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ON REGULARITY OF SOLUTIONS OF EQUATIONS OF STEADY-STATE HEAT CONDUCTION IN CONTIGUOUS MEDIA

By

Nobunori Ikebe

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Introduction. On regularity properties of the strong solutions of elliptic equations in contiguous domains D. P. Squier [8] has given several results. In his paper the boundary value problem with the equations whose principal part are Laplacian and with the boundary conditions whose coefficients are constant has been treated.

When, with his methods, one tries to extend his results to the general equations of steady-state heat conduction, one runs into difficulty. In this paper, using the methods of [5], [6], the author will show that the strong solutions of general equations of steady-state heat conduction in cintiguous media are differentiable up to order 2 and that the second derivatives of the solutions satisfy the Hölder condition up to the interface of contiguous media. The results are also extensible to elliptic boundary value problems. The notations of function-spaces and their norms used in this paper owe to [1], [2].

§1. The main theorem

Let $\sum_R^1$ be a semi-sphere $\{(x_1, x_2, x_3) : x_1^2+x_2^2+x_3^2<R^2, x_3>0\}$, $\sum_R^2$ be a semishere $\{(x_1, x_2, x_3) : x_1^2+x_2^2+x_3^2<R^2, x_3<0\}$ and σ_R be a flat boundary of $\sum_R^1$ or $\sum_R^2$, that is, $\sigma_R=\{x_1, x_2, 0) : x_1^2+x_2^2 \leqq R^2$. Let P be a point (x_1, x_2, x_3) in three dimensional Euclidean space. We consider the equations

$$L_k(P, D)u_k(P)=\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(a_{ij}^k(P)\frac{\partial u_k(P)}{\partial x_j}\right)+\sum_{i=1}^{3}b_i^k(P)\frac{\partial u_k(P)}{\partial x_i}+c_k(P)u_k(P)=f_k(P)$$

$$\text{in } \textstyle\sum_R^k,\ k=1,2. \tag{1°}$$

If we presume u_k as temperature, it may be natural to consider that the temperature and the flux are continuous, so that we set the following boundary conditions

$$u_1(P)-u_2(P)=0 \quad \text{on} \quad \sigma_R, \tag{2°}$$

$$\sum_{i=1}^{3} a_{3i}^1(P)\frac{\partial u_1(P)}{\partial x_i}-\sum_{i=1}^{3} a_{3i}^2(P)\frac{\partial u_2(P)}{\partial x_i}=0 \quad \text{on } \sigma_R. \tag{3°}$$

Asumption 1°. $\frac{\partial}{\partial x_i} a_{ij}^k(P),\ b_i^k(P),\ c^k(P),\ f_k(P) \in C^\alpha(\overline{\sum_R^k})$, $a_{3i}^k(P) \in C^{1+\alpha}(\sigma_R)$ for some α, $0<\alpha<1$, and $u_k \in H_{2,L_2}(\sum_R^k)$, $k=1, 2$, $i, j=1, 2, 3$.

Definition 1°. Let $L_k(P, \xi_1, \xi_2, \xi_3)=\sum_{i,j=1}^{3} a_{ij}^k(P)\xi_i\xi_j$, $k=1, 2$, $P \in \sigma_R$, and let τ_1, τ_2 be the roots (in τ) of $L_1(P, \xi_1, \xi_2, \tau)L_2(P, \xi_1, \xi_2, -\tau)=0$ with positive imaginary part (the existence of these roots is guaranteed by the following assumption). Moreover denote $(\tau-\tau_1)(\tau-\tau_2)$ by $M^+(P, \tau)$.

Assumption 2°. There exists a positive constant A such that

$$A^{-1}(\xi_1^2+\xi_2^2+\xi_3^2)\leq\left|\sum_{i,j}a_{ij}^k(P)\xi_i\xi_j\right|\leq A(\xi_1^2+\xi_2^2+\xi_3^2)$$

for any $P \in \overline{\sum_R^k}$ and for any real ξ_1, ξ_2, ξ_3, $k=1, 2$. And the vectors $(L_2(P, \xi_1, \xi_2, -\tau), -L_1(P, \xi_1\ \xi_2, \tau))$, $((a_{31}^1(P)\xi_1+a_{32}^1(P)\xi_2+a_{33}^1(P)\tau)L_2(P, \xi_1, \xi_2, -\tau), -(a_{31}^2(P)\xi_1+a_{32}^2(P)\xi_2-a_{33}^2(P)\tau)L_1(P, \xi_1, \xi_2, \tau))$ are linearly independent modulo $M^+(P, \tau)$ for any $P \in \sigma_R$.

