Regularity of solutions of the Neumann problems in L^(2, λ) spaces

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Regularity of solutions of the Neumann problems in $\mathcal{L}^{(2,\lambda)}$ spaces

By

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Introduction

Recently the characterization of the $\mathcal{L}^{(p,\lambda)}$ spaces have been done by many authors including S. Campanato [1], A. Ono [6, 7, 8], J. Peetre [9, 10], L. C. Piccinini [11] and G. Stampacchia [12, 13], and with the aid of them the regularity of the solutions of the partial differential equations in the $\mathcal{L}^{(p,\lambda)}$ spaces have been studied by S. Campanato [2, 3, 4], J. Peetre [9], and others. Among them S. Campanato ([2]) treated the Dirichlet problems for the elliptic equations with principal part only;

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)=\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}+f \tag{1}$$

and he proved the regularity of solutions in the $\mathcal{L}^{(2,\lambda)}$ spaces $(0\leqq\lambda\leqq n+2)$.

In this paper we shall study the regularity of solutions of the Neumann problems for the elliptic equations in the $\mathcal{L}^{(2,\lambda)}$ spaces with the following form:

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}(x)\frac{\partial u}{\partial x_j}+a_i(x)u\right)+a(x)u=\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}+f\ . \tag{2}$$

As (2) is more general than (1), we can apply a similar device to that of S. Campanato ([2]) only for the case of $0\leqq\lambda\leqq n$.

We organize the paper in the following manner. In § 1, we state the preliminaries. We give the interior estimates in § 2 and the boundary estimates in § 3. With the aid of these results, in § 4, we shall study the regularity theorem of solutions of the Neumann problems.

Here, the authors wish to express their sincere thanks to Professor Akira Ono for his valuable advices and constant encouragement during the preparation of this article.

§1 Preliminaries

Let Ω be a bounded open domain in the n-dimensional Euclidean space R^n, $\partial\Omega$ be its boundary, and $\bar{\Omega}$ be a closure of Ω: $\bar{\Omega}=\Omega\cup\partial\Omega$.

We shall use the notations $D_i=\dfrac{\partial}{\partial x_i}$, $D_i{}^k=\dfrac{\partial^k}{\partial x_i{}^k}$, $i=1, \dots, n$, for $x=(x_1, x_2, \dots, x_n)$ and $D^m=D_1{}^{m_1}D_2{}^{m_2}\dots D_n{}^{m_n}$, $|m|=\sum_{i=1}^{n} m_i$ for each n-tuple $m=(m_1, m_2, \dots, m_n)$ of non-negative integers.

We denote by $H^k(\Omega)$ the Banach space which is obtained as the completion of $C^k(\Omega)$ with respect to the norm

$$||u||_{H^k(\Omega)}=\left\{\sum_{|m|\leq k} ||D^m u||^2_{L_2(\Omega)}\right\}^{1/2}$$

where k is a non-negative integer, and we denote by $H_0^k(\Omega)$ the completion of $C_0^k(\Omega)$ with respect to the above norm.

We denote the set $\{x;|x-x_0|<r\}$ by $I(x_0,r)$, simply $I(0, r)$ by $I(r)$ and the set $I(x_0, r)\cap\Omega=\Omega(x_0, r)$.

The spaces which we shall use in this paper are the following.

DEFINITION 1.1. (Morrey space $L^{(2,\lambda)}(\Omega)$) A function u is said to belong to Morrey space $L^{(2,\lambda)}(\Omega)$, where $0\leq\lambda\leq n$, if u belongs to $L^2(\Omega)$ and the relation

$$||u||_{L^{(2,\lambda)}(\Omega)}=\sup_{x_0\in\Omega, r>0}\{r^{-\lambda}\int_{\Omega(x_0,r)}|u(y)|^2dy\}^{1/2}<\infty \tag{1.1}$$

holds. The space $L^{(2,\lambda)}(\Omega)$ with the above norm is a Banach space.

One can verify that the following relations hold:

$L^{(2,0)}(\Omega)=L^2(\Omega)$; $L^{(2,\lambda)}(\Omega)\subset L^2(\Omega)$ $(0<\lambda<n)$; $L^{(2,n)}(\Omega)=L^\infty(\Omega)$,

furthermore

$L^{(2,\lambda)}(\Omega)\subset L^{(2,\mu)}(\Omega)$ and $||u||_{L^{(2,\mu)}(\Omega)}\leq C||u||_{L^{(2,\lambda)}(\Omega)}$ if $\mu\leq\lambda$.

If a function u is integrable on Ω, we shall denote the mean-value of u over $\Omega(x_0, r)$ by $u_{x_0,r}$ or simply u_r :

$$u_r=u_{x_0,r}=\frac{1}{|\Omega(x_0,r)|}\int_{\Omega(x_0,r)}u(y)\ dy$$

where $|\Omega(x_0, r)|$ is the measure of the set $\Omega(x_0, r)$.

DEFINITION 1.2. (the space $\mathcal{L}^{(2,\lambda)}(\Omega)$) A function u is said to belong to the space $\mathcal{L}^{(2,\lambda)}(\Omega)$, where $0\leq\lambda<\infty$, if u belongs to $L^2(\Omega)$ and the relation

(1. 2) $$[u]_{\mathcal{L}^{(2,\lambda)}(\Omega)}=\sup_{x_0\in\Omega\, r>0}\{r^{-\lambda}\int_{\Omega(x_0,r)}|u(y)-u_r|^2dy\}^{1/2}<\infty$$

holds. $[u]_{\mathcal{L}^{(2,\lambda)}(\Omega)}$ is a semi-norm in the space $\mathcal{L}^{(2,\lambda)}(\Omega)$. We define a norm of the space $\mathcal{L}^{(2,\lambda)}(\Omega)$ by

(1. 3) $$||u||_{\mathcal{L}^{(2,\lambda)}(\Omega)}=||u||_{L^2(\Omega)}+[u]_{\mathcal{L}^{(2,\lambda)}(\Omega)}.$$

This norm renders the space $\mathcal{L}^{(2,\lambda)}(\Omega)$ a Banach space.

It is well known that the following isomorphism theorem which characterizes the spaces $\mathcal{L}^{(2,\lambda)}(\Omega)$ holds. (See [6], [9], [11], [12])

ISOMORPHISM THEOREM

We assume that the domain Ω is of type (A)[1], *then the following isomorphic relations hold:*

(*i*) (*S. Campanato*) *If $0\leqq\lambda<n$, then the space $\mathcal{L}^{(2,\lambda)}(\Omega)$ is isomorphic to the Morrey space $L^{(2,\lambda)}(\Omega)$ with the equivalent norm.*

(*ii*) (*F. John - L. Nirenberg*) *If $\lambda=n$, then the space $\mathcal{L}^{(2,n)}(\Omega)$ is isomorphic to the John-Nirenberg space ε_0.*

(*iii*) (*S. Campanato - G. N. Meyers*) *If $n<\lambda\leqq n+2$, then the space $\mathcal{L}^{(2,\lambda)}(\Omega)$ is isomorphic to the space $C^{0,\alpha}(\overline{\Omega})$, $\alpha=\dfrac{\lambda-n}{2}$, with the equivalent norm.*

(*iv*) *If $\lambda>n+2$, then the space $\mathcal{L}^{(2,\lambda)}(\Omega)$ consists of constant functions.*

It is well known that the space $L^{(2,n)}(\Omega)=L^\infty(\Omega)$ is a proper subspace of the space $\mathcal{L}^{(2,n)}(\Omega)$ (See [5]).

Between the space $\mathcal{L}^{(2,\lambda)}(\Omega)$ the same inclusion relation verified between Morrey spaces holds, that is, if $\mu\leqq\lambda$ then

(1. 4) $$\mathcal{L}^{(2,\lambda)}(\Omega)\subset\mathcal{L}^{(2,\mu)}(\Omega) \text{ and } [u]_{\mathcal{L}^{(2,\mu)}(\Omega)}\leqq C[u]_{\mathcal{L}^{(2,\lambda)}(\Omega)}$$

DEFINITION 1. 3. (The space $H^{k,\lambda}(\Omega)$) A function u is said to belong to the space $H^{k,\lambda}(\Omega)$, where k is a non-negative integer and $0\leqq\lambda\leqq n+2$, if u belongs to $H^k(\Omega)$ and $D^m u$ belongs to $\mathcal{L}^{(2,\lambda)}(\Omega)$ for each m, $|m|=k$. The space $H^{k,\lambda}(\Omega)$ with a norm

(1. 5) $$||u||_{\bar{H}^{k,\lambda}(\Omega)}=||u||_{H^k(\Omega)}+\sum_{|m|=k}[D^m u]_{\mathcal{L}^{(2,\lambda)}(\Omega)}$$

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1) A bounded domain Ω is said to be of type (A), if there exists a positive constant K such that for any $x\in\Omega$ and any $r>0$ the following relation holds: $|\Omega(x_0, r)|\geqq Kr^n$.
A domain Ω is of type (A) if it has a cone property.

becomes a Banach space.

The following inclusion relation holds.

(1.6) $H^{k,\lambda}(\Omega)\subset H^{k-1,\lambda+2}(\Omega)$ and $||u||_{H^{k-1,\lambda+2}(\Omega)}\leqq C||u||_{H^{k,\lambda}(\Omega)}$ if $0\leqq\lambda<n$, $k\geqq 1$.

Finally, to prove the theorems we need the following Fundamental Lemma due to S. Campanato.

