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On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$

Dedicated to Professor Kiyoshi Aoki on his 60th birthday

By

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Let $U_*(G)$ be the bordism group of principal unitary G-manifolds which is isomorphic to the group $U_{*-\dim G}(B_G)$ where $U_*(B_G)$ is the unitary bordism group of the classifying space for G [2]. Denote by $S^1 \cdot Z_2$ a semi-direct product of a 1-dimensional unitary group $U(1)=S^1$ by a cyclic group Z_2 of order 2 with the product rule

$$[g', t^i][g, t^j] = [g' g^{(-1)^i}, t^{i+j}]$$

where t is a generator of Z_2 and $g, g' \in S^1$. The purpose of the present paper is to study the bordism group $U_*(S^1 \cdot Z_2)$. In [6] H. Minami and the author have discussed about the complex bordism group of dihedral group D_p of order $2p$ and obtained an isomorphism for p an odd prime

$$\tilde{U}_*(Z_p)^{Z_2} \oplus \tilde{U}_*(Z_2) \cong \tilde{U}_*(D_p) \ .$$

In this paper we obtain that $U_{2n+2}(S^1 \cdot Z_2)$ is isomorphic to $U_{2n+1}(Z_2)$ but $\tilde{U}_{2n+1}(S^1 \cdot Z_2)$ is not isomorphic to $\tilde{U}_{2n+1}(S^1)^{Z_2} \oplus \tilde{U}_{2n}(Z_2) \cong \tilde{U}_{2n+1}(S^1)^{Z_2}$.

1. The classifying space $B_{S^1 \cdot Z_2}$

Let S^{2n+1} be the $(2n+1)$-sphere in the $(n+1)$-dimensional complex plane C^{n+1} with the coordinate $(z_0, \cdots, z_n)$ and let S^m be the m-sphere in the $(m+1)$-dimensional real plane with the coordinate $(x_0, \cdots, x_m)$. Consider an action of $S^1 \cdot Z_2$ upon $S^m \times S^{2n+1}$ given by

$$[g, t^i](x, z) = ((-1)^i x, \ \ g c^i(z)),$$

where for $x=(x_0, \cdots, x_m) \in S^m$ $(-1)x=(-x_0, \cdots, -x_m)$, for $z=(z_0, \cdots, z_n) \in S^{2n+1}$ $c(z)=(\bar{z}_0, \cdots, \bar{z}_n)$ and $g(z_0, \cdots, z_n)=(gz_0, \cdots, gz_n)$ and we define $(-1)^i x$ and $c^i(z)$ inductively by setting

$$(-1)^i(x) = (-1)((-1)^{i-1}x) \ , \ c^i(z) = c(c^{i-1}(z)) \ .$$

Then we have a principal bundle

$$S^1\cdot Z_2 \to S^m\times S^{2n+1}\to S^m\times S^{2n+1}/S^1\cdot Z_2 \ .$$

We put $D'(m, n)=S^m\times S^{2n+1}/S^1\cdot Z_2$. The classifying space of $S^1\cdot Z_2$ is the limit space concerned with the inclusion $D'(m, m)\to D'(m+1, m+1)$, that is,

$$B_{S^1\cdot Z_2}=\varinjlim_{m} D'(m, m)$$

The orbit space $D'(m, n)$ is homeomorphic to the Dold manifold $D(m, n)$ which is the quotient space obtained from $S^m\times CP^n$ by identifying $(x, [z])$ with $(-x, [c(z)])$ where CP^n is the n-dimensional complex projective space [3].

