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the initial-boundary value problem for $D^2_t u - \Delta u + (D_t u)^3 = 0$

Kato, Hisako
College of General Education, Kyushu University

Nanbu, Tokumori
College of General Education, Kyushu University

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**The existence of a local classical solution
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 for $D_t^2 u - \Delta u + (D_t u)^3 = 0$**

by

Hisako KATO and Tokumori NANBU

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§ 1 Problems and notations

J. Sather proved the existence of a global classical solution for the initial-boundary value problem for $D_t^2 u - \Delta u + u^3 = f$ ([5]). For the equation considered here, will be able to verified the results analogous to the ones that J. Sather established? This problem was proposed by Lions in his book ([2]). In this paper we will show the existence of the classical but local solution $u = u(x, t)$ satisfying the following equation and conditions:

- (1.1) $D_t^2 u - \Delta u + (D_t u)^3 = 0, \quad x \in \mathcal{Q}, \quad t > 0$
- (1.2) $u(x, 0) = u_0(x) \quad x \in \mathcal{Q}$
- (1.3) $D_t u(x, 0) = (D_t u)_0(x) \quad x \in \mathcal{Q}$
- (1.4) $u(x, t) = 0 \quad x \in \partial \mathcal{Q}, \quad t > 0.$

Here $u_0(x)$, $(D_t u)_0(x)$ are the data of the problem, and \mathcal{Q} is a bounded domain in R^3 with a sufficiently smooth boudary $\partial \mathcal{Q}$.

In this paper we shall use the following notations:

$$(u, v) = \int_{\mathcal{Q}} u(x)v(x) \, dx, \quad \|u\| = \sqrt{(u, u)}$$

$$(\nabla u, \nabla v) = \sum_{i=1}^3 \int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx, \quad \|\nabla u\| = \sqrt{(\nabla u, \nabla u)}$$

$$\|u\|_p = \left\{ \int_{\mathcal{Q}} |u|^p \, dx \right\}^{1/p} \quad (p > 1, p \neq 2).$$

Sobolev space $W_2^1(\mathcal{Q})$ is the Hilbert space with the inner product $(u, v) + (\nabla u, \nabla v)$. The space $\overset{\circ}{W}_2^1(\mathcal{Q})$ is the closure of $C_0^\infty(\mathcal{Q})$ in $W_2^1(\mathcal{Q})$.

§ 2 A priori estimates

We have the following a priori estimates.

(2.1) $\|D_t u\|^2 + \|\nabla u\|^2 \leq \|(D_t u)_0\|^2 + \|\nabla u_0\|^2 \equiv C_0.$

$$(2.2) \quad \|D_t^2 u\|^2 + \|\nabla D_t u\|^2 \leq \| (D_t^2 u)_o \|^2 + \|\nabla (D_t u)_o\|^2 \equiv C_1.$$

$$(2.3) \quad \|D_t^3 u\|^2 + \|\nabla D_t^2 u\|^2 \leq$$

$$\frac{\| (D_t^3 u)_o \|^2 + \|\nabla (D_t^2 u)_o\|^2}{\left\{ 1 - 6 \left(\frac{4}{\sqrt{3}} \right)^3 T \left(\| (D_t^2 u)_o \|^2 + \|\nabla (D_t u)_o\|^2 \right)^{1/2} \left(\| (D_t^2 u)_o \|^2 + \|\nabla (D_t^2 u)_o\|^2 \right)^{1/2} \right\}^2} \equiv C_2,$$

where a positive number T is chosen sufficiently small to hold the inequality

$$1 - 6 \left(\frac{4}{\sqrt{3}} \right)^3 T \left(\| (D_t^2 u)_o \|^2 + \|\nabla (D_t u)_o\|^2 \right)^{1/2} \left(\| (D_t^2 u)_o \|^2 + \|\nabla (D_t^2 u)_o\|^2 \right)^{1/2} > 0.$$

$$(2.4) \quad \|D_t^4 u\|^2 + \|\nabla D_t^3 u\|^2 \leq \{C_3 T + \| (D_t^4 u)_o \|^2 + \|\nabla (D_t^3 u)_o\|^2\} e^{C_3 T} \equiv C_3,$$

where C_3' and C_3'' are constants depending only on C_1 and C_2 .

$$(2.5) \quad \|D_t^5 u\|^2 + \|\nabla D_t^4 u\|^2 \leq \{C_4 T + \| (D_t^5 u)_o \|^2 + \|\nabla (D_t^4 u)_o\|^2\} e^{C_4 T} \equiv C_4,$$

where C_4' and C_4'' are constants depending only on C_1 , C_2 and C_3 .

In the above $(D_t^k u)_o$ denotes the function $D_t^k u(x, 0)$.

PROOF. Inequalities (2.1) and (2.2) are easily verified. Now, differentiate the equation (1.1) two times with respect to t , multiply by $D_t^3 u$ and integrate over Ω . Then it follows that

$$(D_t^4 u, D_t^3 u) - (\Delta D_t^2 u, D_t^3 u) + (D_t^2 (D_t u)^3, D_t^3 u) = 0.$$

Hence we have

$$\begin{aligned} & \frac{d}{dt} \left\{ \|D_t^3 u\|^2 + \|\nabla D_t^2 u\|^2 \right\} + 6 \int_{\Omega} (D_t u)^2 (D_t^3 u)^2 dx \\ &= -12 \int_{\Omega} D_t u (D_t^2 u)^2 D_t^3 u dx \\ &\leq 12 \|D_t u\|_6 \|D_t^2 u\|_6^2 \|D_t^3 u\| \\ &\leq 12 \left(\frac{4}{\sqrt{3}} \right)^3 \|\nabla D_t u\| \|\nabla D_t^2 u\|^2 \|D_t^3 u\| \\ &\leq 12 \left(\frac{4}{\sqrt{3}} \right)^3 \left(\| (D_t^2 u)_o \|^2 + \|\nabla (D_t u)_o\|^2 \right)^{1/2} \left(\|D_t^3 u\|^2 + \|\nabla (D_t^2 u)_o\|^2 \right)^{3/2}. \end{aligned}$$

In the above we integrated by parts, since u vanishes on the boundary, and applied Hölder's inequality and Sobolev's imbedding theorem ([3]).

We set $C = 12 \left(\frac{4}{\sqrt{3}} \right)^3 \left(\| (D_t^2 u)_o \|^2 + \|\nabla (D_t u)_o\|^2 \right)^{1/2}$ and $y(t) = \|D_t^3 u\|^2 + \|\nabla D_t^2 u\|^2$. Then it follows that $-\frac{dy}{dt} \leq C(y(t))^{3/2}$, which implies

$$\sqrt{y(t)} \leq \frac{\sqrt{y(0)}}{1 - 1/2 CT\sqrt{y(0)}} .$$

This inequality gives (2.3). Next we show the inequality (2.4). Differentiate the equation (1.1) three times, multiply by $D_t^4 u$ and integrate over Ω , then it follows that

$$(D_t^5 u, D_t^4 u) - (\Delta D_t^3 u, D_t^4 u) + (D_t^3(D_t u))^3, D_t^4 u = 0.$$

Hence we have

$$\begin{aligned} & -\frac{d}{dt} \left\{ \|D_t^4 u\|^2 + \|\nabla D_t^3 u\|^2 \right\} + 6 \int_{\Omega} (D_t u)^2 (D_t^4 u)^2 dx \\ & \leq 12 \|D_t^2 u\|_6^3 \|D_t^4 u\| + 36 \|D_t u\|_6 \|D_t^2 u\|_6 \|D_t^3 u\|_6 \|D_t^4 u\| \\ & \leq 12 \left(4/\sqrt{3}\right)^3 \left\{ \|\nabla D_t^2 u\|^3 \|D_t^4 u\| + 3 \|\nabla D_t u\| \|\nabla D_t^2 u\| \|\nabla D_t^3 u\| \|D_t^4 u\| \right\} \\ & \leq 6 \left(4/\sqrt{3}\right)^3 \left\{ \|\nabla D_t^2 u\|^4 + \|\nabla D_t^2 u\|^2 \|D_t^4 u\|^2 \right. \\ & \quad \left. + 3 \|\nabla D_t u\| \|\nabla D_t^2 u\| (\|\nabla D_t^2 u\|^2 + \|D_t^4 u\|^2) \right\}. \end{aligned}$$

These relations give (2.4). Finally, an argument analogous to the above gives the inequality (2.5).

