SPHERE BUNDLES OVER SPHERES

Sawa, Tatuo
Fukuoka University Institute of mathematics

https://doi.org/10.15017/1448922

出版情報：九州大学教養部数学雑誌. 4 (1), pp.19-22, 1966-05. General Education Department, Kyushu University

Math. Rep.
IV-1, 1966.

SPHERE BUNDLES OVER SPHERES

By

Tatuo SAWA

[1965. 12. 9]

§1. Introduction and results.

The classification (by bundle equivalence relation) problem of the k-sphere bundles over an n-sphere S^n has been completely solved in the sense that the problem reduces to the computation of the homotopy group $\pi_{n-1}(0(k+1))$ and of the well-known operations of $\pi_0(0(k+1))$ on $\pi_{n-1}(0(k+1))$.

The enumeration of homeomorphy-classes of the total spaces of the k-sphere bundles over S^n is, however, neither completely determined, nor is completely clarified the topological structure of the total spaces belonging to each homeomorphy-class, even if dimensions n and k are comparatively low.

Now, in this note, the following results are given.

There exists a doubly indexed family of the 3-sphere bundles over S^4, if we assign to $m\alpha_3+n\beta_3$ the bundle $\mathfrak{B}(m,n)$. These are not all pairwise distinct, but $\mathfrak{B}(m,n)$ and $\mathfrak{B}(m', n')$ are equivalent if and only if $m'=m+n$ and $n'=-n$. α_3 and β_3 are generators of $\pi_3(SO(4))=Z+Z$ represented by ρ and σ respectively, and $\rho,\ \sigma:\ S^3 \longrightarrow SO(4)$ are given by

$$\rho(q)=\begin{pmatrix} x_1^2-x_2^2-x_3^2+x_4^2 & 2(x_1x_2+x_3x_4) & 2(x_1x_3-x_2x_4) & 0 \\ 2(x_1x_2-x_3x_4) & -x_1^2+x_2^2-x_3^2+x_4^2 & 2(x_2x_3+x_1x_4) & 0 \\ 2(x_1x_3+x_2x_4) & 2(x_2x_3-x_1x_4) & -x_1^2-x_2^2+x_3^2+x_4^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma(q)=\begin{pmatrix} x_4 & x_3 & -x_2 & x_1 \\ -x_3 & x_4 & x_1 & x_2 \\ x_2 & -x_1 & x_4 & x_3 \\ -x_1 & -x_2 & -x_3 & x_4 \end{pmatrix},$$

where a point (x_1, x_2, x_3, x_4) of the unit 3-sphere S^3 with the origin as the centre in the 4-space is identified with a quaternion $q=x_4+x_3i+x_2j+x_1k$ of absolute value 1.

Let $B(m, n)$ denote any total space of a bundle $\mathfrak{B}(m, n)$, then $B(m, n)$, $B(m+n, -n)$ and $B(-m, -n)$ are all homeomorphic (See [1], §26.6). Now we obtain the following

Proposition 1. *The total spaces* $B(-1, 2)$, $B(1, 2)$ *are all homeomorphic to the Stiefel manifold* $V(5, 2)$.

There exists the only one non-trivial class of the 4-sphere bundles over S^5. Then we have

Proposition 2. *The total spaces of the non-trivial bundle are all homeomorphic to the Stiefel manifold* $V(6, 2)$ (*Cf.* [1], §26.10).

§2. Proofs.

Proofs are based on [1].

Proof of Proposition 1. Let $\tau(n)=\{p: V(n+1, 2)\longrightarrow S^n\}$ be the tangent $(n-1)$-sphere bundle consisting of vectors of length 1 tangent to the n-sphere S^n, where $V(n+1, 2)$ is the Stiefel manifold of orthonormal 2-frames in the $(n+1)$-dimensional Euclidean space. The characteristic map for a normal form of the tangent bundle $\tau(n)$ of S^n is a map $T_{n+1}: S^{n-1}\longrightarrow SO(n)$ given by [1], §23.3, (6).

