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Existence of the admissible multiplication in η^2-coefficient cohomology theories

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1. Introduction

Let $\tilde{h}$ be a reduced cohomology (defined on the category of finite *CW*-complexes) and be equipped with a multiplication μ, i.e., μ is a map: $\tilde{h}^i(X) \otimes \tilde{h}^j(Y) \to \tilde{h}^{i+j}(X \wedge Y)$ for all i and j, which is i) linear, ii) natural, iii) has a bilateral unit $1 \in \tilde{h}^0(S^0)$ and iv) is compatible with suspension σ in $\tilde{h}$ in the sense that $\sigma\mu(x \otimes y) = (1 \wedge T)^*\mu(\sigma x \otimes y) = (-1)^i \mu(x \otimes \sigma y)$ for deg $x = i$, where T is a map switching factors $T = T(Y, S^1)$. If the map μ is commutative (in the general sense), or associative, we say that multiplication μ is commutative, or associative.

Let η be a stable class of the Hopf map $S^3 \to S^2$ and $\eta^2 = \eta \cdot (S\eta)$ be a generator of stable homotopy group $\{S^{n+2}, S^n\}$. The (reduced) *η^2-coefficient cohomology* $\tilde{h}(\ ;\eta^2)$ of a cohomology theory $\tilde{h}$ is defined by

$$\tilde{h}^i(X;\eta^2) = \tilde{h}^{i+5}(X \wedge Q) \text{ for all } i,$$

where $Q = S^2 \cup_{\eta^2} e^5$. And the *suspension* isomorphism $\sigma_{\eta^2}: \tilde{h}^i(X;\eta^2) \to \tilde{h}^{i+1}(SX;\eta^2)$ is defined as the composition $\sigma_{\eta^2} = (1_X \wedge T)^*\sigma$, where $T = T(S^1, Q)$. Let i: $S^2 \to Q$, $\pi: Q \to S^5$ be the inclusion and the map collapsing S^2. We define

$$\rho_{\eta^2}: \tilde{h}^i(X) \to \tilde{h}^i(X;\eta^2), \textit{ reduction mod } \eta^2,$$
$$\delta: \tilde{h}^i(X;\eta^2) \to \tilde{h}^{i+3}(X), \textit{ Bockstein homomorphism},$$

respectively by $\rho_{\eta^2} = (-1)^i(1 \wedge \pi)^*\sigma^5$, $\delta = (-1)^i\sigma^{-2}(1 \wedge i)^*$. We call $\delta_{\eta^2} = \rho_{\eta^2} \cdot \delta$ the mod η^2 *Bockstein homomorphism.*

The multiplication μ induces the following multiplications

$$\mu_R: \tilde{h}(\ ;\eta^2) \otimes \tilde{h} \to \tilde{h}(\ ;\eta^2),$$
$$\mu_L: \tilde{h} \otimes \tilde{h}(\ ;\eta^2) \to \tilde{h}(\ ;\eta^2)$$

in a natural way (c. f. [**3**]). A multiplication μ_{η^2} in $\tilde{h}(\ ;\eta^2)$ is said to be *admissible* if it satisfies

(A_1) $\mu_R=\mu_{\eta 2}(1\otimes\rho_{\eta 2})$, $\mu_L=\mu_{\eta 2}(\rho_{\eta 2}\otimes 1)$,

(A_2) $\delta_{\eta 2}\mu_{\eta 2}(x\otimes y)=\mu_L(\delta x\otimes y)+(-1)^i\mu_R(x\otimes\delta y)$ for deg $x=i$ and

(A_3) if x or y, or z, is $\rho_{\eta 2}$-images, then

$$\mu_{\eta 2}(\mu_{\eta 2}(x\otimes y)\otimes z)=\mu_{\eta 2}(x\otimes\mu_{\eta 2}(y\otimes z)).$$

In this note we discuss the admissible multiplications in the η^2-coefficient cohomology theories.

