On commutativity and associativity of multiplications in $\eta$-coefficient cohomology theories

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https://doi.org/10.15017/1448950
On commutativity and associativity of multiplications
in \( \gamma \)-coefficient cohomology theories

By

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(Received Dec. 25, 1972)

In the previous paper [2], we discussed the admissible multiplications in the cohomology theories with coefficient maps and obtain some uniqueness type theorems at the admissible multiplications. And, by constructing the multiplication, we have the existence theorem in the cohomology theory with coefficient the Hopf map \( \eta(S^3 \to S^2) \). This note contains some corrections to [2] (Lemma 3.9 and (3.19)) and the theorems on the commutativity and associativity of the admissible multiplications in the \( \gamma \)-coefficient cohomology theory.

1. Preliminaries.

Let \( (\tilde{h}, \sigma) \) be a reduced cohomology (defined on the category of finite CW-complexes) and be equipped with an associative and commutative multiplication \( \mu \).

The (reduced) \( \gamma \)-coefficient cohomology \((\tilde{h}(\ ; \gamma), \sigma_\gamma)\) is defined by
\[
\tilde{h}(X; \gamma) = \tilde{h}^{i+1}(X \wedge P) \quad \text{for all } i \text{ and } X,
\]
\[
\sigma_\gamma = (1_\mathbb{Z} \wedge T)^* \sigma,
\]
where \( P = S^2 \cup e^4 \) and \( T = T(S^i, P): S^i \wedge P \to P \wedge S^i \). Denote by
\[
i: S^i \to P \quad \text{and} \quad \pi: P \to S^i,
\]
the canonical inclusion and the map collapsing \( S^2 \) to a point. Then the reduction \( \rho_\gamma = (1 \wedge \pi)^* \sigma \) and the Bockstein homomorphism \( \delta_\gamma = (1 \wedge \pi)^* \sigma^2 (1 \wedge \delta)^* \) are defined.

A multiplication in \( \tilde{h}(\ ; \gamma) \) is said to be admissible if it is compatible with the reduction, quasi associative and the Bockstein homomorphism works as a derivation in a certain sense (cf. [2], 1.6.).

We have the following theorems in [2].

1.1 Theorem. If \( 3\gamma^{**} = 0 \) in \( \tilde{h} \) then admissible multiplication \( \mu_\gamma \) exists in \( \tilde{h}(\ ; \gamma) \).
(1. 2) **Theorem.** If there exists an admissible multiplication in \( h(\; \tau) \) then admissible multiplications are in one-to-one correspondence with the elements of \( h^{\ast}(S^\tau; \tau) \).

(1. 3) **Theorem.** If there exists an admissible multiplication in \( h(\; \tau) \) then either there is no commutative one, or every one is commutative.

And see the formula \((\#)\) in the proof of Theorem 2. 5 in [2], we obtain

(1. 4) **Theorem.** If there exists an associative and commutative admissible multiplication in \( h(\; \tau) \) then every one is associative.

Furthermore in this note we obtain the following theorems.

(1. 5) **Theorem.** The admissible multiplication \( \mu_\tau \) which is given by (1. 1) is commutative, i.e., if \( 3\mu^{\ast \ast} = 0 \) in \( h \) then the admissible multiplications in \( h(\; \tau) \) are commutative.

(1. 6) **Theorem.** The admissible multiplication \( \mu_\tau \) which is given by (1. 1) is associative, i.e., if \( 3\mu^{\ast \ast} = 0 \) in \( h \) then the admissible multiplications in \( h(\; \tau) \) are associative.

From Theorem 5. 3 in [2], we obtain

(1. 7) **Corollary.** Through the Wodd isomorphism \( \widetilde{K}\ell^{\ast}(\; ) \rightarrow \widetilde{K}O^{\ast}(\; \tau) \), there exist the admissible multiplications in KU-theory and they are commutative and associative.

In this note we devoted the proofs of (1. 5) and (1. 6). Throughout this note we use the same notations as [1] and [2].

2. **Correction and enlargement to the article [2].**

2. 1. Put

\[ P = S^1 \cup e^1, \quad Q = S^1 \cup e^2, \quad N_s = (S^1 P \vee S^1) \cup e^4 \text{ and } N_v = (S^1 \vee S^1) \cup e^4, \]

where \( \gamma, \nu \) are 1-stem and 3-stem Hopf maps respectively and \( i : S^1 \rightarrow P \) is natural inclusion (cf. [2], § 3). We have following cofibration sequences,

\[
\begin{align*}
(2. 1) & \quad S^3 \xrightarrow{\gamma} S^2 \xleftarrow{i} P \xrightarrow{\pi} S^1, \\
(2. 2) & \quad S^3 \xrightarrow{3\nu} S^1 \xleftarrow{i'} SQ \xrightarrow{\pi'} S^0, \\
(2. 3) & \quad S^1 P \xrightarrow{3(S^1 \nu)(S^1 \pi)} S^1 P \xleftarrow{i_0} \widetilde{N}_s \xrightarrow{\pi_0} S^1 P, \\
(2. 4) & \quad S^1 P \xrightarrow{3(S^1 \pi)} S^1 \xleftarrow{i_0} N_v \xrightarrow{\pi_0} S^1 P, \\
(2. 4') & \quad Q \xrightarrow{\pi_1'} S^0 \xleftarrow{i_1} N_v \xrightarrow{\pi_1} SQ, \\
(2. 5) & \quad S^1 \xrightarrow{i_0 \gamma} N_s \xleftarrow{i} \widetilde{N}_s \xrightarrow{\pi} S^1.
\end{align*}
\]
And we have relations

\[(2.6) \quad \pi_i l_i = S' l_i, \quad (S' \pi)_{l_i} = \pi'_{l_i} \quad \text{and} \quad \pi_0 l_0 = l_0'.\]