FUNDAMENTAL LEMMA. (*S. Campanato* [2])

Let $\phi(t)$ and $B(t)$ *be non-negative functions defined for* $t>0$ *and* $t>1$ *respectively,* A *be a constant* >1, α *and* β *be real numbers satisfying the relation* $0<\beta<\alpha$.

Suppose that, for any $p>1$, *there exists a constant* $t(p)>0$ *such that the inequality*

$$\phi(\rho)\leqq A\left(\frac{\rho}{r}\right)^{\alpha}\phi(r)+B(p)\rho^{\beta}$$

holds for any pair of numbers ρ, $r\in(0,\ t(p)]$ *satisfying* $1<\dfrac{r}{\rho}<p$.

Then for arbitrary $\varepsilon\in(0,\ \alpha-\beta)$ *and* ρ, r satisfying $0<\rho<r<t(A^{1/\varepsilon})$, *the following inequality holds*:

$$\phi(\rho)\leqq A\left(\frac{\rho}{r}\right)^{\alpha-\varepsilon}\phi(r)+B(A^{1/\varepsilon})\frac{A^{\frac{\alpha-\beta}{\varepsilon}}}{A^{\frac{\alpha-\beta}{\varepsilon}}-A}\rho^{\beta}.$$

§2 Interior estimates

Let Ω be a bounded open domain R^n with cone property.

In this section, we shall treat the uniformly elliptic differential operator E of the second order:

$$Eu\equiv\sum_{i=1}^{n}D_i\left(\sum_{j=1}^{n}a_{ij}(x)D_ju+a_i(x)u\right)+\sum_{i=1}^{n}b_i(x)D_iu+a(x)u.$$

We set the following assumptions for E.

(A. 2. 1) There exists a positive constant ν such that the following inequalities hold for any $x\in\Omega$ and any non-zero real vector $\xi=(\xi_1, \xi_2, \dots, \xi_n)\in R^n$

(2.1) $$\nu^{-1}|\xi|^2\leqq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leqq\nu|\xi|^2.$$

(A. 2. 2) $a_{ij}(x)$ are symmetric: $a_{ij}(x)=a_{ji}(x)$.

We denote the homogeneous differential operators with constant coefficients by E_0

$$E_0 u \equiv \sum_{i,j=1}^{n} D_i(a_{ij}(x_0) D_j u) \qquad \text{for any fixed } x_0 \in \Omega\ .$$

In this section we shall estabilish the interior estimates for a "weak solution" of the equation

$$Eu = \sum_{i=1}^{n} D_i f_i + f \tag{2. 2}$$

where we set the following assumption (A. 2. 3) for the functions f_i and f:
(A. 2. 3) f_i $(i=1, 2, \ldots, n)$, f are given functions in $L^2(\Omega)$.
Here, the definition of a "weak solution" of the equation (2. 2) is the following

DEFINITION 2. 1. A function $u \in H^1(\Omega)$ is said to be a weak solution of the equation (2. 2) in Ω, if u satisfies the following equality for every $\phi \in H_0^1(\Omega)$

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\Omega} \left\{ \sum_{j=1}^{n} a_{ij}(x) D_j u(x) + a_i(x) u(x) \right\} D_i \phi(x) dx \\ &\quad - \int_{\Omega} \left\{ \sum_{i=1}^{n} b_i(x) D_i u(x) + a(x) u(x) \right\} \phi(x) dx \\ &= \sum_{i=1}^{n} \int_{\Omega} f_i(x) D_i \phi(x) dx - \int_{\Omega} f(x) \phi(x) dx. \end{aligned} \tag{2. 3}$$

LEMMA 2. 1. *Let $u \in H_0^1(\Omega)$ be a weak solution of $E_0 u = \sum_{i=1}^{n} D_i f_i + f$ in Ω, that is, u satisfies for every $\phi \in H_0^1(\Omega)$*

$$\begin{aligned} &\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x_0) D_j u(x) D_i \phi(x) dx \\ &\quad = \sum_{i=1}^{n} \int_{\Omega} f_i(x) D_i \phi(x) dx - \int_{\Omega} f(x) \phi(x) dx\ . \end{aligned} \tag{2. 4}$$

Then for arbitrary real vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^n$ the following inequality holds

$$\sum_{i=1}^{n} \| D_i u \|_{L^2(\Omega)}^2 \leqq C \left\{ \sum_{i=1}^{n} \| f_i - \gamma_i \|_{L^2(\Omega)}^2 + \| f \|_{L^2(\Omega)}^2 \right\} \tag{2. 5}$$

where C is a positive constant depending only on ν, Ω, but not on u.

PROOF. Putting $\phi = u$ particularly in (2. 4), we have for any real vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$

$$\sum_{i,j=1}^{n}\int_{\Omega}a_{ij}(x_0)D_ju(x)D_iu(x)dx$$
$$=\sum_{i=1}^{n}\int_{\Omega}(f_i(x)-\gamma_i)D_iu(x)dx-\int_{\Omega}f(x)u(x)dx\ .$$

By using (2. 1), Schwarz's inequality and Poincarè's inequality, we obtain

$$\nu^{-1}\sum_{i=1}^{n}\int_{\Omega}|D_iu|^2dx$$
$$\leq\left(\sum_{i=1}^{n}||f_i-\gamma_i\ ||^2_{L^2(\Omega)}\right)^{1/2}\left(\sum_{i=1}^{n}||D_iu\ ||^2_{L^2(\Omega)}\right)^{1/2}$$
$$+C(\Omega)\ ||f||_{L^2(\Omega)}\left(\sum_{i=1}^{n}||D_iu\ ||^2_{L^2(\Omega)}\right)^{1/2}.$$

The inequality (2. 5) follows from this.

To state the lemmas we assume that the coefficients $a_{ij}(x)$, $b_i(x)$ are continuous functions on $\overline{I(r)}$ and we put

$$\omega^2(r)=\max_{i,j}\ \{\sup_{x\in\overline{I(r)}}|a_{ij}(x)-a_{ij}(0)|^2,\ \sup_{x\in\overline{I(r)}}|b_i(x)-b_i(0)\,|^2\}\ .$$

LEMMA 2. 2. *Under the conditions that* a_{ij}, $b_i\in C^0(\overline{I(r)})$, and a_i, $a\in L^\infty(I(r))$, $(i,\ j=1,\ 2,\ \ldots,\ n)$, *if* $u\in H^1(I(r))$ *is a weak solution of the equation* (2. 2) *then for any number* $\rho\in(0,\ r]$ *the following inequality holds*

$$\sum_{i=1}^{n}\int_{I(\rho)}|D_iu|^2dx$$
$$\leq C\left\{\left[\left(\frac{\rho}{r}\right)^n+\omega^2(r)\right]\sum_{i=1}^{n}\int_{I(r)}|D_iu|^2dx+\int_{I(r)}|u|^2dx+\right. \tag{2. 6}$$
$$\left.+\sum_{i=1}^{n}\int_{I(r)}|f_i|^2dx+\int_{I(r)}|f|^2dx\right\}.$$

Where C is a positive constant depending only on ν *and the coefficients of E but not on u.*

PROOF. This lemma is proved by the same argument as the proof of Lemma 8.I of Campanato ([2]).

We shall decompose the function u to the sum $u=v+w$ where the functions v and w are weak solutinos of the following Dirichlet problems in $I(r)$ respectively:

$$\begin{cases} v-u \in H_0^1(I(r)) \\ E_0 v=0 \end{cases}$$

and

$$\begin{cases} w \in H_0^1(I(r)) \\ E_0 w = \sum_{i=1}^n D_i[f_i + \sum_{j=1}^n (a_{ij}(0)-a_{ij}(x))D_j u-(a_i(x)+b_i(0))u] + \\ \qquad + f - \sum_{i=1}^n (b_i(x)-b_i(0))D_i u - au \end{cases}$$

where $E_0 u \equiv \sum_{i,j=1}^n D_i(a_{ij}(0)\, D_j u)$.

Applying Corollary 7. I of Campanato ([2]) to the function v, we obtain for any $\rho \in (0, r)$

$$\sum_{i=1}^n \int_{I(\rho)} |D_i v|^2 dx \leq C(\nu) \left(\frac{\rho}{r}\right)^n \sum_{i=1}^n \int_{I(r)} |D_i v|^2 dx \tag{2. 7}$$

where $C(\nu)$ is a positive constant depending only on ν.
While, we apply Lemma 2. 1 with $\gamma=0$ to the function w, so we have for any $\rho \in (0, r]$

$$\sum_{i=1}^n \int_{I(\rho)} |D_i w|^2 dx \leq C\, \{\omega^2(r) \sum_{i=1}^n \int_{I(r)} |D_i u|^2 dx + \int_{I(r)} |u|^2 dx + \\ + \sum_{i=1}^n \int_{I(r)} |f_i|^2 dx + \int_{I(r)} |f|^2 dx\} . \tag{2. 8}$$

The inequality (2. 6) follows from (2. 7) and (2. 8).

