The existence of a local classical solution of the initial-boundary value problem for D^2_tu−⊿u+(D_tu)^3=0

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The existence of a local classical solution of the initial-boundary value problem for $D_t^2u - \Delta u + (D_tu)^3 = 0$

by

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§ 1 Problems and notations

J. Sather proved the existence of a global classical solution for the initial-boundary value problem for $D_t^2u - \Delta u + u^3 = f$ ([5]). For the equation considered here, will be able to verified the results analogous to the ones that J. Sather established? This problem was proposed by Lions in his book ([2]). In this paper we will show the existence of the classical but local solution $u = u(x, t)$ satisfying the following equation and conditions:

$$D_t^2u - \Delta u + (D_tu)^3 = 0, \quad x \in \Omega, \ t > 0 \tag{1.1}$$

$$u(x, 0) = u_0(x) \qquad x \in \Omega \tag{1.2}$$

$$D_tu(x, 0) = (D_tu)_0(x) \quad x \in \Omega \tag{1.3}$$

$$u(x, t) = 0 \qquad x \in \partial\Omega, \ t > 0. \tag{1.4}$$

Here $u_0(x)$, $(D_tu)_0(x)$ are the data of the problem, and Ω is a bounded domain in R^3 with a sufficiently smooth boudary $\partial\Omega$.

In this paper we shall use the following notations:

$$(u, v) = \int_\Omega u(x)v(x)\ dx, \qquad \| u \| = \sqrt{(u, u)}$$

$$(\nabla u, \nabla v) = \sum_{i=1}^{3} \int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \qquad \| \nabla u \| = \sqrt{(\nabla u, \nabla u)}$$

$$\| u \|_p = \left\{ \int_\Omega |u|^p \ dx \right\}^{1/p} \quad (p>1, \ p \neq 2).$$

Sobolev space $W_2^1(\Omega)$ is the Hilbert space with the inner product $(u, v) + (\nabla u, \nabla v)$. The space $\overset{\circ}{W}{}_2^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_2^1(\Omega)$.

§ 2 A priori estimates

We have *the following a priori estimates.*

$$\| D_tu \|^2 + \| \nabla u \|^2 \leq \| (D_tu)_0 \|^2 + \| \nabla u_0 \|^2 \equiv C_0. \tag{2.1}$$

(2.2) $$\| D_t^2 u \|^2 + \| \nabla D_t u \|^2 \leq \| (D_t^2 u)_o \|^2 + \| \nabla (D_t u)_o \|^2 \equiv C_1.$$

(2.3) $$\| D_t^3 u \|^2 + \| \nabla D_t^2 u \|^2 \leq \frac{\| (D_t^3 u)_o \|^2 + \| \nabla (D_t^2 u)_o \|^2}{\left\{1 - 6\left(4/\sqrt{3}\right)^3 T (\| (D_t^2 u)_o \|^2 + \| \nabla (D_t u)_o \|^2)^{1/2} (\| (D_t^3 u)_o \|^2 + \| \nabla (D_t^2 u)_o \|^2)^{1/2}\right\}^2} \equiv C_2,$$

where a positive number T is chosen sufficiently small to hold the inequality

$$1 - 6\left(4/\sqrt{3}\right)^3 T (\| (D_t^2 u)_o \|^2 + \| \nabla (D_t u)_o \|^2)^{1/2} (\| (D_t^3 u)_o \|^2 + \| \nabla (D_t^2 u)_o \|^2)^{1/2} > 0.$$

(2.4) $$\| D_t^4 u \|^2 + \| \nabla D_t^3 u \|^2 \leq \{C_3' T + \| (D_t^4 u)_o \|^2 + \| \nabla (D_t^3 u)_o \|^2\} e^{C_3'' T} \equiv C_3,$$

where C_3' and C_3'' are constants depending only on C_1 and C_2.

(2.5) $$\| D_t^5 u \|^2 + \| \nabla D_t^4 u \|^2 \leq \{C_4' T + \| (D_t^5 u)_o \|^2 + \| \nabla (D_t^4 u)_o \|^2\} e^{C_4'' T} \equiv C_4,$$

where C_4' and C_4'' are constants depending only on C_1, C_2 and C_3.

In the above $(D_t^k u)_o$ denotes the function $D_t^k u(x, 0)$.

PROOF. Inequalities (2.1) and (2.2) are easily verified. Now, differentiate the equation (1.1) two times with respect to t, multiply by $D_t^3 u$ and integrate over Ω. Then it follows that

$$(D_t^4 u, D_t^3 u) - (\Delta D_t^2 u, D_t^3 u) + (D_t^2 (D_t u)^3, D_t^3 u) = 0.$$

Hence we have

$$\frac{d}{dt}\left\{ \| D_t^3 u \|^2 + \| \nabla D_t^2 u \|^2 \right\} + 6 \int_\Omega (D_t u)^2 (D_t^3 u)^2 \, dx$$

$$= -12 \int_\Omega D_t u (D_t^2 u)^2 \, D_t^3 u \, dx$$

$$\leq 12 \| D_t u \|_6 \| D_t^2 u \|_6^2 \| D_t^3 u \|$$

$$\leq 12 \left(4/\sqrt{3}\right)^3 \| \nabla D_t u \| \, \| \nabla D_t^2 u \|^2 \| D_t^3 u \|$$

$$\leq 12 \left(4/\sqrt{3}\right)^3 (\| (D_t^2 u)_o \|^2 + \| \nabla (D_t u)_o \|^2)^{1/2} (\| D_t^3 u \|^2 + \| \nabla (D_t^2 u) \|^2)^{3/2}.$$

In the above we integrated by parts, since u vanishes on the boundary, and applied Hölder's inequality and Sobolev's imbedding theorem ([3]). We set $C = 12 \left(4/\sqrt{3}\right)^3 (\| (D_t^2 u)_o \|^2 + \| \nabla (D_t u)_o \|^2)^{1/2}$ and $y(t) = \| D_t^3 u \|^2 + \| \nabla D_t^2 u \|^2$. Then it follows that $\frac{dy}{dt} \leq C (y(t))^{3/2}$, which implies

$$\sqrt{y(t)} \leq \frac{\sqrt{y(0)}}{1-1/2\ CT\sqrt{y(0)}} \quad .$$

This inequality gives (2.3). Next we show the inequality (2.4). Differentiate the equation (1.1) three times, multiply by $D_t^4 u$ and integrate over Ω, then it follows that

$$(D_t^5 u,\ D_t^4 u) - (\Delta D_t^3 u,\ D_t^4 u) + (D_t^3 (D_t u)^3,\ D_t^4 u) = 0.$$

Hence we have

$$\frac{d}{dt}\left\{ \| D_t^4 u \|^2 + \| \nabla D_t^3 u \|^2 \right\} + 6\int_\Omega (D_t u)^2 (D_t^4 u)^2\ dx$$

$$\leq 12 \| D_t^2 u \|_6^3 \| D_t^4 u \| + 36 \| D_t u \|_6 \| D_t^2 u \|_6 \| D_t^3 u \|_6 \| D_t^4 u \|$$

$$\leq 12 \left(4/\sqrt{3}\right)^3 \{ \| \nabla D_t^2 u \|^3 \| D_t^4 u \| + 3 \| \nabla D_t u \|\ \| \nabla D_t^2 u \|\ \| \nabla D_t^3 u \|\ \| D_t^4 u \| \}$$

$$\leq 6 \left(4/\sqrt{3}\right)^3 \{ \| \nabla D_t^2 u \|^4 + \| \nabla D_t^2 u \|^2 \| D_t^4 u \|^2$$

$$+ 3 \| \nabla D_t u \|\ \| \nabla D_t^2 u \| (\| \nabla D_t^3 u \|^2 + \| D_t^4 u \|^2) \}.$$

These relations give (2.4). Finally, an argument analogous to the above gives the inequality (2.5).