Let $C_i^+ (C_i^-)$ denote an open i-cell of S^m defined by $x_{i+1}=x_{i+2}=\cdots=x_m=0$, $x_i>0 (x_i<0)$ and D_j denote an open $2j$-cell of CP^n defined by $z_j=1$, $z_{j+1}=\cdots=z_n=0$. Denote by Φ: $S^m\times CP^n\to D(m, n)$ the projection and write $(C_i,D_j)=\Phi(C_i^+\times D_j)$. Then $\{(C_i, D_j)\,;\ i=0, 1,\cdots, m,\ j=0, 1,\cdots, n\}$ is a cellular decomposition of $D(m, n)$ whose boundary relation are given by

$$\partial(C_i, D_j)=\pm(1+(-1)^{i+j})(C_{i-1}, D_j) \qquad \cdots(1.\ 1).$$

Therefore, we have the following

PROPOSITION 1. 1. *The integral homology group $H_*(B_{S^1\cdot Z_2}\,;\ Z)$ is a direct sum of the free abelian group generated by*

$$(C_0, D_{2j})\ ,\quad j=0, 1, 2, \cdots$$

and the torsion group generated by

$$(C_i, D_j)\ ,\quad i+j\colon odd$$

whose order are 2.

Let (c^i, d^j) denote the cochain dual to (C_i, D_j) the coboundary relation is given by

$$\delta(c^i, d^j)=\pm(1+(-1)^{i+j+1})(c^{i+1}, d^j) \qquad \cdots(1.\ 2).$$

The following theorem is proved in A. Dold [3].

THEOREM 1. 2. *The mod 2 cohomology ring $H^*(D(m,n)\,;\ Z_2)$ is a truncated polynomial ring $Z_2[c, d]/(c^{m+1}, d^{n+1})$, where $c=(c^1, d^0)_2$ and $d=(c^0, d^1)_2$.*

Denote by Sq^1: $H^*(D(m, n)\,;\ Z_2)\to H^{*+1}(D(m, n)\,;\ Z_2)$ and β_2: $H_*(D(m, n)\,;\ Z_2)\to H_{*-1}(D(m, n)\,;\ Z)$ the Bockstein homomorphisms. From the boundary relations (1.1) and (1.2), it is easily verified that

LEMMA 1. 3. (i) *$Sq^1 d = cd$*

(ii)
$$\beta_2(C_i, D_j)_2 = \begin{cases} (C_{i-1}, D_j) & \text{if } i+j \text{ is even and } i>0 \\ 0 & \text{otherwise.} \end{cases}$$

2. The differentials $d^3_{p,0}$ of the spectral sequence

For a CW-complex X, we denote by X^k the k-skeleton of X. There is a bordism spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ for X defined by

$$E^r_{p,q} = Z^r_{p,q}/B^r_{p,q} \ ,$$

$$Z^r_{p,q} = Im\{U_{p+q}(X^p, X^{p-r}) \xrightarrow{j_*} U_{p+q}(X^p, X^{p-1})\}$$

$$B^r_{p,q} = Im\{U_{p+q+1}(X^{p+r-1}, X^p) \to U_{p+q}(X^p, X^{p-1})\}$$

and for $x \in U_{p+q}(X^p, X^{p-1})$

$$d^r_{p,q}\{j_*(x)\} = \{\partial(x)\} \ ,$$

where $j_* : U_{p+q}(X^p, X^{p-r}) \to U_{p+q}(X^p, X^{p-1})$ and $\partial : U_{p+q}(X^p, X^{p-r}) \to U_{p+q-1}(X^{p-r}, X^{p-r-1})$, where we denote by j_* the homomorphism induced by an injection and by ∂ the boundary homomorphism. Note that $\{E^r_{*,*}\}$ are U_*-modules and differentials are U_*-homomorphisms.

Some results about the differentials of the oriented bordism spectral sequences are obtained by R. O. Burdick [1]. In the same way, we can discuss about that of the unitary bordism spectral sequences. For a CW-complex X there exists homology operations

$$Sq_i : H_n(X; Z_2) \to H_{n-i}(X; Z_2) \qquad i=0, 1, 2, \cdots$$

which are dual to the Steenrod square opration Sq^i that is

$$\langle x, Sq_i(y) \rangle = \langle Sq^i(x), y \rangle$$

$\langle \ , \ \rangle$ is the Kronecker index paring cohomology and homology. Then, we have

THEOREM 2. 1. *For a CW-complex X, the following is a commutative diagram*

$$\begin{array}{ccc} E^3_{p,0} & \xrightarrow{\quad d^3_{p,0} \quad} & E^3_{p-3,2} \\ \mu_1 \downarrow \cong & & \mu_2 \downarrow \cong \\ H_p(X; Z) & \xrightarrow{\beta_2 Sq_2 \rho_2 \otimes [CP^1]} & H_{p-3}(X; Z) \otimes U_2 \end{array}$$

where ρ_2 is the reduction modulo 2 homomorphism.

