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On commutativity and associativity of multiplications in η-coefficient cohomology theories

By

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In the previous paper [**2**], we discussed the admissible multiplications in the cohomology theories with coefficient maps and obtain some uniqueness type theorems at the admissible multiplications. And, by constructing the multiplication, we have the existence theorem in the cohomology theory with coefficient the Hopf map $\eta(S^3 \to S^2)$. This note contains some corrections to [**2**] (Lemma 3. 9 and (3. 19)) and the theorems on the commutativity and associativity of the admissible multiplications in the η-coefficient cohomology theory.

1. Preliminaries.

Let $\{\tilde{h}, \sigma\}$ be a reduced cohomology (defined on the category of finite *CW*-complexes) and be equipped with an associative and commutative multilication μ.

The (reduced) *η-coefficient cohomology* $\{\tilde{h}(\ \ ;\eta), \sigma_\eta\}$ is defined by

$$\tilde{h}^i(X;\eta) = \tilde{h}^{i+4}(X \wedge P) \text{ for all } i \text{ and } X,$$
$$\sigma_\eta = (1_X \wedge T)^* \sigma,$$

where $P = S^2 \cup_\eta e^4$ and $T = T(S^1, P) : S^1 \wedge P \to P \wedge S^1$. Denote by

$$i : S^2 \to P \quad \text{and} \quad \pi : P \to S^4 ,$$

the canonical inclusion and the map collapsing S^2 to a point. Then the *reduction* $\rho_\eta = (1 \wedge \pi)^* \sigma^4$ and the *Bockstein homomorphism* $\delta_\eta = (1 \wedge \pi)^* \sigma^2 (1 \wedge i)^*$ are defined.

A multiplication in $\tilde{h}(\ \ ;\eta)$ is said to be *admissible* if it is compatible with the reduction, quasi associative and the Bockstein homomorphism works as a derivation in a certain sense (cf. [**2**], **1. 6.**).

We have the following theorems in [**2**].

(1. 1) THEOREM. *If $3\nu^{**} = 0$ in $\tilde{h}$ then admissible multiplication μ_η exists in $\tilde{h}(\ \ ;\eta)$.*

(1. 2) THEOREM. *If there exists an admissible multiplication in* $\tilde{h}(\ ;\eta)$ *then admissible multiplications are in one-to-one correspondence with the elements of* $\tilde{h}^{-4}(S^0;\eta)$.

(1. 3) THEOREM. *If there exists an admissible multiplication in* $\tilde{h}(\ ;\eta)$ *then either there is no commutative one, or every one is commutative.*

And see the formula (#) in the proof of Theorem 2. 5 in [**2**], we obtain

(1. 4) THEOREM. *If there exists an associative and commutative admissible multiplication in* $\tilde{h}(\ ;\eta)$ *then every one is associative.*

Furthermore in this note we obtain the following theorems.

(1. 5) THEOREM. *The admissible multiplication* μ_η *which is given by* (1. 1) *is commutarive, i. e., if* $3\nu^{**}=0$ *in* $\tilde{h}$ *then the admissible multiplications in* $\tilde{h}(\ ;\eta)$ *are commutative.*

(1. 6) THEOREM. *The admissible multiplication* μ_η *which is given by* (1. 1) *is associative, i. e., if* $3\nu^{**}=0$ *in* $\tilde{h}$ *then the admissible multiplications in* $\tilde{h}(\ ;\eta)$ *are associative.*

From Theorem 5. 3 in [**2**], we obtain

(1. 7) COROLLARY. *Through the Wodd isomorphism* $\widetilde{KU}^*(\ )\to\widetilde{KO}^*(\ ;\eta)$, *there exist the admissible multiplications in KU-theory and they are commutative and associative.*

In this note we devoted the proofs of (1. 5) and (1. 6). Throughout this note we use the same notations as [**1**] and [**2**].

2. Correction and enlargement to the article [2].

2. 1. Put

$P=S^2\cup_\eta e^4$, $Q=S^3\underset{3\nu}{\cup} e^7$, $\bar{N}_\eta=(S^3P\vee S^6)\underset{3i\nu\vee\eta}{\cup} e^8$ and $N_\eta=(S^4\vee S^6)\underset{3\nu\vee\eta}{\cup} e^8$,

where η, ν are 1-stem and 3-stem Hopf maps respectively and $i: S^2\to P$ is natural inclusion (cf. [**2**], § **3**). We have following cofibration sequences,

(2. 1) $S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} P \xrightarrow{\pi} S^4$,

(2. 2) $S^7 \xrightarrow{3\nu} S^4 \xrightarrow{i'} SQ \xrightarrow{\pi'} S^8$,

(2. 3) $S^3P \xrightarrow{3(S^2i)\nu(S^3\pi)} S^2P \xrightarrow{\bar{i}_0} \bar{N}_\eta \xrightarrow{\bar{\pi}_0} S^4P$,

(2. 4) $S^3P \xrightarrow{3\nu(S^3\pi)} S^4 \xrightarrow{i_0} N_\eta \xrightarrow{\pi_0} S^4P$,

(2. 4′) $Q \xrightarrow{\eta\pi'} S^6 \xrightarrow{i_1} N_\eta \xrightarrow{\pi_1} SQ$,

(2. 5) $S^5 \xrightarrow{i_0\eta} N_\eta \xrightarrow{j} \bar{N}_\eta \xrightarrow{p} S^6$.

And we have relations

(2. 6) $\pi_0 i_1 = S^4 i,\ (S^4\pi)\pi_0 = \pi'\pi_1$ *and* $\pi_1 i_0 = i'$.