The main purpose of this paper is to prove the following theorem and to show the related results.

Theorem 1°. Under the assumptions 1° and 2°, the strong solution u_k of the equations (1), (2), (3) belongs to $C^{2+\alpha}(\overline{\sum_{R'}^k})$, $k=1, 2$, for any R', $0<R'<R$.

2. Change to linear elliptic systems in a domain

To prove theorem 1°, we shall change the boundary value problem. By the change of the variable (x_1, x_2, x_3) to $(x_1, x_2, -x_3)$, the function f_2 and the coefficients of $L_2(P, D)$ become to the functions defined on $\sum_R^1$, and u_2 does so. From now on $\sum_R^1$ is denoted by $\sum_R$. Hence the equations(1°), (2°), (3°), the assumptions 1°, 2° and theorem 1° become to the following.

$$L_k(P, D)u_k=\sum_{i,j=1}^{3} a_{ij}^k(P)\frac{\partial^2 u_k}{\partial x_i\partial x_j}+\sum_{i=1}^{3} a_i^k(P)\frac{\partial u_k}{\partial x_i}+a_k(P)u_k=f_k(P)$$

$$\text{in } \textstyle\sum_R,\ k=1, 2. \tag{1}$$

$$u_1(P)-u_2(P)=0 \quad \text{on} \quad \sigma_R, \tag{2}$$

$$\sum_{i=1}^{3} a_{3i}^1(P)\frac{\partial u_1}{\partial x_i}+\sum_{i=1}^{3} a_{3i}^2(P)\frac{\partial u_2}{\partial x_i}=0 \quad \text{on} \quad \sigma_R. \tag{3}$$

For reasons of notational simplicity we have used the same letters $a_{ij}^2(P)$, $f_2(P)$ with those of (1).

Assumption 1[(1)]. $a_{ij}^k(P)$, $a_i^k(P)$, $a_k(P)$, $f_k(P) \in C^\alpha(\overline{\sum_R})$, $a_{3i}^k(P) \in C^{1+\alpha}(\sigma_R)$, $0<\alpha<1$, $i=1, 2, 3$. $u^k \in H_{2,L_2}(\sum_R)$, $k=1, 2$.

Definition 1. Let $L_k(P, \xi_1, \xi_2, \xi_3)=\sum_{i,j=1}^{3} a_{ij}^k(P)\xi_i\, \xi_j$, $P \in \sigma_R$, and let τ_1, τ_2 be the roots (in τ) of $L_1(P, \xi_1, \xi_2, \tau)L_2(P, \xi_1\, \xi_2, \tau)=0$(the existence of these roots is guaranteed by the following assumption). Moreover denote $(\tau-\tau_1)(\tau-\tau_2)$ by $M_+(P, \tau)$.

Assumption 2. There exists a positive constant A such that

$$A^{-1}(\xi_1^2+\xi_2^2+\xi_3^2)\leqq\Big|\sum_{i,j}a_{i,j}^k(P)\xi_i\xi_j\Big|\leqq A(\xi_1^2+\xi_2^2+\xi_3^2)$$

for any $P \in \overline{\sum_R}$ and for any real ξ_1, ξ_2, ξ_3, $k=1, 2$. And the vectors $(L_2(P, \xi_1, \xi_2, \tau), -L_1(P, \xi_1, \xi_2, \tau))$, $((a_{31}^1(P)\xi_1+a_{32}^1(P)\xi_2+a_{33}^1(P)\tau)L_2(P, \xi_1, \xi_2, \tau), (a_{31}^2(P)\xi_1+a_{32}^2(P)\xi_2+a_{33}^2(P)\tau)L_1(P, \xi_1, \xi_2, \tau))$ are linearly independent modulo $M^+(P, \tau)$ for any $P \in \sigma_R$.