PROOF. The isomorphism μ_2 is induced by the isomorphism

$$\hat{\mu}_2 : U_{p-1}(X^{p-3}, X^{p-4}) \cong H_{p-3}(X^{p-3}, X^{p-4}) \otimes U_2,$$

for every element $[M, f]$ in $U_{p-1}(X^{p-3}, X^{p-4})$

$$\hat{\mu}_2 [M, f] = \frac{1}{2} f_* \{c_1(M) \cap \sigma(M)\} \otimes [CP^1]$$

where $c_1(M)$ is the 1-st Chern class of the tangent bundle of M and $\sigma(M)$ is the fundamental class of M. The isomorphism μ_1 is given by $\hat{\mu}_1[M, f] = f_*\sigma(M)$, where $\hat{\mu}_1 : U_p(X^p, X^{p-1}) \to H_p(X^p, X^{p-1})$.

We note that the differential $d^3_{p,0}$ is concerned with the following homomorphisms

$$\begin{array}{ccc} U_p(X^p, X^{p-3}) & \xrightarrow{\partial} & U_{p-1}(X^{p-3}, X^{p-4}) \\ \downarrow{\scriptstyle j_*} & & \\ U_p(X^p, X^{p-1}) & & \end{array} .$$

Consider an element $[M, f]$ in $U_p(X^p, X^{p-3})$ then we have

$$\hat{\mu}_1 j_* [M, f] = j_* f_* \sigma(M)$$

and

$$\hat{\mu}_2 \partial [M, f] = \frac{1}{2} \partial f_* \{c_1(M) \cap \sigma(M)\} \otimes [CP^1] ,$$

$$f_* \{c_1(M) \cap \sigma(M)\} \in H_{p-2}(X^p, X^{p-3}) .$$

Let $\hat{j}_* : H_*(X^{p-2}, X^{p-3}) \to H_*(X^p, X^{p-3})$ be a homomorphism induced by an injection $\hat{j} : (X^{p-2}, X^{p-3}) \to (X^p, X^{p-3})$, then there is an element y in $H_{p-2}(X^{p-2}, X^{p-3})$ such that

$$\hat{j}_*(y) = f_*(c_1(M) \cap \sigma(M)) .$$

Here, we regard the groups $H_p(X^p, X^{p-1})$ and $H_p(X^p, X^{p-1}; Z_2)$ as chain groups $C_p(X)$ and $C_p(X; Z_2)$ of X respectively. Denote by $[x] \in H_p(X)$ ($H_p(X; Z_2)$) the homology class of the cycle x in $C_p(X)$ ($C_p(X; Z_2)$). Then we have

$$\beta_2 [\rho_2 y] = \left[\frac{1}{2} \partial f_* \{c_1(M) \cap \sigma(M)\} \right] .$$

Since $\rho_2 \{c_1(M) \cap \sigma(M)\} = Sq_2 \sigma(M)$, it follows that

$$[\hat{\mu}_2 \partial [M, f]] = \beta_2 Sq_2 \rho_2 [j_* f_* \sigma(M)] \otimes [CP^1] . \qquad \text{Q. E. D.}$$

3. The structure of $U_*(B_{S^1 \cdot Z_2})$

In M. Fujii [4] and [5], it is proved that there exist homeomorphisms

$$D(m, n)/D(m-1, n) \approx S^m \wedge (CP^n)^+$$

where X^+ is the disjoint union of X and a point, and

$$D(k, n)/D(k, n-1) \approx S^n \wedge (RP^{n+k}/RP^{n-1})$$

where RP^n is the n-dimensional real projective space. Using the bordism group exact sequences for CW-pairs $(D(m, n), D(m-1, n))$ and $(D(k, n), D(k, n-1))$, it follows that

PROPOSITION 3. 1.