2.2. Making use of Lemma 3.1 in [2] and Puppe’s sequence associated with the cofibration \((2.1)\), we have following exact sequence

\[
0 \rightarrow (S'P, S') \xrightarrow{i_*} (S'P, P) \xrightarrow{\pi_*} (S'P, S') \rightarrow 0,
\]

\[
\begin{array}{c}
\mathbb{Z}_{12} \\
\mathbb{Z}_{24} \\
(\nu)
\end{array}
\]

where \((\nu)\), \((\nu \pi)\) are represented by the generators of above groups. Denote by \((\alpha, \beta, \gamma)\) the Toda bracket [4], we consider an element \(\nu \pi\) of \((S'P, P)\) which is an extension of \(\nu \pi\), then

\[12\nu \pi \in -i_* (\gamma, \nu, 12\pi).\]

Since \((\gamma, \nu, 12\pi)\) consists of a single element,

\[\{(\gamma, \nu, 12\pi) = (\gamma, \nu, \pi) = (\gamma, \nu, \pi) = 12\nu.\]

Thus,

\[12\nu \pi = -12\nu \pi \quad \text{and} \quad 12(i \nu + \nu \pi) = 12(1 \nu / \nu) = 0.\]

Similarly, we have the following exact sequence

\[
\begin{array}{c}
\mathbb{Z}_{12} \\
\mathbb{Z}_{24} \\
(\nu)
\end{array}
\]

Since \(1 \nu / \nu = 3 \nu (S\pi)\) and

\[\eta \nu = \nu (1 \nu / \nu) = 3 \nu (S\pi) = 3 \nu (S\pi) = \nu (S\pi),\]

\(i_*\) is trivial in the above sequence. Then we have

\[
(2.7) \quad \text{PROPOSITION.} \quad (S'P, P) \cong \mathbb{Z}_{12} + \mathbb{Z}_{24} \quad : \quad \text{with generators } 1 \nu, \quad i \nu (or \nu \pi),
\]

\[1 \nu / \nu = 0.\]

2.3. By (3.25) in [2], \(\omega'_1 (= \alpha'_2) \in (S^4, P \wedge P)\) is of type \((1, 1)\). Exchanging the generator, we obtain
PROPOSITION. \((S^6, P \wedge P) \cong Z \oplus Z\): with generators \(\alpha_i'\) and \(\xi \wedge i\), where \(\alpha_i'\) satisfies \((1_p/\pi)\alpha_i' = (1_p/\pi)T\alpha_i' = S^4_i\) \((T = T(P, P))\).

The element of \((S^6, P \wedge P)\) is determined by its type. Since \(i \wedge \xi\) is of type \((0, 2)\), we obtain

\[\begin{equation} \tag{2. 8'} \end{equation}\]

\[i \wedge \xi = 2\alpha_i' - \xi \wedge i.\]

There are some mistakes on the generators of Lemma 3.6 in [2], then we revise as follows.

PROPOSITION. The groups \((S^{2-k}, P \wedge P)\) and \((P \wedge P, S^{4+k}P)\) are isomorphic to the corresponding groups in the following table:

<table>
<thead>
<tr>
<th>(k \geq 3)</th>
<th>(k = 2)</th>
<th>(k = 1)</th>
<th>(k = 0)</th>
<th>(k = -1)</th>
<th>(k = -2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\xi \wedge i)</td>
<td>\pi</td>
<td>1_p/\pi, \alpha_i'(S^2i), \xi \wedge i(\text{or } \xi \wedge i)</td>
<td>\xi(S^2i) \wedge i, 1_p/\pi, \xi, \xi(\text{or } \xi)\rangle</td>
<td>\xi(S^2i) \wedge i, 1_p/\pi, \xi, \xi(\text{or } \xi)\rangle</td>
</tr>
</tbody>
</table>

\(\xi = 0\), \(\beta_i = 0(1_p/\pi)\alpha_i = \xi(1_p/\pi)\alpha_i = \xi \wedge i\), \(\beta_i(1_p/\pi)\alpha_i = \xi \wedge i\).

\[\begin{equation} \tag{2. 9'} \end{equation}\]

\[\pi: (S^6, P \wedge P) \cong (S^2, P \wedge P) \cong (S^{44}, P \wedge P)\]

where \(\alpha_i, \beta_i, \tilde{\alpha}_i, \tilde{\beta}_i\), and \(\gamma_i\) are elements satisfying

\[(1_p/\pi)\tilde{\gamma}_i = \xi \wedge i, (1_p/\pi)\tilde{\alpha}_i = \xi \wedge i, (1_p/\pi)\tilde{\gamma}_i(1_p/\pi)\tilde{\alpha}_i = \xi \wedge i, \text{ and } (1_p/\pi)\tilde{\gamma}_i(1_p/\pi)\tilde{\alpha}_i = \xi \wedge i.\]

PROOF. Making use of (2. 8') and the relation \(\xi + \xi = 2 \cdot 1_p\), we obtain

\[\pi^*(\xi \wedge \xi) = 2 \cdot 1_p \wedge \xi - (i \wedge \xi)(S^4i) = 2(1_p/\pi - \alpha_i'(S^4i)) + \xi(S^2i) \wedge i.\]

On the other hand, since

\[(1_p/\pi) \times (1_p/\xi - \alpha_i'(S^4i) - \tilde{\alpha}_i) = 2 - 1_p + S^4(i \wedge \xi - \xi) = 0 ,\]

we have

\[1_p/\xi - \alpha_i'(S^4i) = l'(1_p/\pi)(S^2i) (S^4i) + \tilde{\alpha}_i,\]

for some integer \(l\). Then,

\[\pi^*(\xi \wedge \xi) = (2l' + 1) \xi(S^2i) \wedge i + 2\tilde{\alpha}_i.\]

Put \(i\tilde{\alpha}_i = i(\xi \wedge i) + m \alpha_i'\) (for some integers \(l, m\),

\[\]
\[ 0 = i^* \pi^* (\tilde{\xi} \wedge \tilde{\xi}) = (2l' + 1)i^* (\tilde{\xi} \wedge i)(S'\tilde{\xi}) + 2i^* \tilde{\alpha}_i \]
\[ = (2(2l' + 1) + 2l)\tilde{\xi} \wedge i + m\alpha_i. \]

Thus \( l = -(2l' + 1) \) and \( m = 0. \)

Therefore, we put \( \tilde{\alpha}'_i = (l' + 1)\tilde{\xi}(S'\tilde{\xi}) \wedge i + \tilde{\alpha}_i \), and take \( \tilde{\alpha}_i = \tilde{\alpha}'_i \) as a generator of \((S', P \wedge P)\) then \( \tilde{\alpha}_i \) satisfies (2.9').