Theorem 1. Under the assumption 1, 2 the strong solutions of the boundary value problem (1), (2), (3) belong to $C^{2+\alpha}(\sum_{R'})$ for any R', $0<R'<R$.

Since the proof of theorem 1 is somewhat long, we shall divide it into several sections.

§3. Integral estimates

Theorem 2. The boundary conditions (2), (3) satisfy Complementing Boundary Condition (see p. 42[2]).

Proof. By assumption 2 it is evident.

(1) Note that the assumption 1 is weaker than the assumption 1°. If we replace (2), (3) with

$$\sum_k b_k(p)u_k=\varphi_1(p) \tag{2$'$}$$

$$\sum_{i,k} b_{i,k}(p)\frac{\partial u^k}{\partial x^i}+\sum C_k(p)u_k=\varphi_2(p) \tag{3$'$}$$,

the smoothness assumption on the boundary condition becomes

$$b_k,\ \varphi_1 \in C^{2+\alpha}(\sigma_R),\ b_{ik},\ C_k,\ \varphi_2 \in C^{1+\alpha}(\sigma_R)$$

in place of $a_{3i}{}^k(p) \in C^{1+\alpha}(\sigma_R)$.

As it is convenient to denote $x_3=t$, we use both x_3 and t in view of circumstances. Without loss of generality we may assume that $u_k(p)$, $f_k(p)$ vanish near the curved boundary of $\sum_R$.

We write the operators (1) and the boundary operators (2), (3) as follows:

$$L_k(P, D)u_k=L_k'(P_0, D)u_k+(L_k'(P, D)-L_k'(P_0, D))u_k+L_k''(P, D)u_k, \tag{4}$$

$$B_1(D)u=u_1-u_2 \tag{5}$$

$$B_2(x, D)u=B_2(x_0, D)+(B_2(x, D)-B_2(x_0, D)u, \ (P=(x, t), \ P_0=(x_0, t_0)) \tag{6}$$

where $L_k'(P, D)$ is the principal part of $L_k(P, D)$, $L_k''(P, D)=L_k(P, D)-L_k'(P, D)$ and

$$B_2(x, D)u=\sum_{i=1}^{3} a_{3i}^1(x, 0)\frac{\partial u_1}{\partial x_i}+\sum_{i=1}^{3} a_{31}^2(x, 0)\frac{\partial u_2}{\partial x_i} \quad \text{on } \sigma_R.$$

Denote by $\Gamma_k(P_0, P-Q)$ the fundamental solution for the operator $L_k'(P_0, D)$ with singularity at $Q=P$. Let $K_{k1}(P_0, x-y, t)$ $K_{k2}(P_0, x-y, t)$ be the Poisson kernels (see [2]) for the operators $L_k'(P_0, D)$, $B_1(D)$, $B_2(x_0, D)$ with singularities at $y=x$, $t=0$ respectively. Existence of the Poisson kernels is guaranteed by theorem 2.

To estimate u_k we first give the integral representations of the second order derivatives of u_k.

For any functions $u_k \in C^\infty(\overline{\sum_R})$ which vanish near the curved boundary of $\sum_R$, applying Representation formula 6.7 (see corollary p 69[2]) we obtain

$$\begin{aligned} D^2u_j(x, t)=D^2\int &\Gamma_j(P_0, P-Q)[L_j'(P_0, D)u_j(Q)]_R dQ \\ &+\int D^2K_{j1}(P_0, x-y, t)(B_1(D)u(y, 0)-B_1(D)v(y, 0))dy \\ &+\int D^2K_{j2}(P_0, x-y, t)(B_2(x_0, D)u(y, 0)-B_2(x_0, D)v(y, 0))dy, \end{aligned} \tag{7}$$

$$j=1, 2,$$

where

$$P=(x_1, x_2, t), \ Q=(y_1, y_2, t), \ x=(x_1, x_2), \ y=(y_1, y_2),$$
$$v=(v_1, v_2),$$
$$v_j(P)=\int \Gamma_j(P_0, P-Q)[L_j'(P_0, D)u_j(Q)]_R dQ$$

and $[\ \ ]_R$ denote the extension of function $[\ \ ]$ by (4.4) in [1].