(i) *If* $i < m-1$ *then* $U_i(D(m, n)) \cong U_i(B_{S^1 \cdot Z_2})$.

(ii) *If* $i < n-1$ *then* $U_i(D(m, n)) \cong U_i(B_{S^1 \cdot Z_2})$.

Since there is a cross section r: $RP^m \to D(m, n)$ defined by $r([x]) = [x, [1, 0, \cdots, 0]]$ [4], we have immediately the following

LEMMA 3. 2. *The homomorphism* r_*: $U_*(RP^m) \to U_*(D(m, n))$ *is monomorphic and* r_*: $U_*(B_{Z_2}) \to U_*(B_{S^1 \cdot Z_2})$ *is monomorphic.*

We apply results in section 2 to the unitary bordism spectral sequence for $B_{S^1 \cdot Z_2}$.

LEMMA 3. 3. *If* $(C_i, D_j) \in E^3_{p,0} = H_p(B_{S^1 \cdot Z_2}; Z)$ *then*

$$d^3_{p,0}(C_i, D_j) = \begin{cases} (C_{i-1}, D_{j-1}) \otimes [CP^1] & \textit{if } i \textit{ is odd and } j \textit{ is even } (>0) \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. From Lemma 1.3 (i), we have

$$Sq^2 c^i d^j = \left\{ \binom{i}{2} + ij + \binom{j}{2} \right\} c^{i+2} d^j + j c^i d^{j+1}$$

Noting that Sq_2 is dual to Sq^2, we have

$$Sq_2(C_i, D_j)_2 = \left\{ \binom{i-2}{2} + (i-2)j + \binom{j}{2} \right\} (C_{i-2}, D_j)_2 + (j-1)(C_i, D_{j-1})_2$$

By Lemma 1.3 (ii) the lemma follows.

Then, we have a following table about E^3-term

$$
\begin{array}{llll}
(C_0, D_0) & & & \\
(C_1, D_0) & & & \\
(C_3, D_0) & & & \\
 & (C_0, D_1)\otimes[CP^1]\nwarrow & (C_0, D_2) & \\
(C_5, D_0) & \searrow & (C_1, D_2) & \\
 & (C_2, D_1)\otimes[CP^1]\nwarrow & & \\
(C_7, D_0) & \searrow & (C_3, D_2) & \\
 & (C_4, D_1)\otimes[CP^1]\nwarrow & & (C_0, D_3)\otimes[CP^1]\nwarrow \quad (C_0, D_4) \\
(C_9, D_0) & \searrow & (C_5, D_2) & \searrow (C_1, D_4) \\
 & (C_6, D_1)\otimes[CP^1)\nwarrow & & (C_2, D_3)\otimes[CP^1)\nwarrow \\
(C_{11}, D_0) & \searrow & (C_7, D_2) & \searrow (C_3, D_4)
\end{array}
$$

THEOREM 3. 4. *$U_{2n+1}(B_{S^1 \cdot Z_2})$ is isomorphic to $U_{2n+1}(B_{Z_2})$ and (C_{2i+1}, D_0) $i=0, 1, \cdots$ are permanent cycles in the spectral sequence of $B_{S^1 \cdot Z_2}$.*

PROOF. From Lemma 3.3, it follows that

$$d^3_{2k+1,0}(C_{2k+1}, D_0) = 0$$

and $\sum_{i+j:odd} E^5_{i,j}$ is the U_*-module generated by (C_{2k+1}, D_0) $k=0, 1, 2, \cdots$. Denote by $\{J_n\}$ the filtrtaion of $U_{2n+1}(B_{S^1 \cdot Z_2})$ such that

$$U_{2n+1}(B_{S^1 \cdot Z_2}) = J_{2n+1,0} \supset J_{2n-1,2} \supset \cdots \supset J_{1,2n} \text{ and } J_{i,j}/J_{i-1,j+1} \cong E^\infty_{i,j}.$$