§ 3 Estimates of approximate solutions

Consider the eigenfunctions $\{\psi_k\}$ for the Laplace operator Δ with zero boundary conditions; $\psi_k \in \overset{\circ}{W}{}_2^1(\Omega)$,

$$\Delta \psi_k = \mu_k\ \psi_k \text{ in } \Omega\ (k = 1, 2, \ldots\ldots).$$

With respect to the regularity of the eigenfunctions ψ_k, it is well known that $\{\psi_k\}$ is involved in the space $W_2^6(\Omega)$ for the bounded domain Ω with sufficiently smooth boundary. $\{\psi_k:\ k = 1, 2, \ldots\ldots\}$ is a complete orthonormal system in $L^2(\Omega)$

Next we introduce the spaces of admissible initial data.

$\{\phi_k:\ k = 1, 2, \ldots\ldots\} \equiv$ orthonormalization of $\{\psi_k\}$ in $W_2^6(\Omega)$.

$\{\chi_k:\ k = 1, 2, \ldots\ldots\} \equiv$ orthonormalization of $\{\psi_k\}$ in $W_2^4(\Omega)$

$V_0 \equiv$ closure of $\left\{\sum_{k=1}^m \alpha_k \phi_k:\ \alpha_k \text{ is a real number}\right\}$ in $W_2^6(\Omega)$

$V_1 \equiv$ closure of $\left\{\sum_{k=1}^m \beta_k \chi_k:\ \beta_k \text{ is a real number}\right\}$ in $W_2^4(\Omega)$

For any element u_0 belonging to the space V_0 it holds that

$$u_0=\sum_{k=1}^{\infty}(u_0,\ \phi_k)_{W_2^6(\Omega)}\ \phi_k,$$

where ϕ_k is a linear combination of $\psi_j(j=1,2,\ldots\ldots,k)$. Hence there exists the linear combination of $\psi_k(k=1,2,\ldots\ldots,m)$ such that

$$\|\sum_{k=1}^{m}\lambda_k^m\psi_k-u_0\|_{W_2^6(\Omega)}\longrightarrow 0\ \ (m\longrightarrow\infty).$$

By the argument analogous to the above it follows that for any element $(D_tu)_0$ belonging to the space V_1 there exists the linear combination of $\psi_k(k=1,2,\ldots\ldots,m)$ such that

$$\|\sum_{k=1}^{m}\nu_k^m\psi_k-(D_tu)_0\|_{W_2^4(\Omega)}\longrightarrow 0\ \ (m\longrightarrow\infty)$$

V_0 and V_1 are Hilbert spaces, and are dense linear subsets in $L^2(\Omega)$.

We turn to estimate the approximate solutions;

$$u_m=u_m(x,\ t)=\sum_{k=1}^{m}\lambda_k^m(t)\psi_k(x)\ \ (m=1,2,\ldots\ldots),$$

which are determined by the following system of differential equations.

$$(D_t^2u_m,\ \psi_k)+(\nabla u_m,\ \nabla\psi_k)+((D_tu_m)^3,\ \psi_k)=0,\ \ (k=1,2,\ldots\ldots m), \tag{3.1}$$

or

$$\frac{d^2\lambda_k^m}{dt^2}+\sum_{j=1}^{m}\lambda_j^m(t)(\nabla\psi_j,\ \nabla\psi_k)+\left(\left\{\sum_{j=1}^{m}\frac{d\lambda_j^m}{dt}\psi_j\right\}^3,\ \psi_k\right)=0,\ \ (k=1,2,\ldots\ldots m). \tag{3.2}$$

Here, initial date $\lambda_k^m(0)$, $D_t\lambda_k^m(0)$ are chosen in such a way that as $m\to\infty$ we have

$$u_{m0}=u_m(x,\ 0)=\sum_{k=1}^{m}\lambda_k^m(0)\psi_k(x)\longrightarrow u_o(x)\ \text{ in } W_2^6(\Omega), \tag{3.3}$$

$$(D_tu_m)_0=D_tu_m(x,\ 0)=\sum_{k=1}^{m}D_t\lambda_k^m(0)\psi_k(x)\to(D_tu)_o(x)\ \text{ in } W_2^4(\Omega) \tag{3.4}$$

$$u_o\in V_0,\ \ (D_tu)_o\in V_1. \tag{3.5}$$

As is indicated later, we have

$$\left|\frac{d\lambda_k^m}{dt}\right|,\ \left|\sum_{j=1}^{m}\lambda_j^m(t)(\nabla\psi_j,\ \nabla\psi_k)\right|,\ \left|(\sum_{j=1}^{m}\frac{d\lambda_j^m}{dt}\psi_j,\ \psi_k)\right|\leqq C, \tag{3.6}$$

where C is a constant independent of k, m and t.
Hence $\lambda_k^m(t)$ $(k=1,2,\ldots\ldots,m)$ are defined in $[0,\ T]$ for any $T>0$.

We have *the following estimations.*

(3.7) $$\| (D_t^{k+1} u_m)_o \|, \| \nabla (D_t^k u_m)_o \| \leq C, \quad (k = 0, 1, 2, 3, 4),$$

where C is a constant independent of m.

PROOF. Denote by P_m the orthogonal projection onto m dimensinal subspace of $L^2(\Omega)$ with basis $\{\psi_1, \psi_2, \ldots\ldots, \psi_m\}$.
Since $u_{m0} \to u_o$ in $W_2^6(\Omega)$ and $(D_t u_m)_0 \to (D_t u)_o$ in $W_2^4(\Omega)$, we have

(3.8) $$\| (D_t u_m)_0 \|, \quad \| \nabla u_{m0} \| \leq C,$$

(3.9) $$| D_x^\alpha u_{mo}(x) | \leq C \| u_{m0} \|_{W_2^6(\Omega)} \leq C, \qquad x \in \overline{\Omega} \quad |\alpha| \leq 4,$$

(3.10) $$| D_x^\alpha (D_t u_m)_0(x) | \leq C \| (D_t u_m)_0 \|_{W_2^4(\Omega)} \leq C, \qquad x \in \overline{\Omega} \quad |\alpha| \leq 2.$$

Here C is a constant independent of m. To prove (3.7), we will frequently use (3.9) and (3.10).
The differential equations (3.1) give the following equality.

$$(D_t^2 u_m)_0 - \Delta u_{m0} + P_m (D_t u_m)_0^3 = 0.$$

Here we applied the equality $P_m \Delta w = \Delta P_m w$ for any element $w \in W_2^2(\Omega) \cap \overset{o}{W}{}_2^1(\Omega)$. Hence we have

