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https://doi.org/10.15017/1448959

出版情報：九州大学教養部数学雑誌. 9 (2), pp.43-65, 1974-12. College of General Education, Kyushu University

Math. Rep.
IX-2, 1974.

On the Local Classical Solution of the Mixed Problem for Nonlinear Wave Equation

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(Received Aug. 23, 1974)

Introduction

In present paper we study the Initial-Boundary Value Problem

$$(1)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + C_0(u)^p + C_1\left(\dfrac{\partial u}{\partial t}\right)^q + C_2|\nabla u|^{2r} = f \\ \qquad\qquad\qquad\qquad in \quad \Omega\times(O,T) \\ u(x,0)=u_0(x)\,, \qquad \dfrac{\partial u(x,0)}{\partial t} = u_1(x) \qquad in \quad \Omega \\ u(x,t)|_{\partial\Omega} = 0 \qquad for \quad t\in(0,T) \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$, whose boundary $\partial\Omega$ is sufficiently smooth and T is some positive number, p, q, r are arbitrarily fixed positive integers, $C_i(i=0,1,2)$ are constants, and $f(x,t)$, $u_0(x)$, $u_1(x)$ are given functions.

In [1] J. Sather proved the existence of the global classical solution for the problem (1) in the case $n=3$, $C_0=1, C_1=C_2=0$ and $p=3$, *i, e.*

$$(2)\qquad \frac{\partial^2 u}{\partial t^2} - \Delta u + (u)^3 = f\,.$$

After then, J. L. Lions [2] suggested a question if we could discuss the analogous argument for the equation

$$(3)\qquad \frac{\partial^2 u}{\partial t^2} - \Delta u + \left(\frac{\partial u}{\partial t}\right)^3 = f\,.$$

We can easily verify the existence of the global weak solution but in the classical sense the argument is very difficult.

Quite recently H. Kato and T. Nanbu [3] obtained classical solution in the local sense under the same conditions of (1) for the equation (3) in the case $n=3$.

Our main theorem in this paper states that there exist a positive number δ and $u(x, t)$ which solves the Problem (1) classically in $[0, \delta)$ for the arbitrary spatial dimension n.

In section 1, we mention some lemmas and abstract propositions which are essential in this paper and in section 2, we prove our main theorem by the specialized Galerkin's method, in [Remark 1] made after the proof of the theorem we generalize our results for some non-linear evolution equations of the form

$$\frac{\partial^2 u}{\partial t^2}+L(D)u+G(u, Du, \ldots\ldots, D^m u, D_t u)=f$$

here $L(D)=\sum_{|\alpha|\leq 2m} a_\alpha D^\alpha$ is uniformly elliptic and coercive on $\mathring{H}^m(\Omega)$ and $G=G(u, Du, \ldots, D^m u, D_t u)$ is a polynomial in $u, Du, \ldots, D^m u, D_t u$. In general we can not expect the global existence theorems for above equations, but for specialized equations the author believes the global existence of a classical solution by giving special initial data.

§ 1. Lemmas and Abstract Propositions

Throughout this paper we assume all function spaces are considered over real field and their notations and definitions are same as those Mizohata [4] and S. Agmon [5] etc.

Now we mention some lemmas which play an essential role in this paper.

[LEMMA 1]

If a function $\varphi(t)$ *belonging to* $C^1[0, T]$ *satisfies for given* $f(t)$, $g(t)\in C^0[0, T]$

$$\begin{cases} \varphi'(t)\leq f(t)\varphi(t)+g(t) & t\in[0, T] \\ \varphi(0)=\varphi_0 \end{cases}$$

then we have for $t\in[0, T]$

$$\varphi(t)\leq F(t)\left[\int_0^t g(\tau)\,[F(\tau)]^{-1}\,d\tau+\varphi_0\right]$$

where $F(t)=e^{\int_0^t f(\theta)d\theta}$.

(proof)

We put $\psi(t)$ the solution of

$$\begin{cases} y'(t) = f(t)y(t)+g(t) \\ y(0) = \varphi_0 \end{cases}$$

then we know

$$\psi(t) = F(t)\left[\int_0^t g(\tau)\,[F(\tau)]^{-1}d\tau+\varphi_0\right].$$

Now if we assume that there exists $t_0 \in [0, T]$ such that $\varphi(t_0)=\psi(t_0)$ and $\varphi'(t_0)>\psi'(t_0)$, then we have from the latter relation,

$$f(t_0)\psi(t_0)+g(t_0) < f(t_0)\varphi(t_0)+g(t_0)$$

that is $f(t_0)\psi(t_0) < f(t_0)\varphi(t_0)$
this contradicts the former. (q. e. d)

[LEMMA 2]

If $\varphi(t) \in C^1[0, T)$ *satisfies*

$$\begin{cases} \varphi'(t) \leqq C\cdot\{\varphi(t)\}^N & \textit{for } t \in (0, T) \\ \varphi(0) = \varphi_0 \end{cases}$$

for given positive constants C, φ_0, N *with* $N \geqq 1$, *then it holds*

$$\varphi(t) \leqq \frac{\varphi_0}{[1-C(N-1)\varphi_0^{N-1}t]^{\frac{1}{N-1}}}$$

for any $t \in [0, T)$, *where*

$$T_0 = min\left\{T, \frac{1}{C(N-1)\varphi_0^{N-1}}\right\}$$

Proof of this lemma 2 is obviously obtained by a similar way as the proof of lemma 1.

[LEMMA 3] (SOBOLEV'S IMBEDDING THEOREM)

Let Ω *be an open set in* R^n *with* C^n*-class boundary and* $u \in \mathring{H}^n(\Omega)$ *then we have*

(1) *if* $0 \leqq m < \frac{n}{2}$, *u should belong to* $L^p(\Omega)$ *and it holds*

$$||u||_{L^p(\Omega)} \leqq C(n, m, \Omega)\,||u||_{\mathring{H}^m(\Omega)}$$

where $\dfrac{1}{p} = \dfrac{1}{2} - \dfrac{m}{n}$,

(2) *if* $m=\left[\frac{n}{2}\right]+1+r, u$ *should belong to* $\beta^r(\bar{\Omega})$ *and it holds*

$$||u||_{\beta^r(\bar{\Omega})} \leqq C(n, m, r, \Omega) \; ||u||_{\dot{H}^m(\Omega)} .$$

Obviously we have, from (2) of this lemma, if u, $v \in \mathring{H}^{\left[\frac{n}{2}\right]+1}(\Omega)$, then for any positive integers p, q, $u^p \cdot v^q \in \mathring{H}^{\left[\frac{n}{2}\right]+1}(\Omega)$ and it holds

$$||u^p \cdot v^q||_{\dot{H}^{\left[\frac{n}{2}\right]+1}(\Omega)} \leqq C(p, q, n, \Omega) ||u||^p_{\dot{H}^{\left[\frac{n}{2}\right]+1}(\Omega)} ||v||^q_{\dot{H}^{\left[\frac{n}{2}\right]+1}(\Omega)}$$

Next lemma 4 plays most important role to verify regularity of weak solutions defined in this paper.

[LEMMA 4]

Le H *be a separable Hilbert Space, and* $\{u_{p0}(t)\}$ *be a sequence of mappings*

$$u_{p0}(t): \; [0, T] \longrightarrow H$$

satisfying next two conditions;

(1) $\sup\limits_{p_0} \sup\limits_{t \in [0, T]} ||u_{p0}(t)||_H < +\infty$

(2) *for any* $t_1, t_2 \in [0, T]$, *there exist positive numbers* K, α *with* $0 < \alpha \leqq 1$, *such that the following inequality holds independently of* p_0

$$||u_{p_0}(t_1) - u_{p_0}(t_2)||_H \leqq K|t_1 - t_2|^\alpha$$

then $\{u_{p_0}(t)\}$ *has a weakly convergent subsequence* $\{u_{pk}(t)\}$ *and its weak limit* $u(t)$ *belongs to* $\mathcal{E}^0_{[0,T]}[H]$ *and the convergence is uniformly in* t.