2. 2. Making use of Lemma 3. 1 in [**2**] and Puppe's sequence associated with the cofibration (2. 1), we have following exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \{S^3P,\ S^2\} & \xrightarrow{i_*} & \{S^3P,\ P\} & \xrightarrow{\pi_*} & \{S^3P,\ S^4\} & \longrightarrow 0\ , \\ & \wr\| & & & & \wr\| & \\ & Z_{24} & & & & Z_{12} & \\ & (\bar{\nu}) & & & & (\nu\pi) & \end{array}$$

where $(\bar{\nu})$, $(\nu\pi)$ are represented by the generators of above groups. Denote by $\{\alpha,\ \beta,\ \gamma\}$ the *Toda bracket* [**4**], we consider an element $\widetilde{\nu\pi}$ of $\{S^3P,\ P\}$ which is an extension of $\nu\pi$, then

$$12\widetilde{\nu\pi} \in -i_*\{\eta,\ \nu,\ 12\pi\}\ .$$

Since $\{\eta,\ \nu,\ 12\pi\}$ consists of a single element,

$$\{\eta,\ \nu,\ 12\pi\} = \{\eta,\ \eta^3,\ \pi\} = \{\eta^3,\ \eta,\ \pi\} = 12\bar{\nu}\ .$$

Thus,

$$12\widetilde{\nu\pi} = -12i\bar{\nu} \text{ and } 12(i\bar{\nu} + \widetilde{\nu\pi}) = 12(1_P \wedge \nu) = 0\ .$$

Similarly, we have the following exact sequence

$$\begin{array}{ccccccc} \rightarrow \{S^4P,\ S^3\} & \xrightarrow{\eta_*} & \{S^4P,\ S^2\} & \xrightarrow{i_*} & \{S^4P,\ P\} & \xrightarrow{\pi_*} 0\ . \\ \wr\| & & \wr\| & & & \\ Z_{24} & & Z_2 & & & \\ (\bar{\nu}) & & (\nu^2\pi) & & & \end{array}$$

Since $1_P \wedge \eta = 3i\nu(S\pi)$ and

$$\eta\bar{\nu} = \bar{\nu}(1_P \wedge \eta) = 3\bar{\nu}i\nu(S\pi) = 3\nu^2\pi = \nu^2\pi\ ,$$

i_* is trivial in the above sequence. Then we have

(2. 7) Proposition. $\{S^3P,\ P\} \cong Z_{12} + Z_{24}$: *with generators* $1_P \wedge \nu$, $i\bar{\nu}$ *(or* $\widetilde{\nu\pi}$*)* ,
$\{S^4P,\ P\} = 0$.

2. 3. By (3. 25) in [**2**], $\omega_0'(=\alpha_0') \in \{S^6,\ P \wedge P\}$ is of type (1, 1). Exchanging the generator, we obtain

(2. 8) Proposition. *$\{S^6, P\wedge P\}\cong Z+Z$: with generators α_0' and $\tilde{\zeta}\wedge i$, where α_0' satisfies $(1_P\wedge\pi)\alpha_0'=(1_P\wedge\pi)T\alpha_0'=S^4 i$ $(T=T(P, P))$.*

The element of $\{S^6, P\wedge P\}$ is determined by its type. Since $i\wedge\tilde{\zeta}$ is of type (0, 2), we obtain

$$i\wedge\tilde{\zeta}=2\alpha_0'-\tilde{\zeta}\wedge i. \tag{2. 8'}$$

There are some mistakes on the generators of Lemma 3. 6 in **[2]**, then we revise as follows.

(2. 9) Proposition. *The groups $\{S^{2-k}, P\wedge P\}$ and $\{P\wedge P, S^{4+k}P\}$ are isomorphic to the corresponding groups in the following table:*

	$k\geqq 3$	$k=2$	$k=1$	$k=0$	$k=-1$	$k=-2$
	0	Z	0	$Z+Z+Z$	Z_3	$Z+Z+Z$
generators of $\{S^{2-k}P, P\wedge P\}$		$(i\wedge i)\pi$		$1_P\wedge i$, $\alpha_0'(S^2\pi)$, $i\bar{\zeta}\wedge i$(*or* $\tilde{\zeta}\pi\wedge i$)	$i\nu(S\pi)\wedge i$	$\tilde{\zeta}(S^2\bar{\zeta})\wedge i$, $1_P\wedge\tilde{\zeta}$, $\tilde{\alpha}_1$(*or* $\bar{\alpha}_1$)
generators of $\{P\wedge P, S^{4+k}P\}$		$(S^2 i)\pi\wedge\pi$		$1_P\wedge\pi$, $(S^4 i)\beta_0$, $i\bar{\zeta}\wedge\pi$(*or* $\tilde{\zeta}\pi\wedge\pi$)	$S^3 i(\nu\pi\wedge\pi)$	$S^2\tilde{\zeta}(\bar{\zeta}\wedge\pi)$, $1_P\wedge\bar{\zeta}$, $\tilde{\beta}_1$(*or* $\bar{\beta}_1$)

where $\tilde{\alpha}_1$, $\tilde{\beta}_1$, $\bar{\alpha}_1$ and $\bar{\beta}_1$ are elements satisfying

$$(1_P\wedge\pi)\tilde{\alpha}_1=\tilde{\zeta}\pi,\ \tilde{\beta}_1(1_P\wedge i)=\tilde{\zeta}\pi,\ (1_P\wedge\pi)\bar{\alpha}_1=i\bar{\zeta}\ \text{and}\ \bar{\beta}_1(1_P\wedge i)=i\bar{\zeta}.$$

(2. 9′) Proposition. *We can choose the element $\tilde{\alpha}_1$ as the generator of $\{S^4, P\wedge P\}$ such that*

$$\tilde{\alpha}_1(S^4 i)=\tilde{\zeta}\wedge i \text{ and } (1_P\wedge\pi)\tilde{\alpha}_1=\tilde{\zeta}\pi.$$

Proof. Making use of (2. 8′) and the relation $i\bar{\zeta}+\tilde{\zeta}\pi = 2\cdot 1_P$, we obtain

$$\pi^*(\tilde{\zeta}\wedge\tilde{\zeta})=2\cdot 1_P\wedge\tilde{\zeta}-(i\wedge\tilde{\zeta})(S^4\bar{\zeta})=2(1_P\wedge\bar{\zeta}-\alpha_0'(S^4\bar{\zeta}))+\tilde{\zeta}(S^2\bar{\zeta})\wedge i.$$