2. 4. The results (3.19) in [2] are incorrect, they should be replaced by following:

(2.10) PROPOSITION.

<table>
<thead>
<tr>
<th>groups</th>
<th>( i = 7 )</th>
<th>( i = 6 )</th>
<th>( i = 5 )</th>
<th>( i = 4 )</th>
<th>( i = 3 )</th>
<th>( i = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (N, S'P) \approx )</td>
<td>( Z )</td>
<td>( Z )</td>
<td>( Z )</td>
<td>( \mathbb{Z} )</td>
<td>( \mathbb{Z} )</td>
<td>( \mathbb{Z} )</td>
</tr>
<tr>
<td>generators</td>
<td>( i\pi_0 )</td>
<td>( \pi_0, \tilde{\zeta}_0 \pi_0 = \tilde{\zeta}'_0 \pi_1 )</td>
<td>( i\nu \pi_0 )</td>
<td>( i\nu \pi_0 )</td>
<td>( \tilde{\xi}(S'\tilde{\xi}) \pi_0, \varepsilon_0 )</td>
<td></td>
</tr>
<tr>
<td>( = i\pi' )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \( \varepsilon_0 \) is defined by \( \varepsilon_0 = 2 - i. \)

2. 5. Making use of Lemma 3.5 in [2] and (2.9), we have following exact sequence associated with (2.5),

\[ 0 \to (S'P, P \wedge P) \to (N, P \wedge P) \overset{i_0^*}{\to} (S', P \wedge P) \to 0. \]

Then we obtain that \((N, P \wedge P)\) is a free group with generators \( (\tilde{\xi}(S'\tilde{\xi}) \wedge i) \)
\( \pi_0, (1_p \wedge \tilde{\zeta}) \pi_0, \tilde{\alpha}_i \pi_0 \) and \( w \), where \( w \) is an extension of \( i \wedge i \) (cf. [2], (3.22)).

By (2.10) and the relation \( 1_p \wedge \gamma = 3\nu (S\pi) \), the map \((1_p \wedge \gamma)_* : (N, S'P) \to (N, S'P)\) is trivial. And since \((1_p \wedge \pi)_* (\tilde{\xi}(S'\tilde{\xi}) \wedge i) = 0, (1_p \wedge \pi)_* (1_p \wedge \tilde{\zeta}) \pi_0 = 2 \pi_0 \) and \((1_p \wedge \pi)_* \tilde{\alpha}_i \pi_0 = \tilde{\xi} \pi_0, \) we can put \((1_p \wedge \pi)_* \pi_0 = (2a + 1) \pi_0 + b \xi \pi_0 \) for some integers \( a \) and \( b. \) Here we put

\[ w' = w - a(1_p \wedge \tilde{\zeta}) \pi_0 - b \tilde{\alpha}_i \pi_0 \]

then \( w'i = w'i - i \wedge i, (1_p \wedge \pi)w' = \pi_0. \) Thus we have the following

(2.11) PROPOSITION.

\((N, P \wedge P) \approx Z + Z + Z + Z: \) with generators \( (\tilde{\xi}(S'\tilde{\xi}) \wedge i) \pi_0, (1_p \wedge \tilde{\zeta}) \pi_0, \tilde{\alpha}_i \pi_0 \) and \( w, \)
where \( w \) is an element satisfying \( w'i = i \wedge i \) and \((1_p \wedge \pi)w = \pi_0. \)
2. 6. We consider the ordinary homology maps induced by elements of \((N_n, P\wedge P)\). Let \(e'_n\) be a generator of group \(\tilde{H}_n(N_n)\) \((k=4, 6, 8)\) and \(e_i\wedge e_j\) be generators of group \(\tilde{H}_i(P\wedge P)\) \((i, j=2, 4; i+j=k)\) given by 3. 5 in [2]. For the generators, induced homology maps can be expressed as

\[
\begin{align*}
(\tilde{\xi}(S^2)\wedge l)_{\#}\pi_{06} : (e'_n, e'_n, e'_n) \rightarrow (0, 4e_i\wedge e_j, 0), \\
(1_P\wedge \tilde{\xi})_{\#}\pi_{06} : (e'_n, e'_n, e'_n) \rightarrow (0, 2e_i\wedge e_j, 2e_i\wedge e_j), \\
\tilde{\alpha}_{1*}\pi_{06} : (e'_n, e'_n, e'_n) \rightarrow (0, 2e_i\wedge e_j, 2e_i\wedge e_j), \\
w_* : (e'_n, e'_n, e'_n) \rightarrow (e_i\wedge e_j, ke_i\wedge e_j + e_i\wedge e_j, e_i\wedge e_j),
\end{align*}
\]

for some integer \(k\).