LEMMA 1. (H. Toda [5] Lemma 3.5) $1_Q\wedge\eta^2=0$ *in the stable homotopy group* $\{S^4Q, S^2Q\}$.

Then we have homotopy equivarence $Q\wedge Q\sim S^2Q\vee S^5Q$ in the stable range. We obtain

PROPOSITION 2. *Let* ν *be a generator of* $\{S^{n+3}, S^n\}$ *and* G *be a subgroup of* $\{S^5Q, S^5Q\}$ *generated by* $S^5(i\nu\pi)$ *then there exist an element* $\gamma\in\{S^5Q, Q\wedge Q\}$ *satisfying the relations*

i) $-(1_Q\wedge\pi)\gamma=(1_Q\wedge\pi)T\gamma=1_{S5Q}$ *in* $\{S^5Q, S^5Q\}$ mod G

and

ii) $T(1_Q\wedge i)=1_Q\wedge i-\gamma(S^5i)(S^2\pi)$ *in* $\{S^2Q, Q\wedge Q\}$, *where* T *is a map switching factors* $T=T(Q, Q)$.

Making use of γ we define a map $\mu_{\eta 2}$ as the composition;

$$\mu_{\eta 2}=(-1)^i\sigma^{-5}(1\wedge\gamma)^*(1\wedge T\wedge 1)^*\mu:$$

$$\begin{aligned}\tilde{h}^i(X;\eta^2)\otimes\tilde{h}^j(Y;\eta^2)=\tilde{h}^{i+5}(X\wedge Q)&\otimes\tilde{h}^{j+5}(Y\wedge Q)\\ &\xrightarrow{\mu}\tilde{h}^{i+j+10}(X\wedge Q\wedge Y\wedge Q)\\ &\xrightarrow{(1\wedge T\wedge 1)^*}\tilde{h}^{i+j+10}(X\wedge Y\wedge Q\wedge Q)\\ &\xrightarrow{(1\wedge\gamma)^*}\tilde{h}^{i+j+10}(X\wedge Y\wedge S^5Q)\\ &\xrightarrow{(-1)^i\sigma^{-5}}\tilde{h}^{i+j+5}(X\wedge Y\wedge Q)=\tilde{h}^{i+j}(X\wedge Y;\eta^2).\end{aligned}$$

Then we obtain

THEOREM 3. *If* $(1_X\wedge i\nu\pi)^*=0$ *in* $\tilde{h}^*$ *for any* X *and if* $\tilde{h}$ *is equipped with an associative multiplication, then there exists an admissible multiplication* $\mu_{\eta 2}$ *in* $\tilde{h}(\ ;\eta^2)$.

Throughout this note we use the same notations as [1] and [3].

2. Stable homotopy groups of some elementary complexes

New we compute some stable homotopy groups for the proof of Proposition 2. Let $Q=S^2\cup_{\eta 2}C(S^4)$. We have a cofiblation

$$S^2\xrightarrow{i}Q\xrightarrow{\pi}S^5,$$

where i is the inclution and π is the map collapsing S^2 to a point.

From the Puppe's exact sequence associated with above cofiblation, we obtain

(2.1) *the groups* $\{S^{n+i}, S^nQ\}$ *and* $\{S^nQ, S^{n+7-i}\}$ $(i\leq 8)$ *are both isomorphic to the corresponding groups in the following table*;

	$i\leq 1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
$\{S^{n+i}, S^nQ\}\simeq$ $\{S^nQ, S^{n+7-i}\}\simeq$	0	Z	Z_2	0	$Z+Z_{12}$	0	Z_2	Z_2+Z_{24}
generators of $\{S^{n+i}, S^nQ\}$		i	$i\eta$		$\xi,\ i\nu$		$\tilde{\eta}^2$	$i\nu^2,\ \tilde{\nu}$
generators of $\{S^nQ, S^{n+7-i}\}$		π	$\eta\pi$		$\zeta,\ \nu\pi$		$\bar{\eta}^2$	$\nu^2\pi,\ \bar{\nu}$

where ξ, ζ, $\tilde{\eta}^2$, $\bar{\eta}^2$, $\tilde{\nu}$, *and* $\bar{\nu}$ *are elements satisfying*

$$\pi\xi=2\cdot 1_{S^5},\ \zeta i=2\cdot 1_{S^2},\ \pi\tilde{\eta}^2=\eta^2,\ \bar{\eta}^2 i=\eta^2,\ \pi\tilde{\nu}=\nu,\ \bar{\nu}i=\nu,$$

and we have relations $\xi\eta=0$, $\eta\zeta=0$.