§ 3 Estimates of approximate solutions

Consider the eigenfunctions $\{\psi_k\}$ for the Laplace operator Δ with zero boundary conditions; $\psi_k \in \overset{\circ}{W}_2^1(\Omega)$,

$$\Delta \psi_k = \mu_k \psi_k \text{ in } \Omega \quad (k = 1, 2, \dots).$$

With respect to the regularity of the eigenfunctions ψ_k , it is well known that $\{\psi_k\}$ is involved in the space $W_2^s(\Omega)$ for the bounded domain Ω with sufficiently smooth boundary. $\{\psi_k: k = 1, 2, \dots\}$ is a complete orthonormal system in $L^2(\Omega)$

Next we introduce the spaces of admissible initial data.

$$\begin{aligned} \{\phi_k: k = 1, 2, \dots\} & \equiv \text{orthonormalization of } \{\psi_k\} \text{ in } W_2^s(\Omega). \\ \{\chi_k: k = 1, 2, \dots\} & \equiv \text{orthonormalization of } \{\psi_k\} \text{ in } W_2^1(\Omega) \\ V_0 & \equiv \text{closure of } \left\{ \sum_{k=1}^m \alpha_k \phi_k: \alpha_k \text{ is a real number} \right\} \text{ in } W_2^s(\Omega) \\ V_1 & \equiv \text{closure of } \left\{ \sum_{k=1}^m \beta_k \chi_k: \beta_k \text{ is a real number} \right\} \text{ in } W_2^1(\Omega) \end{aligned}$$

For any element u_0 belonging to the space V_0 it holds that

$$u_0 = \sum_{k=1}^{\infty} \lambda_k (u_0, \phi_k)_{W_2^6(\mathcal{Q})} \phi_k,$$

where ϕ_k is a linear combination of $\psi_j (j = 1, 2, \dots, k)$. Hence there exists the linear combination of $\phi_k (k = 1, 2, \dots, m)$ such that

$$\left\| \sum_{k=1}^m \lambda_k^m \phi_k - u_0 \right\|_{W_2^6(\mathcal{Q})} \longrightarrow 0 \quad (m \longrightarrow \infty).$$

By the argument analogous to the above it follows that for any element $(D_t u)_0$ belonging to the space V_1 there exists the linear combination of $\phi_k (k = 1, 2, \dots, m)$ such that

$$\left\| \sum_{k=1}^m \nu_k^m \phi_k - (D_t u)_0 \right\|_{W_2^4(\mathcal{Q})} \longrightarrow 0 \quad (m \longrightarrow \infty)$$

V_0 and V_1 are Hilbert spaces, and are dense linear subsets in $L^2(\mathcal{Q})$.

We turn to estimate the approximate solutions;

$$u_m = u_m(x, t) = \sum_{k=1}^m \lambda_k^m(t) \phi_k(x) \quad (m = 1, 2, \dots),$$

which are determined by the following system of differential equations.

$$(3.1) \quad (D_t^2 u_m, \phi_k) + (\nabla u_m, \nabla \phi_k) + ((D_t u_m)^3, \phi_k) = 0, \quad (k = 1, 2, \dots, m),$$

or

$$(3.2) \quad \frac{d^2 \lambda_k^m}{dt^2} + \sum_{j=1}^m \lambda_j^m(t) (\nabla \psi_j, \nabla \phi_k) + \left(\left\{ \sum_{j=1}^m \frac{d\lambda_j^m}{dt} \psi_j \right\}^3, \phi_k \right) = 0, \quad (k = 1, 2, \dots, m).$$

Here, initial data $\lambda_k^m(0)$, $D_t \lambda_k^m(0)$ are chosen in such a way that as $m \rightarrow \infty$ we have

$$(3.3) \quad u_{m0} = u_m(x, 0) = \sum_{k=1}^m \lambda_k^m(0) \phi_k(x) \longrightarrow u_0(x) \text{ in } W_2^6(\mathcal{Q}),$$

$$(3.4) \quad (D_t u_m)_0 = D_t u_m(x, 0) = \sum_{k=1}^m D_t \lambda_k^m(0) \phi_k(x) \rightarrow (D_t u)_0(x) \text{ in } W_2^4(\mathcal{Q})$$

$$(3.5) \quad u_0 \in V_0, \quad (D_t u)_0 \in V_1.$$

As is indicated later, we have

$$(3.6) \quad \left| \frac{d\lambda_k^m}{dt} \right|, \left| \sum_{j=1}^m \lambda_j^m(t) (\nabla \psi_j, \nabla \phi_k) \right|, \left| \left(\sum_{j=1}^m \frac{d\lambda_j^m}{dt} \psi_j, \phi_k \right) \right| \leq C,$$

where C is a constant independent of k , m and t .

Hence $\lambda_k^m(t)$ ($k = 1, 2, \dots, m$) are defined in $[0, T]$ for any $T > 0$.

We have the following estimations.

$$(3.7) \quad \| (D_t^{k+1} u_m)_o \|, \| \nabla (D_t^k u_m)_o \| \leq C, \quad (k = 0, 1, 2, 3, 4),$$

where C is a constant independent of m .

PROOF. Denote by P_m the orthogonal projection onto m dimensional subspace of $L^2(\mathcal{Q})$ with basis $\{\psi_1, \psi_2, \dots, \psi_m\}$.

Since $u_{m_0} \rightarrow u_o$ in $W_2^6(\mathcal{Q})$ and $(D_t u_m)_o \rightarrow (D_t u)_o$ in $W_2^4(\mathcal{Q})$, we have

$$(3.8) \quad \| (D_t u_m)_o \|, \| \nabla u_{m_0} \| \leq C,$$

$$(3.9) \quad |D_x^\alpha u_{m_0}(x)| \leq C \| u_{m_0} \|_{W_2^6(\mathcal{Q})} \leq C, \quad x \in \overline{\mathcal{Q}} \quad |\alpha| \leq 4,$$

$$(3.10) \quad |D_x^\alpha (D_t u_m)_o(x)| \leq C \| (D_t u_m)_o \|_{W_2^4(\mathcal{Q})} \leq C, \quad x \in \overline{\mathcal{Q}} \quad |\alpha| \leq 2.$$

Here C is a constant independent of m . To prove (3.7), we will frequently use (3.9) and (3.10).