Thus the element of stable homotopy group \((N_n, P\wedge P)\) is determined by its ordinary homology map type. Particularly the homology map induced by \(\alpha\) (given by Proposition 3. 8 in [2]) is

\[(2. 12) \quad \alpha_* : (e'_n, e'_n, e'_n) \rightarrow (me_i\wedge e_j, e_i\wedge e_j + e_i\wedge e_j, e_i\wedge e_j),\]

for some integer \(m\). And \(T\alpha\) has of same homology map type (where \(T=T(P, P)\)). Then we obtain

\[(2. 13) \quad \text{LEMMA. } T\alpha=\alpha \text{ in } (N_n, P\wedge P), \text{ where } \alpha \text{ is given by Proposition 3. 8 in [2] and } T=T(P, P).\]

3. Proof of Theorem (1. 5).

Let \(\mu\) be an associative and commutative multiplication in \(\tilde{h}\), and assume that \((1_x\wedge 3\nu)^* = 0\) in \(\tilde{h}\) for any finite CW-complex \(X\). And let \(\mu_*\) be the admissible multiplication in \(\tilde{h}\) \((; \gamma)\) constructed in [2]. That is, for \(x \in \tilde{h}(X; \gamma) = \tilde{h}^{++}(X\wedge P)\) and \(y \in \tilde{h}(Y; \gamma) = \tilde{h}^{++}(Y\wedge P)\), we have

\[
\mu_*(x \otimes y) = \sigma^{-1}\gamma_*((1_x\wedge \alpha)^*(1_x\wedge T'\wedge 1_x)^*\mu(x \otimes y))
\]

where \(W=X\wedge Y\) and \(T'=T(Y, P)\).

Put

\[
\mu'_*(x \otimes y) = (-1)^{T'\mu_*}\mu_*(y \otimes x)
\]

for \(T'=T(X, Y)\). \(\mu'_*\) is also an admissible multiplication and by a routine calculation making use of the naturality of \(\gamma\) etc., we see that

\[
\mu'_*(x \otimes y) = \sigma^{-1}\gamma_*((1_x\wedge (T\alpha)^*)(1_x\wedge T'\wedge 1_x)^*\mu(x \otimes y)),
\]

where \(T=T(P, P)\). From (2.13), \(T\alpha=\alpha\) in \((N_n, P\wedge P)\) thus \(\mu'_* = \mu_*\), it follows Theorem (1. 5).
4. Stable homotopy of some elementary complexes.

4.1. Making use of Lemma 3.1 in [2], (2.6) and Puppe's exact sequences associated with (2.2) and (2.4'), we obtain following tables (4.1) and (4.2).

(4.1) Proposition.

<table>
<thead>
<tr>
<th>groups</th>
<th>$i \leq -1$</th>
<th>$i = 0$</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
<th>$i = 5$</th>
<th>$i = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S^iP, SQ)^\prime$</td>
<td>0</td>
<td>$Z$</td>
<td>0</td>
<td>$Z$</td>
<td>$Z_3$</td>
<td>$Z$</td>
<td>$Z_6$</td>
<td>$Z$</td>
</tr>
<tr>
<td>generators</td>
<td></td>
<td>$i'\pi$</td>
<td></td>
<td>$i'(S^3\xi')$</td>
<td>$i'\nu(S^3\pi)$</td>
<td>$\xi$</td>
<td>$i'\nu$</td>
<td>$\xi$</td>
</tr>
</tbody>
</table>

where $\tilde{\pi}$, $\tilde{\xi}$ are elements satisfying $\pi'\tilde{\pi} = 4(S^i\pi)$ and $\pi'\tilde{\xi} = 4(S^i\xi)$.

(4.2) Proposition.

<table>
<thead>
<tr>
<th>groups</th>
<th>$i \leq -1$</th>
<th>$i = 0$</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S^iP, N^\prime)$</td>
<td>0</td>
<td>$Z$</td>
<td>0</td>
<td>$Z+Z$</td>
<td>$Z_3$</td>
<td>$Z+Z$</td>
</tr>
<tr>
<td>generators</td>
<td></td>
<td>$i_0\pi$</td>
<td></td>
<td>$i_0(S^3\xi)$, $i_0(S^3\pi)$</td>
<td>$i_0(S^3\xi)$, $\xi_0$</td>
<td></td>
</tr>
</tbody>
</table>

where $\xi_0$ is defined by $\pi, \xi_0 = \tilde{\pi}$.

4.2. Consider the Puppe's exact sequence associated with a cofiberation

(4.3) $S^iP \rightarrow P \wedge SQ \rightarrow S^iP$.

From results of Lemmas 3.1, 3.2 in [2], (2.7), (4.1) and (4.2), we have following tables (4.4)~(4.7).

(4.4) Proposition.

<table>
<thead>
<tr>
<th>groups</th>
<th>$i \geq 13$</th>
<th>$i = 12$</th>
<th>$i = 11$</th>
<th>$i = 10$</th>
<th>$i = 9$</th>
<th>$i = 8$</th>
<th>$i = 7$</th>
<th>$i = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(P \wedge SQ, S^i)$</td>
<td>0</td>
<td>$Z$</td>
<td>0</td>
<td>$Z$</td>
<td>$Z_3$</td>
<td>$Z$</td>
<td>$Z_6$</td>
<td>$Z$</td>
</tr>
<tr>
<td>generators</td>
<td></td>
<td>$\pi \wedge \pi'$</td>
<td>$\xi \wedge \pi'$</td>
<td>$\nu \pi \wedge \pi'$</td>
<td>$\tilde{\pi}$</td>
<td>$\tilde{\nu} \wedge \pi'$</td>
<td>$\tilde{\xi}$</td>
<td></td>
</tr>
</tbody>
</table>

where $\tilde{\pi}$, $\tilde{\xi}$ are defined by $\tilde{\pi}(1_\pi \wedge i') = 4(S^i\pi)$ and $\tilde{\xi}(1_\pi \wedge i') = 4(S^i\xi)$.
(4. 5) Proposition.