(2.2) *The groups* $\{S^nQ, S^{n+i}Q\}$ $(i\geq -2)$ *are as follows*;

	$i\geq 4$	$i=3$	$i=2$	$i=1$	$i=0$	$i=-1$	$i=-2$
$\{S^nQ, S^{n+i}Q\}\simeq$	0	Z	Z_2	0	$Z+Z+Z_{12}$	Z_2	Z_2
generators		$(S^3i)\pi$	$(S^2i)\eta\pi$		1_Q, $\xi\pi$ *or* $i\zeta$, $i\nu\pi$	$1_Q\wedge\eta$	$\tilde{\eta}^2\pi=i\bar{\eta}^2$

and we have relation $i\zeta+\xi\pi=2\cdot 1_Q$.

From Lemma 1 (Toda [**5**] Lemma 3.5) we have a homotopy equivarence

$$Q\wedge Q\sim S^2Q\vee S^5Q=N$$

in the stable range. Hereafter, we use the following notations:

$i_0: S^2Q\to N$, $i_1: S^5Q\to N$ the inclusions,

$\pi_0: N\to S^5Q$, $\pi_1: N\to S^2Q$ the map collapsing S^2Q or S^5Q

and these mappings will be fixed so as to satisfy the relations:

$$\pi_1 i_0=1_{S^2Q},\ \pi_0 i_1=1_{S^5Q}.$$

(2.3) *There exists an element* α *of* $\{N, Q\wedge Q\}$ *satisfying the following three relations*:

i) α *is a homotopy equivarence; i. e., there is a inverse* $\beta\in\{Q\wedge Q, N\}$ *of* α *such that* $\alpha\beta=1$ *and* $\beta\alpha=1$,

ii) $\alpha i_0=1_Q\wedge i$ *thus* $\beta(1_Q\wedge i)=i_0$,

iii) $(1_Q\wedge\pi)\alpha=\pi_0$ *thus* $\pi_0\beta=1_Q\wedge\pi$.

Put $\alpha_0=\alpha i_1(S^5 i)\in\{S^7, Q\wedge Q\}$ and $\beta_0=(S^2\pi)\pi_1\beta\in\{Q\wedge Q, S^7\}$. It follows from ii), iii) of (2.3) that

$$(1_Q\wedge\pi)\alpha_0=S^5 i \quad\text{and}\quad \beta_0(1_Q\wedge i)=S^2\pi.$$

For any CW-complex W, we have the short exact sequences

$$0\to\{W, S^2Q\}\xrightarrow{(1_Q\wedge i)_*}\{W, Q\wedge Q\}\xrightarrow{(1_Q\wedge\pi)_*}\{W, S^5Q\}\to 0,$$

$$0\to\{S^5Q, W\}\xrightarrow{(1_Q\wedge\pi)^*}\{Q\wedge Q, W\}\xrightarrow{(1_Q\wedge i)^*}\{S^2Q, W\}\to 0,$$

associated with the cofiberation

$$S^2Q\xrightarrow{1_Q\wedge i}Q\wedge Q\xrightarrow{1_Q\wedge\pi}S^5Q$$

since $(1_Q\wedge\eta^2)_*$ and $(1_Q\wedge\eta^2)^*$ are trivial.