(proof)

Since H is separable, there exists a countable set $\{\varphi_k\}$ whose finite linear combinations is dense in H.

First we put $F_{p_0}(t, \varphi_1) = (u_{p_0}(t), \; \varphi_1)_H$ $(p_0 = 1, 2, \ldots\ldots)$.

Then from our hypotheses (1) (2) we can easily verify that the functions $\{F_{p_0}(t, \varphi_1)\}$ is uniformly bounded and equi-continuous in $[0, T]$.

Therefore by Ascoli-Arzela's Theorem, we have a subsequence $\{u_{p_1}(t)\}$ of $\{u_{p_0}(t)\}$ such that $\{F_{p_1}(t, \varphi_1)\}$ is uniformly convergent in $[0, T]$.

Next let us set $F_{p_1}(t, \varphi_2) = (u_{p_1}(t), \varphi_2)_H$. Then by an analogous arguement we obtain a subsequence $\{u_{p_2}(t)\}$ of $\{u_{p_1}(t)\}$ such that $\{F_{p2}(t, \varphi_2)\}$ and $\{F_{p2}(t, \varphi_1)\}$ are uniformly convergent in $[0, T]$.

By continuing these procedure, we can see that $\{F_{pm}(t, \varphi_j)\}$ $(j = 1, 2, 3, \ldots, m)$ are uniformly convergent in $[0, T]$.

Consequently, by taking diagonal element of these subsequences, we have for any $\varphi_k \in \{\varphi_k\}$, $\{F_{p_p}(t, \varphi_k)\}$ is uniformly convergent in $[0, T]$.

Therefore we can conclude that $\{F_{p_p}(t, \varphi)\}$ is uniformly convergent in $[0, T]$ for an $\varphi \in H_0$, where H_0 is the totality of finite linear combinations of $\{\varphi_k\}$.

Now we set for each $\varphi \in H_0$

$$L_t(\varphi) = \lim_{p_p \to \infty} F_{p_p}(t, \varphi)$$

then we see $L_t(\varphi)$ is linear functional on H_0 for each $t \in [0, T]$, and we have by Schwarz inequality,

$$|L_t(\varphi)| \leqq M\|\varphi\|_H \qquad \text{for} \quad \varphi \in H_0$$

where $M = \sup_{p0} \sup_{t \in [0,T]} \|u_{p0}(t)\|_H$.

Since H_0 is dense in H it must follow that $L_\cdot(\varphi)$ is continuous linear functional on H.

Therefore by Riesz Theorem, there exists a function $u(t)$ in H such that $L_\cdot(\varphi) = (u(t), \varphi)_H$ for each $\varphi \in H$ and $t \in [0, T]$.

And since $\|u(t)\|_H \leqq \underline{\lim}_{pp \to \infty} \|u_{pp}(t)\|$, we may assert that $u(t) \in \mathcal{E}^0_{[0,T]}[H]$ from our hypothesis (2) as $p_p \to \infty$.

The uniform convergence is obvious from the above arguement. This completes the proof of lemma 4. (q, e, d,)

[LEMMA 5]

If the linear partial differential operator $L(D) = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha$ is coercive on $\mathring{H}^1(\Omega)$, that is there exists $C(>0)$ satisfying for any $\varphi \in \mathring{H}^m(\Omega) \cap H^{2m}(\Omega)$

$$\langle L(D)\varphi, \varphi \rangle_{L^2(\Omega)} \geqq C\|\varphi\|^2_{\dot{H}^m(\Omega)}$$

and uniformly elliptic in Ω,

then for any positive integer p the operator $L^p(D)$ is coercive on $\mathring{H}^{pm}(\Omega)$ and the eigen functions $\{\varphi_j\}$ of $L^p(D)$ consists of a complete base in $\mathring{H}^{pm}(\Omega)$.

(proof)

Coercivity of $L^p(D)$ is easily verified by induction and the remaining part of this lemma is a direct conclusion of Riesz Theorem.

Now, we consider abstract problems. Let H, V_0, V_1 be separable Hilbert Spaces with

(1. 1) *$V_1 \subset V_0 \subset H$ where the inclusions are contiuuous*

(1.2) *V_1 is dense in H.*

Here we treat the abstract second order equation

$$\text{(a)}\quad \frac{d^2}{dt^2}u(t)+Lu(t)+G(u(t))=0$$

where

(1.3) *$u(t)$ be an unknown function from $(0,T)\to V_0$, and $\frac{d}{dt}u(t)$, $\frac{d^2}{dt^2}u(t)$ mean usual strong derivatives of $u(t)$ in H, and*

(1.4) *L be a densely defined self-adjoint operator in H with*

$$\frac{1}{c_0}||u||_{V_0}^2\leqq (Lu,u)_H=[u,u]\leqq c_0||u||_{V_0}^2 \qquad for \quad u\in V_0,$$

$$\frac{1}{c_1}||v||_{V_1}\leqq ||Lv||_H\leqq c_1||v||_{V_1} \qquad for \quad v\in V_1$$

where the constants c_0, c_1 are positive constants independent of u, v respectively,

(1.5) *$G(u)$ be a nonlinear operator satisfying*

(1.5.1) *$G(u)\in H$ if $u\in V_0$ and*

(1.5.2) *$(G(u_j),\varphi)_H\longrightarrow (G(u),\varphi)_H$ for each $\varphi\in V_1$ if $u_j\to u$ weakly in V_0 and moreover*

(1.5.3) *for u, $v\in V_0$*

$$||G(u)-G(v)||_H\leqq C(u,v)\,||u-v||_{V_0}$$

where the constant $C(u,v)$ depends u,v continuously.

Now we formulate the abstract Cauchy-Problem. We call a function $u(t)$ a solution of the problem (A) when it satisfies

$$(A)\quad \begin{cases} \dfrac{d^2}{dt^2}(u(t),\varphi)_H+[u(t),\varphi]+(G(u(t)),\varphi)_H=0 \\ \qquad for\ every\quad t\in(0,T)\ and\ any\ \varphi\in V_1 \\ \qquad u(0)=u_0,\quad u'(0)=u_1 \qquad (u_0,u_1:\ given) \end{cases}$$

Here we have

[THEOREM 1] (UNIQUENESS).

Let $u(t)$, $v(t)\in\mathcal{E}^0_{[0,T]}[V_1]\cap\mathcal{E}^1_{[0,T]}[V_0]\cap\mathcal{E}^2_{[0,T]}[H]$ be solutions of the problem (A) with initial data $u_0, u_1, v_0, v_1\in V_0$ respectively, then it follows

$$(1.6)\quad |u'(t)-v'(t)||_H^2+\frac{1}{c_0}||u(t)-v(t)||_{V_0}^2$$

$$\leqq k(t)\left\{||u_1-v_1||_H^2+\frac{1}{c_0}||u_0-v_0||_{V_0}^2\right\}$$

for any $t \in [0, T]$, *where the function* $k(t)$ *depends on* $u(t)$, $v(t)$ *continuously.*

(proof)

By setting $w(t)=u(t)-v(t)$ and $F(t)=(w'(t), w'(t))_H+[w(t), w(t)]$ then $F(t)$ should satisfy

$$F'(t) = 2\{(w''(t), w'(t))_H+[w(t), w'(t)]\}$$
$$= 2(G(v(t))-G(u(t)), w'(t))_H .$$

Thus by utilizing Schwarz inequality and (1.5.3), we have for $t \in (0, T)$

$$F'(t) \leqq 2\|w'(t)\|_H \|G(v(t))-G(u(t))\|_H$$
$$\leqq \|w'(t)\|_H^2+\{C(u(t), v(t))\}^2\|u(t)-v(t)\|_{V_0}^2$$
$$\leqq \|w'(t)\|_H^2+C_0\{C(u(t), v(t))\}^2[w(t), w(t)]$$
$$\leqq k_0(t)\{\|w'(t)\|_H^2+[w(t), w(t)]\}$$

where $k_0(t)=max\{1, C_0\{C(u(t), v(t))\}^2\}$,

Therefore it holds $F'(t) \leqq k_0(t) F(t)$ for $t \in (0, T)$, and by lemma 1, we have

$$F(t) \leqq e^{\int_0^t k_0(\tau)d\tau} F(0)$$

and setting $k(t) = e^{\int_0^t k_0(\tau)d\tau}$, (1.6) follows by (1.4). (q. e. d)

We make next assumptions for L and G.