(3.11) $$\| (D_t^2 u_m)_0 \| \leq \| \Delta u_{m0} \| + \| (D_t u_m)_0^3 \| \leq C + \left(4/\sqrt{3}\right)^3 \| \nabla (D_t u_m)_0 \|^3 \leq C,$$

(3.12) $$\| \nabla (D_t u_m)_0 \| \leq C.$$

Also, (3.1) gives

$$(D_t^3 u_m)_0 - \Delta (D_t u_m)_0 + 3P_m (D_t u_m)_0^2 (D_t^2 u_m)_0 = 0$$

and hence we have

(3.13) $$\| (D_t^3 u_m)_0 \| \leq \| \Delta (D_t u_m)_0 \| + C \| (D_t^2 u_m)_0 \| \leq C$$

(3.14) $$\begin{aligned} \| \nabla (D_t^2 u_m)_0 \| &\leq C \| \Delta (D_t^2 u_m)_0 \| = C \| \Delta \{ \Delta u_{m0} - P_m (D_t u_m)_0^3 \} \| \\ &\leq C \{ \| \Delta^2 u_{m0} \| + \| \Delta (D_t u_m)_0^3 \| \} \leq C + C \{ \| \Delta (D_t u_m)_0 \} (D_t u_m)_0^2 \| \\ &+ \| (D_t u_m)_0 \{ \nabla (D_t u_m)_0 \}^2 \| \leq C. \end{aligned}$$

In (3.14) we applied the relation $\| \nabla w \| \leq C \| \Delta w \|$ for any element $w \in W_2^2(\Omega) \cap \overset{o}{W}{}_2^1(\Omega)$.
Similarly, (3.1) gives the following relation

$$(D_t^4 u_m)_0 - \Delta (D_t^2 u_m)_0 + 6P_m (D_t u_m)_0 (D_t^2 u_m)_0^2 + 3P_m (D_t u_m)_0^2 (D_t^3 u_m)_0 = 0.$$

Hence we have

(3.15) $$\begin{aligned} \| (D_t^4 u_m)_0 \| &\leq \| \Delta (D_t^2 u_m)_0 \| + C \{ \| (D_t u_m)_0 (D_t^2 u_m)_0^2 \| + \| (D_t u_m)_0^2 (D_t^3 u_m)_0 \| \} \\ &\leq C + C \{ \| (D_t^2 u_m)_0^2 \| + \| (D_t^3 u_m)_0 \| \} \leq C + C \| \nabla (D_t^2 u_m)_0 \|^2 \leq C, \end{aligned}$$

$$(3.16)\qquad \|\nabla(D_t^3u_m)_0\| \leqq C\|\Delta(D_t^3u_m)_0\|$$
$$\leqq C\|\Delta\{\Delta(D_tu_m)_0-3P_m(D_tu_m)_0^2(D_t^2u_m)_0\}\|$$
$$\leqq C\{\|\Delta^2(D_tu_m)_0\|+\|\Delta[(D_tu_m)_0^2(D_t^2u_m)_0]\|\}$$
$$\leqq C+C\{\|\{\Delta(D_tu_m)_0\}(D_tu_m)_0(D_t^2u_m)_0\|$$
$$+\|(D_tu_m)_0^2\Delta(D_t^2u_m)_0\|+\|(D_tu_m)_0\{\nabla(D_tu_m)_0\nabla(D_t^2u_m)_0\}\|$$
$$+\|(D_t^2u_m)_0\{\nabla(D_tu_m)_0\}^2\|\}$$
$$\leqq C+C\{\|(D_t^2u_m)_0\|+\|\Delta(D_t^2u_m)_0\|+\|\nabla(D_t^2u_m)_0\|\}\leqq C,$$

Finally, (3.1) gives the equality;

$$(D_t^5u_m)_0-\Delta(D_t^3u_m)_0+6P_m(D_t^2u_m)_0^3+18P_m(D_tu_m)_0(D_t^2u_m)_0(D_t^3u_m)_0$$
$$+3P_m(D_tu_m)_0^2(D_t^4u_m)_0=0.$$

Hence we have firstly

$$(3.17)\qquad \|(D_t^5u_m)_0\|\leqq\|\Delta(D_t^3u_m)_0\|+C\{\|(D_t^2u_m)_0^3\|$$
$$+\|(D_tu_m)_0(D_t^2u_m)_0(D_t^3u_m)_0\|+\|(D_tu_m)_0^2(D_t^4u_m)_0\|\}$$
$$\leqq C+C\{\|\nabla(D_tu_m)_0\|\ \|\nabla(D_t^2u_m)_0\|\ \|\nabla(D_t^3u_m)_0\|$$
$$+\|\nabla(D_t^2u_m)_0\|^3+\|(D_t^4u_m)_0\|\}\leqq C.$$

Secondly we have

$$(3.18)\qquad \|\nabla(D_t^4u_m)_0\|\leqq C\|(\Delta D_t^4u_m)_0\|$$
$$\leqq C\{\|\Delta^2(D_t^2u_m)_0\|+\|\Delta P_m[(D_tu_m)_0(D_t^2u_m)_0^2]\|+\|\Delta P_m[(D_tu_m)_0^2(D_t^3u_m)_0]\|\}$$

Here, each term of the right side is estimated as follows.
Noting that

$$\Delta(D_tu_m)_0^3=3\{\Delta(D_tu_m)_0\}(D_tu_m)_0^2+6(D_tu_m)_0\{\nabla(D_tu_m)_0\}^2\in W_2^2(\Omega)\cap\mathring{W}_2^1(\Omega),$$

we have

$$\|\Delta^2(D_t^2u_m)_0\|\leqq\|\Delta^3u_{m0}\|+\|\Delta^2P_m(D_tu_m)_0^3\|\leqq C+\|\Delta^2(D_tu_m)_0^3\|$$
$$\leqq C+C\|(D_tu_m)_0\|_{W_2^4(\Omega)}\leqq C,$$
$$\|\Delta P_m[(D_tu_m)_0^2(D_t^3u_m)_0]\|\leqq\|\Delta[(D_tu_m)_0^2(D_t^3u_m)_0]\|\leqq C,$$

and by an application of Sobolev's imbedding theorem:
$$\|\Delta(D_t^2u_m)_0\|_6\leqq C\|\nabla\{\Delta(D_t^2u_m)_0\}\|;\quad \|\nabla(D_t^2u_m)_0\|_6\leqq C\|\nabla(D_t^2u_m)_0\|_{W_2^1(\Omega)},$$

we have

$$\|\Delta P_m[(D_tu_m)_0(D_t^2u_m)_0^2\|\leqq\|\Delta[(D_tu_m)_0(D_t^2u_m)_0^2]\|$$
$$\leqq\|\{\Delta(D_tu_m)_0\}(D_t^2u_m)_0^2\|+2\|(D_tu_m)_0(D_t^2u_m)_0\Delta(D_t^2u_m)_0\|$$
$$+2\|(D_tu_m)_0\{\nabla(D_t^2u_m)_0\}^2\|+4\|(D_tu_m)_0\{\nabla(D_tu_m)_0\nabla(D_t^2u_m)_0\}\|$$

$$\leqq C\{ \| \nabla(D_t^2u_m)_0 \|^2 + \| \nabla(D_t^2u_m)_0 \| \ \| \nabla\{\Delta(D_t^2u_m)_0\} \|$$
$$+ \| \nabla(D_t^2u_m)_0 \|^2_{W_2^1(\Omega)} + \| \nabla(D_t^2u_m)_0 \| \}$$
$$\leqq C + C\{ \| \Delta^2(D_t^2u_m)_0 \| + \| (D_t^2u_m)_0 \|^2_{W_2^2(\Omega)} \leqq C + C \| \Delta(D_t^2u_m)_0 \|^2 \leqq C.$$

Consequently estimates (3.7) are established.

