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On the complex K-group of certain manifold

By

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1. Let D_p be a dihedral group of order $2p$ which is generated by g of order p and t of order 2 such that $tgt=g^{-1}$. We consider an action of D_p over a product space $S^{2l+1}\times S^m\subset C^{l+1}\times R^{m+1}$ of spheres, given by

$$g^k t^j(z, x)=(\rho^k c^j(z), (-1)^j x), \tag{1.1}$$

where c is the conjugation and $\rho=exp2\pi\sqrt{-1}/p$. Denote by $D_p(l, m)=(S^{2l+1}\times S^m)/D_p$ the orbit space [3]. We are concerned with the complex K-group of $D_p(2k+1, 2n+1)$ where p is an odd prime.

Consider an action of Z_2, a cyclic group of order 2, over the complex K-group $K(X)$, given by the conjugation automorphism $\overline{[\xi]}=[\bar{\xi}]$. Let $K(X)^{Z_2}$ be the invariant subgroup of $K(X)$ under the involution. In this paper we obtain

THEOREM 1.1. *Suppose that p is an odd prime. Then, there exists an isomorphism*

$$\tilde{K}(D_p(2k+1, 2n+1)) \simeq Z\oplus \tilde{K}(L^{2k+1}(p))^{Z_2}\oplus \tilde{K}(RP^{2n+1}),$$

where $L^{2k+1}(p)$ is a standard $(4k+3)$-dimensional lens space and RP^{2n+1} is a $(2n+1)$-dimensional real projective space.

2. Denote by $K_G(X)$ the equivariant K-group of G-space X. It is well-known that if the action of G is free then $K_G(X)\cong K(X/G)$. There is a canonical homomorphism from the representation ring $R(G)$ to $K_G(X)$ which maps a representation space M to a equivariant G-bundle $X\times M$. If X is free G-space then there is a homomorphism

$$\pi:\ R(G)\longrightarrow K_G(X)\cong K(X/G).$$

Let α: $H\longrightarrow G$ be a homomorphism and f: $Y\longrightarrow X$ be an equivariant map from H-space Y to G-space X, that is, $f(h\cdot y)=\alpha(h)\cdot f(y)$. The equivariant map f induces the homomorphism

$$f!: K_G(X) \longrightarrow K_H(Y) \quad [6].$$

We take a Z_2-action on S^m by

$$t^i \cdot x = (-1)^i x, \ t \in Z_2 \text{ a generator},$$

a Z_p-action on S^{2l+1} by

$$g^j \cdot z = \rho^j z, \ \rho = exp 2\pi\sqrt{-1}/p, \ g \in Z_p \text{ a generator}$$

and a D_p-action on $S^{2l+1} \times S^m$ given by (1.1) in section 1. We have these orbit spaces an m-dimensional real projective space RP^m, a $(2l+1)$-dimensional lens space $L^l(p)$ and $D_p(l, m)$. There exist equivariant maps

$$i: S^{2l+1} \longrightarrow S^{2l+1} \times S^m, \ i(z) = (z, (1, 0, \cdots, 0)),$$
$$j: S^m \longrightarrow S^{2l+1} \times S^m, \ j(x) = ((1, 0, \cdots, 0), x))$$

and

$$p: S^{2l+1} \times S^m \longrightarrow S^m, \ p(z, x) = x$$

compatible with injections $\tilde{i}: Z_p \longrightarrow D_p$, $\tilde{j}: Z_2 \longrightarrow D_p$ and a projection $\tilde{p}: D_p \longrightarrow Z_2$ respectively. It follows immediately that

(2.1) $$j^! p^! = 1.$$

Let H be a normal subgroup of a finite group G and A be a unitary representation of H. The induced representation A^G is defined as follows

(2.2) $$A^G(g) = \begin{pmatrix} A(t_1 g t_1) & A(t_1 g t_2) & \cdots & A(t_1 g t_n) \\ A(t_2 g t_1) & A(t_2 g t_2) & \cdots & A(t_2 g t_n) \\ \vdots & \vdots & & \vdots \\ A(t_n g t_1) & A(t_n g t_2) & \cdots & A(t_n g t_n) \end{pmatrix}$$

where $G/H = \{\{t_1\}, \{t_2\}, \cdots, \{t_n\}\}$ and $A(t_i g t_j) = 0$ if $t_i g t_j \notin H$.