(1.7) *For the operator L there exists a countable sequence* $\{\varphi_k\}_{k=1}^{\infty}$ *in* V_1 *with the condition that it is L-invariant and totality* Φ *of its finite linear combinations is dense in* V_1.

Here we set V_2 as a Hilbert space endowed with a norm

$$\|\varphi\|_{V_2} = [L\varphi, L\varphi]^{\frac{1}{2}} = (L^3\varphi, \varphi)_H^{\frac{1}{2}} .$$

(1.8) $G(\cdot)$ *satisfies*

(1.8.1) $[G(u), G(u)] \leqq const\cdot\|u\|_{V_2}^m$

for $u \in V_2$ *where m is some positive number independent of u,*

(1.8.2) *for* $u(t) \in \mathcal{E}^2_{[0,T]}[V_0]$, *it holds*

$G(u(t)) \in \mathcal{E}^2_{[0,T]}[H]$ *and,*

$$\|G(u(t))\|_H \leqq const\ (1+\|u(t)\|_{V_0}^2)^{\frac{N}{2}} \quad with \quad N \geqq 1,$$

$$\left\|\frac{d}{dt} G(u(t))\right\|_H \leqq f(u(t)) \left\|u'(t)\right\|_{V_0}$$

here $f(u(t))$ *is positive function of* $u \in V_0$ *satisfying*

$$\sup_{\|u\|_{V_0}\leq M} f(u) = k(M) < +\infty\,, \quad \textit{and}$$

$$\left\| \frac{d^2}{dt^2} G(u(t)) \right\|_H \leq g(u(t),\ u'(t)) \left\| u''(t) \right\|_{V_0}$$

here $g(u, v)$ is a positive funktion of $(u, v) \in V_0 \times V_0$ satisfying

$$\sup_{\|u\|_{V_0}\leq M_1,\, \|v\|_{V_0}\leq M_2} g(u, v) = k(M_1, M_2) < +\infty\,.$$

Here we have a existence theorem for the problem (A).

[THEOREM 2]

There exists one and only one function $u(t)$ in $\mathcal{E}^0_{[0,T_1]}[V_1] \cap \mathcal{E}^1_{[0,T_1]}[V_0] \cap \mathcal{E}^2_{[0,T_1]}[H]$ which solves the problem (A) for the initial data $u_0 \in V_2$, $u_1 \in V_1$ where the number T $(0 < T_1 \leq T)$ depends on u_0, u_1, N and other given constants.

(proof)

By the usual Galerkin's method we shall make approximating solutions $\{u_p(t)\}$. That is, let $\{\lambda_{p,k}(t)\}$ $(k = 1, 2, \dots, p)$ be solutions of the Ordinary Cauchy Problem;

$$(A')\quad \begin{cases} \sum_{k=1}^{p} \{\lambda''_{p,k}(t)(\varphi_k, \varphi_j)_H + \lambda_{p,k}(t)[\varphi_k, \varphi_j]\} & \\ \qquad + \left(G\left(\sum_{k=1}^{p} \lambda_{p,k}(t)\varphi_k\right), \varphi_j\right)_H = 0 & (j = 1, 2, \dots, p) \\ \lambda_{p,k}(0) = \alpha_k \qquad \lambda'_{p,k}(0) = \beta_k & (k = 1, 2, \dots, p) \end{cases}$$

where $\{\varphi_k\}$ be an L-invariant base in V_1 and $\{\alpha_k\}$, $\{\beta_k\}$ be sequences of numbers satisfying

$$u_p(0) = \sum_{k=1}^{p} \alpha_k \varphi_k \longrightarrow u_0 \qquad (s) \text{ in } V_2$$

$$u_p'(0) = \sum_{k=1}^{p} \beta_k \varphi_k \longrightarrow u_1 \qquad (s) \text{ in } V_1$$

It is well-known that one can find $T_p > 0$ such that the solution with the form $u_p(t) = \sum_{k=1}^{p} \lambda_{p,k}(t)\varphi_k$ of the problem (A′) exists in the interval $[0, T_p)$, i. e. for $t \in (0, T_p)$ the function $u_p(t)$ satisfies

$$(1.9)\qquad (u_p''(t), \varphi_j)_H + [u_p(t), \varphi_j] + (G(u_p(t)), \varphi_j)_H = 0 \qquad (j = 1, 2, \dots, p)$$

and

$$(1.10)\qquad u_p(0) = \sum_{k=1}^{p} \alpha_k \varphi_k,\ u_p'(0) = \sum_{k=1}^{p} \beta_k \varphi_k\,.$$

From (1.9), we have for $t \in (0, T_p)$

$$(u_p'(t), u_p''(t))_H + [u_p(t), u_p'(t)] + (G(u_p(t)), u_p'(t))_H = 0$$

and therefore

$$\frac{d}{dt}\{\|u_p'(t)\|_H^2 + [u_p(t), u_p(t)]\} \leq \|u_p'(t)\|_H^2 + \|G(u_p(t))\|_H^2 .$$

Hence by (1.82) the following inequality holds for $t \in (0, T_p)$

$$\frac{d}{dt}\{\|u_p'(t)\|_H^2 + [u_p(t), u_p(t)]\} \leq \|u_p(t)\|_H^2 + \mathrm{const}(1 + \|u_p(t)\|_{V_0}^2)^N$$
$$\leq const(1 + \|u_p'\|_H^2 + \|u_p\|_{V_0}^2)^N .$$

Therefore the function

$$f_p(t) = 1 + \|u_p'(t)\|_H^2 + [u_p(t), u_p(t)]$$

should satisfy for $t \in [0, T_p]$

$$f_p(t) \leq \frac{f_p(0)}{[1 - c(N-1)\{f_p(0)\}^{N-1} t]^{\frac{1}{N-1}}} \quad \text{by lemma 2 .}$$

Since $f_p(0) \leq 1 + const\|u_1\|_H^2 + const[u_0, u_0]$, we have

$$f_p(t) \leq \frac{c(u_0, u_1)}{[1 - c(N, u_0, u_1)t]^{\frac{1}{N-1}}} \qquad \text{for} \quad t \in [0, T_1]$$

where the number T_1 is the inverse of $c(N, u_0, u_1)$.
Consequently by fixing $T_0 (0 < T_0 < T_1)$ arbitrarily, we have

(1.11) $\sup_p \sup_{t \in [0, T_0]} [u_p(t), u_p(t)] < +\infty$

(1.12) $\sup_p \sup_{t \in [0, T_0]} \|u_p'(t)\|_H < +\infty$

Secondly by differentiating (1.9) we have for $t \in [0, T_0]$

(1.13) $$(u_p^{(3)}(t), \varphi_j)_H + [u_p'(t), \varphi_j] + \left(\frac{d}{dt} G(u_p(t)), \varphi_j\right)_H = 0$$

$(j = 1, 2, \dots, p)$.