<table>
<thead>
<tr>
<th>groups</th>
<th>$i \geq 10$</th>
<th>$i = 9$</th>
<th>$i = 8$</th>
<th>$i = 7$</th>
<th>$i = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(P \land SQ, S'</td>
<td>Q)$</td>
<td>$0$</td>
<td>$Z$</td>
<td>$0$</td>
<td>$Z$</td>
</tr>
<tr>
<td>generators</td>
<td>$i' \pi \land \pi'$</td>
<td>$i' (\xi \land \pi')$</td>
<td>$i'' (\xi' \land \pi')$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>groups</th>
<th>$i = 5$</th>
<th>$i = 4$</th>
<th>$i = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(P \land SQ, S'</td>
<td>Q)$</td>
<td>$Z + Z$</td>
<td>$Z_6$</td>
</tr>
<tr>
<td>generators</td>
<td>$(1_{SQ} \land \pi) T, \bar{\pi} (1_p \land \pi')$</td>
<td>$i' \bar{\nu} \land \pi'$</td>
<td>$(1_{SQ} \land \bar{\pi}) T, \bar{\zeta} (1_p \land \pi')$</td>
</tr>
</tbody>
</table>

where $T = T(P, SQ)$.

(4. 6) Proposition.

<table>
<thead>
<tr>
<th>groups</th>
<th>$i \geq 11$</th>
<th>$i = 10$</th>
<th>$i = 9$</th>
<th>$i = 8$</th>
<th>$i = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(P \land SQ, S'</td>
<td>P)$</td>
<td>$0$</td>
<td>$Z$</td>
<td>$0$</td>
<td>$Z + Z$</td>
</tr>
<tr>
<td>generators</td>
<td>$i (\pi \land \pi')$</td>
<td>$i \bar{\zeta} \land \pi', 1_p \land \pi'$</td>
<td>$i' \bar{\nu} (1_p \land \pi')$</td>
<td>$(1_{SQ} \land \bar{\pi}) T, \bar{\zeta} (1_p \land \pi')$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>groups</th>
<th>$i = 6$</th>
<th>$i = 5$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(P \land SQ, S'</td>
<td>P)$</td>
<td>$Z + Z$</td>
<td>$Z_3 + Z_6$</td>
</tr>
<tr>
<td>generators</td>
<td>$i_\pi, \bar{\zeta} (S'</td>
<td>\bar{\pi}) (1_p \land \pi')$</td>
<td>$1_p \land \nu \pi', i_\bar{\nu} (1_p \land \pi')$</td>
</tr>
</tbody>
</table>

where $\bar{\pi}_i, \bar{\xi}_i, \bar{\zeta}_i$ are defined by $\bar{\pi}_i (1_p \land \pi') = 2i \pi_i, \bar{\xi}_i (1_p \land \pi') = 4 \cdot 1_{SQ}$ and $\bar{\zeta}_i (1_p \land \pi') = 4\bar{\zeta}_i$.

(4. 7) Proposition.

<table>
<thead>
<tr>
<th>groups</th>
<th>$i \geq 9$</th>
<th>$i = 8$</th>
<th>$i = 7$</th>
<th>$i = 6$</th>
<th>$i = 5$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(P \land SQ, S'</td>
<td>N_s)$</td>
<td>$0$</td>
<td>$Z$</td>
<td>$0$</td>
<td>$Z + Z$</td>
<td>$Z_3$</td>
</tr>
<tr>
<td>generators</td>
<td>$i_\pi \land \pi'$</td>
<td>$i_\xi (\bar{\zeta} \land \pi'), i_\zeta (\pi \land \pi')$</td>
<td>$i_\nu (\pi \land \pi')$</td>
<td>$i_\zeta (\bar{\xi} \land \pi'), \bar{\zeta} (1_p \land \pi'), p_6$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $p_6$ is an element satisfying
On commutativity and associativity of multiplications

(4. 8) \[ p_\nu(1_{\nu}/\nu') = S'(1_{\nu}/\nu) \text{ and } (S'_{\pi_1}) p_\nu(1_{\nu}/\nu') = 1_{\nu}/\nu' \]

**Proof of (4. 8).** We take \( p_\nu' \) as a generator of \((P \wedge SQ, S'N_\nu)\) such that \( p_\nu'(1_{\nu}/\nu') = S'(1_{\nu}/\nu) \). From (4. 5), we can put
\[ (S'_{\pi_1}) p_\nu''(1_{\nu}/\nu') = a(1_{\nu}/\nu) T + b \pi(1_{\nu}/\nu') \]
for some integers \( a \) and \( b \). Since \( T(S', S') = 1 \) and \( \pi, \iota = i' \),
\[ (S'_{\pi_1}) p_\nu''(1_{\nu}/\nu') = a(1_{\nu}/\nu) T(1_{\nu}/\nu') = aS'(1_{\nu}/\nu) \]
and \( a = 1 \). Put
\[ p_\nu' = p_\nu'' - b \pi(1_{\nu}/\nu') \]
then we have the following relations
\[ (S'_{\pi_1}) p_\nu' = (1_{\nu}/\nu) T, \quad p_\nu'(1_{\nu}/\nu') = p_\nu''(1_{\nu}/\nu') = S'(1_{\nu}/\nu) . \]

We discuss the ordinary homology maps induced by the elements of \((P \wedge SQ, SP)\) and \((P \wedge SQ, S'N_\nu)\). Since \( p_\nu' \) satisfies the relations (*), the map \( p_\nu' \) is rephrased as
\[ p_\nu' : (e_1 \wedge e_2, e_3 \wedge e_4, e_5 \wedge e_6, e_7 \wedge e_8) \rightarrow (0, \sigma^l e'_1, l^2 e'_2, \sigma^l e'_3) , \]
for some integer \( l \), where \( e_1 \wedge e_2 \) is a generator of \( H_{\iota, i}(P \wedge SQ) \) and \( e'_1 \) is a generator of \( H_{\iota, i}(N_\nu) \). From (4. 6), the element of \((P \wedge SQ, SP)\) is determined by its homology map type. Then considering the homology maps, we obtain
\[ (S'_{\pi_1}) p_\nu' = 1_{\nu}/\nu' + l(i_{\nu}/\nu) \]
for some integer \( l \) and \( l' = 2l + 1 \).