Denote by L a standard representation space with $(exp 2\pi\sqrt{-1}/p)$. For a standard Z_p-space S^{2l+1} with $S^{2l+1}/Z_p = L^l(p)$, we put

$$\xi_l = S^{2l+1} \times_{Z_p} L.$$

N. Mahammed [5] obtained that

$$K_{Z_p}(S^{2l+1}) \cong K(L^l(p)) \cong Z[\xi_l]/(\xi_l^p - 1, \ (\xi_l - 1)^{l+1}).$$

Hence, the homomorphism $\pi: R(Z_p) \longrightarrow K_{Z_p}(S^{2l+1})$ is surjective. Then, we define the homomorphism

$$i_*: K_{Z_p}(S^{2l+1}) \longrightarrow K_{D_p}(S^{2l+1}\times S^m)$$

by $i_*(S^{2l+1}\times M)=S^{2l+1}\times S^m\times M^{D_p}$, where M is representation space of Z_p and M^{D_p} is the induced representation space.

We consider a Z_2-action over $K_{Z_p}(S^{2l+1})$ given by

$$t(S^{2l+1}\times M) = S^{2l+1}\times \overline{M},$$

where $\overline{M}$ is a conjugate representation space of M and t is a generator of Z_2.

Let $K_{Z_p}(S^{2l+1})^{Z_2}$ be the invariant subgroup under the Z_2-action. Then, we have

PROPOSITION 2.1. *For* $\eta \in K_{Z_p}(S^{2l+1})^{Z_2}$,

$$i^! i_*(\eta) = 2\eta.$$

PROOF. Suppose that $\eta = S^{2l+1}\times M \in K_{Z_p}(S^{2l+1})^{Z_2}$, that is, $\overline{M} = M$. Let A be the representation of M. Then, $i_*(\eta) = S^{2l+1}\times S^m\times M^{D_p}$, where M^{D_p} is the induced representation space of M. The representation A^{D_p} of M^{D_p} is given as follows:

$$A^{D_p}(g)=\begin{pmatrix} A(g) & 0 \\ 0 & \overline{A}(g)\end{pmatrix}, \quad g \in Z_p.$$

Since $\overline{M} = M$ and $\overline{A} = A$,

$$i^! i_*(\eta) = 2\eta. \qquad \text{q. e. d.}$$

THEOREM 2.2. *The homomorphism*

$$\theta: \tilde{K}_{Z_p}(S^{2l+1})^{Z_2}\oplus\tilde{K}_{Z_2}(S^m) \longrightarrow \tilde{K}_{D_p}(S^{2l+1}\times S^m)$$

given by $\theta(\eta, \nu) = i_*(\eta)+p^!(\nu)$ *is injective.*

PROOF. Suppose that $\theta(\eta, \nu) = 0$. Since $i^! p^! = 0$, we have $i^! i_*(\eta) = 0$. Hence, from Proposition 2.1, it follows that $\eta = 0$. On the other hand, from (2.1), we have $\nu = j^! p^!(\nu) = 0$. q. e. d.

3. The manifold $D_p(l, m)$ is homeomorphic to an orbit space $(L^l(p)\times S^m)/Z_2$, where a Z_2-action on $L^l(p)\times S^m$ is given by

$$t^j([z], x) = ([c^j(z)], (-1)^j x),$$

where t is a generator of Z_2. Denote by (C_i, D_j) a cell of $(L^l(p)\times S^m)/Z_2$ represented by a standard cell C_i of $L^l(p)$ and a standard cell D_j of S^m

and by (c^i, d^j) a dual cochain of (C_i, D_j). The coboundary relations are given by

$$\delta(c^{2i+1}, d^j) = p(c^{2(i+1)}, d^j) + \{(-1)^i + (-1)^j\}(c^{2i+1}, d^{j+1}),$$
$$\delta(c^{2i}, d^j) = \{(-1)^i + (-1)^{j+1}\}(c^{2i}, d^{j+1}).$$

Therefore, we have the following.

PROPOSITION 3.1. *The integral cohomology group $\tilde{H}^*(D_p(2k+1, 2n+1); Z)$ is a direct sum of the following groups:*
free groups generated by (c^0, d^{2n+1}), (c^{4k+3}, d^0) and (c^{4k+3}, d^{2n+1}),
torsion groups generated by (c^0, d^{2j}) and (c^{4k+3}, d^{2j}) whose orders are 2
and torsion groups generated by (c^{4i}, d^0) and (c^{4i}, d^{2n+1}) whose orders are p, where $1 \leq j \leq n$ and $1 \leq i \leq k$.

