Interrelation Between Two Problems

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Interrelation Between Two Problems

By

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1. Do you know "Text Book of Algebla by G. Chrystal" which was edited in 1889 for the higher classes of secondary schools and for colleges ?

Early this century it was read widely and well known in our country. Our library has its fifth edition of 1904 and there lies the following problem in it.

(A) The sides of a triangle are the roots of

$$x^3-ax^2+bx-c=0\ .$$

Show that its area is

$$\frac{1}{4}\sqrt{a(4ab-a^3-8c)}\ .$$

The other day we met a next problem.

(B) The graph of $y=x^4+ax^3+bx^2+cx+d$ is symmetric with respect to a line parallel to y-axis.

Show that there exists

$$a^3-4ab+8c=0\ .$$

These two problems seem quite different, but the formulas in them are too much similar. So we decided to examine why this similarity appears.

2. We can prove them easily.

(A) Let the roots of the equation

$$x^3-ax^2+bx-c=0 \qquad \cdots\cdots\cdots(1)$$

be α, β and γ. Then we have

$$\left.\begin{aligned} \alpha+\beta+\gamma &= a \\ \alpha\beta+\beta\gamma+\gamma\alpha &= b \\ \alpha\beta\gamma &= c \end{aligned}\right\} . \qquad \cdots\cdots\cdots(2)$$

The area of a triangle whose sides are α, β and γ is

$$S = \sqrt{s(s-\alpha)(s-\beta)(s-\gamma)}$$

where $2s = \alpha+\beta+\gamma$.

Since $s = \frac{a}{2}$ by (2)

$$\begin{aligned} S^2 &= s(s-\alpha)(s-\beta)(s-\gamma) \\ &= \frac{a}{16}(a-2\alpha)(a-2\beta)(a-2\gamma) \\ &= \frac{a}{16}\{a^3-2(\alpha+\beta+\gamma)a^2+4(\alpha\beta+\beta\gamma+\gamma\alpha)a-8\alpha\beta\gamma\} . \end{aligned}$$

Here we apply (2) again, then

$$\begin{aligned} S^2 &= \frac{a}{16}(a^3-2a^3+4ab-8c) \\ &= \frac{a}{16}(4ab-a^3-8c) \end{aligned}$$

$$\therefore \quad S = \frac{1}{4}\sqrt{a(4ab-a^3-8c)} .$$

(B) We assume the graph of

$$y = f(x) = x^4+ax^3+bx^2+cx+d \qquad \cdots\cdots\cdots(3)$$

is symmetric with respect to a line $x=k$.

It means, if we put

$$x = X+k, \quad y = Y$$

then (3) is changed into the form

$$Y = X^4+mX^2+n . \qquad \cdots\cdots\cdots(4)$$

On the otherhand (3) is transformed into

$$Y = X^4+\frac{f'''(k)}{3!}X^3+\frac{f''(k)}{2!}X^2+f'(k)X+f(k) \qquad \cdots\cdots\cdots(5)$$

Compairing (4) and (5), there must be

$$\frac{f'''(k)}{3!} = 4k+a = 0 \qquad \cdots\cdots\cdots(6)$$

$$f'(k) = 4k^3+3ak^2+2bk+c = 0 . \qquad \cdots\cdots\cdots(7)$$

Eliminating k with (6) and (7), the result is

$$a^3-4ab+8c=0\,.$$

We think this proof of (B) is popular, but it does not explain the similarity at all.

3. Reviewing the problems thoroughly, we noticed the corresponding parts in them, namely

$$x^3-ax^2+bx-c=0 \quad \text{in (A)} \qquad \cdots\cdots\cdots(8)$$

and

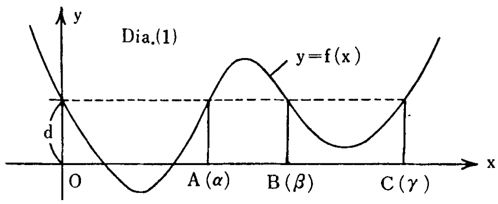
$$\begin{aligned} y &= x^4+ax^3+bx^2+cx+d \\ &= x(x^3+ax^2+bx+c)+d \quad \text{in (B)}\,. \end{aligned} \qquad \cdots\cdots\cdots(9)$$

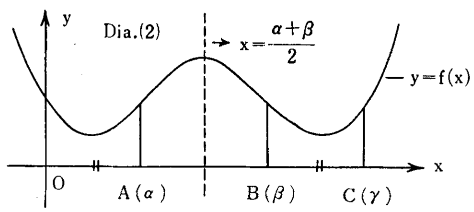
So we suppose the condition of symmetry in (B) must be connected with the roots of the equation

$$x^3+ax^2+bx+c=0 \qquad \cdots\cdots\cdots(10)$$

which is picked up from (9).

Therefore we name the roots of the equation (10) α, β and $\gamma(\alpha<\beta<\gamma)$ again.





Then (3) is written as

$$f(x)=x(x-\alpha)(x-\beta)(x-\gamma)+d\,.$$

Hence

$$f(0)=f(\alpha)=f(\beta)=f(\gamma)=d$$

and the graph of $f(x)$ is the curve in the diagram (1). By the diagram (2), symmetry of the graph indicates

$$\alpha-0=\gamma-\beta$$

namely

$$\alpha+\beta=\gamma \qquad \cdots\cdots\cdots(11)$$

and the axis of symmetry is the line $x=\dfrac{\alpha+\beta}{2}$.

So (11) is the necessary condition of symmetry and its sufficiency is as follows.

By the transformation

$$x = X + \frac{\alpha+\beta}{2}, \quad y = Y$$

we have $(-p, 0)$, $(-q, 0)$, $(q, 0)$ and $(p, 0)$ as new coordinates of the points O, A, B and C, where

$$p = \frac{\alpha+\beta}{2}, \quad q = \frac{\beta-\alpha}{2}.$$

Thus (3) is written

$$\begin{aligned} Y &= (X+p)(X+q)(X-q)(X-p)+d \\ &= X^4-(p^2+q^2)X^2+p^2q^2+d \end{aligned}$$

and the graph is symmetric with respect to Y-axis.

4. Now it remains to find the relation between a, b and c which is led from (11).

Here

$$x^3+ax^2+bx+c = (x-\alpha)(x-\beta)(x-\gamma)$$

and

$$\alpha+\beta = \gamma .$$

So we have

$$\begin{cases} \alpha+\beta+\gamma = 2(\alpha+\beta) = -a \\ \alpha\beta+\beta\gamma+\gamma\alpha = (\alpha+\beta)^2+\alpha\beta = b \\ \alpha\beta\gamma = \alpha\beta(\alpha+\beta) = -c . \end{cases}$$

Eliminating $\alpha+\beta$ and $\alpha\beta$, we obtain

$$\left(\frac{-a}{2}\right)^2+\frac{2c}{a} = b .$$

This changes into

$$a^3-4ab+8c = 0$$

and (B) is proved.

In the same way if the relation $\alpha+\beta=\gamma$ exists among the roots of (8), we have

$$4ab-a^3-8c = 0 .$$

This means a triangle in (A) degenerates to a line segment and its area vanishes.

Thus we have an example where two different problems with external resemblance are related deeply inside.

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