Hence we take
\[ p_\nu = p_\nu' - l \cdot S'_{i}(i_{\nu}/\nu) \]
then it satisfies the relations (4. 8) and we can choose it as a generator of \((P \wedge SQ, S'N_\nu)\). Q. E. D.

4. 3. Since \( i^* : (SP, S') \rightarrow (S', S') \) is isomorphic and \( i_{\nu} = 2 \nu \), we have \( i_{\nu} = 2 \nu \). Then
\[ (1_{\nu}/\nu \times 3 \nu)(i_{\nu}/\nu) = 3 \cdot S'(i_{\nu}/\nu) \]

We consider the following exact sequence associated with the cofibration (4. 3).
\[ 0 \rightarrow (P \wedge SQ, SP) \rightarrow (P \wedge SQ, P \wedge SQ) \rightarrow (P \wedge SQ, SP) \rightarrow \cdots \]
Making use of (4.6) and above relation, \((1_p \wedge 3 \nu)_*\) is trivial in this sequence and we obtain the following

(4.9) PROPOSITION. \((P \wedge SQ, P \wedge SQ) \cong Z + Z + Z + Z\); with generators \((i \wedge f^\cdot) \tilde{\xi}, (1_p \wedge f^\cdot) \tilde{\xi}, \lambda P_{ASQ}\) and \(\tilde{\xi} \wedge 1_{SQ}\).

4.4. The following lemma will be used in the next section.

(4.10) LEMMA. For any \(\alpha \in (N, P\wedge P)\) satisfying \((1_p \wedge \pi)\alpha = \pi_0\) there exists an element \(\kappa = \kappa_\alpha \in (P \wedge SQ, P \wedge N_i)\) such that

\[(1_p \wedge \pi_\alpha)\kappa = (S^\nu \alpha) p_0 \in (P \wedge SQ, P \wedge S^P)\]

and

\[(1_p \wedge \pi_\alpha)\kappa = 1_{P_{ASQ}} \in (P \wedge SQ, P \wedge SQ)\,.

PROOF. Since \((1_p \wedge S^\nu \pi)(S^\nu \alpha) p_0 = 1_p \wedge \pi'\) (by (4.8)),

\[(1_p \wedge 3 \nu \pi)_* (S^\nu \alpha) p_0 = (1_p \wedge 3 \nu) (1_p \wedge \pi') = 3 \cdot 1_p \wedge \nu \pi' = 0\]

in \((P \wedge SQ, S^P)\). Thus there exists an element \(\kappa' \in (P \wedge SQ, P \wedge N_i)\) such that

\[(1_p \wedge \pi_\alpha)\kappa' = (S^\nu \alpha) p_0\,.

And we have

\[(1_p \wedge \pi_\alpha)(1_p \wedge \pi_\alpha)\kappa = (1_p \wedge S^\nu \pi)(1_p \wedge \pi_\alpha)\kappa' = (1_p \wedge \pi') 1_{P_{ASQ}}\,.

Therefore,

\[1_{P_{ASQ}} = (1_p \wedge \pi_\alpha)\kappa + (1_p \wedge \pi') x\,.

Put \(\kappa = \kappa' + (1_p \wedge \iota) x\) then

\[(1_p \wedge \pi_\alpha)\kappa = (1_p \wedge \pi_\alpha)\kappa' = (S^\nu \alpha) p_0\,\]

\[(1_p \wedge \pi_\alpha)\kappa = (1_p \wedge \pi_\alpha)\kappa' + (1_p \wedge \iota) x = 1_{P_{ASQ}}\]

because \(\pi_\alpha \iota = \iota'\).

Q. E. D.

4.5. We shall discuss some structure of \(P \wedge P \wedge P\).

(4.11) PROPOSITION. There exists a (stable) homotopy equivalence

\[e \in (S^P \wedge (P \wedge SQ), P \wedge N_i)\]

such that \((1_p \wedge \pi_\alpha) e\) is the projection of \(S^p \wedge (P \wedge SQ)\) onto \(P \wedge SQ\).

PROOF. Consider the cofibration (2.4') we have

\[P \wedge N_i = P \wedge (S^p \cup \nu_{\pi_\alpha}) = S^p \cup C(P \wedge Q)\,.

By (4.6), \(1_p \wedge \pi_\alpha = (1_p \wedge \gamma)(1_p \wedge \pi') = 3 \nu(S_p)(1_p \wedge \pi') = 0\) in \((P \wedge SQ, S^P)\). Thus, by
general argument we can conclude the proposition.

From the complex structure \( N_n \), \( 3i \nu \) is homotopic to \( i, \gamma \) then
\[
1_p \setminus i \gamma = (1_p \setminus i) S^4(3i \nu (S^4)) = (i \setminus 3i \nu) S^4 \pi
= (i \setminus i; \gamma) S^4 \pi = (1_p \setminus i; \gamma) S^4 (\nu (S^4)) = 0
\]
in \( (S^4P, P \setminus N_n) \). Thus
\[
P \setminus N_n = P \setminus (N_n \cup C(S^4P)) = (P \setminus N_n) \cup C(S^4P)
\]
is homotopic to \( (P \setminus N_n) \setminus S^4P \), i.e., there exists a homotopy equivalence \( \epsilon \in (P \setminus N_n, (P \setminus N_n) \setminus S^4P) \).