Moreover, we have

$$(3.19)\quad \begin{cases} (D_t^2u_m)_0 = \Delta u_{m0} - P_m (D_tu_m)_0^3 \longrightarrow \Delta u_0 - (D_tu)_0^3 \ in \ L^2(\Omega), \\ \nabla(D_tu_m)_0 \longrightarrow \nabla(D_tu)_0 \ in \ L^2(\Omega), \\ (D_t^3u_m)_0 = \Delta(D_tu_m)_0 - 3P_m\{(D_tu_m)_0^2(D_t^2u_m)_0\} \\ \longrightarrow \Delta(D_tu)_0 - 3(D_tu)_0^2\{\Delta u_0 - (D_tu)_0^3\} \ in \ L^2(\Omega), \\ and \\ \nabla(D_t^2u_m)_0 = \nabla\{\Delta u_{m0} - P_m(D_tu_m)_0^3\} \longrightarrow \nabla(\Delta u_0) - \nabla(D_tu)_0^3 \ in \ L^2(\Omega), \end{cases}$$

Let's verify (3.19)

$$\| (D_t^2u_m)_0 - \{\Delta u_0 - (D_tu)_0^3\} \| = \| \Delta u_{m0} - P_m(D_tu_m)_0^3 - \{\Delta u_0 - (D_tu)_0^3\} \|$$
$$\leqq \| \Delta u_{m0} - \Delta u_0 \| + \| P_m(D_tu_m)_0^3 - P_m(D_tu)_0^3 \| + \| P_m(D_tu)_0^3 - (D_tu)_0^3 \| .$$

Here, $\| P_m(D_tu_m)_0^3 - P_m(D_tu)_0^3 \| \leqq C \| (D_tu_m)_0 - (D_tu)_0 \|$.

$$\| (D_t^3u_m)_0 - [\Delta(D_tu)_0 - 3(D_tu)_0^2\{\Delta u_0 - (D_tu)_0^3\}] \|$$
$$\leqq \| \Delta(D_tu_m)_0 - \Delta(D_tu)_0 \| + C \| P_m((D_tu_m)_0^2(D_t^2u_m)_0) - (D_tu)_0^2\{\Delta u_0 - (D_tu)_0^3\} \|$$
$$\leqq \| \Delta(D_tu_m)_0 - \Delta(D_tu)_0 \| + C\{ \| (D_tu_m)_0^2(D_t^2u_m)_0 - (D_tu)_0^2\{\Delta u_0 - (D_tu)_0^3\} \|$$
$$+ \| P_m[(D_tu)_0^2\{\Delta u_o - (D_tu)_0^3\}] - (D_tu)_0^2\{\Delta u_0 - (D_tu)_0^3\} \| \}.$$

Here,

$$\| (D_tu_m)_0^2 (D_t^2u_m)_0 - (D_tu)_0^2\{\Delta u_0 - (D_tu)_0^3\} \|$$
$$\leqq \| \{(D_t^2u_m)_0 - (\Delta u_0 - (D_tu)_0^3\}(D_tu_m)_0^2 \| + \| \{(D_tu_m)_0^2 - (D_tu)_0^2\}\{\Delta u_0 - (D_tu)_0^3\} \|$$
$$\leqq C\{ \| (D_t^2u_m)_0 - \{\Delta u_0 - (D_tu)_0^3\} \| + \| (D_tu_m)_0 - (D_tu)_0 \| \}.$$
$$\| \nabla(D_t^2u_m)_0 - \{\nabla(\Delta u_0) - \nabla(D_tu)_0^3\} \|$$
$$\leqq \| \nabla(\Delta u_{m0} - \Delta u_0) \| + \| \nabla P_m(D_tu_m)_0^3 - \nabla(D_tu)_0^3 \|$$
$$\leqq \| \nabla(\Delta u_{m0} - \Delta u_0) \| + C\{ \| \Delta P_m(D_tu_m)_0^3 - \Delta(D_tu)_0^3 \|$$
$$\leqq \| \nabla(\Delta u_{m0} - \Delta u_0) \| + C\{ \| P_m\{3\Delta(D_tu_m)_0(D_tu_m)_0^2 + 6(D_tu_m)_0\{\nabla(D_tu_m)_0\}^2$$
$$-3\Delta(D_tu)_0(D_tu)_0^2 - 6(D_tu)_0\{\nabla(D_tu)_0\}^2 \| + \| P_m\{\Delta(D_tu)_0^3\} - \Delta(D_tu)_0^3 \| \}.$$

Estimates of the above give (3.19).

By a priori estimates in § 2, estimates (3.7) and relations (3.19), the approximate solutions are estimated as follows.

PROPOSITION 1. *It holds that for* $k=0, 1, 3, 4$

$$\| D_t^{k+1}u_m \|^2 + \| \nabla D_t^k u_m \|^2 \leq C\{ \| (D_t^{k+1}u_m)_0 \|^2 + \| \nabla (D_t^k u_m)_0 \|^2\} + C \leq C$$

and specially for $k=2$

$$\| D_t^3 u_m \|^2 + \| \nabla D_t^2 u_m \|^2$$

$$\leq \frac{\| (D_t^3 u_m)_0 \|^2 + \| \nabla (D_t^2 u_m)_0 \|^2}{\{1-6\left(4/\sqrt{3}\right)^3 T_0(\| (D_t^2 u_m)_0 \|^2 + \| \nabla (D_t u_m)_0 \|^2)^{1/2}(\| (D_t^3 u_m)_0 \|^2 + \| \nabla D_t^2 u_m)_0 \|^2)^{1/2}\}^2},$$

where a positive number T_0 *is chosen sufficiently small to satisfy the inequality*

$$1-6\left(4/\sqrt{3}\right)^3 T_0(\| \Delta u_0 - (D_t u)_0^3 \|^2 + \| \nabla (D_t u)_0 \|^2)^{1/2}$$

$$\times (\| \Delta (D_t u)_0 - 3(D_t u)_0^2\{\Delta u_0 - (D_t u)_0^3\} \|^2 + \| \nabla (\Delta u_0) - \nabla (D_t u)_0^3 \|^2)^{1/2} > 0.$$

REMARK. As is explained before, in the differential equations (3.2) $\lambda_k^m(t)$ $(k=1, 2, \ldots\ldots, m)$ are determined in $[0, T]$ for any finite $T>0$, from the following boundedness

$$\left|\frac{d\lambda_k^m(t)}{dt}\right|^2 = \| D_t u_m \|^2 \leq C, \ t \in [0, T], \ (m=1, 2, \ldots\ldots, \ k=1, 2, \ldots\ldots, \ m),$$

$$\left|\sum_{j=1}^m \lambda_j^m(t)(\nabla\psi_j, \nabla\psi_k)\right| = |(\nabla u_m, \nabla\psi_k)| \leq \| \Delta u_m \| \ \| \psi_k \|$$

$$\leq \| D_t^2 u_m \| + \| P_m(D_t u_m)^3 \| \leq C + C \| \nabla (D_t u_m) \|^3 \leq C$$

and

$$\left|\left(\left\{\sum_{j=1}^m \frac{d\lambda_j^m}{dt}\psi_j\right\}^3, \ \psi_k\right)\right| = |(\{D_t u_m\}^3, \ \psi_k)| \leq \| (D_t u_m)^3 \| \ \| \psi_k \| \leq C.$$

Here, we applied the proposition 1 for $k=0$ and 1.