The differential equations (3.1) give the following equality.

$$(D_t^2 u_m)_o - \Delta u_{m_0} + P_m (D_t u_m)_o^3 = 0.$$

Here we applied the equality $P_m \Delta w = \Delta P_m w$ for any element $w \in W_2^2(\mathcal{Q}) \cap \dot{W}_2^1(\mathcal{Q})$. Hence we have

$$(3.11) \quad \| (D_t^2 u_m)_o \| \leq \| \Delta u_{m_0} \| + \| (D_t u_m)_o^3 \| \leq C + \left(4/\sqrt{3}\right)^3 \| \nabla (D_t u_m)_o \|^3 \leq C,$$

$$(3.12) \quad \| \nabla (D_t u_m)_o \| \leq C.$$

Also, (3.1) gives

$$(D_t^3 u_m)_o - \Delta (D_t u_m)_o + 3P_m (D_t u_m)_o^2 (D_t^2 u_m)_o = 0$$

and hence we have

$$(3.13) \quad \| (D_t^3 u_m)_o \| \leq \| \Delta (D_t u_m)_o \| + C \| (D_t^2 u_m)_o \| \leq C$$

$$(3.14) \quad \begin{aligned} \| \nabla (D_t^2 u_m)_o \| &\leq C \| \Delta (D_t^2 u_m)_o \| = C \| \Delta (\Delta u_{m_0} - P_m (D_t u_m)_o^3) \| \\ &\leq C \{ \| \Delta^2 u_{m_0} \| + \| \Delta (D_t u_m)_o^3 \| \} \leq C + C \{ \| \Delta (D_t u_m)_o \} \| (D_t u_m)_o^2 \| \\ &\quad + \| (D_t u_m)_o \{ \nabla (D_t u_m)_o \}^2 \| \leq C. \end{aligned}$$

In (3.14) we applied the relation $\| \nabla w \| \leq C \| \Delta w \|$ for any element $w \in W_2^2(\mathcal{Q}) \cap \dot{W}_2^1(\mathcal{Q})$.

Similarly, (3.1) gives the following relation

$$(D_t^4 u_m)_o - \Delta (D_t^2 u_m)_o + 6P_m (D_t u_m)_o (D_t^2 u_m)_o^2 + 3P_m (D_t u_m)_o^2 (D_t^3 u_m)_o = 0.$$

Hence we have

$$(3.15) \quad \begin{aligned} \| (D_t^4 u_m)_o \| &\leq \| \Delta (D_t^2 u_m)_o \| + C \{ \| (D_t u_m)_o (D_t^2 u_m)_o^2 \| + \| (D_t u_m)_o^2 (D_t^3 u_m)_o \| \} \\ &\leq C + C \{ \| (D_t^2 u_m)_o \| + \| (D_t^3 u_m)_o \| \} \leq C + C \| \nabla (D_t^2 u_m)_o \|^2 \leq C, \end{aligned}$$

$$\begin{aligned}
(3.16) \quad & \| \mathcal{F}(D_t^2 u_m)_0 \| \leq C \| \mathcal{A}(D_t^2 u_m)_0 \| \\
& \leq C \| \mathcal{A}(\mathcal{A}(D_t u_m)_0 - 3P_m(D_t u_m)_0^2)(D_t^2 u_m)_0 \| \\
& \leq C \{ \| \mathcal{A}^2(D_t u_m)_0 \| + \| \mathcal{A}[(D_t u_m)_0^2](D_t^2 u_m)_0 \| \} \\
& \leq C + C \{ \| \mathcal{A}(D_t u_m)_0 \| \| (D_t u_m)_0 \| \| (D_t^2 u_m)_0 \| \\
& \quad + \| (D_t u_m)_0^2 \| \| \mathcal{A}(D_t^2 u_m)_0 \| + \| (D_t u_m)_0 \| \| \mathcal{F}(D_t u_m)_0 \| \| \mathcal{F}(D_t^2 u_m)_0 \| \\
& \quad + \| (D_t^2 u_m)_0 \| \| \mathcal{F}(D_t u_m)_0 \|^2 \} \\
& \leq C + C \{ \| (D_t^2 u_m)_0 \| + \| \mathcal{A}(D_t^2 u_m)_0 \| + \| \mathcal{F}(D_t^2 u_m)_0 \| \} \leq C,
\end{aligned}$$

Finally, (3.1) gives the equality;

$$\begin{aligned}
& (D_t^5 u_m)_0 - \mathcal{A}(D_t^3 u_m)_0 + 6P_m(D_t^2 u_m)_0^3 + 18P_m(D_t u_m)_0(D_t^2 u_m)_0(D_t^3 u_m)_0 \\
& \quad + 3P_m(D_t u_m)_0^2(D_t^4 u_m)_0 = 0.
\end{aligned}$$

Hence we have firstly

$$\begin{aligned}
(3.17) \quad & \| (D_t^5 u_m)_0 \| \leq \| \mathcal{A}(D_t^3 u_m)_0 \| + C \{ \| (D_t^2 u_m)_0^3 \| \\
& \quad + \| (D_t u_m)_0(D_t^2 u_m)_0(D_t^3 u_m)_0 \| + \| (D_t u_m)_0^2(D_t^4 u_m)_0 \| \} \\
& \leq C + C \{ \| \mathcal{F}(D_t u_m)_0 \| \| \mathcal{F}(D_t^2 u_m)_0 \| \| \mathcal{F}(D_t^3 u_m)_0 \| \\
& \quad + \| \mathcal{F}(D_t^2 u_m)_0 \|^3 + \| (D_t^4 u_m)_0 \| \} \leq C.
\end{aligned}$$

Secondly we have

$$\begin{aligned}
(3.18) \quad & \| \mathcal{F}(D_t^4 u_m)_0 \| \leq C \| \mathcal{A}(D_t^4 u_m)_0 \| \\
& \leq C \{ \| \mathcal{A}^2(D_t^2 u_m)_0 \| + \| \mathcal{A}P_m[(D_t u_m)_0(D_t^2 u_m)_0^2] \| + \| \mathcal{A}P_m[(D_t u_m)_0^2(D_t^3 u_m)_0] \| \}
\end{aligned}$$

Here, each term of the right side is estimated as follows.

Noting that

$$\mathcal{A}(D_t u_m)_0^3 = 3\mathcal{A}(D_t u_m)_0(D_t u_m)_0^2 + 6(D_t u_m)_0(\mathcal{F}(D_t u_m)_0)^2 \in W_2^1(\Omega) \cap \dot{W}_2^1(\Omega),$$

we have

$$\begin{aligned}
& \| \mathcal{A}^2(D_t^2 u_m)_0 \| \leq \| \mathcal{A}^3 u_m \| + \| \mathcal{A}^2 P_m(D_t u_m)_0^3 \| \leq C + \| \mathcal{A}^2(D_t u_m)_0^3 \| \\
& \leq C + C \| (D_t u_m)_0 \|_{W_2^1(\Omega)} \leq C, \\
& \| \mathcal{A}P_m[(D_t u_m)_0^2(D_t^3 u_m)_0] \| \leq \| \mathcal{A}[(D_t u_m)_0^2(D_t^3 u_m)_0] \| \leq C,
\end{aligned}$$

and by an application of Sobolev's imbedding theorem:

$$\| \mathcal{A}(D_t^2 u_m)_0 \|_6 \leq C \| \mathcal{F}(\mathcal{A}(D_t^2 u_m)_0) \|; \quad \| \mathcal{F}(D_t^2 u_m)_0 \|_6 \leq C \| \mathcal{F}(D_t^2 u_m)_0 \|_{W_2^1(\Omega)},$$

we have

$$\begin{aligned}
& \| \mathcal{A}P_m[(D_t u_m)_0(D_t^2 u_m)_0^2] \| \leq \| \mathcal{A}[(D_t u_m)_0(D_t^2 u_m)_0^2] \| \\
& \leq \| \mathcal{A}(D_t u_m)_0 \| \| (D_t^2 u_m)_0^2 \| + 2 \| (D_t u_m)_0(D_t^2 u_m)_0 \mathcal{A}(D_t^2 u_m)_0 \| \\
& \quad + 2 \| (D_t u_m)_0(\mathcal{F}(D_t^2 u_m)_0)^2 \| + 4 \| (D_t u_m)_0(\mathcal{F}(D_t u_m)_0)\mathcal{F}(D_t^2 u_m)_0 \|
\end{aligned}$$

$$\begin{aligned}
 &\leq C(\| \nabla(D_t^2 u_m)_0 \|^2 + \| \nabla(D_t^2 u_m)_0 \| \| \nabla(\Delta(D_t^2 u_m)_0) \| \\
 &\quad + \| \nabla(D_t^2 u_m)_0 \|_{W_2^1(\Omega)}^2 + \| \nabla(D_t^2 u_m)_0 \|) \\
 &\leq C + C(\| \Delta^2(D_t^2 u_m)_0 \| + \| (D_t^2 u_m)_0 \|_{W_2^2(\Omega)}^2) \leq C + C \| \Delta(D_t^2 u_m)_0 \|^2 \leq C.
 \end{aligned}$$

Consequently estimates (3.7) are established.