When we consider the equation (2. 2) in Ω, we shall put for $x_0 \in \Omega$, $r>0$

$$\omega^2(x_0, r) = \max_{i,j} \left\{ \sup_{x \in \overline{\Omega(x_0,r)}} |a_{ij}(x)-a_{ij}(x_0)|^2,\ \sup_{x \in \overline{\Omega(x_0,r)}} |b_i(x)-b_i(x_0)|^2 \right\} .$$

THEOREM 2. 1. *We assume that* a_{ij}, $b_i \in C^0(\overline{\Omega})$, a_i, $a \in L^\infty(\Omega)$, f_i, $f \in \mathcal{L}^{(2,\lambda)}(\Omega)$ $0 \leq \lambda < n$.
Let $u \in H^1(\Omega)$ *be a weak solution of the equation* (2. 2), *then* $D_i u \in \mathcal{L}^{(2,\lambda)}(\Omega_0)$, $i=1$, 2, ..., n, *for any subdomain* $\Omega_0 \subset\subset \Omega$, *and the following estimate holds*:

$$\sum_{i=1}^n ||D_i u||^2_{\mathcal{L}^{(2,\lambda)}(\Omega_0)} \leq C\left\{ ||u||^2_{H^1(\Omega)} + \sum_{i=1}^n ||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)} + ||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)} \right\} . \tag{2. 9}$$

Where C is a positive constant depending only on the coefficients of E, Ω_0, and λ *but independent of u.*

PROOF. Let $\Omega_0 \subset\subset \Omega$ and δ_0 be a distance from $\partial\Omega$ to $\overline{\Omega}_0$, $x_0 \in \Omega_0$ and 0

$<r\leqq\frac{\delta_0}{2}$.

At first we shall give the proof of this theorem of the case $0\leqq\lambda\leqq 2$.

By Lemma 2.2, for any $\rho\in(0, r)$

$$\sum_{i=1}^{n}\int_{I(x_0,\rho)}|D_iu|^2dx\leqq C\left\{\left[\left(\frac{\rho}{r}\right)^n+\omega^2(x_0\ ,r)\right]\sum_{i=1}^{n}\int_{I(x_0,r)}|D_iu|^2dx+\right.$$
$$\left.+\int_{I(x_0,r)}|u|^2dx+\sum_{i=1}^{n}\int_{I(x_0,r)}|f_i|^2dx+\int_{I(x_0,r)}|f|^2dx\right\} \tag{2.10}$$

where C is independent of ρ, r, and u.

Since $u\in H^1(\Omega)$ and $0\leqq\lambda\leqq 2$, by (1.4) and (1.6),

$$[u]^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\leqq C[u]^2_{\mathcal{L}^{2,2)}(\Omega)}\leqq C\sum_{i=1}^{n}||D_iu||^2_{L^2(\Omega)}$$

and hence

$$\int_{I(x_0,r)}|u|^2dx\leqq Cr^\lambda||u||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\leqq Cr^\lambda||u||^2_{H^1(\Omega)}\ . \tag{2.11}$$

Moreover, f_i, f being in the space $\mathcal{L}^{(2,\lambda)}(\Omega)$, from (2, 10) (2.11) we obtain

$$\sum_{i=1}^{n}\int_{I(x_0,\rho)}|D_iu|^2dx\leqq C\left[\left(\frac{\rho}{r}\right)^n+\omega^2(x_0,r)\right]\sum_{i=1}^{n}\int_{I(x_0,r)}|D_iu|^2dx+$$
$$+C\left\{||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\right\}r^\lambda\ . \tag{2.12}$$

Since the coefficients $a_{ij}(x)$, $b_i(x)$ are continuous on $\overline{\Omega}$, for arbitrary fixed number $p>1$ there is a positive number $r(p)\leqq\frac{\delta_0}{2}$ such that for any $x_0\in\overline{\Omega_0}$ and for any pair of numbers ρ, r satisfying the relations

$$0<\rho<r\leqq r(p)\ ;\ 1<\frac{r}{\rho}\leqq p$$

the following inequality holds:

$$\omega^2(x_0,r)\leqq M\left(\frac{\rho}{r}\right)^n.$$

Consequently, from (2.12), for any $x_0\in\overline{\Omega}_0$ and for any pair of the numbers ρ, r satisfying the above relations we obtain

$$\sum_{i=1}^{n}\int_{I(x_0,\rho)}|D_iu|^2dx\leqq C\left(\frac{\rho}{r}\right)^n\sum_{i=1}^{n}\int_{I(x_0,r)}|D_iu|^2dx+$$
$$+Cp^\lambda\left[||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\right]\rho^\lambda\ .$$

Here we may apply Fundamental Lemma, so we can show that there exists a positive number $r(\lambda)\leqq\frac{\delta_0}{2}$ such that for any $x_0\in\overline{\Omega_0}$ and any number $\rho\in(0,r(\lambda)]$ the following inequality holds

$$\rho^{-\lambda}\sum_{i=1}^{n}\int_{I(x_0,\rho)}|D_iu|^2dx\leqq C\sum_{i=1}^{n}||D_iu||^2_{L^2(\Omega)}+$$
$$+C(\lambda)\ [\,||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}].$$

(See Campanato [2] in detail).

On the other hand, if $\rho\geqq r(\lambda)$ we obtain immediately

$$\rho^{-\lambda}\sum_{i=1}^{n}\int_{\Omega(x_0,\rho)}|D_iu|^2dx\leqq r(\lambda)^{-\lambda}\sum_{i=1}^{n}\int_{\Omega}|D_iu|^2dx\ .$$

From these two inequalities we complete the proof of Theorem 2.1 of the case $0\leqq\lambda\leqq2$.

Next, we shall give the proof of the case $2\leqq\lambda<n$.

Let h be a positive integer such as $2^h\leqq\lambda<2^{h+1}$, and let's take subdomains Ω^i $(i=1,2,\dots,h)$ of Ω which satisfy

$$\Omega_0\subset\subset\Omega^h\subset\subset\Omega^{h-1}\subset\subset\dots\subset\subset\Omega^1\subset\subset\Omega\ .$$

Because f_i, $f\in\mathcal{L}^{(2,\lambda)}(\Omega)\subset\mathcal{L}^{(2,2)}(\Omega)$, by the preceding result for $0\leqq\lambda\leqq2$, $D_iu\in\mathcal{L}^{(2,2)}(\Omega^1)$ and

$$\sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,2)}(\Omega^1)}\leqq C\Big\{\,||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\Big\}\,.$$

Since $u\in H^{1,2}(\Omega^1)\subset\mathcal{L}^{(2,4)}(\Omega^1)$, f_i, $f\in\mathcal{L}^{(2,\lambda)}(\Omega)\subset\mathcal{L}^{(2,4)}(\Omega^1)$, in the same way for the ease of $0\leqq\lambda\leqq2$, it follows that $D_iu\in\mathcal{L}^{(2,4)}(\Omega^2)$ and

$$\sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,4)}(\Omega 2)}\leqq C\Big\{\,||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\Big\}\,.$$

By repeating this procedure $h-1$ times, we know that $D_iu\in\mathcal{L}^{(2,2^h)}(\Omega^h)$ and

$$\sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,2^h)}(\Omega^h)}\leqq C\Big\{\,||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\Big\}\,.$$

Thus, noting that f_i, $f\in\mathcal{L}^{(2,\lambda)}(\Omega)\subset\mathcal{L}^{(2,\lambda)}(\Omega^h)$, and $u\in\mathcal{L}^{(2,2^{h+1})}(\Omega^h)\subset\mathcal{L}^{(2,\lambda)}(\Omega^h)$, finally we arrive at a conclusion of this theorem.

LEMMA 2. 3. *We assume the same hypothesis as the preceding lemma for u and for the coefficients of E.*
Then for any $\rho\in(0,r)$ and for any real vectors $\beta=\{\beta_i\}$, $\gamma=\{\gamma_i\}$ the following

estimate holds:

$$\sum_{i=1}^{n}\int_{I(\rho)}|D_iu-\{D_iu\}_\rho|^2dx\leqq C\Big\{\sum_{i=1}^{n}\Big(\frac{\rho}{r}\Big)^{n+2}\int_{I(r)}|D_iu-\beta_i|^2dx+$$

(2. 13)

$$+\omega^2(r)\sum_{i=1}^{n}\int_{I(r)}|D_iu|^2dx+\int_{I(r)}|u|^2dx+\sum_{i=1}^{n}\int_{I(r)}|f_i-\gamma_i|^2dx+\int_{I(r)}|f|^2dx\Big\}.$$

Where C is a positive constant depending only on ν and the coefficients of E but independent of u.

PROOF. We decompose $u=v+w$, where v and w are same as in the proof of Lemma 2. 2.

By applying Corollary 7. II of Campanato [2] to the function v, we have for any $\rho\in(0, r)$ and for any $\beta=\{\beta_i\}\in R^n$

$$\sum_{i=1}^{n}\int_{I(\rho)}|D_iv-\{D_iv\}_\rho|^2dx\leqq C(\nu)\Big(\frac{\rho}{r}\Big)^{n+2}\sum_{i=1}^{n}\int_{I(r)}|D_iv-\beta_i|^2dx\ . \tag{2. 14}$$

On the other hand we may apply Lemma 2. 1 to the function w, so we obtain

$$\sum_{i=1}^{n}\int_{I(r)}|D_iw|^2dx\leqq C\Big\{\sum_{i=1}^{n}\int_{I(r)}|f_i-\gamma_i|^2dx+\omega^2(r)\sum_{i=1}^{n}\int_{I(r)}|D_iu|^2dx+$$

$$+\int_{I(r)}|u|^2dx+\int_{I(r)}|f|^2dx\Big\}. \tag{2. 15}$$

From (2. 14) and (2. 15) we obtain (2.13).

