On certain positively curved submanifolds in Kaehler manifolds.

Tsukamoto, Yotaro
College of General Education, Kyushu University

Yamaguchi, Tadashi
College of General Education, Kyushu University

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On certain positively curved submanifolds in Kaehler manifolds.

by

Y. TSUKAMOTO and T. YAMAGUCHI

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In a recent paper [4] K. Nomizu proved the following interesting theorem;

Theorem

Let $P^{n+1}(C)$ be a complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 1, and M be a non-singular compact complex hypersurface in $P^{n+1}(C)$.

If $n \geqq 3$ and if the sectional curvature of M with respect to the induced Kaehler metric is $\geqq 1/4$ for every tangent 2-planes, then M is imbedded as a projective hyperplane.

The purpose of this note is to generalize the above theorem into the case for submanifolds.

We shall prove the following theorem;

Theorem

Let M' be a Kaehler manifold with holomorphic sectional curvature $\leqq 1$, and M be a compact complex submanifold of M'.

If complex dimension of M is $\geqq 2$ and if the sectional curvature of M with respect to the induced Kaehler metric is $\geqq 1/4$ for every tangent 2-planes, then M is totally geodesic submanifold and is holomorphically isometric to a complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 1.

Corollary

Let $P^n(C)$ be a complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 1, and M be a compact complex submanifold in $P^n(C)$.

If complex dimension of M is $\geqq 2$ and if the sectional curvature of M with respect to the induced Kaehler metric is $\geqq 1/4$ for every tangent

2-planes, then M is totally geodesic.

1. Statement of known results.

We need the following propositions.

Proposition 1 ([3])

Let M be a complex submanifold of a Kaehler manifold M' with second fundamental form α. Let R and R' be the Riemannian curvature tensor fields of M and M' respectively. Then

$$(*)\quad R(X, JX, X, JX)=R'(X,JX, X, JX)-2g'(\alpha(X, X), \alpha(X, X))$$

for all vector fields X on M, where g' is Kaehler metric on M' and J is complex structure on M.

Proposition 2 ([1], [2])

Let K and hol-K be the sectional curvature and the holomorphic sectional curvature on a Kaehler manifold M.

(1) If complex dimension of M is $\geqq 2$ and if the inequalities $0\leqq\delta\leqq K\leqq 1$ are satisfied for every tangent 2-planes, then we have the inequalities $\delta(8\delta+1)/(1-\delta)\leqq$ hol-$K\leqq 1$ for every non-zero tangent vectors.

(2) If the inequalities $0\leqq\lambda\leqq$ hol-$K\leqq 1$ are satisfied for every non-zero tangent vectors, then we have the inequalities $(3\lambda-2)/4\leqq K\leqq 1$ for every tangent 2-planes.

Proposition 3 ([5])

A complete δ-holomorphically pinched Kaehler manifold with $\delta>0$ is compact and simply connected.

Proposition 4 ([3])

Any two simply connected complete Kaehler manifolds of the same constant holomorphic sectional curvature are holomorphically isometric to each other.

2. Proof of Theorem

By using proposition 1 and proposition 2 (2) we find easily that the sectional curvature of M with respect to the induced Kaehler metric is $\leqq 1$. Therefore M is 1-holomorphically pinched and simply connected by proposition 2 (1) and proposition 3, so we can apply proposition 4.

Now we use again propotion 1. By the inequality (*) for every unit vector X tangent to M, where $R(X, JX, X, JX)=1$ and $R'(X, JX, X, JX)\leqq 1$, we have $\alpha(X, X)=0$. As α is symmetric bilinear, $\alpha\equiv 0$ i. e. M is totally geodesic.

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