Moreover, we have

$$(3.19) \quad \begin{cases} (D_t^2 u_m)_0 = \Delta u_{m0} - P_m(D_t u_m)_0^3 \longrightarrow \Delta u_0 - (D_t u)_0^3 \text{ in } L^2(\Omega), \\ \nabla(D_t u_m)_0 \longrightarrow \nabla(D_t u)_0 \text{ in } L^2(\Omega), \\ (D_t^2 u_m)_0 = \Delta(D_t u_m)_0 - 3P_m((D_t u_m)_0^2 (D_t^2 u_m)_0) \\ \longrightarrow \Delta(D_t u)_0 - 3(D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\} \text{ in } L^2(\Omega), \\ \text{and} \\ \nabla(D_t^2 u_m)_0 = \nabla(\Delta u_{m0} - P_m(D_t u_m)_0^3) \longrightarrow \nabla(\Delta u_0) - \nabla(D_t u)_0^3 \text{ in } L^2(\Omega), \end{cases}$$

Let's verify (3.19)

$$\begin{aligned}
 \| (D_t^2 u_m)_0 - \{\Delta u_0 - (D_t u)_0^3\} \| &= \| \Delta u_{m0} - P_m(D_t u_m)_0^3 - \{\Delta u_0 - (D_t u)_0^3\} \| \\
 &\leq \| \Delta u_{m0} - \Delta u_0 \| + \| P_m(D_t u_m)_0^3 - P_m(D_t u)_0^3 \| + \| P_m(D_t u)_0^3 - (D_t u)_0^3 \|.
 \end{aligned}$$

Here, $\| P_m(D_t u_m)_0^3 - P_m(D_t u)_0^3 \| \leq C \| (D_t u_m)_0 - (D_t u)_0 \|$.

$$\begin{aligned}
 \| (D_t^2 u_m)_0 - [\Delta(D_t u)_0 - 3(D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\}] \| & \\
 &\leq \| \Delta(D_t u_m)_0 - \Delta(D_t u)_0 \| + C \| P_m((D_t u_m)_0^2 (D_t^2 u_m)_0) - (D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\} \| \\
 &\leq \| \Delta(D_t u_m)_0 - \Delta(D_t u)_0 \| + C(\| (D_t u_m)_0^2 (D_t^2 u_m)_0 - (D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\} \| \\
 &\quad + \| P_m[(D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\}] - (D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\} \|).
 \end{aligned}$$

Here,

$$\begin{aligned}
 &\| (D_t u_m)_0^2 (D_t^2 u_m)_0 - (D_t u)_0^2 \{\Delta u_0 - (D_t u)_0^3\} \| \\
 &\leq \| \{(D_t^2 u_m)_0 - \{\Delta u_0 - (D_t u)_0^3\}\} (D_t u_m)_0^2 \| + \| \{(D_t u_m)_0^2 - (D_t u)_0^2\} \{\Delta u_0 - (D_t u)_0^3\} \| \\
 &\leq C(\| (D_t^2 u_m)_0 - \{\Delta u_0 - (D_t u)_0^3\} \| + \| (D_t u_m)_0 - (D_t u)_0 \|). \\
 &\| \nabla(D_t^2 u_m)_0 - \{\nabla(\Delta u_0) - \nabla(D_t u)_0^3\} \| \\
 &\leq \| \nabla(\Delta u_{m0} - \Delta u_0) \| + \| \nabla P_m(D_t u_m)_0^3 - \nabla(D_t u)_0^3 \| \\
 &\leq \| \nabla(\Delta u_{m0} - \Delta u_0) \| + C(\| \Delta P_m(D_t u_m)_0^3 - \Delta(D_t u)_0^3 \| \\
 &\leq \| \nabla(\Delta u_{m0} - \Delta u_0) \| + C(\| P_m(3\Delta(D_t u_m)_0 (D_t u_m)_0^2 + 6(D_t u_m)_0 \{\nabla(D_t u_m)_0\}^2 \\
 &\quad - 3\Delta(D_t u)_0 (D_t u)_0^2 - 6(D_t u)_0 \{\nabla(D_t u)_0\}^2 \| + \| P_m\{\Delta(D_t u)_0^3\} - \Delta(D_t u)_0^3 \|).
 \end{aligned}$$

Estimates of the above give (3.19).

By a priori estimates in § 2, estimates (3.7) and relations (3.19), the approximate solutions are estimated as follows.

PROPOSITION 1. *It holds that for $k=0, 1, 3, 4$*

$$\|D_t^{k+1}u_m\|^2 + \|\nabla D_t^k u_m\|^2 \leq C(\|(D_t^{k+1}u_m)_0\|^2 + \|\nabla(D_t^k u_m)_0\|^2) + C \leq C$$

and specially for $k=2$

$$\begin{aligned} & \|D_t^3 u_m\|^2 + \|\nabla D_t^2 u_m\|^2 \\ \leq & \frac{\|(D_t^3 u_m)_0\|^2 + \|\nabla(D_t^2 u_m)_0\|^2}{(1-6(4/\sqrt{3})^3 T_0(\|(D_t^2 u_m)_0\|^2 + \|\nabla(D_t u_m)_0\|^2)^{1/2}(\|(D_t^3 u_m)_0\|^2 + \|\nabla D_t^2 u_m)_0\|^2)^{1/2}}, \end{aligned}$$

where a positive number T_0 is chosen sufficiently small to satisfy the inequality

$$\begin{aligned} & 1-6\left(4/\sqrt{3}\right)^3 T_0(\|\Delta u_0 - (D_t u)_0^3\|^2 + \|\nabla(D_t u)_0\|^2)^{1/2} \\ & \times (\|\Delta(D_t u)_0 - 3(D_t u)_0^2(\Delta u_0 - (D_t u)_0^3)\|^2 + \|\nabla(\Delta u_0) - \nabla(D_t u)_0^3\|^2)^{1/2} > 0. \end{aligned}$$

REMARK. As is explained before, in the differential equations (3.2) $\lambda_k^m(t)$ ($k=1, 2, \dots, m$) are determined in $[0, T]$ for any finite $T > 0$, from the following boundedness

$$\begin{aligned} \left| \frac{d\lambda_k^m(t)}{dt} \right|^2 &= \|D_t u_m\|^2 \leq C, \quad t \in [0, T], \quad (m=1, 2, \dots, k=1, 2, \dots, m), \\ \left| \sum_{j=1}^m \lambda_j^m(t) (\nabla \psi_j, \nabla \psi_k) \right| &= |(\nabla u_m, \nabla \psi_k)| \leq \| \Delta u_m \| \| \psi_k \| \\ &\leq \| D_t^2 u_m \| + \| P_m(D_t u_m) \|^3 \leq C + C \| \nabla(D_t u_m) \|^3 \leq C \end{aligned}$$

and

$$\left| \left(\sum_{j=1}^m \frac{d\lambda_j^m}{dt} \psi_j \right)^3, \psi_k \right| = |(\{D_t u_m\}^3, \psi_k)| \leq \| (D_t u_m)^3 \| \| \psi_k \| \leq C.$$

Here, we applied the proposition 1 for $k=0$ and 1.

