On the boundary value problems for a semilinear elliptic equation with discontinuous coefficients

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On the boundary value problems for a semilinear elliptic equation with discontinuous coefficients

By

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In this paper we study the boundary value problems of the semilinear elliptic equation of the second order with discontinuous coefficients.

The linear elliptic boundary value problems with discontinuous coefficients have been treated by many mathematicians (Cf. [1], [2], [5], [6], [7], [9] and the others.).

In § 1 we describe the uniqueness theorem of the linear elliptic boundary value problem of the second order with discontinuous coefficients and in §2 the a priori estimates of the solutions will be given.

Finally in §3 we shall show the existence theorem of the boundary value problem for a semilinear elliptic equation with discontinuous coefficients by virtue of the Ikebe's Main Theorem([1]) and the Kusano's iteration method([3]).

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§ 1. Preliminaries and Uniqueness Theorem

We consider two bounded domains Ω_1 and Ω_2 in n-dimensinal Euclidean spase R_n ($n\geqq 3$) with boundaries $\partial\Omega_1$ and $\partial\Omega_2$ of $n-1$ dimension. We assume that $\Omega_1\cap\Omega_2=\phi$, $\partial\Omega_1\cap\partial\Omega_2=\Gamma$, $\partial\Omega_k-\Gamma=\Gamma_k(k=1,\ 2)$, $\Gamma_k\cap\Gamma=\phi(k=1.2)$ and each Γ_k and Γ are closed sets and sufficiently smooth surfaces.

Throughout this paper all the functions are assumed to be real valued function.

Let us consider the following problem (1.1)-(1.4) in the closed domain $\overline{\Omega}_1+\overline{\Omega}_2$:

$$L_{(k)}u(x)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_1}\left(a_{ij}^{(k)}(x)\frac{\partial u}{\partial x_j}\right)+c^{(k)}(x)u(x)$$

$$=f^{(k)}(x) \quad \text{in} \quad \Omega_k(k=1.2) \qquad \cdots(1.\ 1)$$

$$u(x)|_{\Gamma 1}=0,\ u(x)|_{\Gamma 2}=0 \qquad \cdots(1.\ 2)$$

$$[u(x)]|_{\Gamma}=u(x)|_{\Gamma-0}-u(x)|_{\Gamma+0}=0 \qquad \cdots(1.\ 3)$$

$$\left[\frac{\partial u}{\partial N}\right]|_{\Gamma}=\sum_{i,j=1}^{n}a_{ij}^{(1)}(x)\frac{\partial u}{\partial x_j}\cos(N,\ x_i)|_{\Gamma-0}$$

$$-\sum_{i,j=1}^{n}a_{ij}^{(2)}(x)\frac{\partial u}{\partial x_j}\cos(N,\ x_i)|_{\Gamma+0}=g(x) \qquad \cdots(1.\ 4)$$

where N is the exterior normal to the domain Ω_1; the symbols $\Gamma-0$ and $\Gamma+0$ denote that limiting values are taken on the interior and exterior sides respectively, of the surface Γ relative to the domain Ω_1. Furtermore we assume that the limiting values of the coefficients are not identical on Γ. And we set the following assumptions;

(A. 1) There is a constant $\nu^{(k)}>0$ such that

$$\nu^{(k)}\sum_{i=1}^{n}\xi_i^2\leqq\sum_{i,j=1}^{u}a_{ij}^{(k)}(x)\xi_i\xi_j \qquad \cdots(1.\ 5)$$

for all x in $\overline{\Omega}_k$ and for all real vector $\xi=(\xi_1,\ \xi_2,\cdots\cdots,\xi_n)(i,\ j=1,\ 2,\cdots\cdots,\ n)$ $(k=1,\ 2)$

(A. 2) $c^{(k)}(x)<0$ holds for all x in $\overline{\Omega_k}(k=1,\ 2)$ $\cdots(.\ 6)$

Lemma

Assume that $a_{ij}^{(k)}(x)$ belong to class $C^1(\overline{\Omega}_k)$ $(k=1,\ 2)$ and assumptions (A.1) and (A. 2) are satiesfied for the operator $L_{(k)}$.

If $u(x)$ is of class $C^2(\Omega_1)\cap C^2(\Omega_2)\cap C^1(\Omega_1+\Gamma)\cap C^1(\Omega_2+\Gamma)\cap C^\circ(\overline{\Omega_1}+\overline{\Omega_2})$ and satiesfies the inequalities and equalities:

$$L_{(k)}u(x)\geqq 0\ [\leqq 0] \text{ in } \Omega_k(k=1,\ 2),\ u(x)|_{\Gamma 1}=u(x)|_{\Gamma 2}\leqq 0\ [\geqq 0],$$

$$[(x)]|_{\Gamma}=0 \text{ and } \left[\frac{\partial u}{\partial N}\right]]|_{\Gamma}\leqq 0\ [\geqq 0],$$

then we have $u(x)\leqq 0$ $[u(x)\geqq 0]$ in $\overline{\Omega}_1+\overline{\Omega}_2$.

(proof) If $u(x)$ attains the positive maximum value at the interior point x_0 of Ω_1 or Ω_2, then we obtain the contradiction $L_{(k)}u(x_0)<0$.

Furthermore if $u(x)$ attains the positive maximum value at the point x_0 on Γ, applying the boundary point maximum principle of O. A. Oleinik

(Cf. [8] Th. 1), we can assert that $\left[\frac{\partial u}{\partial N}\right]>0$ at the point x_0. This result is contrary to the given boundary condition $\left[\frac{\partial u}{\partial N}\right]|_\Gamma \leq 0$.