On the other hand, since

$$(1_P\wedge\pi)_*(1_P\wedge\tilde{\zeta}-\alpha_0'(S^4\bar{\zeta})-\tilde{\alpha}_1)=2\cdot 1_{S^4P}-S^4(i\bar{\zeta}+\tilde{\zeta}\pi)=0\ ,$$

we have

$$1_P\wedge\tilde{\zeta}-\tilde{\alpha}_0'(S^4\bar{\zeta})=l'(1_P\wedge i)(S^2\tilde{\zeta})(S^4\bar{\zeta})+\tilde{\alpha}_1\ ,$$

for some integer l'. Then,

$$\pi^*(\tilde{\zeta}\wedge\tilde{\zeta})=(2l'+1)\tilde{\zeta}(S^2\bar{\zeta})\wedge i+2\tilde{\alpha}_1.$$

Put $i^*\tilde{\alpha}_1=l(\tilde{\zeta}\wedge i)+m\alpha_0'$ (for some integers l, m),

$$0=i^*\pi^*(\tilde{\zeta}\wedge\tilde{\zeta})=(2l'+1)i^*(\tilde{\zeta}\wedge i)(S^4\bar{\zeta})+2i^*\tilde{\alpha}_1$$
$$=(2(2l'+1)+2l)\tilde{\zeta}\wedge i+m\alpha_0'.$$

Thus $l=-(2l'+1)$ and $m=0$.
Therefore, we put $\tilde{\alpha}_1'=(l'+1)\tilde{\zeta}(S^2\bar{\zeta})\wedge i+\tilde{\alpha}_1$ and take $\tilde{\alpha}_1=\tilde{\alpha}_1'$ as a generator of $\{S^4, P\wedge P\}$ then $\tilde{\alpha}_1$ satisfies (2. 9′).

2. 4. The results (3.19) in **[2]** are incorrect, they should be replaced by following ;

(2.10) PROPOSITION.

groups	$i\geqq 7$	$i=6$	$i=5$	$i=4$	$i=3$	$i=2$
$\{N_\eta, S^iP\}\cong$	0	Z	0	$Z+Z$	Z_3	$Z+Z$
generators		$i\pi\pi_0$ $=i\pi'\pi_1$		π_0, $\tilde{\zeta}\pi\pi_0=\tilde{\zeta}\pi'\pi_1$ (or $i\tilde{\zeta}\pi_0$)	$i\nu\pi\pi_0$ $=i\nu\pi'\pi_1$	$\tilde{\zeta}(S^2\bar{\zeta})\pi_0$, ε_0

where ε_0 *is defind by* $\varepsilon_0 i_0=2\cdot i$.

2. 5. Making use of Lemma 3.5 in **[2]** and (2.9), we have following exact sequence associated with (2.5),

$$0\longrightarrow\{S^4P,\ P\wedge P\}\xrightarrow{\pi_0^*}\{N_\eta,\ P\wedge P\}\xrightarrow{i_0^*}\{S^4,\ P\wedge P\}\longrightarrow 0.$$
$$\begin{array}{ccc} \{S^4P,\ P\wedge P\} & & \{S^4,\ P\wedge P\} \\ \wr\| & & \wr\| \\ Z+Z+Z & & Z \\ (\tilde{\zeta}(S^2\bar{\zeta})\wedge i, 1_P\wedge\tilde{\zeta},\ \tilde{\alpha}_1) & & (i\wedge i) \end{array}$$

Then we obtain that $\{N_\eta, P\wedge P\}$ is a free group with generators $(\tilde{\zeta}(S^2\bar{\zeta})\wedge i)\pi_0$, $(1_P\wedge\tilde{\zeta})\pi_0$, $\tilde{\alpha}_1\pi_0$ and w, where w is an extension of $i\wedge i$ (cf. **[2]**, (3.22)). By (2.10) and the relation $1_P\wedge\eta=3i\nu(S\pi)$, the map $(1_P\wedge\eta)_*: \{N_\eta, S^4P\}\to\{N_\eta, S^3P\}$ is trivial. And since $(1_P\wedge\pi)_*(\tilde{\zeta}(S^2\bar{\zeta})\wedge i)\pi_0=0$, $(1_P\wedge\pi)_*(1_P\wedge\tilde{\zeta})\pi_0=2\pi_0$ and $(1_P\wedge\pi)_*\tilde{\alpha}_1\pi_0=\tilde{\zeta}\pi\pi_0$, we can put $(1_P\wedge\pi)_*w=(2a+1)\pi_0+b\tilde{\zeta}\pi\pi_0$ for some integers a and b. Here we put

$$w'=w-a(1_P\wedge\tilde{\zeta})\pi_0-b\tilde{\alpha}_1\pi_0$$

then $w'i_0=wi_0=i\wedge i$, $(1_P\wedge\pi)w'=\pi_0$. Thus we have the following

(2. 11) PROPOSITION.

$\{N_\eta, P\wedge P\}\cong Z+Z+Z+Z$: with generators $(\tilde{\zeta}(S^2\bar{\zeta})\wedge i)\pi_0$, $(1_P\wedge\tilde{\zeta})\pi_0$, $\tilde{\alpha}_1\pi_0$ and w, where w is an element satisfying $wi=i\wedge i$ and $(1_P\wedge\pi)w=\pi_0$.

2. 6. We consider the ordinary homology maps induced by elements of $\{N_\eta, P\wedge P\}$. Let e_k' be a generator of group $\tilde{H}_k(N_\eta)$ $(k=4, 6, 8)$ and $e_i \wedge e_j$ be generators of group $\tilde{H}_k(P\wedge P)$ $(i, j=2$, or 4, $i+j=k)$ given by **3. 5** in [**2**]. For the generators, induced homology maps can be expressed as

$$(\tilde{\zeta}(S^2\bar{\zeta})\wedge i)_*\pi_{0*}: (e_4', e_6', e_8') \longmapsto (0, 4e_4\wedge e_2, 0),$$
$$(1_P\wedge\tilde{\zeta})_*\pi_{0*}: (e_4', e_6', e_8') \longmapsto (0, 2e_2\wedge e_4, 2e_4\wedge e_4),$$
$$\tilde{\alpha}_{1*}\pi_{0*}: (e_4', e_6', e_8') \longmapsto (0, 2e_4\wedge e_2, 2e_4\wedge e_4),$$
$$w_*: (e_4', e_6', e_8') \longmapsto (e_2\wedge e_2, ke_4\wedge e_2+e_2\wedge e_4, e_4\wedge e_4),$$

for some integer k.