§ 4 Compactness of approximate solutions and some properties of the limit function

Now we introduce some conceptions. Let \mathcal{H} be a separable Hilbert space. Here $L^2(\mathcal{Q})$, Sobolev spaces $W_2^m(\mathcal{Q})$ ($m=1, 2, 3, \dots$) are considered. We denote by $w: [0, T] \rightarrow \mathcal{H}$ the function defined in $[0, T]$ with values $w(t)$ in \mathcal{H} . The function w is said to have the *strong derivative* in $[0, T]$ if there exists a function $v: [0, T] \rightarrow \mathcal{H}$ such that $\| \frac{1}{h} \{w(t+h) - w(t)\} - v(t) \|_{\mathcal{H}} \rightarrow 0$ for any $t \in [0, T]$, as $h \rightarrow 0$. The function w is said to be in the class $C^n([0, T]: \mathcal{H})$ if there exist strong derivatives $D^k w$ ($k=1, 2, \dots$

m) and they are continuous i.e. $\|D^k w(t+h) - D^k w(t)\|_{\mathcal{X}} \rightarrow 0$ for any $t \in [0, T]$, as $h \rightarrow 0$. In this section we will apply *the following relations*. ([1], [7], [9].)

If $w \in C([0, T] : \mathcal{X})$ it holds $\frac{d}{dt} (B) \int_0^t w(s) ds = w(t)$ for any $t \in [0, T]$, which implies $\|\frac{1}{h} (B) \int_0^{t+h} w(s) ds - w(t)\|_{\mathcal{X}} \rightarrow 0$, as $h \rightarrow 0$. Here, (B) denotes the Bochner integral.

In the first place we have *the following convergence*:

$$(4.1) \quad D_t^k u_m \longrightarrow D_t^k u \text{ in } L^2(\Omega \times [0, T_0]), \text{ as } m \longrightarrow \infty (0 \leq k \leq 4),$$

where $D_t^k u$ is the generalized derivative.

In fact, the proposition 1 gives the boundedness:

$$\|D_t^k u_m\|_{L^2(\Omega \times [0, T_0])}, \|D_t(D_t^k u_m)\|_{L^2(\Omega \times [0, T_0])}, \|\nabla(D_t^k u_m)\|_{L^2(\Omega \times [0, T_0])} \leq C \quad (0 \leq k \leq 4).$$

Hence by Rellich's theorem it is possible to extract a subsequence which is Cauchy sequence in $L^2(\Omega \times [0, T_0])$.

In the next place we have *the following relations*.

$$(4.2) \quad \begin{cases} D_t^k u_m(\cdot, t) \longrightarrow D_t^k u(\cdot, t) \text{ in } L^2(\Omega), \text{ uniformly for } t \in [0, T_0]. (0 \leq k \leq 3) \\ D_t^k u(\cdot, t) - (D_t^k u)_0 = (B) \int_0^t D_t^{k+1} u(\cdot, s) ds \quad (0 \leq k \leq 2). \end{cases}$$

Here, $D_t^k u(\cdot, t)$ denotes the strong derivative in $L^2(\Omega)$ and (B) denotes the Bochner integral in $L^2(\Omega)$

$$(4.3) \quad \begin{cases} D_t^k u(\cdot, t) \in \overset{\circ}{W}_2^1(\Omega) \text{ for any } t \in [0, T_0], (0 \leq k \leq 3) \\ \nabla D_t^k u_m(\cdot, t) \longrightarrow \nabla D_t^k u(\cdot, t) \text{ weakly in } L^2(\Omega), (0 \leq k \leq 3) \\ \|\nabla D_t^k u(\cdot, t) - \nabla D_t^k u(\cdot, \tau)\| \leq C|t - \tau|, (0 \leq k \leq 3). \end{cases}$$

$$(4.4) \quad D_t^k (D_t u_m)^3 \longrightarrow D_t^k (D_t u)^3 \text{ in } L^2(\Omega), \text{ uniformly for } t \in [0, T_0] \quad (0 \leq k \leq 2).$$

$$(4.5) \quad \begin{cases} D_t^k u_m(\cdot, t) \longrightarrow D_t^k u(\cdot, t) \text{ weakly in } L^2(\Omega), \text{ as } m \rightarrow \infty \\ \|\nabla D_t^k u(\cdot, t) - \nabla D_t^k u(\cdot, \tau)\| \leq C|t - \tau| \\ D_t^3 u(\cdot, t) - D_t^3 u(\cdot, 0) = (B) \int_0^t D_t^4 u(\cdot, s) ds \end{cases}$$

PROOF. From the equality $D_t^k u_m(x, t) = D_t^k u_m(x, 0) + \int_0^t D_t^{k+1} u_m(x, s) ds$ it follows that

$$\begin{aligned} \int_{\Omega} |D_t^k u_m(x, t) - D_t^k u_n(x, t)|^2 dx &\leq 2 \int_{\Omega} |D_t^k u_m(x, 0) - D_t^k u_n(x, 0)|^2 dx \\ &\quad + T_0 \int_0^t \int_{\Omega} |D_t^{k+1} u_m(x, s) - D_t^{k+1} u_n(x, s)|^2 dx ds. \end{aligned}$$

Here, by (3.19), $(D_t^k u_m)_0$ ($k = 0, 1, 2, 3$) converge in $L^2(\Omega)$ as $m \rightarrow \infty$. The second term of the right side converges to zero as $m, n \rightarrow \infty$, by (4.1)

when $0 \leq k \leq 3$.

Hence, there exist functions v_k ($0 \leq k \leq 3$) such that

$$D_t^k u_m(\cdot, t) \longrightarrow v_k(\cdot, t) \text{ in } L^2(\Omega), \text{ uniformly for } t \in [0, T_0].$$

If we set $v_0 = u$, then it follows that $v_k = D_t^k u$. In fact

$$u_m(x, t) - u_m(x, 0) = \int_0^t D_t u_m(x, s) ds,$$

or

$$u_m(\cdot, t) - u_{m0} = (B) \int_0^t D_t u_m(\cdot, s) ds.$$

Here

$$\begin{aligned} & \| (B) \int_0^t D_t u_m(\cdot, s) ds - (B) \int_0^t v_1(\cdot, s) ds \| \\ & \leq \int_0^t \| D_t u_m(\cdot, s) - v_1(\cdot, s) \| ds \longrightarrow 0, \text{ as } m \rightarrow \infty. \end{aligned}$$

It gives that $u(\cdot, t) - u_0 = (B) \int_0^t v_1(\cdot, s) ds$. Moreover, $D_t u = v_1$. Similarly (4.2) is shown. With respect to (4.3), we apply the following Lemma:

If w is the weak limit in $L^2(\Omega)$ of a sequence $w_m \in W_2^1(\Omega)$ ($m = 1, 2, \dots$) with bounded norm $\|w_m\|_{W_2^1(\Omega)} \leq C$, then $w \in W_2^1(\Omega)$ and $\nabla w_m \longrightarrow \nabla w$ weakly in $L^2(\Omega)$ ([4]).