We shall first estimate

$$\int D_i D K_{j1}(P_0,\ x-y,\ t) f(y) dy$$

for $f(\in C^{2+\alpha}(\sigma_R))$(2) which vanishes near the boundary of σ_R, where D_i denotes the differentiation with respect to x_1 or x_2 and D denotes any differentiation.

Now

$$\int D_i D K_{j1}(P_0,\ x-y,\ t) f(y) dy = \int D_i D \mathop{\Delta}_{y}^{\frac{2+q}{2}} K_{j1,q}(P_0,\ x-y,\ t) f(y) dy$$

$$= \sum_{k=1}^{2} \int D_i D D_k \mathop{\Delta}_{y}^{\frac{q}{2}} K_{j1,q}(P,\ x-y,\ t) D_k f(y) dy,$$

here $K_{1,q}$ is seen in p. 56 in [2].

Let $K(P_0,\ x,\ t) = D D_k \mathop{\Delta}_{x}^{\frac{q}{2}} K_{j1,q}(P_0,\ x,\ t)$, $g(x) = D_k f(x)$,

$J_{P_0}(P) = D_i(K(P_0,\ x,\ t) * g(x))$, then there exists the function $w(x,\ t)$ such that

$$w(x,\ 0) = g(x),\ \|W\|_{1,L_q} < \infty .$$

Applying the methods used in p. 712 in [1], we obtain

$$-J_{P_0}(P) = \iint_{s>0} D_s w(y,\ s) D_{y_i} K(P_0,\ x-y,\ t+s)\, dyds$$

$$+ \iint_{s>0} D_{y_i} w(y,\ s) D_s K(P_0,\ x-y,\ t+s) dyds .$$

Then the following lemma holds.

Lemma 1. Put $P_0 = P$, then for the function $w \in H_{1,L_q}(\sum_R)$, we have the estimate

$$\| J_P(P) \|_{L_q} \leqq C \| W \|_{1,L_q},$$

Proof. For simplicity demote $D_{y_i} K(P,\ x-y,\ t+s)$ or $D_s K(p,\ x-y,\ t+s)$, by $G(P,\ x-y,\ t+s)$, and define a kernel $N(P,\ x-y,\ t+s)$ as follows:

$$N(P,\ x-y,\ t+s) = G(P,\ x-y,\ t+s) \quad \text{in} \quad t+s \geqq 0$$

and N is odd in $t+s<0$. From lemma 4.1 in p. 58 in [2] and from the definition of $K_{j1,q}$ in [2] it is easily seen that the kernel N satisfies the condition of Calderon-Zygmund theorem 2 in [3] for all $q>1$. Extend the function w to the whole space such that $w=0$ for $s<0$, and integrate over the whole space, then in virtue of the theorem validity of this lemma is evi-

(2) This assumption needs not in later.

dent.

For the term of K_{j2} the similar estimate is possible. The estimate on the term of Γ_j is also possible, or see pp. 720–721 in [1]. Hence if $D^2 = DD_i$ and $u_j \in H_{2,L_q}(\sum_R)$ for $q \geqq 2$, (7) is valid for $P = P_0$ and for u_j which vanishes near the curved boundary of $\sum_R$ (D^2 should not be operated for for P changed from P).

On the other hand for ths solutions u_k of (1), (2), (3)

$$L_k'(P_0, D)u_k(Q) = f_k(Q) - (L_k'(Q, D) - L_k'(P_0, D)u_k(Q) - L_k''(Q, D)u_k(Q),$$

$$B_1(D)u(Q) = 0 \quad \text{on} \quad \sigma_R,$$

$$B_2(x_0, D)u(Q) = -(B_2(y, D) - B_2(x_0, D))u(Q) \quad \text{on} \quad \sigma_R.$$

Put the above into (7), and put $P_0 = P$, then the following theorem holds.