Hence we have $u(x)\leq 0$ in $\overline{\Omega_1}+\overline{\Omega_2}$. The case [ ] can be proved in a similar way.

Theorem 1

Suppose that $a_{ij}^{(k)}(x)$ belong to class $C^1(\overline{\Omega}_k)$ $(k=1, 2)$ and that the assumptions (A. 1) and (A. 2) are satisfied for the operator $L_{(k)}(k=1, 2)$. If $u(x)$ is of class $C^2(\Omega_1)\cap C^2(\Omega_2)\cap C^1(\Omega_1+\Gamma)\cap C^1(\Omega_2+\Gamma)\cap C^0(\overline{\Omega_1}+\overline{\Omega_2})$ and saties-fies the identities:

$L_{(k)}u(x)=0$ in $\Omega_k(k=1, 2)$, $u(x)|_{\Gamma 1}=u(x)|_{\Gamma 2}=0$, $[u(x)]|_\Gamma=0$ and $\left[\frac{\partial u}{\partial N}\right]|_\Gamma=0$, then we have $u(x)=0$ in $\overline{\Omega}_1+\overline{\Omega}_2$.

(proof) Applying the Lemma to the above $u(x)$, we can easily prove the Theorem 1.

Theorem 2

Suppose that $a_{ij}^{(k)}(x)$ belong to class $C^1(\overline{\Omega}_k)(k=1, 2)$ and that the coefficients $a_{ij}^{(k)}(x)$ and $c^{(k)}(x)$ of the operator $L_{(k)}$ satisfy the assumptions (A. 1) and (A. 2). Then the problem (1. 1)-(1. 4) can't possess more than one solution which is of class $C^2(\Omega_1)\cap C^2(\Omega_2)\cap C^1(\Omega_1+\Gamma)\cap C^1(\Omega_2+\Gamma)\cap C^0(\overline{\Omega_1}+\overline{\Omega_2})$.

(proof) We immediately obtain the above conclusion from the Theorem 1.

§2. Estimates

Theorem 3

Consider the problem (1. 1)-(1. 4) under the following assumptions:

1) There are constants $\nu^{(k)}>0$ and $\mu^{(k)}>0$ such that
$\nu^{(k)}\sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}^{(k)}(x)\xi_i\xi_j \leq \mu^{(k)}\sum_{i=1}^n \xi_i^2$ for all x in Ω_k and all real vector $\xi=(\xi_1, \xi_2,\cdots\cdots,\xi_n)$ $(k=1, 2)$
2) $a_{ij}^{(k)}(x)\in C^1(\overline{\Omega}_k)$ and, the functions $c^{(k)}(x)$ and $f^{(k)}(x)$ are bounded and measurable in $\overline{\Omega}_k$ $(k=1, 2)$

3) $g(x)\in C^{1,\alpha}(\Gamma)$ and $g(x)$ is a trace on Γ or the function $G(x)\in C^1(\overline{\Omega}_1+\overline{\Omega}_2)$,

4) The boundary surfaces Γ_1, Γ_2, and Γ belong to C^3 class.

Let us assume that the value of $M=\underset{\overline{\Omega}_1+\overline{\Omega}_2}{Max}|u(x)|$ is already known.

Then a solution $u(x)$ of class $C^2(\Omega_1)\cap C^2(\Omega_2)\cap C^1(\overline{\Omega}_1)\cap C^1(\overline{\Omega}_2)\cap C^0(\overline{\Omega}_1+\overline{\Omega}_2)$ of the problem (1. 1)-(1. 4) satisfies the uniform Hölder condition on $\overline{\Omega}_1+\overline{\Omega}_2$ and its Hölder exponent and constant are estimated in terms of $\nu^{(k)}$, $\mu^{(k)}$, $\sup_{\overline{\Omega}_k}|C^{(k)}(x)|$, $\sup_{\overline{\Omega}_k}|f^{(k)}(x)|$, M, $g(x)$ and geometry of Γ_2, Γ_2 and Γ.

(proof) Any solution $u(x)$ of the problem (1. 1)-(1.4) which is of class $C^2(\Omega_1)\cap C^2(\Omega_2)\cap C^1(\overline{\Omega}_1)\cap C^1(\overline{\Omega}_2)\cap C^0(\overline{\Omega}_1+\overline{\Omega}_2)$ satisfies the following identity for an arbitrary continuous diffentiable function $\varphi(x)$ which vanishes on the neighborhood of Γ_1 and Γ_2:

$$\sum_{k=1}^{2}\int_{\Omega_k}\left\{\sum_{i,j=1}^{u}a_{ij}^{(k)}(x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_j}+[f^{(k)}(x)-c^{(k)}(x)u(x)]\ \varphi(x)\right\}dx$$
$$-\int_{\Gamma}g(x)\varphi(x)ds=0 \qquad \cdots(2.\ 1)$$

If the support of $\varphi(x)$ is contained in the interior of Ω_1 or Ω_2, then in above (2. 1) the surface integral $\int_{\Gamma}g(x)\varphi(x)\ dS$ vanishes and the following identity holds:

$$\sum_{k=1}^{2}\int_{\Omega_k}\left\{\sum_{i,j=1}^{u}a_{ij}^{(k)}(x)\frac{\partial u}{\partial x_1}\frac{\partial \varphi}{\partial x_j}+\right.$$
$$\left.+[f^{(k)}(x)-c^{(k)}(x)u(x)]\varphi(x)\right\}dx=0 \qquad \cdots(2.\ 2)$$