§ 4 Compactness of approximate solutions and some properties of the limit function

Now we introduce some conceptions. Let $\mathscr{H}$ be a separable Hilbert space. Here $L^2(\Omega)$, Sobolev spaces $W_2^m(\Omega)$ $(m=1, 2, 3, \ldots\ldots)$ are considered. We denote by w: $[0, T] \longrightarrow \mathscr{H}$ the function defined in $[0, T]$ with values $w(t)$ in $\mathscr{H}$. The function w is said to have the *strong derivative* in $[0, T]$ if there exists a function v: $[0, T] \longrightarrow \mathscr{H}$ such that $\| \frac{1}{h}\{w(t+h) - w(t)\} - v(t) \|_{\mathscr{H}} \longrightarrow 0$ for any $t \in [0, T]$, as $h \to 0$. The function w is said to be in the *class* $C^m([0, T] : \mathscr{H})$ if there exist strong derivatives $D^k w$ $(k=1, 2, \ldots\ldots$

m) and they are continuous i.e. $\| D^k w\ (t+h) - D^k w(t) \|_{\mathscr{H}} \to 0$ for any $t \in [0, T]$, as $h \to 0$. In this section we will apply *the following relations.* ([1], [7], [9].)

If $w \in C([0, T] : \mathscr{H})$ *it holds* $\frac{d}{dt}$ (B) $\int_0^t w(s)ds = w(t)$ *for any* $t \in [0, T]$, *which implies* $\| \frac{1}{h}$ (B) $\int_t^{t+h} w(s)ds - w(t) \|_{\mathscr{H}} \to 0$, *as* $h \to 0$. *Here,* (B) *denotes the Bochner integral.*

In the first place we have *the following convergence*:

$$D_t^k u_m \longrightarrow D_t^k u \text{ in } L^2(\Omega \times [0, T_0]), \text{ as } m \longrightarrow \infty (0 \leq k \leq 4), \tag{4.1}$$

where $D_t^k u$ *is the generalized derivative.*
In fact, the proposition 1 gives the boundedness:

$$\| D_t^k u_m \|_{L^2(\Omega \times [0, T_0])},\ \| D_t(D_t^k u_m) \|_{L^2(\Omega \times [0, T_0])},\ \| \nabla (D_t^k u_m) \|_{L^2(\Omega \times [0, T_0])} \leq C\ (0 \leq k \leq 4).$$

Hence by Rellich's theorem it is possible to extract a subsequence which is Cauchy sequence in $L^2(\Omega \times [0, T_0])$.

In the next place we have *the following relations.*

$$\begin{cases} D_t^k u_m(\cdot, t) \longrightarrow D_t^k u(\cdot, t) \text{ in } L^2(\Omega), \textit{ uniformly for } t \in [0, T_0]. \ (0 \leq k \leq 3) \\ D_t^k u(\cdot, t) - (D_t^k u)_0 = (B) \int_0^t D_t^{k+1} u(\cdot, s) ds \ (0 \leq k \leq 2). \end{cases} \tag{4.2}$$

Here, $D_t^k u(\cdot, t)$ *denotes the strong derivative in* $L^2(\Omega)$ *and* (B) *denotes the Bochner integral in* $L^2(\Omega)$

$$\begin{cases} D_t^k u(\cdot, t) \in \overset{\circ}{W}{}_2^1(\Omega) \textit{ for any } t \in [0, T_0], \ (0 \leq k \leq 3) \\ \nabla D_t^k u_m(\cdot, t) \longrightarrow \nabla D_t^k u(\cdot, t) \textit{ weakly in } L^2(\Omega), \ (0 \leq k \leq 3) \\ \| \nabla D_t^k u(\cdot, t) - \nabla D_t^k u(\cdot, \tau) \| \leq C|t-\tau|, \ (0 \leq k \leq 3). \end{cases} \tag{4.3}$$

$$D_t^k (D_t u_m)^3 \longrightarrow D_t^k (D_t u)^3 \textit{ in } L^2(\Omega), \textit{ uniformly for } t \in [0, T_0] \ (0 \leq k \leq 2). \tag{4.4}$$

$$\begin{cases} D_t^4 u_m(\cdot, t) \longrightarrow D_t^4 u(\cdot, t) \textit{ weakly in } L^2(\Omega), \textit{ as } m \to \infty \\ \| D_t^4 u(\cdot, t) - D_t^4 u(\cdot, \tau) \| \leq C|t-\tau| \\ D_t^3 u(\cdot, t) - D_t^3 u(\cdot, 0) = (B) \int_0^t D_t^4 u(\cdot, s)\ ds \end{cases} \tag{4.5}$$

PROOF. From the equality $D_t^k u_m(x, t) = D_t^k u_m(x, 0) + \int_0^t D^{k+1} u_m(x, s)\ ds$ it follows that

$$\int_\Omega | D_t^k u_m(x, t) - D_t^k u_n(x, t) |^2 dx \leq 2 \int_\Omega | D_t^k u_m(x, 0) - D_t^k u_n(x, 0) |^2 dx + T_0 \int_0^{T_0} \int_\Omega | D_t^{k+1} u_m(x, s) - D_t^{k+1} u_n(x, s) |^2 dxds.$$

Here, by (3.19), $(D_t^k u_m)_0$ $(k = 0, 1, 2, 3)$ converge in $L^2(\Omega)$ as $m \longrightarrow \infty$. The second term of the right side converges to zero as $m, n \longrightarrow \infty$, by (4.1)

when $0 \leq k \leq 3$.
Hence, there exist functions $v_k (0 \leq k \leq 3)$ such that

$$D_t^k u_m(\cdot,\ t) \longrightarrow v_k(\cdot,\ t) \text{ in } L^2(\Omega), \text{ uniformly for } t \in [0,\ T_0].$$

If we set $v_0 = u$, then it follows that $v_k = D_t^k u$. In fact

$$u_m(x,\ t) - u_m(x,\ 0) = \int_0^t D_t u_m(x,\ s)\ ds,$$

or

$$u_m(\cdot,\ t) - u_{m0} = (\mathrm{B}) \int_0^t D_t u_m(\cdot,\ s)\ ds.$$

Here

$$\| (\mathrm{B}) \int_0^t D_t u_m(\cdot,\ s) ds - (\mathrm{B}) \int_0^t v_1(\cdot, s) ds \|$$
$$\leq \int_0^t \| D_t u_m(\cdot, s) - v_1(\cdot, s) \|\ ds \longrightarrow 0, \text{ as } m \to \infty.$$

It gives that $u(\cdot,t) - u_0 = (\mathrm{B}) \int_0^t v_1(\cdot, s)\ ds$. Moreover, $D_t u = v_1$. Similarly (4.2) is shown. With respect to (4.3), we apply *the following Lemma*:

If w is the weak limit in $L^2(\Omega)$ of a sequence $w_m \in W_2^1(\Omega)$ $(m = 1, 2, \ldots\ldots)$ with bounded norm $\| w_m \|_{W_2^1(\Omega)} \leq C$, then $w \in W_2^1(\Omega)$ and $\nabla w_m \longrightarrow \nabla w$ weakly in $L^2(\Omega)$ ([4]).