Theorem 3. Under the assumptions 1, 2, for the solutions $u_k \in H_{2,L_q}(\sum_R)$, $q \geqq 2$, the following representations are possible.

$$\begin{aligned} D_iDu_j(x, t) = D_iD \int \Gamma_j(P, P-Q)\,[f_j(Q) - (L_j'(Q, D) - L_j'(P, D))u_j(Q) \\ - L_j''(Q, D)u_j(Q)]_R dQ \\ + \int D_iDK_{j1}(P, x-y, t)\{-B_1(D)v(y, 0)\}dy \\ + \int D_iDK_{j2}(p, x-y, t)\{-(B_2(y, D) - B_2(x, D))u(y, 0) \\ - B_2(x, D)v(y, 0))\}dy. \end{aligned} \tag{8}$$

Here DD_i should not be operated for P changed from P_0 and the integrals with Poisson kernels are in the sence of the integral appearing in the proof of lemma 1.

Let us estimate (8).

In virture of the assumptions there exist constants C, C' such that

$$|(L_k'(Q, D) - L_k'(P, D))u_k(Q)| \leqq C|P-Q|^\alpha |D^2u_k(Q)|$$

$$\left|D\left\{(a_{3i}^k(y, 0) - a_{3i}^k(x, 0))\frac{\partial u_k(y, s)}{\partial y_i}\right\}\right| \leqq C|x-y|^\alpha |D^2u_k(y, s)| + C'|Du_k(y, s)|.$$

Note that $D^2\Gamma_j(P, P-Q)$, $N(x, t, x-y, t+s)$ have the homogeneity of order -3 and that they satisfy the conditions of Calderon-Zygmund theorem. From the above inequalities and the Sobolev's theorem 2 (see p. 43 [7], convolutions in (8) with the left hand side of the above inequalities belong to $L_{\frac{3q}{3-q\alpha}}$. Those with L_k'' and with f_k belong to $L_{\frac{3q}{3-q}}$ and L_p (for any $p \geqq 2$), respectively, in virture of Sobolev's theoorem 2 and the theorem (see p. 69[7]). From the above estimates and Calderon-Zygmund theorem, those with v belong to $L_{\frac{3q}{3-q\alpha}}$. Thus we conclude that $DD_iu_k \in L_{\frac{3q}{3-q\alpha}}$. By ellipticity D^2u_k so.

Begin with $q=2$ and repeat the above argument, then we find that $D^2u_k \in L_q$ for any $q \geqq 2$.

By (11.15) in [7] we conclude that $u_k \in C^{2-\varepsilon}(\overline{\sum_R})$ for any $\varepsilon>0$, after correction on the set of measure zero.

We summarize the above results as follows:

Theorem 4. Assume the assumptions of theorem 1 except for f_k, and assume that $f_k \in L_p(\sum_R)$ for all $p\geqq 2$, then u_k belong to $C^{2-\varepsilon}(\overline{\sum_{R'}})$ for any R', $\varepsilon>0$, $0<R'<R$, $k=1, 2$. If u_k and f_k vanish near the curved boundary of $\sum_R$, $\sum_{R'}$ may be replaced by $\sum_R$.

Furthermore if $B_j(P, D)u=\phi_j(P)$, $j=1, 2$, $\phi_j \in C^{2-j}(\sigma_R)$, $D^{2-j}\phi_j \in H_{1-\frac{1}{p}, L_p}$ for any $p\geqq 2$, then $u_k \in C^{2-\varepsilon}(\overline{\sum_{R'}})$. If ϕ_j vanishes near the boundary of σ_R, the second argument is true in this case.

The proof of the last half of this theorem is obtained similarly to that of the first half.