Thus the element of stable homotopy group $\{N_\eta, P\wedge P\}$ is determind by its ordinary homology map type. Particularly the homology map induced by α (given by Proposition 3. 8 in [**2**]) is

(2. 12) $\qquad \alpha_*: (e_4', e_6', e_8') \longmapsto (me_2\wedge e_2, e_4\wedge e_2+e_2\wedge e_4, e_4\wedge e_4),$

for some integer m. And $T\alpha$ has of same homology map type (where $T=T(P, P)$). Then we obtain

(2. 13) LEMMA. *$T\alpha=\alpha$ in $\{N_\eta, P\wedge P\}$, where α is given by Proposition* 3. 8 *in* [**2**] *and $T=T(P, P)$.*

3. Proof of Theorem (1. 5).

Let μ be an associative and commutative multiplcation in $\tilde{h}$, and assume that $(1_X\wedge 3\nu)^*=0$ in $\tilde{h}$ for any finite CW-complex X. And let μ_η be the admissible multiplication in $\tilde{h}(\ ;\eta)$ constructed in [**2**]. That is, for $x \in \tilde{h}^i(X;\eta)=\tilde{h}^{i+4}(X\wedge P)$ and $y\in \tilde{h}^j(Y;\eta)=\tilde{h}^{j+4}(Y\wedge P)$, we have

$$\mu_\eta(x\otimes y)=\sigma^{-4}\gamma_W(1_W\wedge\alpha)^*(1_X\wedge T'\wedge 1_P)^*\mu(x\otimes y)$$

where $W=X\wedge Y$ and $T'=T(Y,P)$.

Put

$$\mu_\eta'(x\otimes y)=(-1)^{ij}T''^*\mu_\eta(y\otimes x)$$

for $T''=T(X, Y)$. μ_η' is also an admissible multiplication and by a rautine calculation making use of the naturality of γ etc., we see that

$$\mu_\eta'(x\otimes y)=\sigma^{-4}\gamma_W(1_W\wedge(T\alpha))^*(1_X\wedge T'\wedge 1_P)^*\mu(x\otimes y),$$

where $T=T(P, P)$. From (2. 13), $T\alpha=\alpha$ in $\{N_\eta, P\wedge P\}$ thus $\mu_\eta'=\mu_\eta$, it followes Theorem (1. 5).

4. Stable homotopy of some elementary complexes.

4. 1. Making use of Lemma 3. 1 in [**2**], (2. 6) and Puppe's exact sequences associated with (2. 2) and (2. 4′), we obtain following tables (4. 1) and (4. 2).

(4. 1) Proposition.

groups	$i\leqq-1$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$\{S^iP,\ SQ\}\cong$	0	Z	0	Z	Z_3	Z	Z_6	Z
generators		$i'\pi$		$i'(S^2\bar{\zeta})$	$i'\nu(S^3\pi)$	$\tilde{\pi}$	$i'\bar{\nu}$	$\tilde{\zeta}$

where $\tilde{\pi}$, $\tilde{\bar{\zeta}}$ *are elements satisfying* $\pi'\tilde{\pi}=4(S^4\pi)$ *and* $\pi'\tilde{\bar{\zeta}}=4(S^6\bar{\zeta})$.

(4. 2) Proposition.

groups	$i\leqq-1$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
$\{S^iP,\ N_\eta\}\cong$	0	Z	0	$Z+Z$	Z_3	$Z+Z$
generators		$i_0\pi$		$i_0(S^2\bar{\zeta})$, $i_1(S^2\pi)$	$i_0\nu(S^3\pi)$	$i_1(S^4\bar{\zeta})$, $\tilde{\xi}_0$

where $\tilde{\xi}_0$ *is defined by* $\pi_1\tilde{\xi}_0=\tilde{\pi}$.

4. 2. Consider the Puppe's exact sequence associated with a cofiberation

$$S^4P\longrightarrow P\wedge SQ\longrightarrow S^8P\ . \tag{4. 3}$$

From results of Lemmas 3. 1, 3. 2 in [**2**], (2. 7), (4. 1) and (4. 2), we have following tables (4. 4)$\sim$(4. 7).

(4. 4) Proposition.

groups	$i\geqq13$	$i=12$	$i=11$	$i=10$	$i=9$	$i=8$	$i=7$	$i=6$
$\{P\wedge SQ,S^i\}\cong$	0	Z	0	Z	Z_3	Z	Z_6	Z
generators		$\pi\wedge\pi'$		$\bar{\zeta}\wedge\pi'$	$\nu\pi\wedge\pi'$	$\bar{\pi}$	$\bar{\nu}\wedge\pi'$	$\bar{\bar{\zeta}}$

wher $\bar{\pi}$,$\bar{\bar{\zeta}}$ *are defined by* $\bar{\pi}(1_P\wedge i')=4(S^4\pi)$ *and* $\bar{\bar{\zeta}}(1_P\wedge i')=4(S^4\bar{\zeta})$.

(4. 5) PROPOSITION.

groups	$i \geqq 10$	$i=9$	$i=8$	$i=7$	$i=6$
$\{P \wedge SQ, S^i Q\} \cong$	0	Z	0	Z	Z_3
generators		$i'\pi \wedge \pi'$		$i'(\bar{\zeta} \wedge \pi')$	$i'\nu(\pi \wedge \pi')$

groups	$i=5$	$i=4$	$i=3$
$\{P \wedge SQ, S^i Q\} \cong$	$Z+Z$	Z_6	$Z+Z$
generators	$(1_{SQ} \wedge \pi)T$, $\tilde{\pi}(1_P \wedge \pi')$	$i'(\bar{\nu} \wedge \pi')$	$(1_{SQ} \wedge \bar{\zeta})T$, $\tilde{\tilde{\zeta}}(1_P \wedge \pi')$

where $T=T(P, SQ)$.