Therefore it follows

$$\frac{d}{dt}\{\|u_p''(t)\|_H^2 + [u_p'(t), u_p'(t)]\} \leq 2\left\|\frac{d}{dt} G(u_p(t))\right\|_H \cdot \left\|u_p''(t)\right\|_H$$
$$\leq \left\|\frac{d}{dt} G(u_p(t))\right\|_H^2 + \left\|u_p''(t)\right\|_H^2$$

By (1.8.2), we have

$$\frac{d}{dt}\{\|u_p''(t)\|_H^2 + [u_p'(t), u_p'(t)]\} \leq f^2(u_p(t))\|u_p'(t)\|_{V_0}^2 + \|u_p''(t)\|_H^2 .$$

By (1.11) and (1.4) it follows

$$\sup_p \sup_{t\in[0,T_0]} f^2(u_p(t)) \leq const .$$

Thus we have for $t\in[0,T_0]$,

$$\frac{d}{dt}\{||u_p''(t)||_H^2+[u_p'(t),u_p'(t)]\} \leqq const\{||u_p''(t)||_H^2+[u_p'(t),u_p'(t)]\}$$

Therefore, using lemma 1 we obtain

$$(☆)\quad ||u_p''(t)||_H^2+[u_p'(t),u_p'(t)] \leqq e^{ct}\{||u_p''(0)||_H^2+[u_p'(0),u_p'(0)]\} \quad \text{for } t\in[0,T] .$$

Now we show the boundedness of the right hand side of this inequality (☆). Since $u_p'(0)\longrightarrow u_1$ (s) in V_1 the boundedness of $\{[u_p'(0),u_p'(0)]\}$ is obvious. Secondly, in (1.9) we have as $t\longrightarrow 0$

$$(1.9)'\quad (u_p''(0),\varphi_j)_H+[u_p(0),\varphi_j]+(G(u_p(0)),\varphi_j)_H=0 .$$

From this we have

$$||u_p''(0)||_H \leqq ||Lu_p(0)||_H+||G(u_p(0))||_H .$$

Since $u_p(0)\longrightarrow u_0$ (s) in V_2,

we have

$$\sup_p\{||Lu_p(0)||_H\}<+\infty$$

and

$$\sup_p\{||Gu_p(0))||_H\} \leqq \sup_p\{const\ (1+||u_p(0)||_{V_2}^2)^{\frac{N}{2}}\}<+\infty .$$

This shows the boundedness of $\{||u_p''(0)||_H\}$, and hence the right hand side of (☆) is bounded.

Consequently, we have by (☆)

$$(1.14)\quad \sup_p \sup_{t\in[0,T_0]} ||u_p''(t)||_H<+\infty$$

$$(1.15)\quad \sup_p \sup_{t\in[0,T_0]} [u_p'(t),u_p'(t)]<+\infty .$$

Now, by differentiating (1.13) we have for $t\in[0,T_0]$

$$(1.16)\quad (u_p^{(4)}(t),\varphi_j)+[u_p''(t),\varphi_j]+\left(\frac{d^2}{dt^2}G(u_p(t)),\varphi_j\right)=0 .$$

Thus we obtain

$$\frac{d}{dt}\{||u_p^{(3)}(t)||_H^2+[u_p''(t), u_p''(t)]\}\leqq\left\|\frac{d^2}{dt^2}G(u_p(t))\right\|_H^2+\left\|u_p^{(3)}(t)\right\|_H^2 .$$

From (1.8.2), it follows

$$\frac{d}{dt}\{||u_p^{(3)}(t)||_H^2+[u_p''(t), u_p''(t)]\}$$
$$\leqq g^2(u_p(t), u_p'(t))\,||u_p''(t)||_{V0}+||u_p^{(3)}(t)||_H^2 .$$

From (1.11), (1.15), we have

$$g(u_p(t), u_p'(t))\leqq const .$$

Thus we have for $t\in[0, T_0]$

$$\frac{d}{dt}\{||u_p^{(3)}(t)||_H^2+[u_p''(t), u_p''(t)]\}$$
$$\leqq const\ \{||u_p^{(3)}(t)||_H^2+[u_p''(t), u_p''(t)]\} .$$

Applying lemma 1 again, we have

$$||u_p^{(3)}(t)||_H^2+[u_p''(t), u_p''(t)]$$
$$\leqq e^{ct}\{||u_p^{(3)}(0)||_H^2+[u_p''(0), u_p''(0)]\} \qquad \text{for} \quad t\in[0, T_0] .$$

Now we show the boundedness of $\{||u_p^{(3)}(0)||_H\}$ and $\{[u_p''(0), u_p''(0)]\}$.
In (1.13) let t tend to 0, we have

$$(1.13)' \quad (u_p^{(3)}(0), \varphi_j)_H+[u_p'(0), \varphi_j]+(G'(u_p(0)), \varphi_j)_H=0$$

Therefore it follows

$$||u_p^{(3)}(0)||_H\leqq||L\,u_p'(0)||_H+||G'(u_p(0))||_H$$
$$\leqq||L\,u_p'(0)||_H+f(u_p(0))\,||u_p'(0)||_{V_0} .$$

Since $u_p\longrightarrow u_0$ (s) in V_2

$u_p'\longrightarrow u_1$ (s) in V_1 ,

the boundedness of the right hand side of this inequality is obvious.
Now, since $\{\varphi_j\}$ is L-invariant we can take $L\varphi_j$ instead of φ_j in (1.9)′ i, e.

$$(u_p''(0), L\varphi_j)+[u_p(0), L\varphi_j]+(G(u_p(0)), L\varphi_j)_H=0 .$$

Therefore, we have

$$[u_p''(0), u_p''(0)]+[u_p(0),\ L\,u_p''(0)]+[G(u_p(0)), u_p''(0)]=0 .$$

Thus it follows

$$[u_p''(0), u_p''(0)]^{\frac{1}{2}}\leqq[L\,u_p(0), L\,u_p(0)]^{\frac{1}{2}}+[G(u_p(0)),\ G(u_p(0))]^{\frac{1}{2}}$$
$$\leqq||u_p(0)||_{V_2}+const||u_p(0)||_{V_2}^{\frac{m}{2}}$$

from (1.81) .

Since $u_p(0) \longrightarrow u_0$ (s) in V_2, we have the bounedness of the right hand side of this inequality.
Consequently we obtain

$$(1.17) \quad \sup_p \sup_{t\in[0,T_0]} ||u_p^{(3)}(t)||_H < +\infty$$

$$(1.18) \quad \sup_p \sup_{t\in[0,T_0]} [u_p''(t), u_p''(t)] < +\infty .$$

By (1.11) and (1.15), there exist a subsequence $\{u_{pk}(t)\}$ of $\{u_p(t)\}$ and $u(t) \in \mathcal{E}^0_{[0,T_0]}[V_0]$ such that

$$(1.19) \quad u_{pk}(t) \longrightarrow u(t) \qquad (w) \ \ in \ \ V_0 ,$$

and by (1.15), (1.18) there exist a subsequence $\{u_{pq}(t)\}$ of $\{u_{pk}(t)\}$ and $v(t) \in \mathcal{E}^0_{[0,T_0]}[V_0]$ such that

$$(1.20) \quad u_{pq}'(t) \longrightarrow v(t) \qquad (w) \ \ in \ \ V_0$$

and by (1.14) and (1.17) there exist a subsequence of $\{u_{pr}(t)\}$ of $\{u_{pq}(t)\}$ and $w(t) \in \mathcal{E}^0_{[0,T_0]}[H]$ such that

$$(1.21) \quad u_{pr}''(t) \longrightarrow w(t) \qquad (w) \ \ \text{in} \ \ H .$$

And since the convergence of these sequences are uniform in t, we have

$$u'(t) = v(t) \qquad in \ \ V_0$$
$$u''(t) = w(t) \qquad in \ \ H .$$

Thus $u(t)$ should belong to

$$\mathcal{E}^0_{[0,T_0]}[V_0] \cap \mathcal{E}^1_{[0,T_0]}[V_0] \cap \mathcal{E}^2_{[0,T_0]}[H]$$

and satifies from (1.52)

$$(u''(t), \varphi_j)_H + [u(t), \varphi_j] + (G(u(t)), \varphi_j)_H = 0$$

for all $\varphi_j \in \{\varphi_j\}$, and $u(0)=u$, $u'(0)=u_1$.
Therefore for each $t \in [0, T]$ $u(t)$ satisfies,

$$u''(t) + L\,u(t) + G(u(t)) = 0 .$$

Since $u''(t) \in \mathcal{E}^0_{[0,T_0]}[H]$, and $G(u(t)) \in \mathcal{E}^0_{[0,T_0]}[H]$ and from our assumtion (1.4), we have $u(t) \in \mathcal{E}^0_{[0,T_0]}[V_1]$.

Here if we take into account of the arbitrariness of $T_0(0<T_0<T_1)$ and the uniqueness of such solution (Theorem 1), $u(t)$ is the only function which solves the problem (A). This completes the proof of the Theorem 2. (q, e, d)

[REMARK 1] We can easily verify that the function $u(t)$ in the Theorem 2 is a strong solution in the space H.
Obviously, by reasoning of our proof of the Theoem 2, if we want to obtain a weaker solution, some of assumptions may be removed.