From (2.1), (2.2) and the above short exact sequence, we obtain

(2.4) *the groups* $\{S^i, Q\wedge Q\}$ *and* $\{Q\wedge Q, S^{14-i}\}$ *are both isomorphic to the corresponding groups in the following table*;

	$i\leq 3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$\{S^i, Q\wedge Q\}\simeq$ $\{Q\wedge Q, S^{14-i}\}\simeq$	0	Z	Z_2	0	$Z+Z+Z_{12}$	Z_2	Z_2
generators of $\{S^i, Q\wedge Q\}$		$i\wedge i$	$i\eta\wedge i$		$\alpha_0,\ \xi\wedge i,\ i\nu\wedge i$	$\widetilde{i\eta}$	$\tilde{\eta}^2\wedge i$
generators of $\{Q\wedge Q, S^{14-i}\}$		$\pi\wedge\pi$	$\eta\pi\wedge\pi$		$\beta_0,\ \zeta\wedge\pi,\ \nu\pi\wedge\pi$	$\overline{\eta\pi}$	$\bar{\eta}^2\wedge\pi$

where $\widetilde{i\eta}$ *and* $\overline{\eta\pi}$ *are elements satisfying*

$$(1_Q\wedge\pi)\widetilde{i\eta}=i\eta,\quad \overline{\eta\pi}(1_Q\wedge i)=\eta\pi,$$

and we have relation $\widetilde{i\eta}\eta^2=\eta^2\overline{\eta\pi}=0$.

(2.5) *The groups* $\{S^iQ, Q\wedge Q\}$ *are as follows*:

		generators
$\{S^iQ, Q\wedge Q\}(i\leq -2)\simeq 0$	;	
$\{S^{-1}Q, Q\wedge Q\}\simeq Z$	;	$(i\wedge i)\pi$;

$\{Q, Q\wedge Q\}\simeq Z_2$; $(i\wedge i)\eta\pi$;
$\{S^1Q, Q\wedge Q\}\simeq 0$;
$\{S^2Q, Q\wedge Q\}\simeq Z+Z+Z+Z_{12}$; $1_Q\wedge i$, $\alpha_0(S^2\pi)$, $\xi\pi\wedge i$, $i\nu\pi\wedge i$;
$\{S^3Q, Q\wedge Q\}\simeq Z_2+Z_2$; $1_Q\wedge i\eta$, $\widetilde{i\eta}(S^3\pi)$;
$\{S^4Q, Q\wedge Q\}\simeq Z_2$; $\widetilde{i\eta^2}\wedge i=\tilde{\eta}^2\pi\wedge i$.

3. Proof of Proposition 2

First we consider the ordinary homology maps induced by the elements of $\{S^7, Q\wedge Q\}$ and $\{S^2Q, Q\wedge Q\}$. Let s_7, $\begin{pmatrix}\sigma^ie_2\\ \sigma^ie_5\end{pmatrix}$ and $\begin{pmatrix} & e_2\wedge e_2 & \\ e_5\wedge e_2, & & e_2\wedge e_5\\ & e_5\wedge e_5 & \end{pmatrix}$ be the generators of the groups $\tilde{H}_7(S^7)$, $\tilde{H}_*(S^iQ)$ and $\tilde{H}_*(Q\wedge Q)$ respectively, where σ^ie_j is a generator of $\tilde{H}_{i+j}(S^iQ)$, $e_i\wedge e_j$ is that of $\tilde{H}_{i+j}(Q\wedge Q)$, $e_5\wedge e_2$ is represented by 7-cell of S^2Q if we put $Q\wedge Q=S^2Q\underset{1\wedge\eta 2}{\cup}C(S^4Q)$ and $e_2\wedge e_5$ is the other 7-dim. generator. The element $f\in\{S^7, Q\wedge Q\}$ is called to be of type (k, l) if the induced homology map is $f_*(s_7)=k(e_5\wedge e_2)+l(e_2\wedge e_5)$ for some integers k and l. From the relations $(\pi\wedge 1_Q)(\xi\wedge i)=2\cdot(1_Q\wedge i)$, $(1_Q\wedge\pi)(\xi\wedge i)=0$ and $(1_Q\wedge\pi)\alpha_0=S^5i$, the maps $\xi\wedge i$, α_0, $i\nu\wedge i$ are of type (2, 0), (n, 1) and (0, 0) respectively. Since $T\alpha_0$ is of type (1, n), n is odd. We put $n=2m-1$. The generator of $\{S^2Q, Q\wedge Q\}$ induce the homology maps $\tilde{H}_*(S^2Q)\to\tilde{H}_*(Q\wedge Q)$;