Using the homotopy equivalence \( \tilde{\alpha} \in (\tilde{N}_n, P \setminus P) \) (Lemma 3.3 in [2]) we put
\[
\tilde{\alpha} = (1_p / \tilde{\alpha}) \epsilon_{1}^{-1} (\epsilon_{S^4P} \setminus P) \in (S^4P \setminus (P \setminus S^4P) \setminus S^4P, P \setminus P \setminus P)
\]
then \( \tilde{\alpha} \) is a homotopy equivalence. Thus we have

(4.12) PROPOSITION. The space \( S^4P \setminus (P \setminus S^4P) \setminus S^4P \) is homotopic to \( P \setminus P \setminus P \) (in stable).

By (4.6) and (4.9), the elements of \( (P \setminus S^4P, S^4P) \) and \( (P \setminus S^4P, P \setminus S^4P) \) are determined by these induced homology map types. Therefore, we obtain

(4.13) PROPOSITION. \( (P \setminus S^4P, P \setminus P \setminus P) \cong Z + Z + Z + Z + Z + Z + Z + Z + Z + Z + Z \) and the element is determined by induced homology map type.

Now we consider the homology maps induced by \( (1_p / \alpha) \kappa \) and \( T'(1_p / \alpha) \kappa \), where \( T' = T(P, P \setminus P) \). Denote by \( e_i / e_i \) generators of \( H_i (P \setminus P \setminus P) \) where \( l = i + j + k \) and \( i, j, k = 2, 4 \). From (2.12) and (4.10), we obtain
\[
(1_p / \alpha) \kappa \ast : (e_i / e_i, e_i / e_i, e_i / e_i, e_i / e_i) \to (m e_i / e_i / e_i, m e_i / e_i / e_i, m e_i / e_i / e_i, m e_i / e_i / e_i, e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_i / e_
sume that $3v^* = 0$ in $\tilde{h}$. Under this assumption the exact sequence of $\tilde{h}$ associated to the cofibration (2.2) breaks into the following short exact sequences

\begin{equation}
0 \to \tilde{h}^*(W \wedge S^0) \xrightarrow{(1 \wedge \pi_0)^*} h^*(W \wedge S^0) \xrightarrow{(1 \wedge f)^*} h^*(W \wedge S^0) \to 0.
\end{equation}

for any $W$ and $k$. In particular, for $W = S^0$ and $k = 4$, we can choose an element $\tau_0 \in h^*(S^0)$ such that

\begin{equation}
\begin{aligned}
(5.2) & \\
& i^* \tau_0 = \sigma^4 1.
\end{aligned}
\end{equation}

Put $\tau_0 = \pi_1^* \tau_1$. Then, by (2.6), $\tau_0$ satisfies the relations

\begin{equation}
\begin{aligned}
& i_0^* \tau_0 = \sigma 1 \quad \text{and} \quad i_1^* \tau_0 = 0.
\end{aligned}
\end{equation}

Hence any multiplication $\mu$, constructed in [2] by making use of this $\tau_0$ is admissible. We discuss the associativity of such a multiplication $\mu$.

Since, for $x \in \tilde{h}^*(W \wedge S^0)$,

\begin{equation}
(1_{\nu} \wedge f)^* \mu (\sigma^{-4}(1_{\nu} \wedge f)^* x \otimes \tau_1) = \mu (\sigma^{-4}(1_{\nu} \wedge f)^* x \otimes f^* \tau_1)
\end{equation}

\begin{equation}
= (1_{\nu} \wedge f)^* x,
\end{equation}

$x - \mu (\sigma^{-4}(1_{\nu} \wedge f)^* x \otimes \tau_1) \in \text{Ker.} (1_{\nu} \wedge f)^*$. By (5.1), $(1_{\nu} \wedge f)^*$ is monomorphic. Thus we can defined a homomorphism

\begin{equation}
\begin{aligned}
\tilde{\tau}_w : \tilde{h}_n (W \wedge S^0) & \to \tilde{h}_w (W \wedge S^0) \\
\end{aligned}
\end{equation}

for any $W$ by

\begin{equation}
\begin{aligned}
(5.3) & \\
& \tilde{\tau}_w (x) = (1_{\nu} \wedge f)^* x = \mu (\sigma^{-4}(1_{\nu} \wedge f)^* x \otimes (1_{\nu} \wedge \pi_0)^* \tau_0).
\end{aligned}
\end{equation}

Similarly as in Lemma 4.3 in [2], we see

\begin{equation}
(5.4) \ \text{Lemmac.} \quad (i) \ \tilde{\tau}_w \text{ is a left inverse of } (1_{\nu} \wedge \pi_0)^*,
\end{equation}

(ii) \ \tilde{\tau}_w \text{ is natural in the sense that}

\begin{equation}
(Sf)^* \tilde{\tau}_w = \tilde{\tau}_w (f \wedge 1_{S^0})^*
\end{equation}

for $f : W \to W'$.

5.2. We define $\tau_w$ by using $\pi_1^* \tau_1$ as $\tau_0$ (cf. [2], 4.2). For any $x \in \tilde{h}^*(W \wedge N_i \wedge S^0)$ and $p_0$ of (4.8) we obtain

\begin{equation}
\begin{aligned}
& (1_{\nu} \wedge f)^* x \sigma^{-4} x = (1_{\nu} \wedge f)^* (1_{\nu} \wedge S^0 \pi_0) x \sigma^{-4} x \otimes (1_{\nu} \wedge \pi_0)^* \tau_1)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
& = p_0^* x - (1_{\nu} \wedge f)^* (1_{\nu} \wedge \pi_0)^* x \tau_1)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
& = p_0^* x - (1_{\nu} \wedge f)^* \mu ((1_{\nu} \wedge \pi_0)^* x \otimes \tau_1)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
& = p_0^* x - \mu ((1_{\nu} \wedge \pi_0)^* x \otimes \tau_1)
\end{aligned}
\end{equation}
On commutativity and associativity of multiplications

\[ p_0^*x = \mu(\sigma^{-1}(1_{W\wedge P}/i')^*(1_{W}/p_0)^*x \otimes \gamma_1) \]
\[ = (1_{W\wedge P}/\pi')^*\tilde{\tau}_{W\wedge P}(1_{W}/p_0)^*x \]