(4. 6) PROPOSITION.

groups	$i \geqq 11$	$i=10$	$i=9$	$i=8$	$i=7$
$\{P \wedge SQ, S^i P\} \cong$	0	Z	0	$Z+Z$	Z_3
generators		$i(\pi \wedge \pi')$		$i\bar{\zeta} \wedge \pi'$, $1_P \wedge \pi'$	$i\nu(S\pi)(1_P \wedge \pi')$

groups	$i=6$	$i=5$	$i=4$
$\{P \wedge SQ, S^i P\} \cong$	$Z+Z$	Z_3+Z_6	$Z+Z$
generators	$\overline{i\pi}$, $\tilde{\zeta}(S^2\bar{\zeta})(1_P \wedge \pi')$	$1_P \wedge \nu\pi'$, $i\bar{\nu}(1_P \wedge \pi')$ (or $\tilde{\nu}\pi(1_P \wedge \pi')$)	$\bar{\xi}_1$, $i\bar{\bar{\zeta}}$(or $\overline{\tilde{\zeta}\pi}$)

where $\overline{i\pi}$, $\bar{\xi}_1$, $\overline{\tilde{\zeta}\pi}$ *are defined by* $\overline{i\pi}(1_P \wedge i')=2i\pi$, $\bar{\xi}_1(1_P \wedge i')=4\cdot 1_{S^4P}$ *and* $\overline{\tilde{\zeta}\pi}(1 \wedge i')=4\tilde{\zeta}\pi$.

(4. 7) PROPOSITION.

groups	$i \geqq 9$	$i=8$	$i=7$	$i=6$	$i=5$	$i=4$
$\{P \wedge SQ, S^i N_\eta\} \cong$	0	Z	0	$Z+Z$	Z_3	$Z+Z+Z$
generators		$i_0\pi \wedge \pi'$		$i_0(\bar{\zeta} \wedge \pi')$, $i_1(\pi \wedge \pi')$	$i_0\nu(\pi \wedge \pi')$	$i_1(\bar{\zeta} \wedge \pi')$, $\tilde{\xi}_0(1_P \wedge \pi')$, p_0

where p_0 *is an element satisfying*

(4. 8) $\qquad p_0(1_P\wedge i')=S^4(i_0\pi),\ (S^4\pi_1)p_0=(1_{SQ}\wedge\pi)T$ *and* $(S^4\pi_0)p_0=1_P\wedge\pi'$.

PROOF OF (4. 8). We take p_0'' as a generator of $\{P\wedge SQ,\ S^4N_\eta\}$ such that $p_0''(1_P\wedge i')=S^4(i_0\pi)$. From (4. 5), we can put

$$(S^4\pi_1)p_0''=a(1_{SQ}\wedge\pi)T+b\tilde{\pi}(1_P\wedge\pi')$$

for some integers a and b. Since $T(S^4,\ S^4)=1$ and $\pi_1 i_0=i'$,

$$(S^4\pi_1)p_0''(1_P\wedge i')=a(1_{SQ}\wedge\pi)T(1_P\wedge i')=aS^4(i'\pi)$$

and $a=1$. Put

$$p_0'=p_0''-b\tilde{\xi}_0(1_P\wedge\pi')$$

then we have the following relations

(*) $\qquad (S^4\pi_1)p_0'=(1_{SQ}\wedge\pi)T,\ p_0'(1_P\wedge i')=p_0''(1_P\wedge i')=S^4(i_0\pi)$.

We discuss the ordinary homology maps induced by the elements of $\{P\wedge SQ,\ S^8P\}$ and $\{P\wedge SQ,\ S^4N_\eta\}$. Since p_0' satisfies the relations (*), the map $p_0'{}_*$ is repraced as

$$p_0'{}_*:\ (e_2\wedge e_4,\ e_4\wedge e_4,\ e_2\wedge e_8,\ e_4\wedge e_8)\mapsto(0,\ \sigma^4e_4',\ l'\sigma^4e_6',\ \sigma^4e_8')\ ,$$

for some integer l', where $e_i\wedge e_j$ is a generator of $\tilde{H}_{i+j}(P\wedge SQ)$ and e_k' is a generator of $\tilde{H}_k(N_\eta)$. From (4. 6), the element of $\{P\wedge SQ,\ S^3P\}$ is determined by its homology map type. Then considering the homology maps, we obtain

$$(S^4\pi_0)p_0'=1_P\wedge\pi'+l(i\bar{\zeta}\wedge\pi')$$

for some integer l and $l'=2l+1$.

Hence we take

$$p_0=p_0'-l\cdot S^4i_1(\bar{\zeta}\wedge\pi')$$

then it satisfies the relations (4. 8) and we can choose it as a generator of $\{P\wedge SQ,\ S^4N_\eta\}$. Q. E. D.

4. 3. Since $i^*:\ \{S^5P,\ S^4\}\to\{S^7,\ S^4\}$ is isomorphic and $\nu\bar{\zeta}i=2\nu$, we have $\nu\bar{\zeta}=2\bar{\nu}$. Then

$$\begin{aligned}(1_P\wedge3\nu)(i\bar{\zeta}\wedge\pi')&=3\cdot S^5(i\nu(S^3\bar{\zeta}))(1_P\wedge\pi')\\&=6\cdot S^5(i\bar{\nu})(1^P\wedge\pi')\ .\end{aligned}$$

We consider the following exact sequence associated with the cofibration (4. 3).

$$0\to\{P\wedge SQ,\ S^4P\}\xrightarrow{(1\wedge i')_*}\{P\wedge SQ,\ P\wedge SQ\}\xrightarrow{(1\wedge\pi')_*}\{P\wedge SQ,\ S^3P\}$$
$$\xrightarrow{(1\wedge3\nu)_*}\{P\wedge SQ,\ S^5P\}\to\cdots\ .$$

Making use of (4.6) and above relation, $(1_P\wedge 3\nu)_*$ is trivial in this sequence and we obtain the following

(4.9) PROPOSITION. *$\{P\wedge SQ, P\wedge SQ\}\cong Z+Z+Z+Z$; with generators $(i\wedge i')\bar{\zeta}$, $(1_P\wedge i')\bar{\xi}_1$, $1_{P\wedge SQ}$ and $i\bar{\zeta}\wedge 1_{SQ}$.*