§4. The Hölder condition

We first show the differentiability of the solution u_k, and second show the Hölder condition of D^2u_k. Choose some α', α'', $0<\alpha'<\alpha''<\alpha$. For any vector $h=(h_1, h_2)$ with sufficiently small $|h|\neq 0$, put

$$U_k(x, t)=\frac{u_k(x+h, t)-u_k(x, t)}{|h|^{\alpha''}},$$

here and hereafter u_k denotes the solution of the equations (1), (2), (3). Then the equations to be satisfied by $U_k(x, t)$ are given in the form

$$\sum_{i,j} a_{ij}^k(x+h, t)\frac{\partial^2 U_k}{\partial x_i \partial x_j}=g_k(x, t) \quad k=1, 2 \tag{1''}$$

$$U_1(x, 0)-U_2(x, 0)=0 \tag{2''}$$

$$\sum_i a_{3i}^1(x+h, 0)\frac{\partial U_1}{\partial x_i}+\sum_i a_{3i}^2(x+h, 0)\frac{\partial U_2}{\partial x_i}=\chi(x) \tag{3''}$$

Here for some R', $0<R'<R(|R-R'|$ is sufficiently small), the coefficients of (1)$''$ $\in C_\alpha(\overline{\sum_{R'}})$, those of (3)$''$ $\in C^{1+\alpha}(\sigma_{R'})$, $g_k \in L_q(\sum_{R'})$, $\chi \in H_{1-\frac{1}{q}, Lq}$ for any

$q \geqq 2$, and the norms corresponding to the function-spaces are independent of h (this is seen from theorem 4). Hence we can apply theorem 4 to U_k and obtain

$$|U_k|_{2-\varepsilon} \leqq C \text{ (independent of } h).$$

From the assumption that u_k vanishes near the curved boundary of $\sum_R$, we may consider that the norm of the above inequality is given for $\overline{\sum_R}$. Combining this with the following lemma we find that $u_k \in C^2(\overline{\sum_R})$ and $D^2 u_k$ satisfies the Hölder condition with exponent α' with respect to x uniformly for t.

Lemma 2. Let $u(x)$ be a continuous function of one variable with compact support. For any h, $k \neq 0$ and for some α, β, $0<\alpha$, $\beta<1$, $\alpha+\beta>1$, we assume

$$\frac{|u(x+h+k)-u(x+k)-(u(x+h)-u(x))|}{|h|^\alpha |k|^\beta} \leqq K,$$

then u is defferentiable and $|u|_{1+(\alpha+\beta-1)}$ is bounded by the constant depending only on K, α, β, $[u]_0$ and the diameter of the support.

The proof is seen in the appendix in [5].

We shall show u_k to satisfy the Hölder condition with respect to t. For this purpose, choose R', $0<R'<R$, such that u_k vanishes near the curved boundary of $\sum_{R'}$. Cover $\sum_R$ by a finite number of small semi-spheres, lying $\sum_R$ with radius ρ and with center $P_j \in \sigma_R$, which we denote by $\sum_{\rho, p_j}$, and an interior domain of $\sum_R$. Furthermore we assume $\sum_{2\rho, p_j} \subset \sum_R$. Let $\xi(P)$ be a C^∞ function such that $\xi(P)=1$ for $P \in \sum_{\rho, q_j}$ and $\xi(P) \equiv 0$ for $P \in \sum_{2\rho, p_j}$. Making $u_k(P)\xi(P)$, we shall show the desired Hölder condition for this function. Since for the interior domain the improved Schauder estimates (theorem 10.7 in p. 82 in [2]) holds, it is sufficient to prove the following theorem.

Theorem 5. Let u_k be a solution of (1). Assume that the coefficients of (1) and $f_k \in C^{\alpha'}(\overline{\sum_R})$ for sufficiently small R. Furthermore assume that u_k is twice differentiable, u_k, f_k vanish near the curved boundary of $\sum_R$, $D^2 u_k$ belongs to $C(\overline{\sum_R})$ and $D^2 u_k$ satisfies the Hölder condition with exponent α' with respect to x uniformly for t, and assume that $B_1(0, D)$ and $B_2(0, D)$ satisfy Complenting Boundary Condition (see theorem 2), then $u_k \in C^{2+\alpha'}(\sum_R)$, $k=1, 2$.