[REMARK 2] For the equation of the form

$$\text{(b)} \quad u''(t)+Lu(t)+G(u(t), u'(t)) = f(t) ,$$

we can prove the local existence of a strong solution under more strict conditions for the operator L and $G(u, v)$.

(Assumptiions for L)

For any positive integer n, $H_n=D(L^n)=\{u \in H;\ L^{\frac{n}{2}}u \in H\}$ is a Hilbert space endowed with a norm

$$||\varphi||_n^2 = (\varphi, \varphi)_n = (L^{\frac{n}{2}}\varphi, L^{\frac{n}{2}}\varphi)_H$$

and there exists L-invariant base $\{\varphi_k\}$ in the space H_n whose finite linear combinations are dense in H.

(Assumptions for $G(u, v)$)

(A. 1) There exist positive integer n_0 and $N>0$, $c>0$ such that

$$|(v, G(u, v))_{n_0}| \leqq c\{1+||v||_{n_0}^2+||u||_{n_0+1}^2\}^N$$
$$\text{for} \quad u \in H_{n_0+1},\ v \in H_{n_0}.$$

(A. 2) For $u(t) \in \mathcal{E}^2_{[0,T]}[H_{n0}] \cap \mathcal{E}^1_{[0,T]}[H_{n0+1}]$, we have

$$|(D_t^2u(t), D_tG(u(t), D_tu(t)))_{n_0}|$$
$$\leqq c_1(u(t), D_tu(t))\{||D_tu(t)||_{n_0+1}^2+||D_t^2u(t)||_{n_0}^2\}$$

where

$$sup\ c_1(u(t), D_tu(t)) = \alpha(M_1, M_2)<+\infty , \quad \text{if}$$

$$\begin{cases} \sup\limits_{t\in[0,T]} ||u(t)||_{n_0+1} < M_1 \\ \sup\limits_{t\in[0,T]} ||D_tu(t)||_{n_0} < M_2 . \end{cases}$$

(A. 3) For $u(t) \in \mathcal{E}^3_{[0,T]}[H_{n0}] \cap \mathcal{E}^2_{[0,T]}[H_{n0+1}]$, we have

$$|(D_t^3u(t), D_t^2G(u(t), D_tu(t)))_{n_0}|$$
$$\leqq c_2(u(t), D_tu(t), D_t^2u(t))\{||D_t^2u(t)||_{n_0+1}^2+||D_t^3u(t)||_{n_0}^2\}$$

where

$$sup\ c_2(u(t), D_tu(t), D_t^2u(t)) = \beta(M_1, M_2, M_3)<+\infty, \quad \text{if}$$

$$\begin{cases} \sup_{t \in [0,T]} \|u(t)\|_{n_0+1} < M_1 \\ \sup_{t \in [0,T]} \|D_t u(t)\|_{n_0+1} < M_2 \\ \sup_{t \in [0,T]} \|D_t^2 u(t)\|_{n_0} < M_3. \end{cases}$$

(A. 4) If $u_n \longrightarrow u$ (w) in H_{n_0+1}, $v_n \longrightarrow v$ (w) in H_{n_0} then $G(u_n, v_n) \longrightarrow G(u, v)$ (w) in H.

Our main theorem proved in the section 2 is a consequence of Remark 2 by using our lemmas and theorems in this section.

§2. Proof of the main theorem

We consider the nonlinear mixed problem

$$(1) \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + c_0 u^p + c_1 \left(\dfrac{\partial u}{\partial t}\right)^q + c_2 |\Delta u|^{2r} = f & \text{in } \Omega \times (0, T) \\ u(x, 0) = u_0(x), \ \dfrac{\partial u(x, 0)}{\partial t} = u_1(x) & \text{in } \Omega \\ \qquad u(x, t)|_{\partial\Omega} = 0 \qquad \text{for } t \geqq 0. \end{cases}$$

Here p, q, r are positive integers and $c_i (i=0, 1, 2)$ are arbitrary numbers, Ω is a bounded domain in R^n and whose boundary $\partial\Omega$ is sufficiently smooth ($c^{[\frac{n}{2}]+3}$ —class) and u_0, u_1 and f are given data, and ∇u means gradient vector function for u.

Now we have

[THEOREM].

If the data $\{f, u_0, u_1\}$ *satisfy, for* $m = \left[\dfrac{n}{2}\right] + 1$,

(2. 1) $f = f(x, t) \in \mathcal{E}^2_{[0,T]}[H^m(\Omega)]$ *and especially when* $t = 0$, $f(x, 0)$ *belongs to* $H^{m+1}(\Omega)$

(2. 2) $u_0(x) \in \mathring{H}^{m+3}(\Omega)$

(2. 3) $u_1(x) \in \mathring{H}^{m+2}((\Omega)$

then we have one and only one function $u(x, t)$ *in* $C^2(\overline{\Omega} \times [0, \delta])$ *which solves the problem* (1) *in a classical sense* for some $\delta (0 < \delta < T)$ *which depends on* $f, u_0, u_1, p, q, r, n, \Omega, c_i$.

(proof)

Since the operator $(-\Delta)$ is uniformly elliptic and coercive on $\mathring{H}^r(\Omega)$, for any positive integer α, $(-\Delta)^\alpha$ is coercive on $\mathring{H}^\alpha(\Omega)$. Therefore there exists a positive constant c_α satisfying for any $u \in \mathring{H}^\alpha(\Omega)$,

$$<(-\Delta)^{\alpha}u,u> = ((-\Delta)^{\frac{\alpha}{2}}u,\ (-\Delta)^{\frac{\alpha}{2}}u)_{L^2(\Omega)} \geqq c_\alpha \|u\|^2_{\mathring{H}^{\alpha}(\Omega)},$$

and the eigen functions $\{\varphi_j\}$ of $(-\Delta)^{\alpha}$ consists of a complete base in $\mathring{H}^{\alpha}(\Omega)$.

Here we put $\alpha=\left[\frac{n}{2}\right]+1+3=m+3$, and from now on we fix $\{\varphi_j\}$ as the eigen functions of $(-\Delta)^{m+3}$ and denote $(\cdot,\cdot)$ as $(\cdot,\cdot)_{L(\Omega)}$.

And we set $G(u,u',\nabla u)=c_0u^p+c_1(u')^q+c_2|\nabla u|^{2r}$.

From our assumptions (2.2), (2.3) there exist sequences of real numbers $\{\alpha_j\}$, $\{\beta_j\}$ such that

$$(2.4)\quad u_k(0)=\sum_{j=1}^{k}\alpha_j\varphi_j \longrightarrow u_0 \qquad (s)\ \ in\ \ \mathring{H}^{m+3}(\Omega)$$

$$(2.4)'\quad u_k'(0)=\sum_{j=1}^{k}\beta_j\varphi_j \longrightarrow u_1 \qquad (s)\ \ in\ \ \mathring{H}^{m+2}(\Omega)$$

Now we construct approximating solutions $u_k(x,t)=\sum_{j=1}^{k}\lambda_{k,j}(t)\varphi_j(x)$ in the following manner. $\{\lambda_{k,j}(t)\}$ is the solution of the ordinary Cauchy Problem of the system:

$$(2.5)\quad \begin{cases} (u_k'',(-\Delta)^m\varphi_j)+(-\Delta u_k,(-\Delta)^m\varphi_j) \\ \qquad +(G(u_k,u_k',\nabla u_k),(-\Delta)^m\varphi_j)=(f,(-\Delta)^m\varphi_j) \\ \lambda_{k,j}(0)=\alpha_j\,,\quad \lambda'_{k,j}(0)=\beta_j \qquad (j=1,2,\dots,k) \end{cases}$$

It is well-known that the problem (2.5) is solvable for some interval $[0,\delta_k]$ where δ_k depends on k, and we know

$$u_k(x,t)\in\mathcal{E}^2_{[0,\delta_k]}[\mathring{H}^{m+3}(\Omega)]$$

because of $\varphi_j\in\mathring{H}^{m+3}(\Omega)$ and $\lambda_{k,j}(t)\in C^2[0,\delta_k]$.