If we progres the investigation in (2. 2) and apply the method of the Ladyzhenskaya-Ural' tseva's estimates to the solution $u(x)$, we conclude that $u(x)$ belongs to the class $C^{0,\beta}(\Omega_k'+\Gamma_k)$ (where Ω_k' denotes arbitrary subdomain of Ω_k which has positive distance to Γ) ($k=1, 2$). (*Cf.* [4], [5])

Now let us do the estimate near the boundary Γ. Let Γ' be any portion of Γ. We straighten out the Γ' by introducing new appropriate coordinates $y_1, y_2, \cdots\cdots, y_n$ so that $y_n=0$ is the equation for Γ'. By this transformation: $x\longleftrightarrow y$, Γ' corresponds to σ and a neighborhood $\mathfrak{A}$ of Γ' changes into a neighborhood Y of the σ.

Let us assume that $x\longleftrightarrow y$ satisfies the following conditions:

$(x_1, x_2, \cdots\cdots, x_n) \in \Omega_1 \longleftrightarrow y_n > 0,$

$(x_1, x_2, \cdots\cdots, x_n) \in \Omega_2 \longleftrightarrow y_n < 0.$

Then (1. 1), (1. 3) and (1. 4) take the forms

$$\sum_{p,q=1}^{n} \sum_{i=1}^{n} \alpha_{pi} \frac{\partial}{\partial y_p}\left(\bar{\beta}_{iq}^{(k)}(y) \frac{\partial u}{\partial y_q}\right) + c^{(k)}(y)u(y) = f^{(k)}(y) \qquad \cdots(2.\ 3)$$

$$\text{for } y \text{ in } \begin{cases} Y \cap (y_n > 0) & k=1 \\ Y \cap (y_n < 0) & k=2 \end{cases}$$

$$[u(y)] \,|\, \sigma \cap (y_n = 0) = 0 \qquad \cdots(2.\ 4)$$

$$\sum_{q=1}^{n} \bar{A}_{nq}^{(1)}(y) \frac{\partial u}{\partial y_q}\Big|\, y_n \to 0+0 - \sum_{q=1}^{n} \bar{A}_{nq}^{(2)}(y) \frac{\partial u}{\partial y_q}\Big|\, y_n \to 0-0$$

$$= \bar{G}(y) \left[\sum_{i=1}^{n} \left(\frac{\partial y_n}{\partial x_i}\right)^2\right]^{1/2}$$

$$= \tilde{G}(y) \qquad \cdots(2.\ 5)$$

where $\alpha_{pi} = \dfrac{\partial y_p}{\partial x_i}$ and $\bar{A}_{pq}^{(k)}(y) = \sum_{i=1}^{n} \alpha_{pi} \bar{\beta}_{iq}^{(k)}$ $(k=1, 2)$.

We may conclude from (2. 3)-(2.5) that $u(y)$ satisfies the integral identity (2. 6)

$$\int_Y \sum_{p,q=1}^{n} \bar{A}_{pq}(y) \frac{\partial u}{\partial y_q} \frac{\partial \varphi}{\partial y_p} dy + \int_Y \Big\{ \sum_{i,p,q=1}^{n} \bar{\beta}_{iq}(y) \frac{\partial u}{\partial y_q} \frac{\partial \alpha_{pi}}{\partial y_p} +$$

$$+ [\bar{f}(y) - \bar{c}(y)u(y)] \Big\} \varphi(y) dy -$$

$$- \int_{\sigma \cap (y_n = 0)} \tilde{G}(y) \varphi(y) dy_1 dy_2 \cdots dy_{n-1} = 0 \qquad \cdots(2.\ 6)$$

for arbitrary $\varphi(y)$ in $C_0^1(Y)$, where $(Y_1 = Y \cap (y_n > 0),\ Y_2 = Y \cap (y_n < 0))$

$$\bar{A}_{qp}(y) = \begin{cases} \bar{A}_{pq}^{(1)}(y) \cdots y \text{ in } Y_1 \\ \bar{A}_{pq}^{(2)}(y) \cdots y \text{ in } Y_2 \end{cases}, \qquad \bar{f}(y) = \begin{cases} \bar{f}^{(1)}(y) \cdots y \text{ in } Y_1 \\ \bar{f}^{(2)}(y) \cdots y \text{ in } Y_2 \end{cases},$$

$$\bar{\beta}_{iq}(y) = \begin{cases} \bar{\beta}_{iq}^{(1)}(y) \cdots y \text{ in } Y_2 \\ \bar{\beta}_{iq}^{(2)}(y) \cdots y \text{ in } Y_2 \end{cases} \text{ and } \qquad \bar{c}(y) = \begin{cases} \bar{c}^{(1)}(y) \cdots y \text{ in } Y_1 \\ \bar{c}^{(2)}(y) \cdots y \text{ in } Y_2 \end{cases}.$$

Since $\displaystyle\int_{\sigma \cap (y_n = 0)} \tilde{G}(y) \varphi(y) dy_1 dy_2 \cdots dy_{n-1} =$

$$= \int_{Y \cap (y_n \geq 0)} \frac{\partial}{\partial y_n}\Big(\tilde{G}(y) \varphi(y)\Big) dy \text{ holds,}$$

(2. 6) changes into $\displaystyle\int_Y \sum_{p,q=1}^{n} \bar{A}_{pq}(y) \frac{\partial u}{\partial y_q} \frac{\partial \varphi}{\partial y_p} dy +$

$$+\int_Y \Big\{\sum_{i,p,q=1}^n \bar{\beta}_{iq}(y)\frac{\partial u}{\partial y_q}\frac{\partial \alpha_{pi}}{\partial y_p}+[f(y)-c(y)u(y)]\Big\}\varphi(y)\,dy-$$

$$-\int_{(y_n\geq 0)\cap Y}\frac{\partial}{\partial y_n}\Big(\tilde{G}(y)\,\varphi(y)\Big)dy=0 \qquad \cdots(2.\ 7)$$

One can easily see that (2. 7) holds for arbitrary $\varphi(y)\epsilon \mathring{W}_2^1(Y)$ by approxlmations.