Proof. By the use of mollifier $j(r)$ (for instance, see p. 637 [1] or p. 13 [6]) we make

$$h_\varepsilon(P)=h_\varepsilon(x,\ t)=\varepsilon^{-1}\int_0^\infty j\left(\frac{t+\varepsilon-s}{\varepsilon}\right)h(x,\ s)ds\,.$$

The equations to be satisfied by $u_{k\varepsilon}(P)$ are

$$\sum_{i,j}a_{ij}^k\frac{\partial^2 u_{k\varepsilon}}{\partial x_i\partial x_j}+\sum_i a_i^k(x,\ t)\frac{\partial u_{k\varepsilon}}{\partial x_i}+a_k(x,\ t)u_{k\varepsilon}$$

$$=f_{k\varepsilon}(x,\ t)+R_k(\varepsilon,\ x,\ t).$$

Let us rewrite these in the form

$$\sum a_{ij}^k(x,\ t)\frac{\partial^2 u_k}{\partial x_i\partial x_j}=f_{k\varepsilon}'(x,\ t)+R_k(\varepsilon,\ x,\ t).$$

By the assumption it is easily seen that $f_{k\varepsilon}'(x,\ t)\in C^{\alpha'}(\sum_R)$, the norm is independent of ε, $D^2u_{k\varepsilon}$ satisfies the Hölder condition with exponent α' with respect to x independent of ε, t, and that for sufficiently small $h=(h_1,\ h_2)$

$$|R_k(\varepsilon,\ x+h,\ t)-R_k(\varepsilon,\ x,\ t)|\leqq C|h|^{\alpha'},$$

C being independent of ε, t.
Furthermore

$$|R_k(\varepsilon,\ x,\ t+h)-R_k(\varepsilon,\ x,\ t)|\leqq C'|h|^{\alpha'},$$

where C' independent of h, but depend on $|u_k|_2$ (this is seen similarly to [1] p. 723)

Now, from (7)

$$D^2u_{j\varepsilon}(x,\ t)=D^2\int\Gamma_j(0,\ P-Q)[f_{j\varepsilon}'(Q)-(L_j'(Q,\ D)-L_j'(0,\ D))u_{j\varepsilon}(Q)]_R\ Q$$

$$+\int D^2K_{j1}(0,\ x-y,\ t)\{B_1(D)u_\varepsilon(y,\ 0)-B_1(D)v(y,\ 0)\}dy \tag{9}$$

$$+\int D^2K_{j2}(0,\ x-y,\ t)\{B_2(0,\ D)u_\varepsilon(y,\ 0)-B_2(0,\ D)v(y,\ 0)\}dy\,.$$

Denote the above quantities by $g_1(P)$, $g_2(P)$, $g_3(P)$, respectively.

For g_1 apply the argument used in [3], then for sufficiently small λ which depends on R, the following estimate holds

$$[g_1(P)]_{\alpha'}\leqq\lambda([D^2u_{1\varepsilon}]_{\alpha'}+[D^2u_{2\varepsilon}]_{\alpha'})+C_1\,, \tag{10}$$

where C_1 is independent of ε but depends on λ and the norm $|u_k|_2$.

For the terms of g_2, g_3 integrated with u_ε, considering that $D^2u_{k\varepsilon}$ has the uniform Hölder condition with respect to x, we also apply theorem 5.1 (a) in [2].

For those of g_2, g_3 integrated with v, we also apply the theorem and use the estimate (10), then we have the estimate

$$[g_1]_{\alpha'} \leqq C_2 \lambda([D^2 u_{1\varepsilon}]_{\alpha'} + [D^2 u_{2\varepsilon}]_{\alpha'}) + C_3, \tag{11}$$

where the constants are independent of ε but depend on $|u_k|_2$ and the Hölder condition of $D^2 u_k$ with respect to x, C_2 is independent of λ but C_3 depends on λ. Combing (10) with (11) we have the estimate

$$[D^2 u_{k\varepsilon}]_{\alpha'} \leqq C_4,$$

where C_4 is independent of ε but depends on the Hölder condition of $D^2 u_k$ with respect to x. Letting $\varepsilon \to 0$, the proof is complete.

To complete the proof of theorem 1, it is sufficient to show that $D^2 u_k$ satisfies the uniform Hölder condition with exponont α with respect to x. When it is done, by the use of theorem 5, the proof will be complete.