Since \((1_{W\wedge P}/\pi')^*\) is monomorphic, we have

(5.5) **Lemma.** For the element \(p_0 \in \{P_{\wedge SQ}, S'N_s\}\) of (4.8) there holds the relation

\[ \tilde{\tau}_{W\wedge P}(1_{W}/p_0)^* = \sigma^* \gamma \sigma^{-1}. \]

For any \(x \in \tilde{\mathcal{H}}(W\wedge P/\wedge N_s)\), we put

\[ x' = \mu(\sigma^{-1}(1_{W\wedge P}/i_0)^*x \otimes \gamma_1) = \sigma^{-1}(1_{W\wedge P}/(i_0 \wedge 1_{S0}) T_i^*)^* \mu(x \otimes \gamma_1). \]

Then we have

\[ (1_{W\wedge P}/\pi_0)^*\tilde{\tau}_{W\wedge P}x = x - (1_{W\wedge P}/\pi_0)^*x' \]

and

\[ (1_{W\wedge P}/i')^*x' = \sigma^{-1}(1_{W\wedge P}/(i_0 \wedge 1_{S0}) (1_{S'}/i'))^* \mu(x \otimes \gamma_1) \]
\[ = \sigma^{-1} \mu((1_{W\wedge P}/i_0)^*x \otimes i'^* \gamma_1) \]
\[ = (1_{W\wedge P}/i_0)^*x. \]

Hence

\[ \tilde{\tau}_{W\wedge P}x' = (1_{W\wedge P}/\pi')^*^{-1}(x' - \mu(\sigma^{-1}(1_{W\wedge P}/i_0)^*x \otimes \gamma_1)) = 0. \]

Thus, by (4.10) and (5.5),

\[ \gamma^*(1_{W}/\alpha)^*\sigma^{-1}\tilde{\tau}_{W\wedge P}x = \sigma^{-1}\tilde{\tau}_{W\wedge P}(1_{W}/p_0)^*x \]

\[ \gamma^*(1_{W}/\alpha)^*\sigma^{-1}\tilde{\tau}_{W\wedge P}(1_{W}/\pi_0)^*\tilde{\tau}_{W\wedge P}x \]
\[ = \sigma^{-1}\tilde{\tau}_{W\wedge P}(1_{W}/\kappa)^*x \]
\[ = \sigma^{-1}\tilde{\tau}_{W\wedge P}(1_{W}/\kappa)^*x \]
\[ = \sigma^{-1}\tilde{\tau}_{W\wedge P}(1_{W}/\kappa)^*x. \]

We have the following

(5.6) **Lemma.** For \(\kappa = \kappa_\alpha\) of (4.10) there holds the relation

\[ \gamma^*(1_{W}/\alpha)^*\sigma^{-1}\tilde{\tau}_{W\wedge P} = \sigma^{-1}\tilde{\tau}_{W\wedge P}(1_{W}/\kappa)^* \]

5. 3. Put \(W=X \wedge Y \wedge Z\), the map \(U: W\wedge P/\wedge P \rightarrow X\wedge P/\wedge Y\wedge P/\wedge Z\wedge P\) is given by a permutation of factors as \(U(x, y, z, p, p', p'') = (x, p, y, p', z, p'')\). And put \(T=T(P, P), T'=T(P, P/\wedge P), T_1=T(Y, P), T_2=T(Z, P), T_3=T_4=T(Z/\wedge P, X\wedge P/\wedge Y\wedge P)\) and \(T_4=T(Y\wedge Z, P)\). From Lemma 4.3 in [2] and (5.6) we obtain
\[
\mu_e(1 \otimes \mu_e) = \sigma^- \gamma w \alpha ** (1_x \wedge T_t / \wedge 1_p) \ast \mu (1 \otimes \sigma^- \gamma \gamma \alpha ** (1_T / \wedge 1_p) \ast \mu)
\]
\[
= \sigma^- \gamma w \alpha ** (1_x \wedge T_t / \wedge 1_p) \ast \sigma^- \gamma \gamma \alpha ** (1_x \wedge \alpha ** (1_T / \wedge 1_p) \ast \mu (1 \otimes \mu)
\]
\[
= \sigma^- \gamma w \alpha ** \sigma^- \gamma \gamma \alpha ** \ast \mu (1 \otimes \mu)
\]
\[
= \sigma^- \gamma w \alpha ** (1_w / \kappa) \ast \mu (1 \otimes \mu).
\]

On the other hand we obtain, from (2.13) and (4.14),
\[
\mu_e(\mu_e \otimes 1) = \sigma^- \gamma w \alpha ** (1_x \wedge \gamma T / \wedge 1_p) \ast \mu (\sigma^- \gamma \gamma \alpha ** (1_x \wedge T / \wedge 1_p) \ast \mu \otimes 1)
\]
\[
= \sigma^- \gamma w \alpha ** (1_x \wedge \gamma T / \wedge 1_p) \ast \sigma^- \gamma \gamma \alpha ** (1_x \wedge \gamma T / \wedge 1_p) \ast \mu (\mu \otimes 1)
\]
\[
= \sigma^- \gamma w \alpha ** (1_w / \gamma T / \wedge T) \ast \sigma^- \gamma \gamma \alpha ** (1_w / \gamma T / \wedge T) \ast \mu (1 \otimes \mu)
\]
\[
= \sigma^- \gamma w \alpha ** (1_w / \gamma T / \wedge T) \ast \mu (1 \otimes \mu)
\]
\[
= \sigma^- \gamma w \alpha ** (1_w / \kappa) \ast \mu (1 \otimes \mu).
\]

Since \( \mu \) is an associative multiplication, we have
\[
\mu_e(1 \otimes \mu_e) = \mu_e(\mu_e \otimes 1).
\]

Thus theorem is followed.

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References.


