the discussion analogous to the above, it follows $|P_n(\pm n, t)| \leq \tilde{K} e^{\tilde{K}}$, $t \in [0, T]$.

Noting the estimate $|P_n(x, 0)| = \left|\frac{d\tilde{v}_n(x)}{dx}\right| \leq K$ $x \in (-\infty, +\infty)$ we have estimations

$$(3.4) \qquad \left|\frac{\partial v_n}{\partial x}\right| \leq C, \text{ or } \left|\frac{\partial u_n^m}{\partial x}\right| \leq C, \text{ in } G_n, \ (n=1, 2, \cdots)$$

where $u_n \equiv v_n^{\frac{1}{m}}$.

Moreover functions $u_n^m (n=1, 2, \cdots)$ satisfy the inequalities

$$(3.5) \qquad |u_n^m(x+h, t) - u_n^m(x, t)| \leq C|h|, \quad n=1, 2, 3, \cdots$$

On the other hand, since $v_n(x, t) \geq v_{n+1}(x, t)$, there exists a limit function $v(x, t)$ of $v_n(x, t)$.

Let $n \to \infty$ in the above (3.5), then we have the relation

$$(3.6) \qquad |v(x+h, t) - v(x, t)| \leq C|h|, \quad (x, t) \in S.$$

Hence, putting $v(x, t)^{\frac{1}{m}} = u(x, t)$, there exists a generalized derivative $(u^m)_x$ and we have

$$(3.7) \qquad \left|\frac{\partial u^m}{\partial x}\right| \leq C, \quad 0 \leq u(x, t) \leq Le^T \text{ in } S,$$

and it is easily verified that the function $u(x, t)$ satisfies the weak equation (1.4).

Finally, we investigate the continuity with respect to t. Let (x_0, t_0) belong to S. If the function $u(x_0, t) = v(x_0, t)^{\frac{1}{m}}$ was not continuous at $t=t_0$, then there would exist a positive number ε_0 and a sequence $t_k (k=1, 2, \cdots)$ tending to t_0 such that $u(x_0, t_k) - u(x_0, t_0) \geq \varepsilon_0$ and $t_k > t_0 (k=1, 2, \cdots)$.

Considering that $u(x, t)$ is continuous with respect to x, there would exist a interval $[\xi_1, \xi_2]$ containing x_0, such that

$$u(x, t_k) - u(x, t_0) \geq \frac{\varepsilon_0}{2} \qquad \text{for any } x \in [\xi_1, \xi_2].$$

Set $G_\lambda=(-\infty, +\infty)\times[\lambda, T]$ $(\lambda\geqq 0)$ and let $\rho(x, t)\in C^\infty$ be a function satisfying the relations

$$\rho(x, T)=0,\ \ \rho(x, t)=0,\ \ x\in(-\infty, +\infty)-[\xi_1, \xi_2],\ \ t\in[0, T],$$

$$\rho(x, t)>0,\ \ \frac{\partial\rho}{\partial t}\geqq 0,\ \ (x, t)\in[\xi_1, \xi_2]\times[t_0, t_0+\delta],$$

(where $\delta>0$ is a sufficiently small number).

Since the function $v_n(x, t)=u_n^m(x, t)$ is a classical solution of the problem (3.1) (3.2) in G_n, we have

$$\iint_{G_{t_0}}\left[\frac{\partial\rho}{\partial t}u_n-\frac{\partial\rho}{\partial x}\frac{\partial u_n^m}{\partial x}+\rho f(u_n)u_n\right]dxdt+\int_{\xi_1}^{\xi_2}\rho(x, t_0)u_n(x, t_0)dx=0\,.$$

Let $n\to\infty$ in the above equality and further considering G_{t_k} instead of G_{t_0}, we have the equality

$$\iint_{G_{t_0}-G_{t_k}}\left[u\frac{\partial\rho}{\partial t}-\frac{\partial u^m}{\partial x}\frac{\partial\rho}{\partial x}+f(u)u\rho\right]dxdt=\int_{\xi_1}^{\xi_2}[u(x, t_k)\rho(x, t_k)-u(x, t_0)\rho(x, t_0)]dx\,.$$

Here, the left-hand side tends to zero as $k\to\infty$, and with respect to the right-hand side we would have the estimate from below

$$\geqq\frac{\varepsilon_0}{2}\int_{\xi_1}^{\xi_2}\rho(x, t_0)dx\,.$$

It is inconsistent. Thus, the proof of the theorem is complete.

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