THEOREM 2. 2. *We assume that a_{ij}, $b_i\in C^{0,\alpha}(\overline{\Omega})(0<\alpha<1)$, a_i, $a\in L^\infty(\Omega)$, $f_i\in\mathcal{L}^{(2,n)}(\Omega)$ and $f\in L^{(2,n)}(\Omega)$.*
Let $u\in H^1(\Omega)$ be a weak solution of the equation (2. 2), then $D_iu\in\mathcal{L}^{(2,n)}(\Omega_0)(i=1, 2, \dots, n)$, for arbitrary subdomain $\Omega_0\subset\subset\Omega$, and the following estimate holds:

$$\sum_{i=1}^{n}[D_iu]^2_{\mathcal{L}^{(2,n)}(\Omega_0)}\leqq C\Big\{||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,n)}(\Omega)}+||f||^2_{L^{(2,n)}(\Omega)}\Big\}. \tag{2. 16}$$

Where C is a positive constant depending only on the coefficients of E and Ω_0 but independent of u.

PROOF. Let $\Omega_0\subset\subset\Omega$, δ_0 be a distance from $\partial\Omega$ to $\overline{\Omega}_0$, $\delta_0=\mathrm{dis}(\overline{\Omega}_0, \partial\Omega)$, and Ω_1 be a subdomain of Ω such that $\Omega_0\subset\subset\Omega_1\subset\subset\Omega$, $\mathrm{dis}(\overline{\Omega}_0, \partial\Omega_1)=\mathrm{dis}(\overline{\Omega}_1, \partial\Omega)=\dfrac{\delta_0}{2}$.

f_i, $f\in\mathcal{L}^{(2,n-2\alpha)}(\Omega)$ since $f_i\in\mathcal{L}^{(2,n)}(\Omega)$ and $f\in L^{(2,n)}(\Omega)$. Therefore by Theorem 2. 1, $D_iu\in\mathcal{L}^{(2,n-2\alpha)}(\Omega_1)$ and

$$\sum_{i=1}^{n}||D_i u||^2_{\mathcal{L}^{(2,n-2\alpha)}(\Omega_1)} \leqq C\Big\{ ||u||^2_{H^1(\Omega)} + \sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,n-2\alpha)}(\Omega)} + ||f||^2_{\mathcal{L}^{(2,n-2\alpha)}(\Omega)}\Big\}$$

$$\leqq C\Big\{ ||u||^2_{H^1(\Omega)} + \sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,n)}(\Omega)} + ||f||^2_{L^{(2,n)}(\Omega)}\Big\}.$$

Hence for any $x_0 \in \Omega_0$ and for $r \in \left(0, \frac{\delta_0}{2}\right]$ we obtain

$$(2.17)\quad \sum_{i=1}^{n}\int_{I(x_0,r)} |D_i u|^2 dx \leqq C\{ ||u||^2_{H^1(\Omega)} + \sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,n)}(\Omega)} + ||f||^2_{L^{(2,n)}(\Omega)}\} r^{n-2\alpha}$$

If we put $M = \max_{i,j}\{[a_{ij}]^2_{C^{0,\alpha}(\overline{\Omega})},\ [b_i]^2_{C^{0,\alpha}(\overline{\Omega})}\}$, then obviously $\omega^2(x_0, r) \leqq M r^{2\alpha}$. By Lemma 2.3 with $\beta_i = \{D_i u\}_r$, $\gamma_i = \{f_i\}_r$ we have for any $x_0 \in \Omega_0$ and for any $\rho \in (0, r)$

$$\sum_{i=1}^{n}\int_{I(x_0,\rho)} |D_i u - \{D_i u\}_\rho|^2 dx$$

(2. 18)

$$\leqq C\Big\{\left(\frac{\rho}{r}\right)^{n+2}\sum_{i=1}^{n}\int_{I(x_0,r)} |D_i u - \{D_i u\}_r|^2 dx + M r^{2\alpha}\sum_{i=1}^{n}\int_{I(x_0,r)} |D_i u|^2 dx + \int_{I(x_0,r)} |u|^2 dx$$

$$+ \sum_{i=1}^{n}\int_{I(x_0,r)} |f_i - \{f_i\}_r|^2 dx + \int_{I(x_0,r)} |f|^2 dx\Big\}$$

$$\leqq C\Big\{\left(\frac{\rho}{r}\right)^{n+2}\sum_{i=1}^{n}\int_{I(x_0,r)} |D_i u - \{D_i u\}_r|^2 dx + \Big[\ ||u||^2_{H^1(\Omega)} + \sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,n)}(\Omega)} +$$

$$+ ||f||^2_{L^{(2,n)}(\Omega)}\Big] r^n + \int_{I(x_0,r)} |u|^2 dx\Big\}.$$

Next we shall estimate the integral $\int_{I(x_0,r)} |u|^2 dx$. Because $D_i u \in \mathcal{L}^{(2,n-2\alpha)}(\Omega_1)$ and $n < n+2-2\alpha < n+2$, by the inclusion relation, we have $u \in \mathcal{L}^{(2,n+2-2\alpha)}(\Omega_1)$ and

$$||u||^2_{\mathcal{L}^{(2,n+2-2\alpha)}(\Omega_1)} \leqq C\,||u||^2_{H^{1,n-2\alpha}(\Omega_1)}$$

$$\leqq C\Big\{ ||u||^2_{H^1(\Omega)} + \sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,n)}(\Omega)} + ||f||^2_{L^{(2,n)}(\Omega)}\Big\}.$$

Furthermore by Isomorphism Theorem in § 1, $u \in C^{0,\alpha}(\overline{\Omega}_1)$ and

$$||u||_{C^{0,\alpha}(\overline{\Omega}_1)} \leqq C\,||u||_{\mathcal{L}^{(2,n+2-2\alpha)}(\Omega_1)}.$$

Consequently we obtain

$$\int_{I(x_0,r)} |u|^2 dx \leqq \int_{I(x_0,r)} \|u\|^2_{C^{0,\alpha}(\bar{\Omega}_1)} dx \leqq C\|u\|^2_{\mathcal{L}^{(2,n+2-2\alpha)}(\Omega_1)} \cdot r^n$$

(2. 19)

$$\leqq C\left\{ \|u\|^2_{H^1(\Omega)} + \sum_{i=1}^n \|f_i\|^2_{\mathcal{L}^{(2,n)}(\Omega)} + \|f\|^2_{L^{(2,n)}(\Omega)} \right\} r^n .$$

Inserting (2. 19) into (2. 18), we have

$$\sum_{i=1}^n \int_{I(x_0,\rho)} |D_i u - \{D_i u\}_\rho|^2 dx \leqq C\left\{ \left(\frac{\rho}{r}\right)^{n+2} \sum_{i=1}^n \int_{I(x_0,r)} |D_i u - \{D_i u\}_r|^2 dx + \right.$$

$$\left. + \left[\|u\|^2_{H^1(\Omega)} + \sum_{i=1}^n \|f_i\|^2_{\mathcal{L}^{(2,n)}(\Omega)} + \|f\|^2_{L^{(2,n)}(\Omega)} \right] r^n \right\} .$$

Hence, by using Fundamental Lemma, we obtain the following inequalities for any ρ, $r \in \left(0, \frac{\delta_0}{2}\right)$, where $0<\rho<r$,

$$\rho^{-n} \sum_{i=1}^n \int_{I(x_0,\rho)} |D_i u - \{D_i u\}_\rho|^2 dx \leqq C(\lambda)\delta_0^{-n} \sum_{i=1}^n \int_{I(x_0, \frac{\delta_0}{2})} |D_i u - \{D_i u\}_{\frac{\delta_0}{2}}|^2 dx +$$

$$+ C(\lambda) \left[\|u\|^2_{H^1(\Omega)} + \sum_{i=1}^n \|f_i\|^2_{\mathcal{L}^{(2,\lambda)}(\Omega)} + \|f\|^2_{\mathcal{L}^{(2,\lambda)}(\Omega)} \right]$$

$$\leqq C\left[\|u\|^2_{H^1(\Omega)} + \sum_{i=1}^n \|f_i\|^2_{\mathcal{L}^{(2,\lambda)}(\Omega)} + \|f\|^2_{\mathcal{L}^{(2,\lambda)}(\Omega)} \right] .$$

While if $\rho \geqq \frac{\delta_0}{2}$, $x_0 \in \overline{\Omega}_0$, then

$$\rho^{-n} \sum_{i=1}^n \int_{\Omega(x_0,\rho)} |D_i u - \{D_i u\}_\rho|^2 dx \leqq C\, \delta_0^{-n} \sum_{i=1}^n \int_{\Omega_0} |D_i u|^2 dx .$$

From these inequalities we conclude the proof of this theorem.

§ 3 Boundary estimates

In this section we denote the set $I(r) \cap \{x=(x_1, \ldots, x_n)\,;\ x_n>0\}$ by $I^*(r)$ and $\partial I^*(r) \cap \{x\,;\ x_n=0\} \cap I(r)$ by Γ_r. We denote by $V(I^*(r))$ the completion of the class of the functions, which are of class $C^1(\overline{I^*(r)})$ and vanish on $\partial I^*(r) - \Gamma_r$, with respect to the norm $|\cdot|_{H^1(I^*(r))}$.

To state the lemmas, we set the assumptions

(A. 3. 1) We asume that $a_{ij}(x) \in C^0(\overline{I^*(r)})$, a_i, $a \in L^\infty(I^*(r))$ and f_i, $f \in L^2(I^*(r))$.