Since, $\|D_t^k u_m(\cdot, t)\|$, $\|\nabla D_t^k u_m(\cdot, t)\| \leq C$ ($0 \leq k \leq 3$) and $D_t^k u_m(\cdot, t) \longrightarrow D_t^k u(\cdot, t)$ in $L^2(\Omega)$, uniformly for $t \in [0, T_0]$ ($0 \leq k \leq 3$) it follows that $D_t^k u(\cdot, t) \in W_2^1(\Omega)$ for $t \in [0, T_0]$ ($0 \leq k \leq 3$), and $\nabla D_t^k u_m(\cdot, t) \longrightarrow \nabla D_t^k u(\cdot, t)$ weakly in $L^2(\Omega)$ ($0 \leq k \leq 3$). By the result of the above, it holds that $D_t^k u_m(\cdot, t) \longrightarrow D_t^k u(\cdot, t)$ weakly in $W_2^1(\Omega)$, as $m \rightarrow \infty$. Noting that $D_t^k u_m(\cdot, t) \in \overset{\circ}{W}_2^1(\Omega)$ for $t \in [0, T_0]$, we have $D_t^k u(\cdot, t) \in \overset{\circ}{W}_2^1(\Omega)$ for $t \in [0, T_0]$ ($0 \leq k \leq 3$) and

$$\begin{aligned} & |(\nabla D_t^k u_m(\cdot, t) - \nabla D_t^k u_m(\cdot, \tau), w)| = |(\int_{\tau}^t \nabla D_t^k u_m(\cdot, s) ds, w)| \\ & \leq \int_{\tau}^t \|\nabla D_t^k u_m(\cdot, s)\| ds \|w\| \\ & \leq C \|w\| |t - \tau| \quad \text{for any } w \in L^2(\Omega), 0 \leq k \leq 3, \end{aligned}$$

which implies $\|\nabla D_t^k u(\cdot, t) - \nabla D_t^k u(\cdot, \tau)\| \leq C|t - \tau|$.

To prove (4.4), we verify the boundedness:

$$|D_t^k u_m(x, t)| \leq C \text{ in } \Omega \times [0, T_0],$$

where C is a constant independent of m, x, t .

From (3.1) it follows that

$$D_t^{k+2} u_m - \Delta D_t^k u_m + P_m D_t^k (D_t u_m)^3 = 0 \quad (0 \leq k \leq 3).$$

$$\begin{aligned}
 \text{Here, } \quad & \| (D_t u_m)^3 \| \leq C \| \nabla (D_t u_m) \|^3 \leq C, \\
 & \| D_t (D_t u_m)^3 \| \leq C \| \nabla D_t u_m \|^2 \| \nabla D_t^2 u_m \| \leq C, \\
 & \| D_t^2 (D_t u_m)^3 \| \leq C \{ \| \nabla D_t u_m \|^2 \| \nabla D_t^2 u_m \|^2 + \| \nabla D_t u_m \|^2 \| D_t^3 u_m \| \} \leq C, \\
 & \| D_t^3 (D_t u_m)^3 \| \leq C \{ \| \nabla D_t^2 u_m \|^3 + \| \nabla D_t u_m \|^2 \| \nabla D_t^2 u_m \| \| \nabla D_t^3 u_m \| \\
 & + \| \nabla D_t u_m \|^2 \| D_t^3 u_m \| \} \leq C.
 \end{aligned}$$

Hence

$$\begin{aligned}
 |D_t^k u_m(x, t)| & \leq C \| D_t^k u_m(\cdot, t) \|_{W_2^k(\Omega)} \leq C \| \Delta D_t^k u_m(\cdot, t) \| \\
 & \leq C \{ \| D_t^{k+2} u_m(\cdot, t) \| + \| P_m D_t^k (D_t u_m)^3 \| \} \leq C (0 \leq k \leq 3).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \| (D_t u_m)^3 - (D_t u)^3 \| & = \| (D_t u_m - D_t u) ((D_t u_m)^2 + (D_t u_m) D_t u + (D_t u)^2) \| \\
 & \leq C \| D_t u_m - D_t u \| \\
 \| D_t (D_t u_m)^3 - 3(D_t u)^2 (D_t^2 u) \| & \leq C \{ \| (D_t u_m)^2 (D_t^2 u_m - D_t^2 u) \| \\
 & + \| ((D_t u_m)^2 - (D_t u)^2) D_t^2 u \| \} \\
 & \leq C \{ \| D_t^2 u_m - D_t^2 u \| + \| D_t u_m - D_t u \| \} \\
 \| D_t^2 (D_t u_m)^3 - 6(D_t u) (D_t^2 u)^2 - 3(D_t u)^2 D_t^3 u \| & \\
 & \leq C \{ \| D_t u_m ((D_t^2 u_m)^2 - (D_t^2 u)^2) \| + \| (D_t u_m - D_t u) (D_t^2 u)^2 \| \\
 & + \| (D_t u_m)^2 (D_t^3 u_m - D_t^3 u) \| + \| ((D_t u_m)^2 - (D_t u)^2) D_t^3 u \| \} \\
 & \leq C \{ \| D_t^2 u_m - D_t^2 u \| + \| D_t u_m - D_t u \| + \| D_t^3 u_m - D_t^3 u \| \}.
 \end{aligned}$$

These inequalities give (4.4) by application of (4.2).

Finally, we will show (4.5).

$$\begin{aligned}
 (D_t^4 u_m(\cdot, t), w) & = ((D_t^4 u_m)_0, w) + \left(\int_0^t D_t^5 u_m(\cdot, s) ds, w \right) \\
 & = ((D_t^4 u_m)_0, w) + \int_0^t (D_t^5 u_m(\cdot, s), w) ds \quad \text{for } w \in L^2(\Omega).
 \end{aligned}$$

Here,

$$\begin{aligned}
 \| (D_t^4 u_m)_0 \| & \leq C \\
 \| D_t^5 u_m \|_{L^2(\Omega) \times [0, T_0]} & \leq C.
 \end{aligned}$$

Hence, it is possible to choose convergent subsequences of $((D_t^4 u_m)_0, w)$, $\int_0^t (D_t^5 u_m(\cdot, s), w) ds$, which implies the subsequence $D_t^4 u_{m_j}(\cdot, t)$ converges weakly in $L^2(\Omega)$. Denote the limit function by $v_4: [0, T_0] \rightarrow L^2(\Omega)$, then

$$\begin{aligned}
 (D_t^3 u_m(\cdot, t) - (D_t^3 u_m)_0, w) & = \left(\int_0^t D_t^4 u_m(\cdot, s) ds, w \right) = \int_0^t (D_t^4 u_m(\cdot, s), w) ds \\
 & \rightarrow \int_0^t (v_4(\cdot, s), w) ds, \quad \text{as } m \rightarrow \infty, \text{ for any } w \in L^2(\Omega),
 \end{aligned}$$

Since it holds $\| v_4(\cdot, t) \| \leq C$ for any $t \in [0, T_0]$, $v_4(\cdot, t)$ is a Bochner integrable function. Hence we have

$$(D_i^2 u(\cdot, t) - (D_i^2 u)_0, w) = ((B) \int_0^t v_1(\cdot, s) ds, w).$$

Consequently (4.5) is completely shown.

§ 5 The existence of the classical solution

First, we show *equalities*

$$(5.1) \quad (D_i^{2+j} u(\cdot, t), w) + (\nabla D_i^j u(\cdot, t), \nabla w) + (D_i^j (D_i u)^3(\cdot, t), w) = 0$$

for any $w \in \overset{\circ}{W}_2^1(\mathcal{Q})$. ($t \in [0, T_0]$, $0 \leq j \leq 2$)

Here, $u(\cdot, t)$ is the limit function in § 4.

