# SMOOTHNESS OF THE GENERALIZED SOLUTIONS OF LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS, continued

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https://doi.org/10.15017/1448920

出版情報：九州大学教養部数学雑誌. 4 (1), pp.11-17, 1966-05. General Education Department, Kyushu University

Math. Rep.
IV-1, 1966.

# SMOOTHNESS OF THE GENERALIZED SOLUTIONS OF LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS, continued

by

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[1966. 2. 28]

## Introduction

The Schauder estimates for linear elliptic partial differential equations are useful tools in the researches on elliptic partial differential equations, which have been given by Schauder for second order linear elliptic equations. For the general boundary value problems of the higher order linear elliptic equations, S. Agmon, A. Douglis and L. Nirenberg, in their excellent paper [1], have given the estimates near the loundary and A. Douglis and L. Nirenberg [2] have given the interior estimates also.

For simplicity, to mention of a single equation $Lu=F$ of order $2m$ in a domain $\mathfrak{D}$, [2] asserts that for $u\in C^{2m+\alpha}$ in $\mathfrak{D}$ and for any compact subdomain $\mathfrak{A}$

$$|u|^{\mathfrak{A}}_{2m+\alpha}\leqq \text{constant } (|F|^{\mathfrak{D}}_{\alpha}+|u|^{\mathfrak{D}}_{0}).$$

But the assumption $u\in C^{2m+\alpha}$ being unnecessary, in [1] it has been reluxed to $u$ has square integrable derivatives up to order $2m$ in the sense of Sobolev (see appendix 5 in [1]).

In this paper the author will show that in the Schauder estimates near the boundary obtained in [1], the assumption $u\in C^{2m+\alpha}$ can be reluxed to $u\in H_{2m,L_2}$ with the aid of the result of the preceding paper [3] having the same title with this, in other words it means that $u\in H_{2m,L_2}$ implies $u\in C^{2m+\alpha}$ up to the boundary.

## § 1. Assumptions and theorems

Let $\sum_R=\{(x_1,\dots,x_n,t):|\sum_1^n x_i^2+t^2|\leqq R^2,\ t\geqq 0\}$ and $\sigma_R$ be a flat boundary of $\sum_R$. The notations of spaces and norms owe to [1].

We consider the following boundary value problem

$$(1.1)\qquad \begin{aligned} &L(P,\ D)u(P)=F(P) \text{ in } \textstyle\sum_R,\\ &B_j(x,\ D)u(P)=\phi_j(x) \text{ on } \sigma_R,\ j=1,\dots,m, \end{aligned}$$

under the conditions of (i), (ii) and (iii) in [3] which are a little stronger assumptions than those of [1], where $P=(x,\ t)$ and the orders of $L$ and $B_j$ are equal to $2m$ and $m_j<2m$, respectively.

(i) *Condition on* $L$.

(ii) *Complementing condition of the boundary operator relative to* $L$.

(iii) *We assume that* $F$ *and the coefficients of* $L$ *belong to* $C^{\alpha}(\sum_R)$, $0<\alpha<1$, *and have* $|\ \ |_{\alpha}$ *norms bounded by* $k$, *while* $\phi_j$ *and the coefficients of* $B_j$ *belong to* $C^{2m-m_j+\alpha}(\sigma_R)$ *and have* $|\ \ |_{2m-m_j+\alpha}$ *norms bounded by* $k$.

Then we have the following theorems.

*Theorem* 1.1. *Let* $u(P)$ *be a solution of* (1.1) *in* $H_{2m,L_2}$ *on* $\sum_R$, *then* $u(P)$ *belongs to* $C^{2m+\alpha}(\sum_r)$ *for any* $r$, $0<r<R$.

*Theorem* 1.2. *Assume the all conditions of theorem* 1.1. *Then for any* $r$, $R'$, $r<R'<R$, *the following estimate holds*:

$$|u|_{2m+\alpha}^{\Sigma_r}\leqq C([F]_{\alpha}^{\Sigma_{R'}}+\sum_j[\phi_j]_{2m-m_j+\alpha}^{\sigma_{R'}}+|u|_0^{\Sigma_{R'}}),$$

*where* $C$ *depends only on* $r,R,L,B_j,k,n$ *(see p.* 664 [1]).