Since, $\| D_t^k u_m(\cdot,t) \|$, $\| \nabla D_t^k u_m(\cdot,t) \| \leq C (0 \leq k \leq 3)$ and $D_t^k u_m(\cdot,t) \longrightarrow D_t^k u(\cdot,t)$ in $L^2(\Omega)$, uniformly for $t \in [0,\ T_0]$ $(0 \leq k \leq 3)$ it follows that $D_t^k u(\cdot,t) \in W_2^1(\Omega)$ for $t \in [0,\ T_0]$ $(0 \leq k \leq 3)$, and $\nabla D_t^k u_m(\cdot,t) \longrightarrow \nabla D_t^k u(\cdot,t)$ weakly in $L^2(\Omega)$ $(0 \leq k \leq 3)$. By the result of the above, it holds that $D_t^k u_m(\cdot,t) \longrightarrow D_t^k u(\cdot,t)$ weakly in $W_2^1(\Omega)$, as $m \to \infty$. Noting that $D_t^k u_m(\cdot,t) \in \mathring{W}_2^1(\Omega)$ for $t \in [0,\ T_0]$, we have $D_t^k u(\cdot,t) \in \mathring{W}_2^1(\Omega)$ for $t \in [0,\ T_0]$ $(0 \leq k \leq 3)$ and

$$|(\nabla D_t^k u_m(\cdot,t) - \nabla D_t^k u_m(\cdot,\tau),\ w)| = |(\int_\tau^t \nabla D_t^k u_m(\cdot,s)\ ds,\ w)|$$
$$\leq \int_\tau^t \| \nabla D_t^k u_m(\cdot,s) \|\ ds \| w \|$$
$$\leq C \| w \|\ |t-\tau| \qquad \text{for any } w \in L^2(\Omega),\ 0 \leq k \leq 3,$$

which implies $\| \nabla D_t^k u(\cdot,t) - \nabla D_t^k u(\cdot,\tau) \| \leq C |t-\tau|$.
To prove (4.4), we verify the boundedness:

$$|D_t^k u_m(x,\ t)| \leq C \text{ in } \Omega \times [0,\ T_0],$$

where C is a constant independent of m,x,t.
From (3.1) it follows that

$$D_t^{k+2} u_m - \Delta D_t^k u_m + P_m D_t^k (D_t u_m)^3 = 0 \ \ (0 \leq k \leq 3).$$

Here, $\| (D_t u_m)^3 \| \leq C \| \nabla (D_t u_m) \|^3 \leq C,$
$$\| D_t (D_t u_m)^3 \| \leq C \| \nabla D_t u_m \|^2 \| \nabla D_t^2 u_m \| \leq C,$$
$$\| D_t^2 (D_t u_m)^3 \| \leq C\{ \| \nabla D_t u_m \| \ \| \nabla D_t^2 u_m \|^2 + \| \nabla D_t u_m \|^2 \| D_t^3 u_m \| \} \leq C,$$
$$\| D_t^3 (D_t u_m)^3 \| \leq C\{ \| \nabla D_t^2 u_m \|^3 + \| \nabla D_t u_m \| \ \| \nabla D_t^2 u_m \| \ \| \nabla D_t^3 u_m \| + \| \nabla D_t u_m \|^2 \| D_t^4 u_m \| \} \leq C.$$

Hence
$$| D_t^k u_m(x, t) | \leq C \| D_t^k u_m(\cdot, t) \|_{W_2^2(\Omega)} \leq C \| \Delta D_t^k u_m(\cdot, t) \|$$
$$\leq C\{ \| D_t^{k+2} u_m(\cdot, t) \| + \| P_m D_t^k (D_t u_m)^3 \| \} \leq C (0 \leq k \leq 3).$$

Consequently,
$$\| (D_t u_m)^3 - (D_t u)^3 \| = \| (D_t u_m - D_t u)((D_t u_m)^2 + (D_t u_m) D_t u + (D_t u)^2 \| \leq C \| D_t u_m - D_t u \|$$
$$\| D_t (D_t u_m)^3 - 3 (D_t u)^2 (D_t^2 u) \| \leq C \{ \| (D_t u_m)^2 \{ D_t^2 u_m - D_t^2 u \} \| + \| \{ (D_t u_m)^2 - (D_t u)^2 \} D_t^2 u \| \}$$
$$\leq C \{ \| D_t^2 u_m - D_t^2 u \| + \| D_t u_m - D_t u \| \}$$
$$\| D_t^2 (D_t u_m)^3 - 6 (D_t u)(D_t^2 u)^2 - 3 (D_t u)^2 D_t^3 u \|$$
$$\leq C \{ \| D_t u_m \{ (D_t^2 u_m)^2 - (D_t^2 u)^2 \} \| + \| (D_t u_m - D_t u)(D_t^2 u)^2 \| + \| (D_t u_m)^2 \{ D_t^3 u_m - D_t^3 u \} \| + \| \{ (D_t u_m)^2 - (D_t u)^2 \} D_t^3 u \| \}$$
$$\leq C \{ \| D_t^2 u_m - D_t^2 u \| + \| D_t u_m - D_t u \| + \| D_t^3 u_m - D_t^3 u \| \}.$$

These inequalities give (4.4) by application of (4.2).
Finally, we will show (4.5).
$$(D_t^4 u_m(\cdot, t), w) = ((D_t^4 u_m)_0, w) + (\int_0^t D_t^5 u_m(\cdot, s) \, ds, w)$$
$$= ((D_t^4 u_m)_0, w) + \int_0^t (D_t^5 u_m(\cdot, s), w) ds \quad \text{for } w \in L^2(\Omega).$$

Here, $\| (D_t^4 u_m)_0 \| \leq C$
$$\| D_t^5 u_m \|_{L^2(\Omega) \times [0, T_0])} \leq C.$$

Hence, it is possible to choose convergent subsequences of $((D_t^4 u_m)_0, w)$, $\int_0^t (D_t^5 u_m(\cdot, s), w) ds$, which implies the subsequence $D_t^4 u_{m_j}(\cdot, t)$ converges weakly in $L^2(\Omega)$. Denote the limit function by v_4: $[0, T_0] \longrightarrow L^2(\Omega)$, then
$$(D_t^3 u_m(\cdot, t) - (D_t^3 u_m)_0, w) = (\int_0^t D_t^4 u_m(\cdot, s) \, ds, w) = \int_0^t (D_t^4 u_m(\cdot, s), w) \, ds$$
$$\longrightarrow \int_0^t (v_4(\cdot, s), w) \, ds, \text{ as } m \to \infty, \text{ for any } w \in L^2(\Omega),$$

Since it holds $\| v_4(\cdot, t) \| \leq C$ for any $t \in [0, T_0]$, $v_4(\cdot, t)$ is a Bochner integrable function. Hence we have

$$(D_t^3 u(\cdot,t) - (D_t^3 u)_0,\ w) = ((\mathrm{B})\int_0^t v_4(\cdot,s)ds,\ w).$$

Consequently (4.5) is completely shown.