In fact, for approximate solutions $u_m(x, t)$ it holds

$$(D_i^{2+j} u_m(\cdot, t), \phi_k) + (\nabla D_i^j u_m(\cdot, t), \nabla \phi_k) + (D_i^j (D_i u_m)^3(\cdot, t), \phi_k) = 0$$

for $m \geq k$.

Hence, there results

$$(D_i^{2+j} u(\cdot, t), \phi_k) + (\nabla D_i^j u(\cdot, t), \nabla \phi_k) + (D_i^j (D_i u)^3(\cdot, t), \phi_k) = 0$$

($k = 1, 2, 3, \dots$).

by the convergence relations (4.2), (4.3), (4.4) and (4.5).

Here, $W_2^2(\mathcal{Q}) \cap \overset{\circ}{W}_2^1(\mathcal{Q})$ is dense in $\overset{\circ}{W}_2^1(\mathcal{Q})$, and finite linear combinations of eigenfunctions $\{\phi_k\}$ is dense in $W_2^2(\mathcal{Q}) \cap \overset{\circ}{W}_2^1(\mathcal{Q})$.

Consequently (5.1) is shown.

Next we have *properties of u as follows*,

$$(5.2) \quad D_i^j u \in C([0, T_0] : W_2^2(\mathcal{Q})) \cap C(\overline{\mathcal{Q}} \times [0, T_0]) \quad (j = 0, 1, 2).$$

Indeed, set $j = 0$ in (5.1), it follows that for any $w \in \overset{\circ}{W}_2^1(\mathcal{Q})$

$$-(\nabla u(\cdot, t), \nabla w) = (D_i^2 u(\cdot, t) + (D_i u)^3(\cdot, t), w).$$

Hence, $u(\cdot, t) \in \overset{\circ}{W}_2^1(\mathcal{Q})$ for $t \in [0, T_0]$ is a weak solution of the equation

$$\Delta v = D_i^2 u(\cdot, t) + (D_i u)^3(\cdot, t) \in L^2(\mathcal{Q}).$$

Consequently

$$u(\cdot, t) \in W_2^2(\mathcal{Q})$$

and

$$\|u(\cdot, t)\|_{W_2^2(\mathcal{Q})} \leq C \{ \|D_i^2 u(\cdot, t)\| + \|(D_i u)^3(\cdot, t)\| + \|u(\cdot, t)\| \}$$

Moreover it holds

$$\begin{aligned} \|u(\cdot, t) - u(\cdot, \tau)\|_{W^2_2(\mathcal{Q})} \leq C \{ & \|D_t^2 u(\cdot, t) - D_t^2 u(\cdot, \tau)\| \\ & + \|(D_t u)^3(\cdot, t) - (D_t u)^3(\cdot, \tau)\| + \|u(\cdot, t) - u(\cdot, \tau)\| \}. \end{aligned}$$

Since u , $D_t^2 u$ and $(D_t u)^3 \in C([0, T_0] : L^2(\mathcal{Q}))$ by the convergence relations (4.2) and (4.4), we have $u \in C([0, T_0] : W^2_2(\mathcal{Q}))$. Also the inequality

$$|u(x, t) - u(\xi, \tau)| \leq |u(x, t) - u(\xi, t)| + C \|u(\cdot, t) - u(\cdot, \tau)\|_{W^2_2(\mathcal{Q})}$$

gives the relation $u(x, t) \in C(\bar{\mathcal{Q}} \times [0, T_0])$ by the notice of $u(x, t) \in C(\bar{\mathcal{Q}})$ for fixed $t \in [0, T_0]$. Similarly, set $j = 1, 2$ in (5.1), then we have (5.2), since $D_t^3 u + D_t(D_t u)^3$, $D_t^4 u + D_t^2(D_t u)^3 \in C([0, T_0] : L^2(\mathcal{Q}))$.

Finally, we have the following properties of u ;

$$(5.3) \quad (D_t u)^3 \in C([0, T_0] ; W^1_2(\mathcal{Q})) \cap C([0, T_0] ; W^2_2(\mathcal{Q}))$$

$$(5.4) \quad u \in C([0, T_0] ; W^4_2(\mathcal{Q}))$$

$$(5.5) \quad D_t u \in C([0, T_0] ; W^3_2(\mathcal{Q}))$$

PROOF. Relations $D_t u \in C(\bar{\mathcal{Q}} \times [0, T_0])$, $\nabla D_t u \in C([0, T_0] ; L^2(\mathcal{Q}))$ and $D_t u \in C([0, T_0] ; W^2_2(\mathcal{Q}))$ imply that equalities $D_{x_k}(D_t u)^3 = 3(D_t u)^2 D_{x_k}(D_t u)$ and $D_{x_j} D_{x_k}(D_t u)^3 = 6(D_t u) D_{x_j}(D_t u) D_{x_k}(D_t u) + 3(D_t u)^2 D_{x_j} D_{x_k}(D_t u)$ hold. Here, we applied the fact that if $f, \nabla f \in L^p(\mathcal{Q})$ and $g, \nabla g \in L^q(\mathcal{Q})$ ($1/p + 1/q = 1$) it holds $\nabla(fg) = (\nabla f)g + f(\nabla g)$. Noting that $fg \in C([0, T] ; L^2(\mathcal{Q}))$ for $f \in C(\bar{\mathcal{Q}} \times [0, T])$ and $g \in C([0, T] ; L^2(\mathcal{Q}))$ we have (5.3).

If we set $j = 0$ in (5.1), $u(\cdot, t)$ is a weak solution of the equation $\Delta u = D_t^2 u(\cdot, t) + (D_t u)^3(\cdot, t)$, where $D_t^2 u + (D_t u)^3 \in C([0, T_0] ; W^2_2(\mathcal{Q}))$. Hence, $u \in C([0, T_0] ; W^4_2(\mathcal{Q}))$. Also we set $j = 1$ in (5.1), then $D_t u(\cdot, t)$ is a weak solution of the equation $\Delta v = D_t^2 u(\cdot, t) + D_t(D_t u)^3(\cdot, t)$. Here $D_t(D_t u)^3 = 3(D_t u)^2(D_t^2 u) \in C([0, T_0] ; L^2(\mathcal{Q}))$ and $D_{x_k}(D_t u)^2 D_t^2 u = 2D_t u D_{x_k}(D_t u) D_t^2 u + (D_t u)^2 D_{x_k} D_t^2 u \in C([0, T_0] ; L^2(\mathcal{Q}))$, $D_t^3 u \in C([0, T_0] ; W^1_2(\mathcal{Q}))$. Hence, $D_t u \in C([0, T_0] ; W^3_2(\mathcal{Q}))$.

Now, we take notice of the following relations (5.6) and (5.7).

$$(5.6) \quad \text{If } u \text{ belongs to the space } C^m([0, T] : L^2(\mathcal{Q})) \text{ and the strong derivatives } D_t^k u \text{ in } L^2(\mathcal{Q}) \text{ (} k=0, 1, 2, \dots, m) \text{ are belong to the space } C([0, T] : W^1_2(\mathcal{Q})) \text{ it follows that } u \in C^m([0, T] : W^1_2(\mathcal{Q})) \text{ and}$$

$$\begin{aligned} \left\| \frac{1}{h} (D_t^{k-1} u(\cdot, t+h) - D_t^{k-1} u(\cdot, t)) - D_t^k u(\cdot, t) \right\|_{W^1_2(\mathcal{Q})} \rightarrow 0 \text{ as } h \rightarrow 0, \text{ (} k=1, 2, \\ \dots, m). \end{aligned}$$