(A. 3. 2) We assume that $a_{nk}(0)=0$ holds $(k=1, 2, \ldots, n-1)$.

LEMMA 3. 1. *We assume that the condition (A. 2. 1) holds and f_i, $f \in L^2(I^*(r))$. Let $u \in V(I^*(r))$ satisfy the following equation for every $\phi \in V(I^*(r))$:*

$$(3.1)\qquad \sum_{i,j=1}^{n}\int_{I^*(r)} a_{ij}(0)D_iu(x)D_j\phi(x)dx=\sum_{i=1}^{n}\int_{I^*(r)} f_i(x)D_i\phi(x)dx-\int_{I^*(r)} f(x)\phi(x)dx.$$

Then there is a positive constant C depending only on ν such that

$$(3.2)\qquad \sum_{i=1}^{n}||D_iu||^2_{L^2(I^*(r))}\leq C\left\{\sum_{i=1}^{n}||f_i||^2_{L^2(I^*(r))}+||f||^2_{L^2(I^*(r))}\right\}.$$

PROOF. If we take $\phi=u$ in (3. 1), by using Poincare's inequality, we obtain

$$\nu^{-1}\sum_{i=1}^{n}\int_{I^*(r)}|D_iu|^2dx\leq C\left\{\sum_{i=1}^{n}||f_i||_{L^2(I^*(r))}+||f||_{L^2(I^*(r))}\right\}\left\{\sum_{i=1}^{n}||D_iu||^2_{L^2(I^*(r))}\right\}^{1/2}.$$

The conclusion follows immediately from this.

LEMMA 3. 2. *We assume that the condition (A. 3. 2) holds and $u \in C^\infty(I^*(r)\cup\Gamma_r)\cap L^2(I^*(r))$ satisfies*

$E_0u=0$ in $I^*(r)$ and $D_nu(x)=0$ on Γ_r .

Then there is a constant C depending only on ν such that for each $\rho \in (0,r)$

$$(3.3)\qquad \sum_{i=1}^{n}\int_{I^*(\rho)}|D_iu|^2dx\leq C\frac{1}{(r-\rho)^2}\int_{I^*(r)}|u|^2dx .$$

PROOF. By (A. 3. 2), for every $\phi \in V(I^*(r))$, u satisfies the following equality

$$\sum_{i,j=1}^{n}\int_{I^*(r)} a_{ij}(0)D_iu(x)D_j\phi(x)dx=0 .$$

Now we shall take $\phi=(\theta^*)^2u$, where $\theta^*(x)$ is the restriction to $I^*(r)$ of the function $\theta(x)$ which is a function belonging to the class $C_0^\infty(I(r))$ with the properties

$$0\leq\theta(x)\leq1\,;\ \theta(x)=1 \text{ on } I(\rho)\,;\ |D_i\theta(x)|\leq\frac{K}{(r-\rho)}\ (i=1,\ 2,\ \ldots,\ n) .$$

Then, for any $\varepsilon>0$, we obtain

$$\nu^{-1}\sum_{i=1}^{n}\int_{I^*(\rho)}|D_iu|^2dx\leq\varepsilon\sum_{i=1}^{n}\int_{I^*(r)}|D_iu|^2(\theta^*)^2dx+C\sum_{i=1}^{n}\int_{I^*(r)}|D_i\theta^*|^2u^2dx .$$

Using the properties of $\theta^*(x)$, we obtain (3. 3) .

LEMMA 3. 3. *We assume that the condition (A. 3. 2) holds and* $u\in C^{\infty}(I^*(r)\cup\Gamma_r)\cap L^2(I^*(r))$ *satisfies*

$E_0u=0$ *in* $I^*(r)$ *and* $D_nu(x)=0$ *on* Γ_r.

Then there is a positive constant C depending only on ν *such that for each* $\rho\in(0, r]$

$$\int_{I^*(\rho)}|u|^2dx\leqq C\left(\frac{\rho}{r}\right)^n\int_{I^*(r)}|u|^2dx \tag{3. 4}$$

holds.

PROOF. Since $D_hu(x)$ satisfies $E_0(D_hu)=D_hE_0u=0$ in $I^*(r)$ and $D_n(D_hu)=D_h(D_nu)=0$ on $\Gamma_r(h=1, 2,\dots, n-1)$, we may use Lemma 3. 2 for D_hu, so that, for each $\rho\in(0, r)$, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\int_{I^*(\rho)}|D_iD_hu|^2dx&\leqq\frac{4C(\nu)}{(r-\rho)^2}\int_{I^*\left(\frac{r+\rho}{2}\right)}|D_hu|^2dx\\&\leqq\frac{4C(\nu)}{(r-\rho)^2}\times\frac{C(\nu)}{(r-\rho)^2}\int_{I^*(r)}|u|^2dx.\end{aligned}$$

Hence we have

$$\sum_{i=1}^{n}\sum_{h=1}^{n-1}\int_{I^*(\rho)}|D_iD_hu|^2dx\leqq\frac{4C(\nu)^2}{(r-\rho)^4}\int_{I^*(r)}|u|^2dx\ . \tag{3. 5}$$

Now if we return to the equation $E_0u=0$ and use (3. 5), then we have

$$\int_{I^*(\rho)}|D_n^2u|^2dx\leqq C\frac{1}{(r-\rho)^4}\int_{I^*(r)}|u|^2dx\ . \tag{3. 6}$$

From (3. 5) and (3. 6) it follows that

$$\sum_{|m|=2}\int_{I^*(\rho)}|D^mu|^2dx\leqq C(\nu)\frac{1}{(r-\rho)^4}\int_{I^*(r)}|u|^2dx\ .$$

Repeating above procedure, we may conclude the following estimate

$$||u||^2_{H^k(I^*(\frac{r}{2}))}\leqq C(\nu, r, k)\int_{I^*(r)}|u|^2dx\ . \tag{3. 7}$$

where k is any positive integer.

If we choose k sufficiently large and use Sobolev's Theorem and (3.7), we know that for each $\rho\in\left(0, \frac{r}{2}\right]$,

$$\int_{I^*(\rho)}|u|^2dx\leqq C\rho^n\max_{x\in\overline{I^*(\rho)}}|u(x)|^2\leqq C(\nu, r)\rho^n\int_{I^*(r)}|u|^2dx\ . \tag{3. 8}$$

Now we shall investigate the dependency of r of $C(\nu, r)$. If we consider the function $v(x)=u(\lambda x)$, where λ is any positive number, we can see that $v(x)$ satisfies $E_0 v=0$ in $I^*(\frac{r}{\lambda})$and $D_n v=0$ on $\Gamma_{\frac{r}{\lambda}}$. Specially taking $\lambda=r$, from (3.8) we obtain

$$\int_{I^*(\frac{\rho}{r})}|v|^2dx\leqq C(\nu,\ 1)\left(\frac{\rho}{r}\right)^n\int_{I^*(1)}|v|^2dy\ ,$$

so that

$$\int_{I^*(\rho)}|u|^2dx\leqq C(\nu)\left(\frac{\rho}{r}\right)^n\int_{I^*(r)}|u|^2dx\ .$$

Moreover above inequality for any $\rho\in\left(\frac{r}{2},\ r\right]$ is obvious .

Corollary 3. 1. *We assume that the condition (A. 3. 2) holds and* $u\in C^\infty(I^*(r)\cup\Gamma_r)\cap H^1(I^*(r))$ *satisfies* $E_0u=0$ *in* $I^*(r)$ *and* $D_nu=0$ *on* Γ_r. *Then there is a positive constant C depending only on* ν *such that for each* $\rho\in(0,\ r]$ *the following inequalities hold*:

$$\sum_{i=1}^{n-1}\int_{I^*(\rho)}|D_iu|^2dx\leqq C\left(\frac{\rho}{r}\right)^n\sum_{i=1}^{n-1}\int_{I^*(r)}|D_iu|^2dx \tag{3. 9}$$

$$\int_{I^*(\rho)}|D_nu|^2dx\leqq C\left(\frac{\rho}{r}\right)^{n+2}\int_{I^*(r)}|D_nu|^2dx\ . \tag{3. 10}$$

Proof. Since the function D_hu satisfies the hypothesis of our Lemma 3. 3, we have (3. 9).

On the other hand, since D_nu satisfies the conditions of Campanato's Lemma 11. I (See [2]), (3. 10) is verified quite analogously to it.