In (2. 7) we take

$$\varphi(y)=\begin{cases}\zeta^2(y;\ \rho;\ \tau)(u(y)-m) & \text{for } u\geqq m\\ 0 & \text{for } u<m\end{cases}$$

where $\zeta(y;\ \rho;\ \tau)$ is a smooth truncating function which vanishes outside a sphere $K(\rho)$ with center on σ and rudius ρ sufficiently small and τ is some number in (0,1].

We take

$$\zeta(y;\ \rho;\ \tau)=\begin{cases}1 & 0\leqq|y-y_0|\leqq\rho(1-\tau)\\ \dfrac{\rho-|y-y_0|}{\tau\rho} & \rho(1-\tau)\leqq|y-y_0|\leqq\rho\\ 0 & |y-y_0|\geqq\rho.\end{cases}$$

Then we have

$$\int_{A_{m,\rho}}\Big\{\sum_{p,q=1}^n \bar{A}_{pq}(y)\frac{\partial u}{\partial y_p}\frac{\partial u}{\partial y_q}\zeta^2+\sum_{p,q=1}^n \bar{A}_{pq}(y)\frac{\partial u}{\partial y_p}\times$$

$$\times 2\zeta\zeta_{y_q}(u-m)+(f(y)-c(y)u(y))\zeta^2(u-m)\Big\}dy+$$

$$+\int_{A_{m,\rho}}\sum_{p,q,i=1}^n \bar{\beta}_{iq}(y)\frac{\partial u}{\partial y_q}\frac{\partial \alpha_{pi}}{\partial y_p}\zeta^2(u-m)dy-$$

$$-\int_{A_{m,\rho}\cap(y_n\geq 0)}\Big\{\frac{\partial\tilde{G}}{\partial y_n}\zeta^2(u-m)+\tilde{G}(y)2\zeta\zeta_{y_n}(u-m)+\tilde{G}(y)\zeta^2\times\frac{\partial u}{\partial y_n}\Big\}dy=0 \qquad \cdots(2.\ 9)$$

where $A_{m,\rho}=\{y\epsilon Y_\cap K(\rho):\ u(y)\geqq m\}$.

By virtue of the assumptions 1), 2), 3) and 4), there exist positive constants λ_k and δ_k such that

$$\lambda_k\sum_{i=1}^n\xi_i^2\leqq\sum_{p,q=1}^n \bar{A}_{pq}^{(k)}(y)\xi_p\xi_q\leqq\delta_k\sum_{i=1}^n\xi_i^2$$

for all y in Y_k and all real vector $\xi=(\xi_1,\ \xi_2,\cdots,\ \xi_n)$ $(k=1,\ 2)$, where $\lambda_k=\lambda_k(\nu^k,\ \Gamma)$ and $\delta_k=\delta_k(\mu^k,\ \Gamma)$.

Now we set $\tilde{\lambda}=Min(\lambda_1,\ \lambda_2)$ and $\tilde{\delta}=Max\ (\delta_1,\ \delta_2)$.

This implies

$$\tilde{\lambda}\int_{A_{m,\rho}}|\nabla u|^2\zeta^2dy \leq \int_{A_{m,\rho}}\sum_{p,q=1}^{n}|A_{pq}(y)\frac{\partial u}{\partial y_p}2\zeta\zeta_{y_n}(u-m)|dy+$$
$$+\int_{A_{m,\rho}}|(f-cu)\zeta^2(u-m)|dy+\frac{1}{2}\int_{A_{m,\rho}}\Big\{(\varepsilon_1^2\cdot\zeta^2\cdot|\nabla u|^2+$$
$$+\frac{1}{\varepsilon_1^2}\sum_{q=1}^{n}\Big(\sum_{p,q=1}^{n}\bar{\beta}_{tq}\frac{\partial\alpha_{tp}}{\partial y_p}\zeta\ (u-m)\Big)^2\Big\}dy+\underset{Y}{Max}\,|\frac{\partial\tilde{G}}{\partial y_n}|\times$$
$$\times\int_{A_{m,\rho}}|\zeta^2\cdot(u-m)\,|dy+\underset{Y}{Max}|\tilde{G}|\int_{A_{m,\rho}}\Big\{|2\zeta\cdot\zeta_{y_n}(u-m)|+|\zeta^2\frac{\partial u}{\partial y_n}|\Big\}dy. \quad \cdots(2.\ 10).$$