This estimate has been obtained by Agmon, Douglis and Nirenberg for $u\in C^{2m+\alpha}(\sum_{R'})$. Hence if we assume theorem 1.1, theorem 1.2 is identical with theorem 7.1 in [1].

## § 2. Proof of theorem 1.1

To prove theorem 1.1 we first show that $D^{2m}u$ satisfies the Hölder condition with exponent $\alpha'$, $0<\alpha'<\alpha$. For this purpose we need the following lemma which has been proved in [3].

*Lemma* 2.1. *Assume the conditions of theorem* 1.1. *Then* $u(P)$ *belongs to* $C^{2m}(\sum_{R'})$ and $D^{2m}u$ *atisfies the Hölder condition with exponent* $\alpha'$ *with respect to* $x$ *uniformly for* $t$ *for any* $\alpha'$ *and* $R'$, $0<\alpha'<\alpha$, $0<R'<R$.

We shall begin with this lemma, and first show that $D^{2m}u$ satisfies the Hölder condition with respect to $t$. Without loss of generality we may assme that $u$, $F$ and $\phi_j$ vanish in some neighborhood of the curved boundaries of $\sum_R$ and $\sigma_R$, respectively.

Let $L$ be in the form

$$\sum_{|\beta|\leq 2m} a^{(\beta)}(x,\ t)D^{\beta}u(x,\ t)=F(x,\ t). \tag{2.1}$$

Let $P_0=(x_0,\ 0)$ be an interior point of $\sigma_R$, $\sum_{r,P_0}$ be a semi-sphere $\{P: |P-P_0|\leqq r,\ t\geqq 0,\ r<\frac{R-|P_0|}{2}\}$ and $\xi(P)$ be an infinitely differentiable positive function such that $\xi(P)=1$ for $|P-P_0|\leqq\rho$ and $\xi(P)=0$ for $|P-P_0|\geqq 2\rho$. Then the equation satisfied by $\xi(P)u(P)$ may be written in the form

$$\sum_{|\beta|=2m} a^{(\beta)}(x,\ t)D^{\beta}(\xi u)=f(x,\ t),\ f\in C^{\alpha'}, \tag{2.2}$$

which is identical with (2.1) in $\sum_{\rho,P_0}$.

Set

$$j(r)=\begin{cases} Ke^{\frac{r^2}{r^2-1}} & \text{for } r^2<1\\ 0 & \text{for } r^2\geqq 1,\end{cases}$$

where $K=\left(\int_{|r|\leq 1} e^{\frac{r^2}{r^2-1}}dr\right)^{-1}$, so that $\int_{-\infty}^{\infty} j(r)dr=1$. For any sufficiently small $\varepsilon>0$ and for given function $h(x,\ t)$ set

$$h_\varepsilon(P)=h_\varepsilon(x,\ t)=\varepsilon^{-1}\int_0^{\infty} j\left(\frac{t+\varepsilon-s}{\varepsilon}\right)h(x,\ s)ds.$$

Write the equation satisfied by $C^{2m+\alpha'}$function $(u(P)\xi(P))_\varepsilon$ in the form

$$\sum_{|\beta|=2m} a^{(\beta)}(x,\ t)\ D^{\beta}(u\xi)_\varepsilon=f_\varepsilon(x,\ t)+R(\varepsilon,\ x,\ t). \tag{2.3}$$

We can easily obtain the following inequalities by the similar method used in [1] p. 723

$$|R(\varepsilon,\ x,\ t)-R(\varepsilon,\ x,\ t')|\leqq C_1|t-t'|^{\alpha'} \tag{2.4}$$

$$|R(\varepsilon,\ x,\ t)-R(\varepsilon,\ x',\ t)|\leqq C_2|x-x'|^{\alpha'}, \tag{2.5}$$

where the constants are independent of $\varepsilon$, $C_1$ depends on $|u|_{2m}$ (see p. 723 [1]) and $C_2$ depends on the Hölder constant of $D^{2m}u$ with respect to $x$.