To this end, put $\dfrac{u_k(x+h,\ t) - u_k(x,\ t)}{|h|^{\alpha-\alpha'}} = U_k(x,\ t)$ The equations to be satisfied by $U_k(x,\ t)$ are in the form (1)″, (2)″, (3)″ in which α'' is replaced by $\alpha - \alpha'$. The coefficients of (1)″ $\in C^{\alpha}(\overline{\sum_R})$, those of (3)″ $\in C^{1+\alpha}(\sigma_R)$, $g_k \in C^{\alpha'}(\overline{\sum_R})$, $\chi \in C^{1+\alpha'}(\sigma_R)$ and the norms corresponding to the function-spaces are independent of h in virtue of theorem 5. Applying the Schauder estimates (theorem 9.1[2]), we obtain

$$|U_k|_{2+\alpha'} \leqq C, \quad k=1,\ 2.$$

C being independent of h. Combining these estimates with lemma 3 we have the desired Hölder condition.

Lemma 3. Assume the all assumptions of lemma 2 except $\alpha+\beta>1$ and assume $\alpha+\beta<1$ then $u \in C^{\alpha+\beta}$ and $[u]_{\alpha+\beta}$ is bounded by a constant depending only on K, α, β.

The proof is seen in [6]. This completes the proof of theorem 1.

§5. Extension

All the theorem except theorem 2 are extensible for the boundary value problem with boundary condition (2), (3), if transformed conditions similar to §1 satisfy Complementing Boundary Condition and the smoothness assumption (see p. 37). Moreover under the appropriate assumption they so for an arbitrary dimension ($\geqq 2$) and for the case in which the order of L_k is larger than 2. Finally for all the case mentioned above we can give the improvement of Schauder estimate near the boundary as obtained in [6].

§6. Equations of steady-state heat conduction in the discontiguous isotropic media

It is interesting to consider the case of this title. In this case the arguments are somewhat simpler than those of the general case. We shall start from §2.

We consider the problem

$$\Delta u_j=\sum_{i=1}^{3} a_i^j(P)\frac{\partial u_j}{\partial x_i}+c_j(P)u_j+f_j(P) \quad \text{in } \textstyle\sum_R .$$

$$u_1-u_2=0 \quad \text{on} \quad \sigma_R ,$$

$$k_1(P)\frac{\partial u_1}{\partial t}+k_2(P)\frac{\partial u_2}{\partial t}=0 \quad \text{on} \quad \sigma_R .$$

we assume all the assumptions except assumption 2 and smoothness assumption on a^k_j, and assume that $|k_i(P)|\geqq C>0$ for any $P\in\overline{\sum}_R$, $k_1(P)+k_2(P)\neq 0$ for σ_R and $k_1(P)$, $k_2(P)\in C^{1+\alpha}(\sigma_R)$.

Let $\Gamma(P-Q)$ be a fundamental solution for Δ, $K_0(x-y,\ t)$ be a Poisson kernel for Δ and for the boundary operator 1 and $K_1(x-y,\ t)$ be that for Δ and for the boundary operator $\partial/\partial t$.

Then

$$D^2u_j(x,\ t)=D^2\int\Gamma(P-Q)\,[f_j(Q)+\sum_{i=1} a_i^j(Q)D_iu_j(Q)+c_j(Q)u_j(Q)]_R\,dQ$$

$$+\int D^2K_0(x-y,\ t)(u_j(y,\ 0)-v_j(y,\ 0))dy\,, \tag{12}$$

$D^2u_j(x,\ t)=$The first term on the right of (12)

$$+\int D^2K_1(x-y,\ t)\left(\frac{\partial}{\partial t}u_j(y,\ 0)-\frac{\partial}{\partial t}v_j(y,\ 0)\right)dy\,, \tag{13}$$

where

$$v_j(P)=\int\Gamma(P-Q)\,[f_j(Q)+\sum_{i=1}^{3} a_i^j(Q)D_iu_j(Q)+c_j(Q)u_j(Q)]_R\,dQ\,.$$

By the use of (12) and of (13), estimate $D^2u_1-D^2u_2$ and $k_1(x)D^2u_1(P)-k_2(x)D^2u_2(P)$, respectively, then theorem 1 in this case holds

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