$$(1_Q\wedge i)_*:\begin{pmatrix}\sigma^2e_2\\ \sigma^2e_5\end{pmatrix}\longmapsto\begin{pmatrix} e_2\wedge e_2\\ e_5\wedge e_2,\ 0\end{pmatrix},$$

$$(\alpha_0(S^2\pi))_*:\begin{pmatrix}\sigma^2e_2\\ \sigma^2e_5\end{pmatrix}\longmapsto\begin{pmatrix} 0\\ (2m-1)(e_5\wedge e_2),\ e_2\wedge e_5\end{pmatrix},$$

$$(\xi\pi\wedge i)_*:\begin{pmatrix}\sigma^2e_2\\ \sigma^2e_5\end{pmatrix}\longmapsto\begin{pmatrix} 0\\ 2(e_5\wedge e_2),\ 0\end{pmatrix},$$

$$(i\nu\pi\wedge i)_*:\begin{pmatrix}\sigma^2e_2\\ \sigma^2e_5\end{pmatrix}\longmapsto\begin{pmatrix} 0\\ 0,\ 0\end{pmatrix}.$$

By (2.5), any element $f\in\{S^2Q, Q\wedge Q\}$ can be expresssed as

$$f=a(1_Q\wedge i)+b\alpha_0(S^2\pi)+c(\xi\pi\wedge i)+d(i\nu\pi\wedge i)$$

for some integers a, b, c and d. Then

$$f_*:\begin{pmatrix}\sigma^2e_2\\ \sigma^2e_5\end{pmatrix}\longmapsto\begin{pmatrix} a(e_2\wedge e_2)\\ (a+(2m-1)b+2c)(e_5\wedge e_2),\ b(e_2\wedge e_5)\end{pmatrix}.$$

Let G' be a subgroup of $\{S^2Q,\ Q\wedge Q\}$ generated by $i\nu\pi\wedge i$, then the element $f \in \{S^2Q,\ Q\wedge Q\}$ mod G' is determined by its homology map. And then we can put

$$T(1_Q\wedge i)\equiv 1_Q\wedge i+\alpha_0(S^2\pi)-m(\xi\pi\wedge i) \text{ mod } G',$$

considering these homology maps.

If

$$T(1_Q\wedge i)=1_Q\wedge i+\alpha_0(S^2\pi)-m(\xi\pi\wedge i)+k(i\nu\pi\wedge i)$$

for some $k \in Z_{12}$, we take $\gamma_0 \in \{S^7,\ Q\wedge Q\}$ as

$$\gamma_0=-\alpha_0+m(\xi\wedge i)-k(i\nu\wedge i)$$

of type $(1,\ -1)$. Then we can put

$$T\gamma_0=\alpha_0-m(\xi\wedge i)+k'(i\nu\wedge i)$$

for some $k' \in Z_{12}$ and $T\gamma_0$ is of type $(-1, 1)$. Thus γ_0 satisfies the relations:

i′) $-(1_Q\wedge\pi)\gamma_0=(1_Q\wedge\pi)T\gamma_0=S^5i$ in $\{S^7,\ S^5Q\}$

and

ii′) $T(1_Q\wedge i)=1_Q\wedge i-\gamma_0(S^2\pi)$ in $\{S^2Q,\ Q\wedge Q\}$.