Since Γ is covered by a finite number of above Γ', we obtain from (2. 10) that

$$\lambda\int_{A_{m,\rho}}|\nabla u|^2\zeta^2dy \leq C_1\int_{A_{m,\rho}}\Big\{\varepsilon_2|\nabla u|^2\zeta^2+\frac{1}{\varepsilon_2}(u-m)^2\times$$
$$\times|\nabla\zeta|^2\Big\}dy+C_2\int_{A_{m,\rho}}\{\zeta^2(u-m)^2+\zeta^2\}\,dy+$$
$$+\frac{C_3}{2}\varepsilon_1^2\int_{A_{m,\rho}}|\nabla u|^2\zeta^2dy+\frac{C_3}{2\varepsilon_1^2}\int_{A_{m,\rho}}\zeta^2(u-m)^2dy+$$
$$+\frac{C_4}{2}\int_{A_{m,\rho}}\{\zeta^2(u-m)^2+\zeta^2\}py+C_5\int_{A_{m,\rho}}\Big\{\zeta^2+|\nabla\zeta|^2(u-m)^2+$$
$$+\varepsilon_3^2\cdot|\nabla u|^2\zeta^2+\frac{1}{\varepsilon_3^2}\zeta^2\Big\}dy. \quad \cdots(2.\ 11)$$

Now taking $\varepsilon_1=\sqrt{\frac{\lambda}{2}}$, $\varepsilon_2=\frac{\lambda}{4C_1}$ and $\varepsilon_3=\sqrt{\frac{\lambda}{4C_5}}$, we obtain

$$\frac{\lambda}{4}\int_{A_{m,\rho}}|\nabla u|^2\zeta^2dy \leq C_6\int_{A_{m,\rho}}|\nabla\zeta|^2(u-m)^2dy+$$
$$+C_7\int_{A_{m,\rho}}\zeta^2(u-m)^2dy+C_8\int_{A_{m,\rho}}\zeta^2dy$$

Taking a number η such that $(u-m)\leq\eta=\sqrt{\frac{\lambda}{2C_7}}$, we have

$$\frac{\lambda}{4}\int_{A_{m,\rho}}|\nabla u|^2\zeta^2dy \leq C_6\underset{A_{m,\rho}}{Max}(u-m)^2\int_{A_{m,\rho}}|\nabla\zeta|^2dy+$$
$$+\frac{\lambda}{4}\int_{A_{m,\rho}}\zeta^2dy+C_8\int_{A_{m,\rho}}\zeta^2dy$$

Remembering the definition of $\zeta(y;\ \rho;\ \tau)$ we now easily see that $u(y)$ satisfies the following inequality

$$\int_{A_{m,\rho-\tau\rho}} |\nabla u|^2 dy \leq \gamma \text{ mes } A_{m,\rho}\left\{\frac{1}{(\tau\rho)^2} \underset{y\in A_{m,\rho}}{Max} [u(y)-m]^2+1 \right. \qquad \cdots(2.\ 12)$$

In the same way we obtain a similar inequality for $B_{m,\rho-\tau\rho}$

$$\int_{B_{m,\rho-\tau\rho}} |\nabla u|^2 dy < \gamma \text{ mes } B_{m,\rho}\left\{\frac{1}{(\tau\rho)^2} \underset{y\in B_{m,\rho}}{Max} [m-u(y)]^2+1 \right. \qquad \cdots(2.\ 13)$$

where $B_{m,\rho}=\{y\in Y_\cap K(\rho)\ ;\ u(y)\leq m\}$.

Furthermore putting

$$\varphi(y)=e^{\lambda u(y)}\ \zeta^2(y;\ \rho;\ 1/2)$$

into (2. 7), where λ is sufficiently large number, we obtain

$$\int_{K(\rho/2)\cap Y} |\nabla u|^2 dy \leq \gamma\rho^{n-2} \qquad \cdots(2.\ 14)$$

Using (2. 12), (2. 13) and (2. 14), we can conclude that $u(x)$ satisfies a uniform Hölder condition in a neighborhood of the boundary Γ (Cf. Lemma 5, 6 of [4], [5]).

This completes the proof of the theorem.

Theorem 4 [Cf. I. A. Sismarev [7]]

We shall consider the problem (1. 1)-(1.4) under the following conditions;

1) There is a constant $\nu^{(k)}>0$ such that

$$\nu_{(k)}\sum_{i=1}^{n}\xi_i^2 \leq \sum_{i,j,=1}^{n} a_{ij}^{(k)}(x)\xi_i\xi_j$$

for all $x\in\overline{\Omega_k}$ and all real vectors $\xi=(\xi_1,\ \xi_2,\cdots,\ \xi_n)$ $(k=1,\ 2)$,

2) $a_{ij}^{(k)}(x)\in C^{1,\alpha}(\overline{\Omega}_k)$ $(k=1,\ 2)$, $c^{(k)}(x)\in C^{\circ,\alpha}(\overline{\Omega}_k)$ and $f^{(k)}(x)$ is a bounded function in $\overline{\Omega}_k$ $(k=1,\ 2)$. $c^{(k)}(x)<0$ in $\overline{\Omega}_k$ $(k=1,\ 2)$,

3) $g(x)\in C^\circ(\Gamma)$,

4) Boudary surfaces Γ_1, Γ_2 and Γ are sufficiently smooth.