4.4. The following lemma will be used in the next section.

(4.10) LEMMA. *For any $\alpha\in\{N_\eta, P\wedge P\}$ satisfying $(1_P\wedge\pi)\alpha=\pi_0$ there exists an element $\kappa=\kappa_\alpha\in\{P\wedge SQ, P\wedge N_\eta\}$ such that*

$$(1_P\wedge\pi_0)\kappa=(S^4\alpha)p_0\in\{P\wedge SQ, P\wedge S^4P\}$$

and

$$(1_P\wedge\pi_1)\kappa=1_{P\wedge SQ}\quad\in\{P\wedge SQ, P\wedge SQ\}\ .$$

PROOF. Since $(1_P\wedge S^4\pi)(S^4\alpha)p_0=1_P\wedge\pi'$ (by (4.8)),

$$(1_P\wedge 3\nu\pi)_*(S^4\alpha)p_0=(1_P\wedge 3\nu)(1_P\wedge\pi')=3\cdot 1_P\wedge\nu\pi'=0$$

in $\{P\wedge SQ, S^5P\}$. Thus there exists an element $\kappa'\in\{P\wedge SQ, P\wedge N_\eta\}$ such that

$$(1_P\wedge\pi_0)\kappa'=(S^4\alpha)p_0\,.$$

And we have

$$(1_P\wedge\pi')(1_P\wedge\pi_1)\kappa'=(1_P\wedge S^4\pi)(1_P\wedge\pi_0)\kappa'=(1_P\wedge\pi')1_{P\wedge SQ}\quad .$$

Therefore,

$$1_{P\wedge SQ}-(1_P\wedge\pi_1)\kappa'\in\mathrm{Image}\{(1_P\wedge i')_*: \{P\wedge SQ, S^4P\}\to\{P\wedge SQ, P\wedge SQ\}\}\ .$$

Thus for some element $x\in\{P\wedge SQ, S^4P\}$,

$$1_{P\wedge SQ}=(1_P\wedge\pi_1)\kappa'+(1_P\wedge i')x\ .$$

Put $\kappa=\kappa'+(1_P\wedge i_0)x$ then

$$(1_P\wedge\pi_0)\kappa=(1_P\wedge\pi_0)\kappa'=(S^4\alpha)p_0\ ,$$

$$(1_P\wedge\pi_1)\kappa=(1_P\wedge\pi_1)\kappa'+(1_P\wedge i')x=1_{P\wedge SQ}$$

because $\pi_1 i_0=i'$. Q. E. D.

4.5. We shall discuss some structure of $P\wedge P\wedge P$.

(4.11) PROPOSITION. *There exists a (stable) homotopy equivarence*

$$\varepsilon\in\{S^6P\vee(P\wedge SQ), P\wedge N_\eta\}$$

such that $(1_P\wedge\pi_1)\varepsilon$ is the projection of $S^6P\vee(P\wedge SQ)$ onto $P\wedge SQ$.

PROOF. Consider the cofibration (2.4′) we have

$$P\wedge N_\eta=P\wedge(S^6\underset{\eta\pi'}{\cup}CQ)=S^6P\underset{1_P\wedge\eta\pi'}{\cup}C(P\wedge Q)\ .$$

By (4.6), $1_P\wedge\eta\pi'=(1_P\wedge\eta)(1_P\wedge\pi')=3i\nu(S\pi)(1_P\wedge\pi')=0$ in $\{P\wedge SQ, S^7P\}$. Thus, by general argument we can conclude the proposition.

From the complex structure N_η, $3i_0\nu$ is homotopic to $i_1\eta$ then

$$1_P\wedge i_0\eta=(1_P\wedge i_0)S^4(3i\nu(S\pi))=(i\wedge 3i_0\nu)S^5\pi$$
$$=(i\wedge i_1\eta)S^5\pi=(1_P\wedge i_1)S^4(i\eta(S\pi))=0$$

in $\{S^5P,\ P\wedge N_\eta\}$. Thus

$$P\wedge\bar{N}_\eta=P\wedge(N_\eta\underset{i_0\eta}{\cup}C(S^5))=(P\wedge N_\eta)\underset{1_P\wedge i_0\eta}{\cup}C(S^5P)$$

is homotopic to $(P\wedge N_\eta)\vee S^6P$, i. e., there exists a homotopy equivarence $\varepsilon_1\in\{P\wedge\bar{N}_\eta,\ (P\wedge N_\eta)\vee S^6P\}$.

Using the homotopy equivarence $\bar{\alpha}\in\{\bar{N}_\eta,\ P\wedge P\}$ (Lemma 3. 3 in [2]) we put

$$\tilde{\alpha}=(1_P\wedge\bar{\alpha})\varepsilon_1^{-1}(\varepsilon\vee 1_{S^6P})\in\{S^6P\vee(P\wedge SQ)\vee S^6P,\ P\wedge P\wedge P\}$$

then $\tilde{\alpha}$ is a homotopy equivarence. Thus we have

(4. 12) Proposition. *The space $S^6P\vee(P\wedge SQ)\vee S^6P$ is homotopic to $P\wedge P\wedge P$ (in stable).*

By (4. 6) and (4. 9), the elements of $\{P\wedge SQ,\ S^6P\}$ and $\{P\wedge SQ,\ P\wedge SQ\}$ are determind by these induced homology map types. Therefore, we obtain

(4. 13) Proposition. *$\{P\wedge SQ,\ P\wedge P\wedge P\}\cong Z+Z+Z+Z+Z+Z+Z+Z$ and the element is determind by induced homology map type.*

Now we consider the homology maps induced by $(1_P\wedge\alpha)\kappa$ and $T'(1_P\wedge\alpha)\kappa$, where $T'=T(P,\ P\wedge P)$. Denote by $e_i\wedge e_j\wedge e_k$ generators of $\tilde{H}_l(P\wedge P\wedge P)$ where $l=i+j+k$ and $i,\ j,\ k=2$, or 4. From (2. 12) and (4. 10), we obtain

$$(1_P\wedge\alpha)_*\kappa_*:\ (e_2\wedge e_4,\ e_4\wedge e_4,\ e_2\wedge e_8,\ e_4\wedge e_8)\mapsto(me_2\wedge e_2\wedge e_2,\ m(e_2\wedge e_2\wedge e_4+e_2\wedge e_4\wedge e_2+e_4\wedge e_2\wedge e_2),\ e_2\wedge e_4\wedge e_4+e_4\wedge e_2\wedge e_4+e_4\wedge e_4\wedge e_2,\ e_4\wedge e_4\wedge e_4),$$

and

$$T'_*(1_P\wedge\alpha)_*\kappa_*=(1_P\wedge\alpha)_*\kappa_*\,.$$

Thus we have the following

(4. 14) Lemma. $(1_P\wedge\alpha)\kappa=T'(1_P\wedge\alpha)\kappa$ *in* $\{P\wedge SQ,\ P\wedge P\wedge P\}$, *where* $T'=T(P,\ P\wedge P)$.