Denote by Y the $(4k+2n+3)$-skeleton of $D_p(2k+1, 2n+1)$. Then, we have

$$(3.1) \qquad \tilde{H}^i(D_p(2k+1, 2n+1)/Y; Z) = \begin{cases} Z & i = 4k+2n+4, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(3.2) \qquad \tilde{H}^i(Y; Z) \sim \begin{cases} \tilde{H}^i(D_p(2k+1, 2n+1); Z) & i \leq 4k+2n+3, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 3.2. *There exists a short exact sequence*

$$0 \longrightarrow Z \longrightarrow \tilde{K}(D_p(2k+1, 2n+1)) \longrightarrow \tilde{K}(Y) \longrightarrow 0.$$

PROOF. Consider the exact sequence of K-groups with respect to a pair $(D_p(2k+1, 2n+1), Y)$,

$$\cdots \longrightarrow \tilde{K}^{-1}(D_p(2k+1, 2n+1)) \xrightarrow{i^!} \tilde{K}^{-1}(Y) \longrightarrow \tilde{K}(D_p(2k+1, 2n+1)/Y)$$
$$\longrightarrow \tilde{K}(D_p(2k+1, 2n+1)) \longrightarrow \tilde{K}(Y) \longrightarrow \tilde{K}^1(D_p(2k+1, 2n+1)/Y) \longrightarrow \cdots$$

Note that from (3.1) we have

$$\tilde{K}^i(D_p(2k+1, 2n+1)/Y) \simeq \begin{cases} Z & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

From the discussion of the Atiyah-Hirzebruch spectral sequence for $\tilde{K}^{-1}(X)$ with $E_2^{s,t}(X) \simeq \tilde{H}^s(X; \tilde{K}^t(S^0))$ [2], $X = D_p(2k+1, 2n+1)$ or Y, we have the following,

$$\text{the free part of } \tilde{K}^{-1}(X) \cong Z \oplus Z.$$

Hence, it follows that $i^!$: $\tilde{K}^{-1}(D_p(2k+1, 2n+1)) \longrightarrow \tilde{K}^{-1}(Y)$ is isomorphic. q. e. d.

PROPOSITION 3. 3. *The order of the group* $\tilde{K}(Y) \leqq p^k 2^n$.

PROOF. The order of $E_\infty^{s,t}(Y)$ is not more than that of $E_2^{s,t}(Y)$, $s+t=$ even. From Proposition 3. 1, the proposition follows.

PROOF OF THEOREM 1. 1.

Denote by c and r the complexification and the real restriction. Put $\sigma_m = \xi_m - 1$, then

$$\sigma_m + \bar{\sigma}_m = cr\sigma_m.$$

In [4], it is proved that $r\sigma_m$, $(r\sigma_m)^2, \cdots, (r\sigma_m)^{(p-1)/2}$ are linearly independent in $\widetilde{KO}(L^m(p))$ and if $m = s(p-1)+t$, $0 \leqq t < p-1$ then

$$\text{the order of } (r\sigma_m)^i = \begin{cases} p^{s+1} & \text{if } 2i \leqq t, \\ p^s & \text{if } 2i > t. \end{cases}$$

We note that if $m=2k+1$, the complexification c is injective and $\sigma_m + \bar{\sigma}_m$ belongs to $\tilde{K}(L^m(p))^{Z_2}$. Therefore, we have that the order of $\tilde{K}(L^{2k+1k}(p))^{Z_2} \geqq p^k$. Since $\tilde{K}_{Z_2}(S^{2n+1}) \cong \tilde{K}(RP^{2n+1}) \cong Z_{2^n}$ [1],

$$\text{the order of } \tilde{K}(L^{2k+1}(p))^{Z_2} \oplus \tilde{K}(RP^{2n+1}) \geqq p^k 2^n.$$

It follows from Theorem 2. 2, Proposition 3. 3 that the torsion part of $\tilde{K}(D_p(2k+1, 2n+1))$ is isomorphic to $\tilde{K}(L^{2k+1}(p))^{Z_2} \oplus \tilde{K}(RP^{2n+1})$. q. e. d.

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