*Lemma* 2.2. For sufficiently small $\rho$

$$[D^{2m}(u(P)\xi(P))]_{\alpha'}^{\Sigma_{\rho,P_0}}\leqq C_3. \tag{2.6}$$

*$C_3$ is independent of $\varepsilon$ but depends on $|u|_{2m}$ and the Hölder constant of $D^{2m}u$ with respect to $x$.*

Proof. Let us rewrite the operators $L'$, the principal part of $L$, and $B_j$ as follows:

$$L'(P, D) = L'(P_0, D) + (L'(P, D) - L'(P_0, D)$$
$$B_j(x, D) = B_j'(x_0, D) + (B_j'(x, D) - B_j'(x_0, D)) - B_j''(x, D), \tag{2.7}$$

where $B_j'$ is the principal part of $B_j$. Using the representation formula (4.7) in [1] for $C^{2m+\alpha'}$, we can write $D^{2m}(u(P)\xi(P))_\varepsilon$ in the form

$$D^{2m}(u(P)\xi(P))_\varepsilon = D^{2m}\int \Gamma(P_0, P-Q)[f_\varepsilon(Q) - (L'(Q, D) - L'(P_0, D))$$
$$\times (u(Q)\xi(Q))_\varepsilon + R(\varepsilon, Q)]_N dQ$$
$$+ \sum_j \int D^{2m}K_j(P_0, x-y, t)[B_j(y, D)(u\xi)_\varepsilon(y, \tau) - B_j'(x_0, D)v(\varepsilon, y, \tau)$$
$$-(B_j'(y, D) - B_j'(x_0, D))(u\xi)_\varepsilon(y, \tau) - B_j''(y, D)(u\xi)_\varepsilon(y, \tau)]_{\tau=0}dy, \tag{2.8}$$

where $Q=(y, \tau)$, $v(\varepsilon, P)$ is the value of the first integral without the operation $D^{2m}$, $\Gamma(P_0, P-Q)$ is the fundamental solution of $L'(P_0, D)$ with singularity at $Q=P$ and $K_j(P_0, x-y, t)$ is the Poisson kernel of the operators $L'(P_0, D)$ and $B_j'(x_0, D)$ with singularity at $y=x$, $t=0$. Denote the first term on the right of (2.8) by $g(P)$. For some positive number $\lambda$ depending on $\rho$ we have

$$[g(P)]_{\alpha'}^{\Sigma_{2\rho}, P_0} \leqq \lambda [D^{2m}(u\xi)_\varepsilon]_{\alpha'}^{\Sigma_{2\rho}, P_0} + C_4, \tag{2.9}$$

where $C_4$ is independent of $\varepsilon$ but will depend on $\lambda$. This estimate is obtained similarly to [2]. Denote the second term on the right of (2.8) by $h(P)$. Taking into account the fact that $D^{2m}(u\xi)_\varepsilon$ satisfies the Hölder condition with respect to $x$ uniformly for $t$ and $\varepsilon$, and using theorem 3.4 ($a$) in [1], (2.4), (2.5) and (2.9), we have

$$[h(P)]_{\alpha'}^{\Sigma_{2\rho}, P_0} \leqq C_5\lambda [D^{2m}(u\xi)_\varepsilon]_{\alpha'}^{\Sigma_{2\rho}, P_0} + C_6. \tag{2.10}$$

Here the constants are independent of $\varepsilon$, $C_5$ is independent of $\lambda$, $C_6$ depends on the Holder constant of $D^{2m}u$ with respect to $x$ but is independent of that with respect to $t$. With the aid of (2.8), (2.9) and (2.10), for sufficiently small $\rho$, there exists a constant $C_7$ which is independent of $\varepsilon$ such that

$$[D^{2m}(u\xi)_\varepsilon(P)]_{\alpha'}^{\Sigma_{2\rho}, P_0} \leqq C_7.$$

Letting $\varepsilon \to 0$, we obtain (2.6). This completes the proof of lemma 2.2.

From theorem A5.1 [1] $D^{2m} u$ satisfies the Hölder condition with exponent $\alpha$ for any subdomain $\mathfrak{A} \subset \sum_R$, distance $(\mathfrak{A}, (\sum_R)^c) > 0$. Choose some domain $\mathfrak{A}$ and a finite number of $\sum_{\rho,P_0}$ in lemma 2.2 such that $\sum_R \supset \bigcup_{P_0} \sum_{\rho,P_0} \cup \mathfrak{A} \supset \sum_{R'}$, then we have the desired Hölder condition with respect to $t$.