5. Proof of Theorem (1. 6).

5. **1.** Let μ be an associative commutative multiplication in $\tilde{h}$, and as-

sume that $3\nu^{**}=0$ in $\tilde{h}$. Under this assumption the exact sequence of $\tilde{h}$ associated to the cofibration (2. 2) brakes into the following short exact sequences

$$(5.\ 1)\qquad 0\to\tilde{h}^k(W\wedge S^8)\xrightarrow{(1\wedge\pi')^*}\tilde{h}^k(W\wedge SQ)\xrightarrow{(1\wedge i')^*}\tilde{h}^k(W\wedge S^4)\to 0\,.$$

for any W and k. In particular, for $W=S^0$ and $k=4$, we can choose an element $\gamma_1\in\tilde{h}^4(SQ)$ such that

$$(5.\ 2)\qquad i'^*\gamma_1=\sigma^4 1\,.$$

Put $\gamma_0=\pi_1{}^*\gamma_1$. Then, by (2. 6), γ_0 satisfies the relations

$$i_0{}^*\gamma_0=\sigma^4 1 \quad\text{and}\quad i_1{}^*\gamma_0=0\,.$$

Hence any multiplication μ_η constructed in [**2**] by making use of this γ_0 is admissible. We discuss the asscoiativity of such a multplication μ_η.

Since, for $x\in\tilde{h}^k(W\wedge SQ)$,

$$\begin{aligned}(1_W\wedge i')^*\mu(\sigma^{-4}(1_W\wedge i')^*x\otimes\gamma_1)&=\mu(\sigma^{-4}(1_W\wedge i')^*x\otimes i'^*\gamma_1)\\&=(1_W\wedge i')^*x\,,\end{aligned}$$

$x-\mu(\sigma^{-4}(1_W\wedge i')^*x\otimes\gamma_1)\in\mathrm{Ker.}\ (1_W\wedge i')^*$. By (5. 1), $(1_W\wedge\pi')^*$ is monomorphic. Thus we can defined a homomorphism

$$\tilde{\gamma}_W:\ \tilde{h}_k(W\wedge SQ)\to\tilde{h}^k(W\wedge S^8)$$

for any W by

$$(5.\ 3)\qquad \tilde{\gamma}_W(x)=(1_W\wedge\pi')^{*-1}(x-\mu(\sigma^{-4}(1_W\wedge i')^*x\otimes\gamma_1))\,.$$

Similarly as in Lemma 4. 3 in [**2**], we see

(5. 4) LEMMA. (i) *$\tilde{\gamma}_W$ is a left inverse of* $(1_W\wedge\pi_0)^*$,

(ii) *$\tilde{\gamma}_W$ is natural in the sense that*

$$(S^8 f)^*\tilde{\gamma}_W=\tilde{\gamma}_{W'}(f\wedge 1_{SQ})^*$$

for f: $W\to W'$.

5. 2. We define γ_W by using $\pi_1{}^*\gamma_1$ as γ_0 (cf. [**2**], **4. 2**). For any $x\in\tilde{h}^k(W\wedge N_\eta\wedge S^4)$ and p_0 of (4. 8) we obtain

$$\begin{aligned}(1_{W\wedge P}\wedge\pi')^*\sigma^4\gamma_W\sigma^{-4}x&=(1_{W\wedge P}\wedge\pi')^*(1_W\wedge S^4\pi_0)^{*-1}(x-\sigma^4\mu(\sigma^{-4}(1_W\wedge i_0)^*\sigma^{-4}x\otimes\gamma_0))\\&=p_0{}^{**}x-(1_W\wedge p_0)^*\sigma^4\mu(\sigma^{-4}(1_W\wedge i_0)^*\sigma^{-4}x\otimes\pi_1{}^*\gamma_1)\\&=p_0{}^{**}x-(1_W\wedge T_1(S^4\pi_1)p_0)^*\mu((1_W\wedge i_0)^*\sigma^{-4}x\otimes\gamma_1)\\&=p_0{}^{**}x-(1_W\wedge\pi\wedge 1_{SQ})^*\mu((1_W\wedge i_0)^*\sigma^{-4}x\otimes\gamma_1)\\&=p_0{}^{**}x-\mu\ (\sigma^{-4}(1_W\wedge S^4(i_0\pi))^*x\otimes\gamma_1)\end{aligned}$$

$$= p_0^{**}x - \mu(\sigma^{-4}(1_{W\wedge P}\wedge i')^*(1_W\wedge p_0)^*x\otimes\gamma_1)$$
$$= (1_{W\wedge P}\wedge\pi')^*\tilde{\gamma}_{W\wedge P}(1_W\wedge p_0)^*x, \qquad \text{where } T_1=T(SQ,\ S^4).$$

Since $(1_{W\wedge P}\wedge\pi')^*$ is monorphic, we have

(5. 5) LEMMA. *For the element* $p_0\in\{P\wedge SQ,\ S^4N_\eta\}$ *of* (4. 8) *there holds the relation*

$$\tilde{\gamma}_{W\wedge P}(1_W\wedge p_0)^*=\sigma^4\gamma_W\sigma^{-4}.$$

For any $x\in\tilde{h}^k(W\wedge P\wedge N_\eta)$, we put

$$x'=\mu(\sigma^{-4}(1_{W\wedge P}\wedge i_0)^*x\otimes\gamma_1)=\sigma^{-4}(1_{W\wedge P}\wedge(i_0\wedge 1_{SQ})T_1)^*\mu(x\otimes\gamma_1)\ .$$