Lemma 3. 4. *We assume that the conditions (A. 2. 1)* and *(A. 2. 2) hold and* $u\in H^1(I^*(r))$ *satisfies the following equality for every* $\phi\in V(I^*(r))$

$$\sum_{i,j=1}^{n}\int_{I^*(r)}a_{ij}(x)D_iu(x)D_j\phi(x)dx+\sum_{i=1}^{n}\int_{I^*(r)}a_i(x)u(x)D_i\phi(x)dx- \tag{3. 11}$$
$$-\int_{I^*(r)}a(x)u(x)\phi(x)dx=\sum_{i=1}^{n}\int_{I^*(r)}f_i(x)D_i\phi(x)dx-\int_{I^*(r)}f(x)\phi(x)dx\ .$$

Then there is a positive constant C depending only on ν, $||a_i||_{L\infty}$ *and* $||a||_{L\infty}$ *such that for each* $\rho\in(0,\ r)$

$$\sum_{i=1}^{n}\int_{I^*(\rho)}|D_iu|^2dx\leqq C\left\{\left[\left(\frac{\rho}{r}\right)^n+\omega^2(r)\right]\sum_{i=1}^{n}\int_{I^*(r)}|D_iu|^2dx\right. \tag{3. 12}$$
$$\left.+\int_{I^*(r)}|u|^2dx+\sum_{i=1}^{n}\int_{I^*(r)}|f_i|^2dx+\int_{I^*(r)}|f|^2dx\right\}.$$

PROOF. We shall decompose the function u to the sum $u=v+w$ where v and w are solutions of the following problems;

$$(3.13)\qquad \begin{cases} u-v\in V(I^*(r)) \\ \sum_{i,j=1}^{n}\int_{I^*(r)} a_{ij}(0)D_ivD_j\phi\,dx=0 \quad \text{for every } \phi\in V(I^*(r)) \end{cases}$$

and

$$(3.14)\qquad \begin{cases} w\in V(I^*(r)) \\ \sum_{i,j=1}^{n}\int_{I^*(r)} a_{ij}(0)D_iwD_j\phi\,dx=\int_{I^*(r)}\left\{\sum_{i=1}^{n}\left[\sum_{j=1}^{n}(a_{ij}(0)-a_{ij}(x))D_ju+\right.\right. \\ \qquad\left.\left.+f_i-a_iu\right]D_i\phi-(f-au)\phi\right\}dx \quad \text{for every } \phi\in V(I^*(r))\ . \end{cases}$$

It may be easily verified that v is of class $C^\infty(I^*(r)\cup\Gamma_r)\cap H^1(I^*(r))$ and v satisfies $E_0v=0$ in $I^*(r)$ and $D_nv=0$ on Γ_r. Since v satisfies the hypothesis of Corollary 3. 1, we have

$$(3.15)\qquad \sum_{i=1}^{n}||D_iv||^2_{L^2(I^*(\rho))}\leqq C\left(\frac{\rho}{r}\right)^n\sum_{i=1}^{n}||D_iv||^2_{L^2(I^*(r))}\ .$$

Moreover we may apply Lemma 3. 1 to the function w, so that we have

$$(3.16)\qquad \sum_{i=1}^{n}||D_iw||^2_{L^2(I^*(\rho))}$$

$$\leqq C\left\{\omega^2(r)\sum_{i=1}^{n}||D_iu||^2_{L^2(I^*(r))}+||u||^2_{L^2(I^*(r))}+\sum_{i=1}^{n}||f_i||^2_{L^2(I^*(r))}+||f||^2_{L^2(I^*(r))}\right\}.$$

From (3. 15) and (3. 16), we have (3. 12).

With the aid of above lemmas, we can prove the following theorem.

THEOREM 3. 1. *We assume that the condition (A. 2. 1),* $a_{ij}\in C^0(\overline{I^*(1)})$, a_i. $a\in L^\infty(I^*(1))$, f_i, $f\in\mathcal{L}^{(2,\lambda)}(I^*(1))$, $0\leqq\lambda<n$, *and* $a_{nk}(x)=0$ *on* $\Gamma_1(k\neq n)$, *and for every* $\phi\in V(I^*(1))$, $u\in H^1(I^*(1))$ *satisfies the following equality*:

$$(3.17)\qquad \sum_{i=1}^{n}\int_{I^*(1)}\left\{\sum_{j=1}^{n}a_{ij}D_ju+a_iu\right\}D_i\phi\,dx-\int_{I^*(1)}au\phi\,dx=$$
$$\sum_{i=1}^{n}\int_{I^*(1)}f_iD_i\phi\,dx-\int_{I^*(1)}f\phi\,dx\ .$$

Then $D_iu\in\mathcal{L}^{(2,\lambda)}(I^*(R))$, *for each* $R\in(0,1)$, *and there is a positive constant* C *depending only on* ν, λ, R *and* $||a_i||_{L^\infty}+||a||_{L^\infty}$ *such that*

$$(3.18) \qquad \sum_{i=1}^{n} \|D_i u\|^2_{\mathcal{L}^{(2,\lambda)}(I^*(R))} \leqq C\Big\{ \|u\|^2_{H^1(I^*(1))} + \sum_{i=1}^{n} \|f_i\|^2_{\mathcal{L}^{(2,\lambda)}(I^*(1))} + + \|f\|^2_{\mathcal{L}^{(2,\lambda)}(I^*(1))} \Big\}.$$

For the proof of this theorem, we refer to that of Theorem 13. I of Campanato ([2]) and our Theorem 2. 1..

Lemma 3. 5. *We assume that the condition (A. 2. 2) holds and* $u \in C^\infty(I^*(r) \cup \Gamma_r) \cap H^1(I^*(r))$ *satisfies*

$E_0 u = 0$ *in* $I^*(r)$ *and* $D_n u(x) = 0$ *on* Γ_r $(0<r<1)$.

Then there is a positive constant C depending only on ν *such that for each* $\rho \in (0, r]$

$$(3.19) \qquad \int_{I^*(\rho)} |D_h u - \{D_h u\}_\rho|^2 dx \leqq C\left(\frac{\rho}{r}\right)^{n+2} \int_{I^*(r)} |D_h u - \{D_h u\}_r|^2 dx. \qquad (h=1, 2, \ldots, n-1)$$

Proof. Let ρ be any number of the interval $\left(0, \frac{r}{2}\right]$.

Carrying out the elementary calculation, and using (3. 7) and Sobelov's Theorem, we have

$$\begin{aligned}
\int_{I^*(\rho)} |D_h u - \{D_h u\}_\rho|^2 dx &\leqq 4\int_{I^*(\rho)} |D_h u - D_h u(0)|^2 dx \\
&\leqq 4\int_{I^*(\rho)} \Big\{|D_h u(x) - D_h u(\bar{x}, 0)|^2 + |D_h u(\bar{x}, 0) - D_h u(0)|^2\Big\} dx \\
&\leqq 4\omega_n \rho^n \Big\{\max_{x\in\overline{I^*(\rho)}} \frac{|D_n{}^2 D_h u(x)|^2}{2}\rho^4 + \sum_{i=1}^{n} \max_{x\in\overline{\Gamma\rho}} |D_i D_h u(\bar{x}, 0)|^2 \rho^2\Big\} \\
&\leqq C\rho^{n+2}\Big[\max_{x\in\overline{I^*(\rho)}} \frac{|D^2{}_n D_h u(x) - D^2{}_n \{D_h u\}_r|^2}{2} + \\
&\qquad + \sum_{i=1}^{n} \max_{x\in\overline{I^*\left(\frac{r}{2}\right)}} |D_i\{D_h u(x) - \{D_h u\}_r\}|^2\Big] \\
&\leqq C\rho^{n+2} \|D_h u - \{D_h u\}_r\|^2_{H^k\left(I^*\left(\frac{r}{2}\right)\right)} \\
&\leqq C(\nu, r)\rho^{n+2} \|D_h u - \{D_h u\}_r\|^2_{L^2(I^*(r))}.
\end{aligned}$$

Here we have used the notation $x = (\bar{x}, 0) = (x_1, x_2, \ldots, x_{n-1}, 0)$.

By the same argument of the proof of Lemma 3. 3, we obtain, for each $\rho \in \left(0, \frac{r}{2}\right]$,

$$(3.20)\qquad \int_{I^*(\rho)} |D_n u-\{D_n u\}_\rho|^2 dx \leqq C\left(\frac{\rho}{r}\right)^{n+2}\int_{I^*(r)} |D_n u-\{D_n u\}_r|^2 dx.$$

Moreover it is easy to verify that the above inequality (3. 20) holds also for each ρ of the interval $\left(\frac{r}{2}, r\right]$.

LEMMA 3. 6. *We assume that the conditions (A. 3. 1) and (A. 3. 2) hold,* and $u\in H^1(I^*(r))$ *satisfies (3. 17) for every* $\phi\in V(I^*(r))$. *Then there is a positive constant C depending only on* ν *and* $\|a_i\|_{L^\infty}+\|a\|_{L^\infty}$, *such that for each* $\rho\in(0, r)$

$$(3.21)\qquad \sum_{i=1}^{n}\int_{I^*(\rho)} |D_i u-\{D_i u\}_\rho|^2 dx$$
$$\leqq C\Big\{\omega^2(r)\sum_{i=1}^{n}\|D_i u\|^2_{L^2(I^*(r))}+\left(\frac{\rho}{r}\right)^{n+2}\sum_{i=1}^{n-1}\int_{I^*(r)}|D_i u-\{D_i u\}_r|^2 dx+$$
$$+\|u\|^2_{L^2(I^*(r))}+\sum_{i=1}^{n}\|f_i\|^2_{L^2(I^*(r))}+\|f\|^2_{L^2(I^*(r))}\Big\}$$

$$(3.22)\qquad \int_{I^*(\rho)} |D_n u|^2 dx$$
$$\leqq C\Big\{\left(\frac{\rho}{r}\right)^{n+2}\int_{I^*(r)}|D_n u|^2 dx+\omega^2(r)\sum_{i=1}^{n}\int_{I^*(r)}|D_i u|^2 dx+\|u\|^2_{L^2(I^*(r))}+$$
$$+\sum_{i=1}^{n}\|f_i\|^2_{L^2(I^*(r))}+\|f\|^2_{L^2(I^*(r))}\Big\}.$$

PROOF. Let's verify similarly to Lemma 3. 4.

We decompose u to the sum $u=v+w$, where v and w satisfy (3. 13) and (3. 14), respectively.