Then for a solution $u(x)$ which is of class $C^2\ (\overline{\Omega}_1)_\cap C^2(\overline{\Omega}_2)_\cap C^1(\overline{\Omega_1})_\cap C^1(\overline{\Omega_2})_\cap C^\circ(\Omega_1+\Omega_2)$, we have the following estimate;

$$\underset{\overline{\Omega}_1+\overline{\Omega}_2}{Max}|u(x)| \leq Max\left(\frac{\underset{\Omega_1}{sup}|f^{(1)}(x)|}{\underset{\Omega_1}{inf}|c^{(1)}(x)|},\ \frac{\underset{\Omega_2}{sup}|f^{(2)}(x)|}{\underset{\Omega_2}{inf}|c^{(2)}(x)|}\right)+C\underset{\Gamma}{Max}|g(x)|$$

Here the constants C depends only on the coefficient $a_{ij}^{(k)}(x)$ and $c^{(k)}$ and on the form of the domain Ω_1 and Ω_2.

(proof) If $u(x)$ satisfies the problem (1. 1)-(1. 4), then $u(x)$ is expressed by the sum of two functions $u_1(x)$ and $u_2(x)$ such that $u_1(x)$ satisfies the following problem (2. 15)-(2. 16) and $u_2(x)$ satisfies the problem (2. 17) -(2. 18);

$$\begin{cases} \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}^{(k)}(x)\dfrac{\partial u_1}{\partial x_j}\right)+c^{(k)}(x)u_1(x)=f^{(k)}(x) \quad \text{in } \Omega_k \quad (k=1,\ 2) & \cdots(2.\ 15) \\ u_1(x)|_{\Gamma_1}=u_1(x)|_{\Gamma_2}=0,\ [u_1(x)]|_{\Gamma}=0,\ \left[\dfrac{\partial u_1}{\partial N}\right]|_{\Gamma}=0 & \cdots(2.\ 16) \end{cases}$$

$$\begin{cases} \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ji}^{(k)}(x)\dfrac{\partial u_2}{\partial x_j}\right)+c^{(k)}(x)u_2(x)=0 \quad \text{in } \Omega_k \quad (k=1,\ 2) & \cdots(2.\ 17) \\ u_2(x)|_{\Gamma_1}=u_2(x)|_{\Gamma_2}=0,\ [u_2(x)]|_{\Gamma}=0,\ \left[\dfrac{\partial u_2}{\partial N}\right]|_{\Gamma}=g(x) & \cdots(2.\ 18). \end{cases}$$

Now the following estimate holds for the solution $u_1(x)$ of the problem (2. 15)-(2. 15);

$$\underset{\overline{\Omega_1}+\overline{\Omega_2}}{Max}\ |u_1(x)|\leqq Max\left(\frac{\underset{\Omega_1}{sup}|f^{(1)}(x)|}{\underset{\Omega_1}{inf}\,|c^{(1)}(x)|},\ \frac{\underset{\Omega_2}{sup}|f^{(2)}(x)|}{\underset{\Omega_2}{inf}\,|c^{(2)}(x)|}\right).$$

On the other hand, for $u_2(x)$ we have the following estimate (see for example the Theorem of [7])

$\underset{\overline{\Omega_1}+\overline{\Omega_2}}{Max}\ |u_2(x)|\leqq C\ Max|g(x)|$, where the constant C depends only on $a_{ij}^{(k)}(x)$ and on $c^{(k)}(x)$.

Thus we have the estimate of the Theorem 4 for the solution of the problem (1. 1)-(1. 4).

§ 3. The Existence Theorem For a Semilinear Equation

In this section we shall use the Ikebe's Main Theorem [Cf. [1] Th. 5] and Schauder estimate (See for example [1] Th. 2). In [1] N. Ikebe proved the Main Existence Theorem for the higher order linear elliptic equations with general boundary conditions. In the following we shall use his Main Theorem and Schauder estimate in the special case $m=1$.

Let us consider the following problem of the semlinear elliptic equation with discontinuous coefficients in the closed domain $\overline{\Omega_1}+\overline{\Omega_2}$,

$$L^{(k)}u(x)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}^{(k)}(x)\frac{\partial u}{\partial x_j}\right)+c^{(k)}(x)u(x)=f^{(k)}(x, u)$$
$$\text{in } \Omega_k \quad (k=1, 2) \qquad \cdots(3.\ 1)$$

$$u(x)|_{\Gamma 1}=u(x)_{\Gamma 2}=0 \qquad \cdots(3.\ 2)$$

$$[u(x)]|_{\Gamma}=u(x)|_{\Gamma+0}-u(x)|_{\Gamma-0}=0 \qquad \cdots(3.\ 3)$$

$$\left[\frac{\partial u}{\partial N}\right]|_{\Gamma}=\sum_{i,j=1}^{n}a_{ij}^{(1)}(x)\frac{\partial u}{\partial x_j}\cos(N, x_i)|_{\Gamma+0}-$$
$$-\sum_{i,j=1}^{n}a_{ij}^{(2)}(x)\frac{\partial u}{\partial x^j}\cos(N, x_i)|_{\Gamma-0}=g(x) \qquad \cdots(3.\ 4)$$

We assume the following assumptions:

1) There are constants $\nu^{(k)}>0$ and $\mu^{(k)}>0$ such that
$$\nu^{(k)}\sum_{i=1}^{n}\xi_i^2\leqq\sum_{i,j=1}^{n}a_{ij}^{(k)}(x)\xi_i\xi_j\leqq\mu^{(k)}\sum_{i=1}^{n}\xi_i^2$$
for every $x\in\Omega_k$ and every real vector $\xi=(\xi_1, \xi_2,\cdots, \xi_n)\neq 0$

2) $a_{ij}^{(k)}(x)\in C^{1,\alpha}(\overline{\Omega}_k)$ $(i, j=1, 2,\cdots, n)$, $c^{(k)}(x)\in C^{0,\alpha}(\overline{\Omega}_k)$ and $c^{(k)}(x)<0$ in $\overline{\Omega}_k$ $(k=1, 2)$