Next we consider the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \{S^{10},\ S^2Q\} & \longrightarrow & \{S^{10},\ Q\wedge Q\} & \longrightarrow & \{S^{10},\ S^5Q\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow (S^5\pi)^* & & \downarrow & & \\
0 & \longrightarrow & \{S^5Q, S^2Q\} & \xrightarrow{(1\wedge i)_*} & \{S^5Q,\ Q\wedge Q\} & \xrightarrow{(1\wedge\pi)_*} & \{S^5Q,\ S^5Q\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow (S^5i)^* & & \downarrow & & \\
0 & \longrightarrow & \{S^7,\ S^2Q\} & \longrightarrow & \{S^7,\ Q\wedge Q\} & \longrightarrow & \{S^7,\ S^5Q\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From (2.2)$\sim$(2.5) we choose $\bar{\xi} \in \{S^3Q,\ Q\}$ and $\tilde{\xi} \in \{S^{10},\ Q\wedge Q\}$ satisfying the relations

$$(1_Q\wedge i)(S^2\bar{\xi})(S^5i)=\xi\wedge i \quad\text{and}\quad (1_Q\wedge\pi)\tilde{\xi}(S^5\pi)=S^5(\xi\pi).$$

Thus the free part of $\{S^5Q,\ Q\wedge Q\}$ is $Z+Z+Z$ generated by αi_1, $\bar{\xi}\wedge i$ and $\tilde{\xi}(S^5\pi)$.

Let H be the torsion subgroup of $\{S^5Q,\ Q\wedge Q\}$. And if we put $f\equiv a\alpha i_1+b(\bar{\xi}\wedge i)+c\tilde{\xi}(S^5\pi)$ mod H, then

$$f_*: \begin{pmatrix}\sigma^5e_2\\ \sigma^5e_5\end{pmatrix} \longmapsto \begin{pmatrix}(an+2b)(e_5\wedge e_2),\ a(e_2\wedge e_5)\\ (a+2c)(e_5\wedge e_5)\end{pmatrix}.$$

Thus the element $f\in\{S^5Q, Q\wedge Q\}$ mod H is determined by its homology map. Since $(S^5 i)^* 1_{S5Q}=S^5 i=(1_Q\wedge\pi)_*(-\gamma_0)$ and since $(1_Q\wedge\pi)_*$ is epimorphic in the above diagram, we can take $\gamma\in\{S^5Q, Q\wedge Q\}$ satisfying the relations

$$(S^5 i)^*\gamma=\gamma_0=-\alpha_0+m(\xi\wedge i)-k(i\nu\wedge i),$$
$$(1_Q\wedge\pi)_*\gamma=-1_{S5Q}.$$

And, considering the homology maps, we obtain

$$\gamma\equiv-\alpha i_1+m(\bar{\xi}\wedge i) \text{ and } T\gamma\equiv\alpha i_1-m(\bar{\xi}\wedge i) \text{ mod } H.$$

Thus γ satisfies

$$(1_Q\wedge\pi)\gamma=-1_{S5Q}\equiv-(1_Q\wedge\pi)T\gamma \text{ mod } G$$

and

$$T(1_Q\wedge i)=(1_Q\wedge i)-\gamma(S^5 i)(S^2\pi) \quad \text{(by ii}')),$$

where G is a subgroup of $\{S^5Q, S^5Q\}$ generated by $S^5(i\nu\pi)$.

4. Proof of Theorem 3

Let μ be an associative multiplication in a cohomology theory $\tilde{h}$, we shall prove that μ_{η^2} is an admissible multiplication.

(4.1) *If* $(1_X\wedge i\nu\pi)^*=0$ *in* $\tilde{h}^*$ *then the map* μ_{η^2} *is a multiplication in* $\tilde{h}^*(\ ;\eta^2)$ *satisfying* (A_1).