Now, we have from (2.5), for $t\in(0,\delta_k)$

$$(u_k'',(-\Delta)^m u_k')+(-\Delta u_k,(-\Delta)^m u_k') \\ \qquad +(G(u_k,u_k',\nabla u_k),(-\Delta)^m u_k')=(f,(-\Delta)^m u_k')$$

From this, it holds by Schwarz inequality

$$\frac{1}{2}\frac{d}{dt}\left\{|(-\Delta)^{\frac{m}{2}}u_k'|^2_{L^2}+|(-\Delta)^{\frac{m+1}{2}}u_k|^2_{L^2}\right\}$$
$$\leqq\frac{1}{2}\left\{|(-\Delta)^{\frac{m}{2}}f|^2_{L^2}+2|(-\Delta)^{\frac{m}{2}}u_k'|^2_{L^2}+|(-\Delta)^{\frac{m}{2}}G(u_k,u_k',\nabla u_k)|^2_{L^2}\right\}$$

Here we have by lemma 3, after considerably tedious calculations,

$$
\begin{aligned}
&|(-\Delta)^{\frac{m}{2}} G(u_k, u_k', \nabla u_k)|_{L^2}^2 \\
&\quad \leqq |c_0| c(p, n, \Omega) |(-\Delta)^{\frac{m}{2}} u_k|_{L^2}^{2p} + |c_1| c(q, n, \Omega) |(-\Delta)^{\frac{m}{2}} u_k'|_{L^2}^{2q} \\
&\qquad + |c_2| c(r, n, \Omega) |(-\Delta)^{\frac{m+1}{2}} u_k|_{L^2}^{4r}
\end{aligned}
$$

Thus we have for $N=\max\{p, q, 2r\}$

$$
\begin{aligned}
&|(-\Delta)^{\frac{m}{2}} G(u_k, u_k', \nabla u_k)|_{L^2}^2 \\
&\quad \leqq c(p, q, r, n, \Omega)\Big(1+|(-\Delta)^{\frac{m+1}{2}} u_k|_{L^2}^2 + |(-\Delta)^{\frac{m}{2}} u_k'|_{L^2}^2\Big)^N
\end{aligned}
$$

Consequently, it follows for $t \in (0, \delta_k)$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{|(-\Delta)^{\frac{m}{2}} u_k'|_{L^2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k|_{L^2}^2\Big\} \\
&\quad \leqq \frac{1}{2} c'(p, q, r, n, \Omega, f)\Big\{1+|(-\Delta)^{\frac{m}{2}} u_k'|_{L^2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k|_{L^2}^2\Big\}^N
\end{aligned}
$$

Therefore by lemma 2, we have

$$
\begin{aligned}
&1+|(-\Delta)^{\frac{m}{2}} u_k'(t)|_{L^2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k(t)|_{L^2}^2 \\
&\leqq \frac{1+|(-\Delta)^{\frac{m}{2}} u_k'(0)|_{L^2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k(0)|_{L^2}^2}{\Big[1-c'(N-1)\Big\{1+|(-\Delta)^{\frac{m}{2}} u_k'(0)|_{L^2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k(0)|_{L^2}^2\Big\}^{N-1} t\Big]^{\frac{1}{N-1}}} \\
&\qquad \text{for} \quad t \in (0, \delta_k)\,.
\end{aligned}
$$

Here we know from our limit relations (2.4), (2.4)′ that there exist positive constants K_1, K_2 such that

$$
\begin{aligned}
&|(-\Delta)^{\frac{m+1}{2}} u_k(0)|_{L^2}^2 \leqq K_1\Big(1+|(-\Delta)^{\frac{m+1}{2}} u_0|_{L^2}^2\Big) \\
&|(-\Delta)^{\frac{m}{2}} u_k'(0)|_{L^2}^2 \leqq K_2\Big(1+|(-\Delta)^{\frac{m}{2}} u_1|_{L^2}^2\Big)
\end{aligned}
$$

independent of k.

From this we obtain,

$$
\begin{aligned}
&|(-\Delta)^{\frac{m+1}{2}} u_k(t)|_{L^2}^2 + |(-\Delta)^{\frac{m}{2}} u_k'(t)|_{L^2}^2 \\
&\leqq \frac{1+K_1+K_2+K_1|(-\Delta)^{\frac{m+1}{2}} u_0|_{L^2}^2 + K_2|(-\Delta)^{\frac{m}{2}} u_1|_{L^2}^2}{\Big[1-c'(N-1)\cdot\Big\{1+K_1+K_2+K_1|(-\Delta)^{\frac{m+1}{2}} u_0|_{L^2}^2 + K_2|(-\Delta)^{\frac{m}{2}} u_1|_{L^2}^2\Big\}^{N-1} t\Big]^{\frac{1}{N-1}}}
\end{aligned}
$$

The last inequality shows $\{u_k(t)\}$ exist and have meaning in the interval $[0, \delta]$ independent of k where δ is a positive constant depending on f, u_0, $u_1, p, q, r, n, \Omega, C_i$, more precisely,

$$\delta=\Big[c'(p,q,r,n,\Omega,f)\cdot(N-1)\Big(1+K_1+K_2+K_1|(-\Delta)^{\frac{m+1}{2}}u_0|_{L2}^2 +K_2|(-\Delta)^{\frac{n}{2}}u_1|_{L2}^2\Big)^{N-1}\Big]^{-1}$$

From above reasoning, we have

$$(2.6)\quad \begin{cases} \sup\limits_k \sup\limits_{t\in[0,r]} |(-\Delta)^{\frac{m+1}{2}}u_k(t)|_{L2}<+\infty \\ \sup\limits_k \sup\limits_{t\in[0,r]} |(-\Delta)^{\frac{u}{2}}u'_k(t)|_{L2}<+\infty \end{cases}$$

where γ is an arbitrarily fixed positive number with $0<\gamma<\delta$.

Secondly, by differentiating both sides of the first equality of (2.5), we get for every $t\in[0,\gamma]$

$$(2.7)\quad (u_k^{(3)}, (-\Delta)^m\varphi_j)+((-\Delta)u_k', (-\Delta)^m\varphi_j)+(D_tG(u_k, u_k', \nabla u_k), (-\Delta)^m\varphi_j) =(f', (-\Delta)^m\varphi_j)\ .$$

From this, we have as in the preceding manner,

$$\frac{1}{2}\frac{d}{dt}\Big\{|(-\Delta)^{\frac{m}{2}}u_k''|_{L2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k'|_{L2}^2\Big\}$$
$$\leqq\frac{1}{2}\Big\{|(-\Delta)^{\frac{m}{2}}f'|_{L2}^2+2|(-\Delta)^{\frac{m}{2}}u_k''|_{L2}^2+|D_t(-\Delta)^{\frac{m}{2}}G(u_k, u_k', \nabla u_k)|_{L2}^2\Big\}.$$