We may apply Lemma 3. 1 to the function w, so we obtain (3. 16). Using Lemma 3. 5 and (3. 10) of Corollary 3. 1, we have the following estimate for the function v, for each $\rho\in(0, r)$,

$$(3.23)\qquad \sum_{i=1}^{n-1}\int_{I^*(\rho)}|D_i v-\{D_i v\}_\rho|^2 dx\leqq C\left(\frac{\rho}{r}\right)^{n+2}\sum_{i=1}^{n-1}\int_{I^*(r)}|D_i v-\{D_i v\}_r|^2 dx,$$
$$\int_{I^*(\rho)}|D_n v|^2 dx\leqq C\left(\frac{\rho}{r}\right)^{n+2}\int_{I^*(r)}|D_n v|^2 dx.$$

Thus we obtain, using (3. 23),

$$\sum_{i=1}^{n-1}\int_{I^*(\rho)}|D_i u-\{D_i u\}_\rho|^2 dx\leqq\sum_{i=1}^{n-1}\int_{I^*(\rho)}|D_i v-\{D_i v\}_\rho|^2 dx+4\sum_{i=1}^{n-1}\int_{I^*(\rho)}|D_i w|^2 dx$$
$$\leqq C\Big\{\left(\frac{\rho}{r}\right)^{n+2}\sum_{i=1}^{n-1}\int_{I^*(r)}|D_i v-\{D_i v\}_r|^2 dx+4\sum_{i=1}^{n-1}\int_{I^*(r)}|D_i w|^2 dx\Big\}$$

$$\leqq C\left\{\left(\frac{\rho}{r}\right)^{n+2}\sum_{i=1}^{n-1}\int_{I^*(r)}|D_iu-\{D_iu\}_r|^2dx+\omega^2(r)\sum_{i=1}^{n}||D_iu||^2_{L^2(I^*(r))}+\right.$$

$$\left.+||u||^2_{L^2(I^*(r))}+\sum_{i=1}^{n}||f_i||^2_{L^2(I^*(r))}+||f||^2_{L^2(I^*(r))}\right\}$$

and

$$\int_{I^*(\rho)}|D_nu|^2dx\leqq C\left(\frac{\rho}{r}\right)^{n+2}\int_{I^*(r)}|D_nu|^2dx+(C+1)\sum_{i=1}^{n}\int_{I^*(r)}|D_iw|^2dx$$

$$\leqq C\left\{\left(\frac{\rho}{r}\right)^{n+2}\int_{I^*(r)}|D_nu|^2dx+\omega^2(r)\sum_{i=1}^{n}\int_{I^*(r)}|D_iu|^2dx+\right.$$

$$\left.+||u||^2_{L^2(I^*(r))}+\sum_{i=1}^{n}||f_i||^2_{L^2(I^*(r))}+||f||_{L^2(I^*(r))}\right\}.$$

THEOREM 3. 2. *We assume that* $a_{ij}\in C^{0,\alpha}(\overline{I^*(1)})$, a_i, $a\in L^\infty(I^*(1))$, f_i, $f\in L^{(2,n)}(I^*(I))$ *and* $a_{nk}(x)=0$ $(k\neq n)$ *on* Γ_1, *and that* $u\in H^1(I^*(1))$ *satisfies (3. 17) for every* $\phi\in V(I^*(1))$. *Then* $D_iu\in\mathcal{L}^{(2,n)}(I^*(R))$ *for each* $R\in(0,1)$, *and there is a positive constant C depending only on* ν, n, R *and* $||a_i||_{L^\infty}+||a||_{L^\infty}$ *such that*

$$\sum_{i=1}^{n}[D_iu]^2_{\mathcal{L}^{(2,n)}(I^*(R))}\leqq C\left\{||u||^2_{H^1(I^*(1))}+\sum_{i=1}^{n}||f_i||^2_{L^{(2,n)}(I^*(1))}+\right.$$
$$\left.+||f||^2_{L^{(2,n)}(I^*(1))}\right\}. \tag{3. 24}$$

PROOF. Let R fix, $0<R<1$, and set $\delta_0=\frac{1-R}{2}$. Let x_0 be any point of $I^*(R)$ and ρ be any number of $(0,\delta_0)$.

We may consider the following two cases;

The first case is that the sphere $I(x_0,\rho)$ is contained in $I^*(1)$, and the second case is that $x_0\in\Gamma_R$ and $I(x_0,\rho)\cap I^*(1)=I^*(x_0,\rho)$.

To the first case, we may apply Theorem 2. 2, so that we have

$$\sum_{i=1}^{n}\frac{1}{\rho^n}\int_{I(x_0,\rho)\cap I^*(R)}|D_iu-\{D_iu\}_\rho|^2dx$$
$$\leqq C\{||u||^2_{H^1(I^*(1))}+\sum_{i=1}^{n}||f_i||^2_{L^{(2,n)}(I^*(1))}+||f||^2_{L^{(2,n)}(I^*(1))}\}. \tag{3. 25}$$

Now we shall consider the second case.

Since $L^{(2,n)}(I^*(1))\subset\mathcal{L}^{(2,n-2\alpha)}(I^*(1))=L^{(2,n-2\alpha)}(I^*(1))$ holds, we may apply Theorem 3. 1, so that D_iu is of class $L^{(2,n-\alpha)}\left(I^*\left(\frac{1+R}{2}\right)\right)$ $(i=1,2,\dots,n)$ and we have

$$\sum_{i=1}^{n}||D_iu||^2_{L^{(2,n-2\alpha)}(I^*(\frac{1+R}{2}))}\leqq C(\nu,\alpha,R)\left\{\sum_{1=i}^{n}||D_iu||^2_{L^2(I^*(1))}+\right. \tag{3. 26}$$

$$+ \|u\|_{H^1(I^*(1))} + \sum_{i=1}^{n} \|f_i\|^2_{L^{(2,n-2\alpha)}(I^*(1))} + \|f\|^2_{L^{(2,n-2\alpha)}(I^*(1))}\Big\}.$$

Using Lemma 3.6 and the condition $a_{ij} \in C^{0,\alpha}(\overline{I^*(1)})$, we have

$$(3.27) \quad \sum_{i=1}^{n-1}\int_{I^*(\rho)} |D_i u - \{D_i u\}_\rho|^2 dx$$

$$\leq C\Big\{\Big(\frac{\rho}{r}\Big)^{n+2}\sum_{i=1}^{n-1}\int_{I^*(r)} |D_i u - \{D_i u\}_r|^2 dx + \|u\|^2_{L^2(I^*(r))} +$$

$$+ r^n \Big[\sum_{i=1}^{n} \|D_i u\|^2_{L^{(2,n-2\alpha)}(I^*(\frac{1+R}{2}))} + \sum_{i=1}^{n} \|f_i\|^2_{L^{(2,n)}(I^*(1))} +$$

$$+ \|f\|^2_{L^{(2,n)}(I^*(1))}\Big]\Big\}.$$

Furthermore, just as we obtained (2, 19), we obtain

$$(3.28) \quad \int_{I^*(x_0,r)} |u|^2 dx \leq C\Big\{ \|u\|_{H^1(I^*(1))} + \sum_{i=1}^{n} \|f_i\|_{L^{(2,n)}(I^*(1))}$$

$$+ \|f\|_{L^{(2,n)}(I^*(1))}\Big\} r^n$$

Inserting (3, 26) and (3, 28) into (3, 27)

$$(3.29) \quad \sum_{i=1}^{n-1}\int_{I^*(x_0,\rho)} |D_i u - \{D_i u\}_\rho|^2 dx$$

$$\leq C\Big\{\Big(\frac{\rho}{r}\Big)^{n+2}\sum_{i=1}^{n-1}\int_{I^*(x_0,r)} |D_i u - \{D_i u\}_r|^2 dx +$$

$$+ \Big[\|u\|^2_{H^1(I^*(1))} + \sum_{i=1}^{n} \|f_i\|^2_{L^{(2,n)}(I^*(1))} + \|f\|^2_{L^{(2,n)}(I^*(1))}\Big] r^n\Big\}.$$

Thus using Fundamental Lemma, we have

$$\sup_{x_0 \in \overline{I^*(R)}, \rho>0} \frac{1}{\rho^n}\sum_{i=1}^{n-1}\int_{I^*(x_0,\rho)\cap I^*(R)} |D_i u - \{D_i u\}_\rho|^2 dx$$

$$\leq C\Big\{ \|u\|^2_{H^1(I^*(1))} + \sum_{i=1}^{n} \|f_i\|^2_{L^{(2,n)}(I^*(1))} + \|f\|^2_{L^{(2,n)}(I^*(1))}\Big\}.$$

Therefore we conclude that $D_i u$ is of class $\mathcal{L}^{(2,n)}(I^*(R))$ ($i=1, 2, \ldots, n-1$).