Let $\pi(k)$ is the number of partitions of k. Since the order of $E^5_{2k+1,2(n-k)} = 2^{\pi(n-k)}$, it follows that

$$\text{order of } J_{2k+1,2(n-k)}/J_{2k-1,2(n-k)+2} \leqq 2^{\pi(n-k)} .$$

Therefore we have

$$\text{order of } U_{2n+1}(B_{S^1 \cdot Z_2}) \leqq 2^t$$

where $t = \sum_{k=0}^n \pi(k)$. On the other hand

$$\text{order of } U_{2n+1}(B_{Z_2}) = 2^t .$$

Hence, from Lemma 3.2 we have that r_*: $U_{2n+1}(B_{Z_2}) \to U_{2n+1}(B_{S^1 \cdot Z_2})$ is isomorphic. Moreover (C_{2k+1}, D_0) is the permanent cycle. Q. E. D.

PROPOSITION 3. 5. *$U_{4n+2\varepsilon}(B_{S^1 \cdot Z_2})$ has rank of $\sum_{k=0}^n \pi(2k+\varepsilon)$, $\varepsilon=0$ or 1.*

PROOF. There exists an isomorphism

$$U_n(B_{S^1 \cdot Z_2}) \otimes Q = \sum_{i+j=n} H_i(B_{S^1 \cdot Z_2}; Q) \otimes U_j .$$

From Proposition 1.1, we have the proposition. Q. E. D.

PROPOSITION 3. 6. *$d^r_{p,q} = 0$ for $r \geqq 5$.*

PROOF. $E^5_{odd} = \sum_{i+j:\, odd} E^5_{i,j}$ is generated by (C_{2i+1}, D_0) $i=0, 1, 2, \cdots$ and by

Theorem 3.4 the elements of E^5_{odd} are permanent cycles in this spectral sequence. Therefore, it follows that $d^r_{p,q}(r \geqq 5)$ is trivial. Q. E. D.

We define an action of Z_2 on $U_*(S^1) \cong U_{*-1}(B_{S^1})$ as follows: For a class $[S^1, M]$ of a S^1-manifold M with an action φ: $S^1 \times M \to M$ and t a generator of Z_2, we denote by $[S^1, M]^t$ the class of the S^1-manifold M with an action φ^t: $S^1 \times M \to M$ given by

$$\varphi^t(g, x) = \varphi(g^{-1}, x) .$$

Let $U_*(S^1)^{Z_2}$ be the invariant subgroup under the Z_2-action. Let i: $S^1 \to S^1 \cdot Z_2$ and j: $Z_2 \to S^1 \cdot Z_2$ be natural injections, which induce the maps $\tilde{i}$: $B_{S^1} \to B_{S^1 \cdot Z_2}$ and $\tilde{j}$: $B_{Z_2} \to B_{S^1 \cdot Z_2}$ respectively. Then we have the homomorphisms

$$i_*: U_*(S^1) \cong U_{*-1}(B_{S^1}) \xrightarrow{\tilde{i}_*} U_{*-1}(B_{S^1 \cdot Z_2}) \cong U_*(S^1 \cdot Z_2)$$

and

$$j_*: U_*(Z_2) \cong U_*(B_{Z_2}) \xrightarrow{\tilde{j}_*} U_*(B_{S^1 \cdot Z_2}) \cong U_{*+1}(S^1 \cdot Z_2) .$$

We have in analogy to [6] the following

THEOREM 3.7. *The homomorphism* Φ_n: $\tilde{U}_n(S^1)^{Z_2} \oplus \tilde{U}_{n-1}(Z_2) \to \tilde{U}_n(S^1 \cdot Z_2)$ given by $\Phi_n(x,y) = i_*(x) + j_*(y)$ *is monomorphic.*

From Theorem 3.4, we have that Φ_{2n} is isomorphic. However, by Proposition 3.6 and the table of E^3-term we can see that Φ_{2n+1} is not isomorphic.

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