We verify for $m = 1$. From the assumption $D_t u \in C([0, T] ; W^1_2(\mathcal{Q}))$,

there exists the Bochner integral $(W_{\frac{1}{2}}^1, B) \int_0^t D_s u(\cdot, s) ds$ in $W_{\frac{1}{2}}^1(\mathcal{Q})$. Hence, the Bochner integral $(L^2, B) \int_0^t D_s u(\cdot, s) ds$ in $L^2(\mathcal{Q})$ exists, and

$$(L^2, B) \int_0^t D_s u(\cdot, s) ds = (W_{\frac{1}{2}}^1, B) \int_0^t D_s u(\cdot, s) ds$$

as an element of $L^2(\mathcal{Q})$. On the other hand

$$(L^2, B) \int_0^t D_s u(\cdot, s) ds = u(\cdot, t) - u(\cdot, 0). \quad (\text{See [7]})$$

By the above equalities we have

$$u(\cdot, t) - u(\cdot, 0) = (W_{\frac{1}{2}}^1, B) \int_0^t D_s u(\cdot, s) ds,$$

as an element of $W_{\frac{1}{2}}^1(\mathcal{Q})$ or

$$\| \frac{1}{h} (u(\cdot, t+h) - u(\cdot, t)) - D_t u(\cdot, t) \|_{W_{\frac{1}{2}}^1(\mathcal{Q})} \rightarrow 0, \text{ as } h \rightarrow 0.$$

$$(5.7) \quad \bigcap_{k=0}^2 C^k([0, T]: W_{\frac{1}{2}}^{4-k}(\mathcal{Q})) \subset C^2(\bar{\mathcal{Q}} \times [0, T])$$

This relation is given by Sather [5] or Wilcox [8]. We give an outline of the proof. Since $u \in C([0, T]: W_{\frac{1}{2}}^4(\mathcal{Q}))$, we have

$$|D_x^\alpha u(x, t)| \leq C \|u(\cdot, t)\|_{W_{\frac{1}{2}}^4(\mathcal{Q})}, \quad (|\alpha| \leq 2, (x, t) \in \bar{\mathcal{Q}} \times [0, T])$$

and

$$|D_x^\alpha u(x, t) - D_x^\alpha u(\xi, \tau)| \leq C \|u(\cdot, t) - u(\cdot, \tau)\|_{W_{\frac{1}{2}}^4(\mathcal{Q})} + |D_x^\alpha u(x, \tau) - D_x^\alpha u(\xi, \tau)|.$$

Hence, $D_x^\alpha u \in C(\bar{\mathcal{Q}} \times [0, T])$ ($|\alpha| \leq 2$). The relation $u \in C^1([0, T]: W_{\frac{1}{2}}^3(\mathcal{Q}))$ gives $D_t u \in C([0, T]: W_{\frac{1}{2}}^3(\mathcal{Q}))$ and $D_x^\alpha D_t u \in C(\bar{\mathcal{Q}} \times [0, T])$ ($|\alpha| \leq 1$).

Hence $D_t u, D_x D_t u \in C(\bar{\mathcal{Q}} \times [0, T])$. The relation $u \in C^2([0, T]: W_{\frac{1}{2}}^2(\mathcal{Q}))$ give $D_t^2 u \in C(\bar{\mathcal{Q}} \times [0, T])$. Moreover

$$\begin{aligned} & | \frac{1}{h} \{ D_x^\alpha u(x, t+h) - D_x^\alpha u(x, t) \} - D_x^\alpha D_t u(x, t) | \\ & \leq C \| \frac{1}{h} \{ u(\cdot, t+h) - u(\cdot, t) \} - D_t u(\cdot, t) \|_{W_{\frac{1}{2}}^3(\mathcal{Q})} \rightarrow 0, \text{ as } h \rightarrow 0, \\ & | \frac{1}{h} \{ D_t u(x, t+h) - D_t u(x, t) \} - D_t^2 u(x, t) | \\ & \leq C \| \frac{1}{h} \{ D_t u(\cdot, t+h) - D_t u(\cdot, t) \} - D_t^2 u(\cdot, t) \|_{W_{\frac{1}{2}}^2(\mathcal{Q})} \rightarrow 0, \text{ as } h \rightarrow 0. \end{aligned}$$

Consequently, (5.7) is verified.

We attain the following existence theorem of the classical solution:

THEOREM. *Let Ω be a bounded domain in R^3 with a sufficiently smooth boundary. Suppose that the initial data u_0 belongs to the space V_0 , and $(D_i u)_0$ belongs to the space V_1 , where V_i ($i = 0, 1$) are defined in § 3. Then there exists a function u belonging to the class $C^2(\bar{\Omega} \times [0, T_0])$, which satisfies the equation (1.1) in $\Omega \times (0, T_0)$ and conditions (1.2)–(1.4). Here T_0 is a positive number depending on u_0 and $(D_i u)_0$.*

PROOF. The equation (5.1) gives the equality

$$(D_i^2 u(\cdot, t), w) - (\Delta u(\cdot, t), w) + ((D_i u)^3(\cdot, t), w) = 0$$

for any $w \in \overset{\circ}{W}_{\frac{1}{2}}(\Omega)$, $t \in [0, T_0]$.

On the other hand, $u \in \bigcap_{k=0}^2 C^k([0, T_0]; W_{\frac{1}{2}-k}(\Omega)) \subset C^2(\bar{\Omega} \times [0, T_0])$.

Indeed, $u \in C^1([0, T_0]; L^2(\Omega))$ by (4.2) and $u, D_i u \in C([0, T_0]; W_{\frac{3}{2}}(\Omega))$ by (5.5). Hence $u \in C^1([0, T_0]; W_{\frac{3}{2}}(\Omega))$ by noting (5.6). Also, $u \in C^2([0, T_0]; L^2(\Omega))$ by (4.2) and $u, D_i u, D_i^2 u \in C([0, T_0]; W_{\frac{3}{2}}(\Omega))$ by (5.2). Hence $u \in C^2([0, T_0]; W_{\frac{3}{2}}(\Omega))$ by noting (5.6). $u \in C([0, T_0]; W_{\frac{1}{2}}(\Omega))$ was shown in (5.4).

Consequently, $D_i^2 u - \Delta u + (D_i u)^3 = 0$ in $\Omega \times (0, T_0)$, since $\overset{\circ}{W}_{\frac{1}{2}}(\Omega)$ is dense in $L^2(\Omega)$. By virtue of (4.2) it holds that $u(x, 0) = u_0(x)$, $D_i u(x, 0) = (D_i u)_0(x)$, $x \in \Omega$. $u(\cdot, t) \in \overset{\circ}{W}_{\frac{1}{2}}(\Omega) \cap C^1(\bar{\Omega})$ implies that u vanishes on the boundary in the natural sense.

College of General Education,
Kyushu University.

References

- [1] E. HILLE and R. S. PHILLIPS, *Functional analysis and semigroups*. Amer. Math. Soc. (1957).
- [2] J. L. LIONS, *Quelques methodes de résolution de problemes aux limites non linéaires*. Paris, Dunod, (1969).
- [3] C. B. MORREY Jr., *Multiple integrals in the calculus of variations*. Springer, Berlin, (1966).
- [4] L. NIRENBERG, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8**, (1955) 649–675.
- [5] J. SATHER, *The existence of a global classical solution of the initial boundary value problem for $\square u + u^3 = f$* . Archive for Rat. Mech. Anal. **22** (1966) 292–307.
- [6] C. H. WILCOX, *Initial-boundary value problems for linear hyperbolic partial differential equations of the second order*. Arch. Rat. Mech. Anal. **10** (1962) 361–400.
- [7] C. H. WILCOX, *Uniform asymptotic estimates for wave packet in quantum theory of scattering*. J. Math. Phys. **6** (1965) 611–620.
- [8] K. YOSHIDA, *Functional analysis*. Springer-Verlag New York Inc. (1968).