3) $g(x)\in C^{1,\alpha}(\Gamma)$ and $g(x)$ is a trace on Γ of function
$$G(x)\in C^1(\overline{\Omega}_1+\overline{\Omega}_2)$$

4) The function $f^{(k)}(x, u)$ is bounded in $D^{(k)}=\{(x, u);\ x\in\overline{\Omega}_k,\ |u|<\infty\}$ and is uniforrmly Hölder continuous with exponent α in $\overline{\Omega}_k$ for each fixed value of $u(k=1, 2)$. Futhermore $\dfrac{\partial f^{(k)}(x, u)}{\partial u}$ is bounded in $\tilde{D}^{(k)}=\{(x, u);\ x\in\overline{\Omega}_k,\ |u|\leqq M+C\gamma\}$ $(k=1.2)$,

where

$$M=Max\left(\frac{\sup\limits_{D^{(1)}}|f^{(1)}(x, u)|}{\inf\limits_{\Omega_1}|c^{(1)}(x)|}, \frac{\sup\limits_{D^{(2)}}|f^{(2)}(x, u)|}{\inf\limits_{\Omega_2}|c^{(2)}(x)|}\right),\quad \gamma=\underset{\Gamma}{Max}\,|g(x)|$$

and C is determined by the Theorem 4.

In fact constant C is determined by the following equation;

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}^{(k)}(x)\frac{\partial U}{\partial x_j}\right)+c^{(k)}(x)U(x)=0 \text{ in } \Omega_k \qquad \cdots(3.\ 5)$$

$$U|_{\Gamma 1}=0,\ U|_{\Gamma 2}=0,\ [U]|_{\Gamma}=0\ \left[\frac{\partial U}{\partial N}\right]|_{\Gamma}=\gamma \qquad \cdots(3.\ 6)$$

For the solution of the problem (3. 5)-(3. 6) we have

$$\underset{\overline{\Omega_1}+\overline{\Omega_2}}{Max}|U| \leq C\gamma$$

by virtue of Theorem 4. This conttant C is one of above condition 4).

Further we assume that the limiting value of the coefficients are not identical on Γ.

Theorem 5

Under the assumptions 1)-4), the problem (3. 1)-(3. 4) has at least one solution.

(proof) Let $U(x)$ be a solution of the following problem;

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{(k)}(x)\frac{\partial U}{\partial x_j}\right)+c^{(k)}(x)U=-N_k \quad \text{in } \Omega_k(k=1,\ 2)$$

$$U|_{\Gamma 1}=U|_{\Gamma 2}=0,\ [U]|_{\Gamma}=0,\ \left[\frac{\partial U}{\partial N}\right]|_{\Gamma}=\gamma$$

where $N_k=\underset{\tilde{D}}{\sup}|f^{(k)}(x,\ u)|\,(k=1,\ 2)$. By the Ikebe's Main Theorem and our Theorem 2 $U(x)$ exists, is unique and belongs to class $C^{2,\alpha}(\overline{\Omega}_1)\cap C^{2,\alpha}(\overline{\Omega}_2)\cap C^{\circ}(\overline{\Omega}_1+\overline{\Omega}_2)$.

Then we shall consider the sequence of functions $\{u_m(x)\}$ defined by the following identities;

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{(k)}(x)\frac{\partial u_m}{\partial x_j}\right)+c^{(k)}(x)u_m-K^{(k)}u_m$$

$$=f^{(k)}(x,\ u_{m-1})-K^{(k)}u_{m-1} \quad \text{in } \Omega_k \qquad \cdots(3.\ 8)$$

$$u_m|_{\Gamma 1}=u_m|_{\Gamma 2}=0,\ \ [u_m]|_{\Gamma}=0,\ \left[\frac{\partial u_m}{\partial N}\right]_{\Gamma}=g(x)$$

where $K^{(k)}=\sup\left|\dfrac{\partial f^{(k)}(x,\ u)}{\partial u}\right|$ for $x\in\overline{\Omega}_k$ and $|u|\leq \underset{\overline{\Omega}_1+\overline{\Omega}_2}{Max}|U|$.

By the Ikebe's Main Theorem ([1]. Th. 5) and our Theorem 2, above problems are solvable. Now we shall prove that the sequence $\{u_m(x)\}$ is monotonically nonincreasing and uniformly bounded.

Since the following equalities hold;

$$L_{(k)}u_1-K^{(k)}u_1=f^{(k)}(x,\ U)-K^{(k)}U=-N_k-K^{(k)}U \text{ in } {}_k(k=1,\ 2)$$

$$U|_{1\Gamma}=u_1|_{\Gamma 1}=U|_{2\Gamma}=u_1|_{\Gamma 2}=0,\ [U]|_{\Gamma}=[u_1]|_{\Gamma}=0,$$

$\left[\dfrac{\partial U}{\partial N}\right]|_{\Gamma}=\gamma \geqq \left[\dfrac{\partial u_1}{\partial N}\right]|_{\Gamma}=g(x)$, by the Lemma we have

$u_1 - U \leqq 0$ in $\overline{\Omega}_1 + \overline{\Omega}_2$. Similarly we get
$u_1 + U \geqq 0$ in $\overline{\Omega}_1 + \overline{\Omega}_2$.