Then we have

$$(1_{W\wedge P}\wedge\pi_0)^*\gamma_{W\wedge P}x=x-(1_{W\wedge P}\wedge\pi_1)^*x'$$

and

$$(1_{W\wedge P}\wedge i')^*x'=\sigma^{-4}(1_{W\wedge P}\wedge(i_0\wedge 1_{SQ})(1_{S^4}\wedge i'))^*\mu(x\otimes\gamma_1)$$
$$=\sigma^{-4}\mu((1_{W\wedge P}\wedge i_0)^*x\otimes i'^*\gamma_1)$$
$$=(1_{W\wedge P}\wedge i_0)^*x\ .$$

Hence

$$\tilde{\gamma}_{W\wedge P}x'=(1_{W\wedge P}\wedge\pi')^{*-1}(x'-\mu(\sigma^{-4}(1_{W\wedge P}\wedge i_0)^*x\otimes\gamma_1))=0\ .$$

Thus, by (4. 10) and (5. 5),

$$\gamma_W(1_W\wedge\alpha)^*\sigma^{-4}\gamma_{W\wedge P}x=\sigma^{-4}\tilde{\gamma}_{W\wedge P}(1_W\wedge p_0)^*(1_W\wedge S^4\alpha)^*\gamma_{W\wedge P}x$$
$$=\sigma^{-4}\tilde{\gamma}_{W\wedge P}(1_W\wedge\kappa)^*(1_{W\wedge P}\wedge\pi_0)^*\gamma_{W\wedge P}x$$
$$=\sigma^{-4}\tilde{\gamma}_{W\wedge P}((1_W\wedge\kappa)^*x-x')$$
$$=\sigma^{-4}\tilde{\gamma}_{W\wedge P}(1_W\wedge\kappa)^*x\ .$$

We have the following

(5. 6) LEMMA. *For* $\kappa=\kappa_\alpha$ *of* (4. 10) *there holds the relation*

$$\gamma_W(1_W\wedge\alpha)^*\sigma^{-4}\gamma_{W\wedge P}=\sigma^{-4}\tilde{\gamma}_{W\wedge P}(1_W\wedge\kappa)^*\ .$$

5. 3. Put $W=X\wedge Y\wedge Z$, the map U: $W\wedge P\wedge P\wedge P\to X\wedge P\wedge Y\wedge P\wedge Z\wedge P$ is given by a permutation of factors as $U(x,\ y,\ z,\ p,\ p',\ p'')=(x,\ p,\ y,\ p',\ z,\ p'')$. And put $T=T(P,\ P)$, $T'=T(P,\ P\wedge P)$, $T_1=T(Y,\ P)$, $T_2=T(Z,\ P)$, $T_3=T(X\wedge Y\wedge P,\ Z\wedge P)$, $T_4=T(Z\wedge P,\ X\wedge P\wedge Y\wedge P)$ and $T_2'=T(Y\wedge Z,\ P)$. From Lemma 4. 3 in [**2**] and (5. 6) we obtain

$$\begin{aligned}
\mu_\eta(1\otimes\mu_\eta) &= \sigma^{-4}\gamma_W\alpha^{**}(1_X\wedge T_2'\wedge 1_P)^*\mu(1\otimes\sigma^{-4}\gamma_{Y\wedge Z}\alpha^{**}(1_Y\wedge T_2\wedge 1_P)^*\mu) \\
&= \sigma^{-4}\gamma_W\alpha^{**}(1_X\wedge T_2'\wedge 1_P)^*\sigma^{-4}\gamma_{X\wedge P\wedge Y\wedge Z}\alpha^{**}(1_{X\wedge P\wedge Y}\wedge T_2\wedge 1_P)^*\mu(1\otimes\mu) \\
&= \sigma^{-4}\gamma_W\alpha^{**}\sigma^{-4}\gamma_{W\wedge P}\alpha^{**}U^*\mu(1\otimes\mu) \\
&= \sigma^{-8}\tilde{\gamma}_{W\wedge P}(1_W\wedge\kappa)^*\alpha^{**}U^*\mu(1\otimes\mu)\ .
\end{aligned}$$

On the other hand we obtain, from (2. 13) and (4. 14),

$$\begin{aligned}
\mu_\eta(\mu_\eta\otimes 1) &= \sigma^{-4}\gamma_W\alpha^{**}(1_{X\wedge Y}\wedge T_2\wedge 1_P)^*\mu(\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T_1\wedge 1_P)^*\mu\otimes 1) \\
&= \sigma^{-4}\gamma_W\alpha^{**}(1_{X\wedge Y}\wedge T_2\wedge 1_P)^*T_3{}^*\sigma^{-4}\gamma_{Z\wedge P\wedge X\wedge Y}\alpha^{**}(1_{Z\wedge P\wedge X}\wedge T_1\wedge 1_P)^*T_4{}^* \\
&\quad\ \mu(\mu\otimes 1) \\
&= \sigma^{-4}\gamma_W\alpha^{**}(1_W\wedge T)^*\sigma^{-4}\gamma_{W\wedge P}\alpha^{**}(1_W\wedge T')^*U^*\mu(\mu\otimes 1) \\
&= \sigma^{-4}\gamma_W\alpha^{**}\sigma^{-4}\gamma_{W\wedge P}\alpha^{**}(1_W\wedge T')^*U^*\mu(\mu\otimes 1) \\
&= \sigma^{-8}\tilde{\gamma}_{W\wedge P}(1_W\wedge\kappa)^*\alpha^{**}(1_W\wedge T')^*U^*\mu(\mu\otimes 1) \\
&= \sigma^{-8}\tilde{\gamma}_{W\wedge P}(1_W\wedge\kappa)^*\alpha^{**}U^*\mu(\mu\otimes 1)\ .
\end{aligned}$$

Since μ is an associative multiplication, we have

$$\mu_\eta(1\otimes\mu_\eta)=\mu_\eta(\mu_\eta\otimes 1)\ .$$

Thus theorem is followed.

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