§ 5 The existence of the classical solution

First, we show *equalities*

$$(5.1)\qquad (D_t^{2+j}u(\cdot,t),\ w) + (\nabla D_t^j u(\cdot,t),\ \nabla w) + (D_t^j (D_t u)^3(\cdot,t),\ w) = 0$$
$$\textit{for any } w \in \overset{\circ}{W}{}_2^1(\Omega).\ (t \in [0,\ T_0],\ 0 \leq j \leq 2)$$

Here, $u(\cdot,t)$ is the limit function in § 4.

In fact, for approximate solutions $u_m(x,\ t)$ it holds

$$(D_t^{2+j}u_m(\cdot,t),\ \psi_k) + (\nabla D_t^j u_m(\cdot,t),\ \nabla\psi_k) + (D_t^j (D_t u_m)^3(\cdot,t),\ \psi_k) = 0$$
$$\text{for } m \geq k.$$

Hence, there results

$$(D_t^{2+j}u(\cdot,t),\ \psi_k) + (\nabla D_t^j u(\cdot,t),\ \nabla\psi_k) + (D_t^j (D_t u)^3(\cdot,t),\ \psi_k) = 0$$
$$(k = 1, 2, 3, \ldots\ldots).$$

by the convergence relations (4.2), (4.3), (4.4) and (4.5).
Here, $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ is dense in $\overset{\circ}{W}{}_2^1(\Omega)$, and finite linear combinations of eigenfunctions$\{\psi_k\}$ is dense in $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$.
Consequently (5.1) is shown.

Next we have *properties of u as follows,*

$$(5.2)\qquad D_t^j u \in C([0,\ T_0] : W_2^2(\Omega)) \cap C(\overline{\Omega} \times [0,\ T_0])\ \ (j = 0, 1, 2).$$

Indeed, set $j = 0$ in (5.1), it follows that for any $w \in \overset{\circ}{W}{}_2^1(\Omega)$

$$-(\nabla u(\cdot,t),\ \nabla w) = (D_t^2 u(\cdot,t) + (D_t u)^3(\cdot,t),\ w).$$

Hence, $u(\cdot,t)(\in \overset{\circ}{W}{}_2^1(\Omega)$ for $t \in [0,\ T_0])$ is a weak solution of the equation

$$\Delta v = D_t^2 u(\cdot,t) + (D_t u)^3(\cdot,t) \in L^2(\Omega).$$

Consequently

$$u(\cdot,t) \in W_2^2(\Omega)$$

and

$$\| u(\cdot,t) \|_{W_2^2(\Omega)} \leq C\{ \| D_t^2 u(\cdot,t) \| + \| (D_t u)^3(\cdot,t) \| + \| u(\cdot,t) \| \}$$

Moreover it holds

$$\| u(\cdot,t)-u(\cdot,\tau) \|_{W_2^2(\Omega)} \leq C\{ \| D_t^2u(\cdot,t)-D_t^2u(\cdot,\tau) \| + \| (D_tu)^3(\cdot,t)-(D_tu)^3(\cdot,\tau) \| + \| u(\cdot,t)-u(\cdot,\tau) \| \}.$$

Since u, D_t^2u and $(D_tu)^3 \in C([0, T_0] : L^2(\Omega))$ by the convergence relations (4.2) and (4.4), we have $u \in C([0, T_0] : W_2^2(\Omega))$. Also the inequality

$$|u(x, t)-u(\xi, \tau)| \leq |u(x, t)-u(\xi, t))| + C \| u(\cdot,t)-u(\cdot,\tau) \|_{W_2^2(\Omega)}$$

gives the relation $u(x, t) \in C(\overline{\Omega} \times [0, T_0])$ by the notice of $u(x, t) \in C(\overline{\Omega})$ for fixed $t \in [0, T_0]$. Similarly, set $j=1,2$ in (5.1), then we have (5.2), since $D_t^3u+D_t(D_tu)^3$, $D_t^4u+D_t^2(D_tu)^3 \in C([0, T_0] : L^2(\Omega))$.

Finally, we have *the following properties of* u;

(5.3) $\qquad (D_tu)^3 \in C([0, T_0] ; W_2^1(\Omega)) \cap C([0, T_0] ; W_2^2(\Omega))$

(5.4) $\qquad u \in C([0, T_0] ; W_2^4(\Omega))$

(5.5) $\qquad D_tu \in C([0, T_0] ; W_2^3(\Omega))$

PROOF. Relations $D_tu \in C(\overline{\Omega} \times [0, T_0])$, $\nabla D_tu \in C([0, T_0] ; L^2(\Omega))$ and $D_tu \in C([0, T_0] ; W_2^2(\Omega))$ imply that equalities $D_{x_k}(D_tu)^3 = 3(D_tu)^2 D_{x_k}(D_tu)$ and $D_{x_j}D_{x_k}(D_tu)^3 = 6(D_tu)D_{x_j}(D_tu)D_{x_k}(D_tu) + 3(D_tu)^2 D_{x_j}D_{x_k}(D_tu)$ hold. Here, we applied the fact that if f, $\nabla f \in L^p(\Omega)$ and g, $\nabla g \in L^q(\Omega)$ $(1/p+1/q=1)$ it holds $\nabla(fg) = (\nabla f)g + f(\nabla g)$. Noting that $fg \in C([0, T] ; L^2(\Omega))$ for $f \in C(\overline{\Omega} \times [0, T])$ and $g \in C([0, T] ; L^2(\Omega))$ we have (5.3).

If we set $j=0$ in (5.1), $u(\cdot,t)$ is a weak solution of the equation $\Delta v = D_t^2u(\cdot,t)+(D_tu)^3(\cdot,t)$, where $D_t^2u+(D_tu)^3 \in C([0, T_0] : W_2^2(\Omega))$. Hence, $u \in C([0. T_0] : W_2^4(\Omega))$. Also we set $j=1$ in (5.1), then $D_tu(\cdot,t)$ is a weak solution of the equation $\Delta v = D_t^3u(\cdot,t)+D_t(D_tu)^3(\cdot,t)$. Here $D_t(D_tu)^3 = 3(D_tu)^2(D_t^2u) \in C([0, T_0] ; L^2(\Omega))$ and $D_{x_k}(D_tu)^2 D_t^2u = 2D_tuD_{x_k}(D_tu)D_t^2u + (D_tu)^2 D_{x_k}D_t^2u \in C([0, T_0] : L^2(\Omega))$, $D_t^3u \in C([0, T_0] : W_2^1(\Omega))$. Hence, $D_tu \in C([0, T_0] : W_2^3(\Omega))$.