Suppose that we have

$$-U(x) \leqq u_m(x) \leqq u_{m-1}(x) \leqq U(x) \quad x \in \overline{\Omega}_1 + \overline{\Omega}_2. \qquad \cdots(3.\ 9)$$

Then we have

$$L_{(k)}(u_{m+1} - u_m) - K^{(k)}(u_{m+1} - u_m)$$
$$= f^{(k)}(x,\ u_m) - f^{(k)}(x,\ u_{m-1}) - K^{(k)}(u_m - u_{m-1}) \leqq 0 \text{ in } \Omega_k \ (k=1,\ 2)$$
$$u_{m+1} - u_m \,|\, \Gamma_1 = 0,\ u_{m+1} - u_m \,|\, \Gamma_2 = 0$$
$$[u_{m+1} - u_m] \,|\, _\Gamma = 0 \ \left[\frac{\partial(u_{m+1} - u_m)}{\partial N}\right] \Big|\, _\Gamma = 0.$$

Hence by the Lemma we have $u_{m+1}(x) \leqq u_m(x)$ in $\overline{\Omega}_1 + \overline{\Omega}_2$.

Therefore $L_{(k)}\ (u_{m+1}) = f^{(k)}(x,\ u_m) + K^{(k)}(u_{m+1} - u_m)$

$$\leqq N_k = L_{(k)}(-U) \text{ in } \Omega_k \ (k=1,\ 2)$$
$$u_{m+1} + U \,|\, _{\Gamma 1} = u_{m+1} + U \,|\, _{\Gamma 2} = 0,$$
$$[u_{m+1} + U] \,|\, _\Gamma = 0,\ \left[\frac{\partial u_{m+1} - u_m)}{\partial N}\right] \Big|\, _\Gamma = g(x) \geqq \left[\frac{\partial - U}{\partial N}\right] \Big|\, _\Gamma = -\gamma\,.$$

Hence we have $-U(x) \leqq u_{m+1}(x)$ in $\overline{\Omega}_1 + \overline{\Omega}_2$ by the Lemma.

We can conclude that the relation (3. 9) holds for all integer $m=1$, 2,$\cdots\cdots$ by the induction, and (3. 9) denotes the monotonicity and uniform boundedness of $\{u_m(x)\}$.

We set

$$u(x) = \lim_{m \to \infty} u_m(x) \quad x \in \overline{\Omega}_1 + \overline{\Omega}_2\,.$$

In last we shall prove that above $u(x)$ is required solution of our problem (3.1)-(3.4).

If we apply the Theorem 3 to the individual functions $u_m(x)$, we may conclude that $u_m(x)$ satisfy a uniform Hölder condition with exponent β and constant independent of m in $\overline{\Omega}_1 + \overline{\Omega}_2$.

Therefore by the Schauder estimate (See for example [1]. Th. 2) we see that $u_m(x)$ belong to the class $C^{2+\varepsilon}(\overline{\Omega}_1) \cap C^{2+\varepsilon}(\overline{\Omega}_2) \cup C^\circ(\overline{\Omega}_1 + \overline{\Omega}_2)$ (where $\varepsilon = \min\ (\alpha.\ \beta)$) and in its estimates constant C is independent of m.

Hence we can select from $\{u_m(x)\}$ a subsequence which converges uniformly in $\overline{\Omega}_1 + \overline{\Omega}_2$. Therefore we can assert that $u(x) \in C^{2+\varepsilon}\ (\overline{\Omega}_1) \cap^{2+\varepsilon}\ (\overline{\Omega}_2) \cap C^\circ$ $(\overline{\Omega}_1 + \overline{\Omega}_2)$ and $u(x)$ actually satisfies the equation (3. 1) and given boundary conditions (3. 2)-(3. 4).

Bibliography

[1] L. Ikebe; On elliptic boundary value problems with discontinuous coefficients. Memoirs of the Faculty of Science Kyushu Univ. Series A. Math. Vol. XXI No. 2 (1967) pp. 167-184.

[2] V. A. Il' in and I. A. Ŝiŝmarev; Some problems for the Lu=div $[p(x)$ grad $u]$-$q(x)u$ operator with discontinuous coefficients. Soviet Math. Doklady Vol. 1 (1960) pp. 1306-1309.

[3] T. Kusano; On bounded solutions of exterior boundary value problems for linear and quasilinear elliptic differential equations, Jap. Jour. Math. Vol 35 (1965) pp. 31-59.

[4] O. A. Ladyzhenskaya and N. N. Ural'tseva; Quasilinear elliptic equations and variational problems with many independent variables, Russian Math. Surveys published by the London Math. Soc. (translation 16 (1961) pp. 17-91.

[5] O. A. Ladyzhenskaya and N. N. Ural' tseva; Linear and Quasilinear Elliptic Equation, Adademic Press (English translation) 1968.

[6] V. A. Il' in; On the solvability of the Dirichlet and Neumann problems for a linear elliptid operator with discontinuous coefficients, Soviet Math. Dok. Vol. 2 (1961) pp. 228-231.

[7] I. A. Ŝiŝmarev; A priori estimation of solutions to the Dirichlet problem for an elliptic operators with discontinuous coefficients, Soviet Math. Dok. Vol. 1 (1960) pp. 249-252.

[8] O. A. Oleinik; On the properties of solutions of some boundary value problems for equations of elliptic type, Matem. Sbornik 30 (72) (1052) pp. 695-702 (Russian).

[9] O. A. Oleinik; Boundary value problems for linear elliptic and parabolic equations with discontinuous coefficients, Amer. Math. Soc. Translation Vol. 42 (1964) pp. 175-194.