Here we estimate the term

$$|D_t(-\Delta)^{\frac{m}{2}}G(u_k, u_k', \nabla u_k)|_{L2}.$$

At first we have

$$|D_t(-\Delta)^{\frac{m}{2}}(u_k)^p|_{L2}=|(-\Delta)^{\frac{m}{2}}\{pu_k^{p-1}u_k'\}|_{L2}$$
$$\leqq c_1(p,n,\Omega)|(-\Delta)^{\frac{m}{2}}u_k|_{L2}^{p-1}\ |(-\Delta)^{\frac{m}{2}}u_k'|_{L2}$$

as u_k, u_k' belong to $\mathring{H}^m(\Omega)$ respectively.
Therefore as a result of (2.6), we obtain,

$$|D_t(-\Delta)^{\frac{m}{2}}(u_k)^p|_{L2}\leqq c_1(p,n,\Omega,\gamma)\ .$$

Followingly it follows by the same reasoning,

$$|D_t(-\Delta)^{\frac{m}{2}}(u_k')^q|_{L2}=|(-\Delta)^{\frac{m}{2}}\{q(u_k')^{q-1}u_k''\}|_{L2}$$

$$\leqq c_2(q,n,\Omega)|(-\Delta)u_k'|_{L2}^{q-1}|(-\Delta)^{\frac{m}{2}}u_k''|_{L2}$$

$$\leqq c_2(q,n,\Omega,\gamma)|(-\Delta)^{\frac{m}{2}}u_k''|_{L2}$$

For the last term we have,

$$|D_t(-\Delta)^{\frac{m}{2}}|\nabla u|^{2r}\;|_{L2}=|(-\Delta)^{\frac{m}{2}}\Big\{r|\nabla u_k|^{2r-2}\Big[\sum_{i=1}^{n}2(u_k)_{xi}(u_k')_{xi}\Big]\Big\}|_{L2}$$

$$\leqq c_3(r,n,\Omega)|(-\Delta)^{\frac{m+1}{2}}u_k|_{L2}^{2r-1}|(-\Delta)^{\frac{m+1}{2}}u_k'|_{L2}$$

$$\leqq c_3(r,n,\Omega,\gamma)|(-\Delta)^{\frac{m+1}{2}}u_k'|_{L2}\,.$$

From these facts, we can obtain

$$|D_t(-\Delta)^{\frac{m}{2}}G(u_k,u_{kt},\nabla u_k)|_{L2}^2$$

$$\leqq c(p,q,r,n,\Omega,\gamma)\Big\{1+|(-\Delta)^{\frac{m}{2}}u_k''|_{L2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k'|_{L2}^2\Big\}$$

Consequently it follows for $t\in[0,\gamma]$,

$$\frac{d}{dt}\Big\{|(-\Delta)^{\frac{m}{2}}u_k''(t)|_{L2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k'(t)|_{L2}^2\Big\}$$

$$\leqq|(-\Delta)^{\frac{m}{2}}f'(t)|_{L2}^2+c+(2+c)\Big\{|(-\Delta)^{\frac{m}{2}}u_k''(t)|_{L2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k'(t)|_{L2}^2\Big\}.$$

Therefore, by lemma 1, we have

$$|(-\Delta)^{\frac{m}{2}}u_k''(t)|_{L2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k'(t)|_{L2}^2$$

$$\leqq e^{(2+C)}\Big[\Big\{|(-\Delta)^{\frac{m}{2}}u_k''(0)|_{L2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k'(0)|_{L2}^2\Big\}$$

$$+\int_0^t e^{-(2+C)\tau}\{|(-\Delta)f'|_{L2}^2+C\}d\tau\Big]$$

$$\text{for}\quad t\in[0,\gamma].$$

By this inequality, if we could show the boundedness of

$$\Big\{|(-\Delta)^{\frac{m+1}{2}}u_k'(0)|_{L2}\Big\}\,,\quad\Big\{|(-\Delta)^{\frac{m}{2}}u_k''(0)|_{L2}^2\Big\}\,,$$

we can conclude that the following inequalities hold;

$$(2.8)\quad \begin{cases} \sup_k \sup_{t\in[0,\gamma]} |(-\Delta)^{\frac{m+1}{2}} u_k'(t)|_{L2} < +\infty \\ \sup_k \sup_{t\in[0,\gamma]} |(-\Delta)^{\frac{m}{2}} u_k''(t)|_{L2} < +\infty \end{cases} .$$

Since the boundedness of $\left\{|(-\Delta)^{\frac{m+1}{2}} u_k'(0)|_{L2}\right\}$ is evident from our assumption, it suffices to verify $\sup_k |(-\Delta)^{\frac{m}{2}} u_k''(0)|_{L2}$ is finite.

Now, in the equality of (2.5) by letting t tend to 0, we have

$$(2.9)\quad (u_k''(0),\ (-\Delta)^m\varphi_j) + ((-\Delta)u_k(0),\ (-\Delta)^m\varphi_j) + (G(u_k(0),\ u_k'(0),\ \nabla u_k(0)),\ (-\Delta)^m\varphi_j) = (f(0),\ (-\Delta)^m\varphi_j)$$

and by substituting $u_k''(0)$ for φ_j it follows

$$(u_k''(0),\ (-\Delta)^m\ u_k''(0)) = -((-\Delta)u_k(0),\ (-\Delta)^m u_k''(0)) - (G(u_k(0), u_k'(0), \nabla u_k(0)), (-\Delta)^m u_k''(0)) + (f(0), (-\Delta)^m u_k''(0)) .$$

Therefore we have by Schwarz inequality

$$|(-\Delta)^{\frac{m}{2}} u_k''(0)|_{L2} \leq |(-\Delta)^{\frac{m+2}{2}} u_k(0)|_{L2} + |(-\Delta)^{\frac{m}{2}} G(u_k(0), u_k'(0), \nabla u_k(0))|_{L2} + |(-\Delta)^{\frac{m}{2}} f(0)|_{L2} .$$

Since the right hand side of this inequality is bounded from our assumptions (2.4), (2.4)′, we have the boundedness of $\left\{|(-\Delta)^{\frac{m}{2}} u_k''(0)|_{L2}\right\}$.

In the next, by differentiating both sides of the equality (2.7), we get

$$(2.10)\quad (u_k^{(4)}, (-\Delta)^m\varphi_j) + ((-\Delta)u_k'', (-\Delta)^m\varphi_j) + (D_t^2 G(u_k, u_k', \nabla u_k), (-\Delta)^m\varphi_j) = (f'', (-\Delta)^m\varphi_j) .$$

From this we have for $t\in[0,\gamma]$

$$\frac{1}{2}\frac{d}{dt}\left\{|(-\Delta)^{\frac{m}{2}} u_k^{(3)}|_{L2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k''|_{L2}^2\right\} \leq \frac{1}{2}\left\{|(-\Delta)^{\frac{m}{2}} f''|_{L2}^2 + 2|(-\Delta)^{\frac{m}{2}} u_k^{(3)}|_{L2}^2 + |D_t^2(-\Delta)^{\frac{m}{2}} G(u_k, u_k', \nabla u_k)|_{L2}^2\right\} .$$

Therefore by utilizing our results (2.6), (2.8) we get for $t\in[0,\gamma]$

$$\frac{d}{dt}\left\{|(-\Delta)^{\frac{m}{2}} u_k^{(3)}(t)|_{L2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k''(t)|_{L2}^2\right\} \leq \text{const}\left\{|(-\Delta)^{\frac{m}{2}} u_k^{(3)}(t)|_{L2}^2 + |(-\Delta)^{\frac{m+1}{2}} u_k''(t)|_{L2}^2\right\} + \text{const} + |(-\Delta)^{\frac{m}{2}} f''(t)|_{L2}^2$$

where the constants depend on $p, q, r, \gamma, n, \Omega$.

Thus it follows by lemma 1,

$$|(-\Delta)^{\frac{m}{2}}u_k^{(3)}(t)|_{L^2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k''(t)|_{L^2}^2$$
$$\leq \text{const}\left\{|(-\Delta)^{\frac{m}{2}}u_k^{(3)}(0)|_{L^2}^2+|(-\Delta)^{\frac{m+1}{2}}u_k''(0)|_{L^2}^2\right\}+\text{const}\,.$$

Here we show the boundedness of $\left\{|(-\Delta)^{\frac{m}{2}}u_k^{(3)}(0)|_{L^2}\right\}$.

In the equality (2.7), by letting t tend to 0, we get

$$(u_k^{(3)}(0),(-\Delta)^m\varphi_j)+((-\Delta)u_k'(0),(-\Delta)^m\varphi_j)$$
$$+(D_tG(u_k,u_k',\nabla u_k)|_{t=0},(-\Delta)^m\varphi_j)=(f'(0),(-\Delta)^m\varphi_j)\,.$$

Therefore it follows by Schwarz inequality

$$|(-\Delta)^{\frac{m}{2}}u_k^{(3)}(0)|_{L^2}\leq|(-\Delta)^{\frac{m+2}{2}}u_k'(0)|_{L^2}+|(-\Delta)^{\frac{m}{2}}f'(0)|_{L^2}$$
$$+|(-\Delta)^{\frac{m}{2}}\{D_tG(u_k,u_k',\nabla u_k)|_{t=0}\}|_{L^2}\,.$$

The limit relations (2.4), (2.4)′ yields the boundedness of the right hand side of this inequality.