PROOF. The linearity and the naturality of μ_{η^2} are obvious. To prove (A_1), put $T=T(Q, Q)$, $T_1=T(Y, Q)$, $T_2=T(Y\wedge Q, S^5)$ and $T'=T(S^5, Q)$. By definitions of μ_{η^2} and ρ_{η^2} we have on $\tilde{h}^i(X)\otimes\tilde{h}^j(Y;\eta^2)$

$$\begin{aligned}
&\mu_{\eta^2}(\rho_{\eta^2}\otimes 1)\\
&=(-1)^{2i}\sigma^{-5}(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T_1\wedge 1_Q)^*\mu((1_X\wedge\pi)^*\sigma^5\otimes 1_{Y\wedge Q})\\
&=\sigma^{-5}(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T_1\wedge 1_Q)^*(1_X\wedge\pi\wedge 1_{Y\wedge Q})^*\mu(\sigma^5\otimes 1)\\
&=\sigma^{-5}(1_{X\wedge Y}\wedge\gamma)^*(1_{X\wedge Y}\wedge\pi\wedge 1_Q)^*(1_{X\wedge Y}\wedge T')^*(1_X\wedge T_2)^*\mu(\sigma^5\otimes 1)\\
&=\sigma^{-5}(1_{X\wedge Y}\wedge\gamma)^*(1_{X\wedge Y}\wedge T)^*(1_{X\wedge Y\wedge Q}\wedge\pi)^*(1_X\wedge T_2)^*\mu(\sigma^5\otimes 1)\\
&=\sigma^{-5}(1_X\wedge T_2)^*\mu(\sigma^5\otimes 1) \qquad \text{by Proposition 2, i)}\\
&=\mu=\mu_L.
\end{aligned}$$

Similarly we see that on $\tilde{h}^i(X;\eta^2)\otimes\tilde{h}^j(Y)$

$$\mu_{\eta^2}(1\otimes\rho_{\eta^2})=(-1)^j(1_X\wedge T_1)^*\mu=\mu_R,$$

i. e., (A_1) was proved.

Since unit 1 is a left unit for μ_L and a right unit for μ_R, then (Λ_1) implies that $\rho_{\eta 2}(1)=1_\eta$ is a bilateral unit of $\mu_{\eta 2}$.

To prove the compatibility of $\mu_{\eta 2}$ with $\sigma_{\eta 2}$, put $T_1=T(Y, Q)$, $T_2=T(S^1, Q)$, $T_3=T(Y\wedge Q, S^1)$, $T=T(Y, S^1)$ and $T_1'=T(SY, Q)$. By definition of $\mu_{\eta 2}$ and $\sigma_{\eta 2}$ we have on $\tilde{h}^i(X;\eta^2)\otimes\tilde{h}^j(Y;\eta^2)$

$$\begin{aligned}\sigma_{\eta 2}\mu_{\eta 2}&=(-1)^i(1_{X\wedge Y}\wedge T_2)^*\sigma\sigma^{-5}(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T_1\wedge 1_Q)^*\mu\\&=(-1)^{i+1}(1_{X\wedge Y}\wedge T_2)^*\sigma^{-5}(1_{X\wedge Y}\wedge S\gamma)^*(1_X\wedge T_1\wedge 1_{SQ})^*(1_{X\wedge Q}\wedge T_3)^*\mu(\sigma\otimes 1)\\&=(-1)^{i+1}(1_X\wedge T\wedge 1_Q)^*\sigma^{-5}(1_{SX\wedge Y}\wedge\gamma)^*(1_{SX}\wedge T_1\wedge 1_Q)^*\mu((1_X\wedge T_2)\sigma\otimes 1)\\&=(1_X\wedge T)^*\mu_{\eta 2}(\sigma_{\eta 2}\otimes 1),\end{aligned}$$

since $T_0=T(S^1, S^5)$ is a map of degree -1.