Now, we take notice of the following relations (5.6) and (5.7).

(5.6) *If* u *belongs to the space* $C^m([0, T] : L^2(\Omega))$ *and the strong derivatives* D_t^ku *in* $L^2(\Omega)$ $(k=0,1,2,\ldots\ldots,m)$ *are belong to the space* $C([0, T] : W_2^l(\Omega))$ *it follows that* $u \in C^m([0, T] : W_2^l(\Omega))$ *and*

$$\left\| \frac{1}{h}(D_t^{k-1}u(\cdot, t+h)-D_t^{k-1}u(\cdot,t))-D_t^ku(,t) \right\|_{W_2^l(\Omega)} \to 0 \text{ as } h \to 0, \ (k=1,2, \ldots\ldots, m).$$

We verify for $m=1$. From the assumption $D_tu \in C([0, T] ; W_2^l(\Omega))$, there exists the Bochner integral $(W_2^l,\ \mathrm{B})\int_0^t D_s u(\cdot,\ s)ds$ in $W_2^l(\Omega)$. Hence, the Bochner integral $(L^2,\ \mathrm{B})\int_0^t D_s u(\cdot,s)ds$ in $L^2(\Omega)$ exists, and

$$(L^2,\ \mathrm{B})\ \int_0^t D_s u(\cdot,s)ds = (W_2^l,\ \mathrm{B})\int_0^t D_s u(\cdot,s)\ ds$$

as an element of $L^2(\Omega)$. On the other hand

$$(L^2,\ \mathrm{B})\ \int_0^t D_s u(\cdot,s)ds = u(\cdot,t)-u(\cdot,0).\quad \text{(See [7])}$$

By the above equalities we have

$$u(\cdot,t)-u(\cdot,0) = (W_2^l,\ \mathrm{B})\int_0^t D_s u(\cdot,\ s)\ ds,$$

as an element of $W_2^l(\Omega)$ or

$$\| \frac{1}{h}(u(\cdot,\ t+h)-u\ (\cdot,\ t))-D_t u(\cdot,\ t)\|_{W_2^l(\Omega)}\to 0,\ \text{as}\ h\to 0.$$

(5.7) $$\bigcap_{k=0}^{2} C^k([0,\ T]:\ W_2^{4-k}(\Omega))\subset C^2(\bar{\Omega}\times[0,\ T])$$

This relation is given by Sather [5] or Wilcox [8]. We give an outline of the proof. Since $u\in C([0,\ T]:\ W_2^4(\Omega))$, we have

$$|D_x^\alpha u(x,\ t)|\leqq C\| u(\cdot,t)\|_{W_2^4(\Omega)},\ (|\alpha|\leqq 2,\ (x,\ t)\in\bar{\Omega}\times[0,\ T])$$

and

$$|D_x^\alpha u(x,\ t)-D_x^\alpha u\ (\xi,\ \tau)|\leqq C\| u(\cdot,t)-u(\cdot,\tau)\|_{W_2^4(\Omega)}+|D_x^\alpha u(x,\ \tau)-D_x^\alpha u(\xi,\ \tau)|.$$

Hence, $D_x^\alpha u\in C(\bar{\Omega}\times[0,\ T])$ $(|\alpha|\leqq 2)$. The relation $u\in C^1([0,\ T]:\ W_2^3(\Omega))$ gives $D_t u\in C([0,\ T]:\ W_2^3(\Omega))$ and $D_x^\alpha\ D_t u\in C(\bar{\Omega}\times[0,\ T])$ $(|\alpha|\leqq 1)$.
Hence $D_t u,\ D_x D_t u\in C(\bar{\Omega}\times[0,\ T])$. The relation $u\in C^2([0,\ T]:\ W_2^2(\Omega))$ give $D_t^2 u\in C(\bar{\Omega}\times[0,\ T])$. Moreover

$$|\frac{1}{h}\{D_x^\alpha u(x,\ t+h)-D_x^\alpha u\ (x,\ t)\}-D_x^\alpha D_t u(x,\ t)|$$

$$\leqq C\|\frac{1}{h}\{u(\cdot,t+h)-u(\cdot,t)\}-D_t u(\cdot,t)\|_{W_2^3(\Omega)}\to 0,\ \text{as}\ h\to 0,$$

$$|\frac{1}{h}\{D_t u(x,t+h)-D_t u(x,\ t)\}-D_t^2 u(x,\ t)|$$

$$\leqq C\|\frac{1}{h}\{D_t u(\cdot,t+h)-D_t u(\cdot,t)\}-D_t^2 u(\cdot,t)\|_{W_2^2(\Omega)}\to 0,\ \text{as}\ h\to 0.$$

Consequently, (5.7) is verified.

We attain the following existence theorem of the classical solution:

THEOREM. *Let Ω be a bounded domain in R^3 with a sufficiently smooth boundary. Suppose that the initial data u_0 belongs to the space V_0, and $(D_tu)_0$ belongs to the space V_1, where V_i $(i=0,1)$ are defined in* § 3. *Then there exists a function u belonging to the class C^2 $(\bar{\Omega}\times[0, T_0])$, which satisfies the equation* (1.1) *in $\Omega\times(0, T_0)$ and conditions* (1.2)–(1.4). *Here T_0 is a positive number depending on u_0* and $(D_tu)_0$.

PROOF. The equation (5.1) gives the equality

$$(D_t^2u(\cdot,t),\ w)-(\Delta u(\cdot,t),\ w)+((D_tu)^3(\cdot,t),\ w)=0$$

for any $w\in \overset{\circ}{W}{}_2^1(\Omega)$, $t\in[0, T_0]$.

On the other hand, $u\in\bigcap_{k=0}^{2} C^k([0, T_0]:W_2^{4-k}(\Omega))\subset C^2(\bar{\Omega}\times[0, T_0])$.

Indeed, $u\in C^1([0, T_0]:L^2(\Omega))$ by (4.2) and u, $D_tu\in C([0, T_0]:W_2^3(\Omega))$ by (5.5). Hence $u\in C^1([0, T_0]; W_2^3(\Omega))$ by noting (5.6). Also, $u\in C^2([0, T_0]:L^2(\Omega))$ by (4.2) and u, D_tu, $D_t^2u\in C([0, T_0]:W_2^2(\Omega))$ by (5.2). Hence $u\in C^2([0, T_0]; W_2^2(\Omega))$ by noting (5.6). $u\in C([0, T_0]:W_2^4(\Omega))$ was shown in (5.4).

Consequently, $D_t^2u-\Delta u+(D_tu)^3=0$ in $\Omega\times(0, T_0)$, since $\overset{\circ}{W}{}_2^1(\Omega)$ is dense in $L^2(\Omega)$. By virtue of (4.2) it holds that $u(x, 0)=u_0(x)$, $D_tu(x, 0)=(D_tu)_0(x)$, $x\in\Omega$. $u(\cdot,t)\in\overset{\circ}{W}{}_2^1(\Omega)\cap C^1(\bar{\Omega})$ implies that u vanishes on the boundary in the natural sense.

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