Secondly we show that there exists a bounded subsequence

$$\left\{|(-\Delta)^{\frac{m+1}{2}}u_{k_l}''(0)|_{L^2}\right\}\quad\text{of}\quad\left\{|(-\Delta)^{\frac{m+1}{2}}u_k''(0)|_{L^2}\right\}.$$

For this purpose by setting $\{\varphi_{k_1},\varphi_{k_2},\dots,\varphi_{k_l}\}$ as the eigen functions of eigen value λ_k we choose this last numbers k_l and we consider the sequence $\left\{|(-\Delta)^{\frac{m+1}{2}}u_{k_l}''(0)|_{L^2}\right\}$.

Here we know $(-\Delta)\varphi_{ki}$ also satisfies $(-\Delta)^{m+3}(-\Delta)\varphi_{ki}=\lambda_k(-\Delta)\varphi_{ki}$ because of $\varphi_j\in C^\infty(\Omega)$. Therefore it follows that there exist suitable numbers $(a_j)_l$ such that

$$(-\Delta)\varphi_{ki}=\sum_{j=1}^{l}a_j\varphi_{k_j}\,.$$

From this, we can substitute $(-\Delta)u_{k_l}''(0)$ for φ_{kl} in the relation (2.9), that is the equality

$$(2.9)'\quad (u_{k_l}''(0),(-\Delta)^{m+1}u_{k_l}''(0))+((-\Delta)u_{k_l}(0),(-\Delta)^{m+1}u_{k_l}''(0))$$
$$+(G(u_{k_l}(0),u_{k_l}'(0),\nabla u_{k_l}(0)),(-\Delta)^{m+1}u_{k_l}''(0))$$
$$=(f(0),(-\Delta)^{m+1}u_{k_l}''(0))$$

holds.

Thus again by Schwarz, we have

$$|(-\Delta)^{\frac{m+1}{2}} u_{k_l}''(0)|_{L^2} \leqq |(-\Delta)^{\frac{m+3}{2}} u_{k_l}(0)|_{L^2}$$
$$+|(-\Delta)^{\frac{m+1}{2}} G(u_{k_l}(0), u_{k_l}'(0), \nabla u_{k_l}(0))|_{L^2}$$
$$+|(-\Delta)^{\frac{m+1}{2}} f(0)|_{L^2}.$$

The boundedness of the right hand side of this inequality is the consequence of (2.4), (2.4)′ and thus we obtain

$$|(-\Delta)^{\frac{m+1}{2}} u_{k_l}''(0)|_{L^2} \leqq \text{const} < +\infty.$$

Consequently we get

$$(2.11)\quad \begin{cases} \sup\limits_{k_l} \sup\limits_{t\in[0,\gamma]} |(-\Delta)^{\frac{m+1}{2}} u_{k_l}''(t)|_{L^2} < +\infty \\ \sup\limits_{k_l} \sup\limits_{t\in[0,\gamma]} |(-\Delta)^{\frac{m}{2}} u_{k_l}^{(3)}(t)|_{L^2} < +\infty \end{cases}$$

Consequently, by substituting $\{k\}$ for $\{k_l\}$ and renewing $\{k_l\}$ to $\{k\}$ in the results (2.6), (2.8), (2.11), we have by lemma 4 that there exist a subsequence $\{u_k^*(t)\}$ of $\{u_k(t)\}$ and $u(t)$ which belongs to $\mathcal{E}^1_{[0,\gamma]}[\mathring{H}^{m+1}(\Omega)] \cap \mathcal{E}^2_{[0,\gamma]}[\mathring{H}^m(\Omega)]$ such that

$$u_k^*(t) \longrightarrow u(t) \qquad (w) \quad in \quad \mathring{H}^{m+1}(\Omega)$$
$$u_k^{*\prime}(t) \longrightarrow u'(t) \qquad (w) \quad in \quad \mathring{H}^{m+1}(\Omega)$$
$$u_k^{*\prime\prime}(t) \longrightarrow u''(t) \qquad (w) \quad in \quad \mathring{H}^{m}(\Omega) .$$

And since $m=\left[\frac{n}{2}\right]+1$, we have

$$G(u_k^*(t), u_k^{*\prime}(t), \nabla u_k^*(t)) \longrightarrow G(u(t), u'(t), \nabla u(t))$$
$$\text{for each } t\in[0,\gamma], \text{ and } x\in\Omega,$$

and furthermore from ellipticity of Δ and our results, $u(t)$ should belong to

$$\mathcal{E}^0_{[0,\gamma]}[\mathring{H}^{m+1}(\Omega)\cap H^{m+2}(\Omega)] \cap \mathcal{E}^1_{[0,\gamma]}[\mathring{H}^{m+1}(\Omega)] \cap \mathcal{E}^2_{[0,\gamma]}[\mathring{H}^m(\Omega)] .$$

Consequently, (2) of lemma 3 yields

$$u(x,t) \in \mathcal{E}^0_{[0,\gamma]}[C_0^1(\overline{\Omega})\cap C^2(\overline{\Omega})] \cap \mathcal{E}^1_{[0,\gamma]}[C_0^1(\overline{\Omega})] \cap \mathcal{E}^2_{[0,\gamma]}[C_0^0(\overline{\Omega})] .$$

that is $u(x,t)$ is a solution of the problem (1), in the interval $[0,\gamma]$.

On the other hand, we can extend its existence interval to $[0, \delta)$ from uniqueness (Theorem 1) and the arbitrariness of $\gamma (0<\gamma<\delta)$.
This completes the proof of the [Theorem]. (q. e. d)

[REMARK 3]

If we consider the evolution equation of the form

$$\text{(a)} \quad u''(t)+L(D)u(t)+G(u, Du, \ldots\ldots, D^m u, u')=f(t)$$

where $L(D)=\sum_{|\alpha|\leq 2m} a_\alpha D^\alpha$ is uniformly elliptic and coercive on $\mathring{H}^m(\Omega)$, and G is a multi-polynomial of $u, Du, \ldots\ldots, D^m u, u'$, we can construct a local classical solution by an analogous arguement.

[REMARK 4]

For the first order evolution equation of the form

$$\text{(b)} \quad u'(t)+L(D)u(t)+G(u, Du, \ldots\ldots, D^m u)=f(t)$$

the local existence of a classical solution has already obtained, (Y. Ebihara [6].)

(Outline of the proof of [Remark 3])

Since $L(D)$ is coercive on $\mathring{H}^m(\Omega)$, $L^p(D)$ is coercive on $\mathring{H}^{pm}(\Omega)$ (lemma 5) where p is a minimum positive integer with

$$mp \geqq \left[\frac{n}{2}\right]+1 .$$

We construct approximating solutions in the following manner

$$\begin{cases} (u_k''(t), L^p\varphi_j)+(Lu_k(t), L^p\varphi_j)+(G(\cdots), L^p\varphi_j) \\ \quad =(f(t), L^p\varphi_j) \qquad (j=1, 2, \ldots, k) \\ u_k(0)=\sum_{j=1}^k \alpha_j\varphi_j, \quad u_k'(0)=\sum_{j=1}^k \beta_j\varphi_j \end{cases}$$

where $\{\varphi_j\}$ are eigen functions of $L^{p+3}(D)$ and

$$\begin{aligned} u_k(0) &\longrightarrow u_0 && (s) \quad in \quad \mathring{H}^{m(p+3)}(\Omega) \\ u_k'(0) &\longrightarrow u_1 && (s) \quad in \quad \mathring{H}^{m(p+2)}(\Omega) . \end{aligned}$$

Then we have necessary estimates for $\{u_k^{(i)}(t)\}$ $(i=0, 1, 2, 3)$ to obtain the conclusion by an analogous caluculation.

Acknowledgement

The author wishes to express his sincere gratitude to Profs. A. Ono,

T. Nanbu and his colleagues for their kind advices and constant encouragements.

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