While, by (3.22) of Lemma 3.6, (3.26), and (3.28), we have

$$\int_{I^*(x_0,\rho)} |D_n u|^2 dx \leq C\Big\{\Big(\frac{\rho}{r}\Big)^{n+2}\int_{I^*(x_0,r)} |D_n u|^2 dx + \|u\|^2_{L^2(I^*(x_0,r))} +$$

$$+ \Big[\sum_{i=1}^{n} \|D_i u\|^2_{L^{(2,n-2\alpha)}(I^*(\frac{1+R}{2}))} + \sum_{i=1}^{n} \|f_i\|^2_{L^{(2,n)}(I^*(1))}$$

$$+\|f\|^2_{L^{(2,n)}(I^*(1))}\Big]r^n\Big\}\leqq C\Big\{\Big(\frac{\rho}{r}\Big)^{n+2}\int_{I^*(x_0,r)}|D_nu|^2dx+$$

$$+\Big[\|u\|^2_{H^1(I^*(1))}+\sum_{i=1}^n\|f_i\|^2_{L^{(2,n)}(I^*(1))}+\|f\|^2_{L^{(2,n)}(I^*(1))}\Big]r^n\Big\}.$$

Therefore we obtain

$$\sup_{x_0\in \overline{I^*(R)},\rho>0}\frac{1}{\rho^n}\int_{I^*(x_0,\rho)\cap I^*(R)}|D_nu-(D_nu)_\rho|^2dx$$

$$\leqq 4\sup_{x_0\in\overline{I^*(R)},\rho>0}\frac{1}{\rho^n}\int_{I^*(x_0,\rho)\cap I^*(R)}|D_nu|^2dx$$

$$\leqq C\Big\{\|u\|^2_{H^1(I^*(1))}+\sum_{i=1}^n\|f_i\|^2_{L^{(2,n)}(I^*(1))}+\|f\|^2_{L^{(2,n)}(I^*(1))}\Big\},$$

Hence we conclude that D_nu is of class $\mathcal{L}^{(2,n)}(I^*(R))$.

§4 Regularity of solutions of the Neumann problems.

In this section we apply the results estabilished in §§ 2, 3 to the Neumann problems.

Let Ω be a bounded open domain in R^n with a boundary $\partial\Omega$ of class C^2.

Our equation in this section is the following form:

$$\text{(4. 1)}\qquad \mathrm{E}u=\sum_{i=1}^n D_i\Big(\sum_{j=1}^n a_{ij}(x)D_ju+a_i(x)u\Big)+a(x)u=\sum_{i=1}^n D_if_i+f\,.$$

To state the theorem, we assume in addition to the assumptions (A. 2. 1), (A. 2. 2) and (A. 2. 3) moreover the following assumptions:

(A. 4. 1) $a_{ij}\in C^1(\overline{\Omega})$, a_i, $a\in L^\infty(\Omega)$ $(i,\ j=1,\ 2,\ \dots,\ n)$.

(A. 4. 2) There is a positive constant m such that $a(x)<-m<0$ for all $x\in\Omega$.

Definition 4. 1. A function $u\in H^1(\Omega)$ is said to be a "generalized solution of the Neumann problems for the equation (4. 1), if u satisfies the following equality for every $\phi\in H^1(\Omega)$

$$\text{(4. 2)}\qquad \sum_{i=1}^n\int_\Omega\Big(\sum_{j=1}^n a_{ij}(x)D_ju(x)+a_i(x)u(x)\Big)D_i\phi(x)\,dx-\int_\Omega a(x)u(x)\phi(x)dx=$$

$$=\sum_{i=1}^n\int_\Omega f_i(x)D_i\phi(x)dx-\int_\Omega f(x)\phi(x)dx\ .$$

We can obtain the following theorem.

Theorem 4. 1. *We assume (A. 2. 1), (A. 2. 2), (A. 2. 3), (A. 4. 1)* and

(A. 4. 2). Let $u\in H^1(\Omega)$ be a generalized solution of the Neumann problems for the equation (4. 1). Then

(*i*) *If f_i, $f\in\mathcal{L}^{(2,\lambda)}(\Omega)$, $0\leqq\lambda<n$, then $D_iu\in\mathcal{L}^{(2,\lambda)}(\Omega)$* and

$$(4.3)\qquad \sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\leqq C\left\{||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\right\}$$

holds.

(*ii*) *If f_i, $f\in L^{(2,n)}(\Omega)$, then $D_iu\in\mathcal{L}^{(2,n)}(\Omega)$ and*

$$(4.4)\qquad \sum_{i=1}^{n}[D_iu]^2_{\mathcal{L}^{(2,n)}(\Omega)}\leqq C\left\{||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{L^{(2,n)}(\Omega)}+||f||^2_{L^{(2,n)}(\Omega)}\right\}$$

holds.

Where C are positive constants depending only on ν, Ω and the coefficients of E but independent of u.

PROOF. Since both (i) and (ii) can be proved in the same manner, we shall give only the proof of (i).

Because the domain Ω has a boundary $\partial\Omega$ of class C^2, there exists a system of the subdomains Ω_0, Ω_1, ..., Ω_l of Ω such that

$$\Omega_0\subset\subset\Omega\,;\ \bigcup_{k=0}^{l}\Omega_k=\Omega\,;\ \partial\Omega_k\cap\partial\Omega\neq\phi\quad(k=1,\ 2,\ \dots,\ l)\ .$$

At first, by Theorem 2. 1, $D_iu\in\mathcal{L}^{(2,\lambda)}(\Omega_0)$ and

$$(4.5)\qquad \sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,\lambda)}(\Omega_0)}\leqq C\left\{||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\right\}.$$

On the other hand, u satisfies the following equation

$$(4.6)\qquad \begin{aligned}&\sum_{i=1}^{n}\int_{\Omega_k}\Big(\sum_{j=1}^{n}a_{ij}(x)D_ju(x)+a_i(x)u(x)\Big)D_i\phi(x)dx-\int_{\Omega_k}a(x)u(x)\phi(x)dx\\&=\sum_{i=1}^{n}\int_{\Omega_k}f_i(x)D_i\phi(x)dx-\int_{\Omega_k}f(x)\phi(x)dx\end{aligned}$$

for every $\phi\in H^1(\Omega_k)$ vanishing on $\partial\Omega_k-\partial\Omega$.

Because the domain Ω has a boundary of class C^2, the boundary patch $\overline{\Omega}_k$ can be mapped in a one-to-one way onto $\overline{I^*(1)}$ with the boundary $\partial\Omega\cap\partial\Omega_k$ being mapped onto Γ_1. Moreover the above mapping $y=\tau^{(k)}(x)$ and its inverse $x=(\tau^{(k)})^{-1}(y)$ are of class C^2. Then the function $v(y)=(u\circ(\tau^{(k)})^{-1})(y)$ defined on $I^*(1)$ satisfies the following equation:

$$(4.7)\qquad \sum_{i=1}^{n}\int_{I^*(1)}\Big(\sum_{j=1}^{n}A_{ij}(y)D_jv(y)+A_i(y)v(y)\Big)D_i\psi(y)dy-\int_{I^*(1)}A(y)v(y)\psi(y)dy$$

$$=\sum_{i=1}^{n}\int_{I^*(1)}F_i(y)D_i\psi(y)dy-\int_{I^*(1)}F(y)\psi(y)dy$$

for every $\psi\in V(I^*(1))$.

Furthermore we note that one can take $y=\tau^{(k)}(x)$ successfully such that

$$A_{nk}(y)=0 \text{ on } \Gamma_1 \ (k\neq n)\ . \tag{4. 8}$$

One can easily verify that

$$A_{ij}\in C^1(\overline{I^*(1)}),\ A_i,\ A\in L^\infty(I^*(1))\ ,$$

and F_i, $F\in \mathcal{L}^{(2,\lambda)}(I^*(1))$ (See Campanato [2]).

Therefore, by Theorem 3. 1, for each $R\in(0,1)$ $D_iv\in\mathcal{L}^{(2,\lambda)}(I^*(R))$ and

$$\sum_{i=1}^{n}||D_iv||^2_{\mathcal{L}^{(2,\lambda)}(I^*(R))}\leqq C\Big\{||v||^2_{H^1(I^*(1))}+\sum_{i=1}^{n}||F_i||^2_{\mathcal{L}^{(2,\lambda)}(I^*(1))}+$$

$$+||F||^2_{\mathcal{L}^{(2,\lambda)}(I^*(1))}\Big\}\ . \tag{4. 9}$$

However, by the mapping $\psi: v(y)\to u(x)=(v\circ\tau^{(k)})(x)$, spaces $\mathcal{L}^{(2,\lambda)}(I^*(1))$ and $\mathcal{L}^{(2,\lambda)}(\Omega_k)$ are isomorphic (See Campanato [2], p. 375).

Therefore by (4. 9) $D_iu\in\mathcal{L}^{(2,\lambda)}(\Omega_{k,0})$ and the following inequality holds:

$$\sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,\lambda)}(\Omega_{k,0})} \tag{4. 10}$$

$$\leqq C\Big\{||u||^2_{H^1(\Omega_k)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega_k)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega_k)}\Big\}\ ,$$

where $\Omega_{k,0}=(\tau^{(k)})^{-1}(I^*(R))$.

By taking R sufficiently near to 1, we may suppose that the system Ω_0, $\Omega_{1,0}$, ..., $\Omega_{l,0}$ covers Ω. Hence from (4. 5) and (4. 10), it follows that $D_iu\in\mathcal{L}^{(2,\lambda)}(\Omega)$ and

$$\sum_{i=1}^{n}||D_iu||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\leqq C\Big\{||u||^2_{H^1(\Omega)}+\sum_{i=1}^{n}||f_i||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}+||f||^2_{\mathcal{L}^{(2,\lambda)}(\Omega)}\Big\}\ .$$

This completes the proof of (i).

(ii) is proved in the same way but except using Theorems 2.2 and 3.2 instead of Theorems 2.1 and 3.1 in the case of (i).

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