Similarly we see that

$$\begin{aligned}\sigma_{\eta 2}\mu_{\eta 2}&=\sigma^{-5}(1_{X\wedge SY}\wedge\gamma)^*(1_X\wedge T_1'\wedge 1_Q)^*\mu(1\otimes(1_Y\wedge T_2)\sigma)\\&=(-1)^i\mu_{\eta 2}(1\otimes\sigma_{\eta 2}).\end{aligned}$$

Q. E. D.

(4.2) *The multiplication $\mu_{\eta 2}$ satisfies (Λ_2).*

PROOF. Put $T=T(Q, Q)$, $T_1=T(Y, Q)$ and $T'=T(Y\wedge Q, S^2)$. We have on $\tilde{h}^i(X;\eta^2)\otimes\tilde{h}^j(Y;\eta^2)$

$$\begin{aligned}&\mu_L(\delta\otimes 1)+(-1)^i\mu_R(1\otimes\delta)\\&=(-1)^i\mu(\sigma^{-2}(1\wedge i)^*\otimes 1)+(-1)^{i+2j+3}(1_X\wedge T_1)^*\mu(1\otimes\sigma^{-2}(1\wedge i)^*)\\&=(-1)^i\sigma^{-2}\{(1_X\wedge T')^*(1_X\wedge i\wedge 1_{Y\wedge Q})^*-(1_X\wedge T_1\wedge 1_{S2})^*(1_{X\wedge Q\wedge Y}\wedge i)^*\}\mu\\&=(-1)^i\sigma^{-2}\{(1_{X\wedge Y\wedge Q}\wedge i)^*(1_{X\wedge Y}\wedge T)^*-(1_{X\wedge Y\wedge Q}\wedge i)^*\}(1_X\wedge T_1\wedge 1_Q)^*\mu\\&=(-1)^i\sigma^{-2}\{(1_{X\wedge Y}\wedge T(1_Q\wedge i))^*-(1_{X\wedge Y}\wedge 1_Q\wedge i)^*\}(1_X\wedge T_1\wedge 1_Q)^*\mu\\&=(-1)^{i+1}\sigma^{-2}(1_{X\wedge Y}\wedge\gamma(S^5 i)(S^2\pi))^*(1_X\wedge T_1\wedge 1_Q)^*\mu\quad\text{by Proposition 2, ii)}\\&=\delta_{\eta 2}\mu_{\eta 2}.\end{aligned}$$

(4.3) *The multiplication $\mu_{\eta 2}$ satisfies (Λ_3).*

PROOF. Since $\mu_{\eta 2}$ satisfies (Λ_1) it is sufficient to prove the following three relations:

i) $\mu_{\eta 2}(\mu_L\otimes 1)=\mu_L(1\otimes\mu_{\eta 2})$ on $\tilde{h}^i(X)\otimes\tilde{h}^j(Y;\eta^2)\otimes\tilde{h}^k(Z;\eta^2)$,

ii) $\mu_{\eta 2}(\mu_R\otimes 1)=\mu_{\eta 2}(1\otimes\mu_L)$ on $\tilde{h}^i(X;\eta^2)\otimes\tilde{h}^j(Y)\otimes\tilde{h}^k(Z;\eta^2)$,

iii) $\mu_R(\mu_{\eta 2}\otimes 1)=\mu_{\eta 2}(1\otimes\mu_R)$ on $\tilde{h}^i(X;\eta^2)\otimes\tilde{h}^j(Y;\eta^2)\otimes\tilde{h}^k(Z)$.

To prove i) putting $T=T_1(Z, Q)$, we have

$$\begin{aligned}\mu_{\eta 2}(\mu_L\otimes 1)&=(-1)^{i+j}\sigma^{-5}(1_{X\wedge Y\wedge Z}\wedge\gamma)^*(1_{X\wedge Y}\wedge T_1\wedge 1_Q)^*\mu(\mu\otimes 1)\\&=(-1)^{i+j}\sigma^{-5}\mu(1\otimes(1_{Y\wedge Z}\wedge\gamma)^*(1_Y\wedge T_1\wedge 1_Q)^*\mu)\end{aligned}$$

$$=(-1)^j\mu(1\otimes\sigma^{-5}(1_{Y\wedge Z}\wedge\gamma)^*(1_Y\wedge T_1\wedge 1_Q)^*\mu)$$
$$=\mu_L(1\otimes\mu_{\eta 2}).$$

In a similar way we can easily see ii) and iii). Q. E. D.

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