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On absolute neighbourhood retracts of metric spaces

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§1. **Introduction.** A separable metric space is said to be an absolute neighbourhood retract, usually by C. Kuratowski [3], whenever it is a neighbourhood retract of *every separable metric* space which contains it as a closed subset. C. H. Dowker [1], however, defined absolute neighbourhood retracts for metric spaces as follows: a metric space is called an absolute neighbourhood retract if it is a neighbourhood retract of *every metric* space containing it as a closed subset. We shall call an absolute neighbourhood retract which was defined by C. Kuratowski a *K-ANR* and one by C. H. Dowker a *D-ANR*. For separable metric spaces it is obvious that property "a *D-ANR*" implies "a *K-ANR*". In this note we shall show that the both definitions are equivalent for separable metric spaces (Theorem 2).

Throughout this note a term "mapping" always means a single-valued continuous transformation.

§2. **Results.** For a compactum A the definition due to K. Borsuk (in the apparently weak sense) that A is an absolute neighbourheed retract is as follows (e. g., Lefshetz [4], p. 322): A is called an absolute neighbourhood retract if it is a neighbourhood retract of *every compactum* which contains it (necessarily as a closed subset). We shall call this absolute neighbourhood retract a *B-ANR*.

Theorem 1. *If a compactum A is a B-ANR, then A is a neighbourhood retract of every normal Hausdorff space which contains A.*

Consequently, for a compactum A the following three conditions are equivalent:

(i) *A is a B-ANR,*

(ii) *A is a K-ANR,*

(iii) *A is a D-ANR.*

Proof. Let X be an arbitrary normal Hausdorff space which contains

A (necessarily as a closed subset). By the embedding theorem (e. g., Lefshetz [4], p. 39) and our assumption, there exists a homeomorphism f of A into the Hilbert parallelotope Q^ω such that $f(A)$ is a neighbourhood retract of Q^ω. Then, there is a neighbourhood V of $f(A)$ in Q^ω with a retraction θ of V onto $f(A)$, Next, by Tietze's extension theorem (e. g., Lefshetz [4], p. 28) f may be extended to a mapping f' of X into Q^ω. Let $U=f'^{-1}[f'(X)\frown V]$, then U is an open subset of X containing A. Moreover a mapping $f^{-1}\theta f'|U$ is a retraction of U onto A. This completes the proof.

Theorem 2. For a separable metric space a K-ANR implies a D-ANR. Therefore, for a separable metric space a K-ANR and a D-ANR are equivalent.

Proof (based on Fox [3]). Let A be an arbitrary separable metric space which is a *K-ANR* and X an arbitrary metric space which contains A as a closed subset. Since A is a *K-ANR*, by *R. H.* Fox [3], there exists a homeomorphism f of A into the Hilbert parallelotope Q^ω such that $f(A)\times[0]$ is a neighbourhood retract of $f(A)\times[0]\smile Q^\omega\times(0,1]$. Therefore there is a neighbourhood V of $f(A)\times[0]$ in $f(A)\times[0]\smile Q^\omega\times(0,1]$ with a retraction θ of V onto $f(A)\times[0]$. Now by Tietze's extension theorem f may be extended to a mapping g of X into Q^ω. Then we shall define a transformation f' of X into $f(A)\times[0]\smile Q^\omega\times(0,1]$ as follows:

$$f'(x)=\begin{cases}\Pi^{-1}g(x), & \text{for } x\in A,\\ \left(g(x),\ \dfrac{\rho(x,A)}{1+\rho(x,A)}\right), & \text{for } x\in X\text{-}A,\end{cases}$$

where Π denotes the projection of $Q^\omega\times[0]$ onto Q^ω and ρ the distance function in X, Since A is a closed subset of X, $A=\{x'|\ x'\in X,\ \rho(x',A)=0\}$. So we have $\rho(x,A)/(1+\rho(x,A))\neq 0$ for $x\in X\text{-}A$. Therefore f' is a mapping of X into $f(A)\times[0]\smile Q^\omega\times(0,1]$, and moreover $f'|A=\Pi^{-1}f$. (Concerning the continuity of f', see J. Dugundji [2].) Now let $U=f'^{-1}[f'(X)\frown V]$, then U is an open subset of X containing A and a mapping $f^{-1}\Pi\theta f'|U$ is a retraction of U onto A. Therefore A is a *D-ANR*. This completes the proof.

From the theorems 1 and 2, a *K-ANR* is an extensive definition of a *B-ANR* for a separable metric space, and a *D-ANR* is an extensive definition of a *K-ANR* for a metric space.

§3. **Remark.** The above statements for absolute neighbourhood retracts hold in parallel for absolute retracts. The details are, however, all omitted, because of their similarity.

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References

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