Now, under the assignment of (equivalent classes of) $(n-1)$-sphere bundles over S^n to equivalent classes (under the operation by the non-trivial element of $\pi_0(0(n))$ on $\pi_{n-1}(SO(n))$) of $\pi_{n-1}(SO(n))$, the class of $\tau(n)$ corresponds to the element $[T_{n+1}]\in\pi_{n-1}(SO(n))$ represented by T_{n+1} (See [1], §§18.4—5). On the other hand, by [1], §23.6, we have

$$[T_5]=-\alpha_3+2\beta_3\,.$$

Therefore, $\mathfrak{B}(-1, 2)$ is represented by $\tau(4)=\{V(5, 2)\longrightarrow S^4\}$. This completes the proof.

Proof of Proposition 2. It is well known (e. g., [1], §27.5) that the tangent bundle $\tau(5)=\{V(6, 2)\longrightarrow S^5\}$ of S^5 is not equivalent to a trivial bundle. Hence the proposition follows immediately.

§3. Remark.

The method used in the proofs are entirely mechanical. If the (n-1)-sphere bundles over S^n consist of more than two equivalent classes, and $\tau(n)$ is not equivalent to a trivial bundle, then the determination of the element $[T_{n+1}]$ in $\pi_{n-1}(SO(n))$ gives the assignment of the bundle whose total spaces are all homeomorphic to the Stiefel manifold $V(n+1, 2)$. (Applying the method to the case $n=2$, by the simple calculation of T_3, we obtain an alternative proof of the well-known fact that $V(3, 2)=SO(3)$ is homeomorphic to the lens space B_2 (See [1], §26.2) of type (2, 1), that is, the 3-dimensional real projective space.)

If the (n-1)-sphere bundles over S^n consist of just two classes especially and $\tau(n)$ is not equivalent to a product bundle, then the total spaces of the non-trivial bundle are all homeomorphic to $V(n+1,2)$.

Now, let $\rho(n)$ be the Hurwitz-Radon-Eckmann-Adams' function, that is, when we write n in the form

$$n = 2^{4a+b}(2c+1)-1$$

(where a, b and c are integers $\geqq 0$ and $0 \leqq b \leqq 3$),

then $\rho(n)$ is given by the following:

$$\rho(n)=8a+2^b-1.$$

$\rho(n)$ is the largest integer such that S^n admits $\rho(n)$ independent vector fields ([2]). If $n \neq 1$, 3 and 7, we have $\rho(n) < n-1$. Then, by [1], §27.6, $\tau(n)$ is equivalent in $SO(n)$ to a bundle with structure group $SO(n-\rho(n))$, but not equivaient to a trivial bundle (Cf. [3]). On the other hand, $\pi_2(SO(3))=\pi_6(SO(7))=0$. Thus, without checking the operation of the non-trivial element of $\pi_0(O(n))$ on $\pi_{n-1}(SO(n))$, we have the following theorem (for testing n).

Theorem. *If $\pi_{n-1}(SO(n))=Z_2$, there just two $(n$-$1)$-sphere bundles over S^n and ——— more generally, if there are just two $(n$-$1)$-sphere bundles over S^n, then ——— the total spaces of the non-trivial bundle are all homeomorphic to the Stiefel manifold $V(n+1, 2)$ (for any integer $n \geqq 2$).*

These are the cases $n=5$ (Proposition 2), $8s+3$, $8s+5$ and $8s+7$ $(s \geqq 1)$ ([4]).

Fukuoka University Institute of mathematics.

References

[1] N. E. Steenrod. *The topology of fibre bundles.* Princeton, (1951)

[2] J. F. Adams. *Vector fields onspheres.* Ann. of Math., vol. **75** (1962), pp. 603-632.

[3] M. A. Kervaire. *Non-parallelizability of the n-sphere for $n>7$.* Proc. Nat. Acad. Sci.,

U. S. A., vol. **44** (1958), pp. 280-283.
[4] M. A. KERVAIRE. *Some nonstable homotopy groups af Lie groups.* Illinois Jour. Math., vol. **4** (1960), pp. 161-169.