*The Hölder condition with exponent $\alpha$ with respect to $x$.*

Now we shall show that $D^{2m}u$ satisfies the Hölder condition with exponent $\alpha$ with respect to $x$. For sufficiently small $|h| \neq 0$, $h=(h_1, \dots, h_n)$, set $\dfrac{u(x+h, t)-u(x, t)}{|h|^{\alpha-\alpha'}} = U(x, t)$, then $U(x, t)$ satisfies

$$
(2.11)\quad
\begin{aligned}
&\sum a^{(\beta)}(x+h, t)D^{\beta}U(x, t)+\sum \frac{a^{(\beta)}(x+h, t)-a^{(\beta)}(x, t)}{|h|^{\alpha-\alpha'}} D^{\beta}u(x, t)\\
&\qquad = \frac{F(x+h, t)-F(x, t)}{|h|^{\alpha-\alpha'}},\\
&\sum b_{j,\gamma}(x+h)D^{\gamma}U(x, 0)+\sum \frac{b_{j,\gamma}(x+h)-b_{j,\gamma}(x)}{|h|^{\alpha-\alpha'}} D^{\gamma}u(x, 0)\\
&\qquad = \frac{\phi_j(x+h)-\phi_j(x)}{|h|^{\alpha-\alpha'}}.
\end{aligned}
$$

To avoid complexity write (2.11) in the form

$$
(2.12)\quad
\begin{aligned}
&\sum_{|\beta|=2m} a^{(\beta)}(x, t)D^{\beta}U(x, t)=f(x, t),\\
&\sum_{|\gamma|=m_j} b_{j,\gamma}(x)D^{\gamma}U(x, 0)=\varphi_j(x),
\end{aligned}
$$

where $a^{(\beta)} \in C^{\alpha}(\sum_{R''})$, $b_{j,\gamma} \in C^{2m-m_j+\alpha}(\sigma_{R''})$, $f \in C^{\alpha'}(\sum_{R''})$, $\varphi_j \in C^{2m-m_j+\alpha'}(\sigma_{R''})$, $R' < R'' < R$, and they have bounded norms independent of $h$ in the corresponding spaces.

We can apply the Schauder estimate (theorem 7.1 [1]) to (2.12), because $U \in C^{2m+\alpha'}$. Hence

$$
(2.13)\qquad |U|_{2m+\alpha'}^{\Sigma_{R''}} \leqq C,
$$

the constant being independent of $h$. With the aid of (2.13) and lemma A in the appendix we can see that $D^{2m}u$ satisfies the desired Hölder condition with respect to $x$ uniformly for $t$.

*The Hölder condition with exponent $\alpha$ with respect to $t$.*

It remains here for us to show that $D^{2m}u$ satisfies the Hölder condition with exponent $\alpha$ with respect to $t$. To this end, using the above result, we can obtain the desired condition by the same method as described in the first part of this section.

This completes the proof of theorem 1.1.

## Appendix

*Lemma A (Ono-Ikebe). Let $u(x)$ be a continuous function of one variable with compact support. For any $h$ $k \neq 0$, and for some $\alpha$, $\beta$, $0<\alpha$, $\beta<1$, $\alpha+\beta<1$, we assume that*

$$\text{(A.1)} \qquad \frac{|u(x+h+k)-u(x+k)-(u(x+h)-u(x))|}{|h|^{\alpha}|k|^{\beta}} \leqq K.$$

*Then $u \in C^{\alpha+\beta}$ and $[u]_{\alpha+\beta}$ is bounded by constant depending only on $K$, $\alpha$ and $\beta$.*

Proof. Put $k=h$ in (A.1), then

$$\text{(A.2)} \qquad \begin{aligned} &\frac{|u(x+h)-u(x)-u(x+2h)+u(x+h)|}{|h|^{\alpha+\beta}} \\ &=\left|2\frac{u(x+h)-u(x)}{|h|^{\alpha+\beta}}-2^{\alpha+\beta}\frac{u(x+2h)-u(x)}{|2h|^{\alpha+\beta}}\right| \leqq K. \end{aligned}$$

Set $\sup\limits_{h} \dfrac{|u(x+h)-u(x)|}{|h|^{\alpha+\beta}}=A(x)$. Then there exists a sequence $\{h_m\}$ such that

$$\lim_{m\to\infty} \frac{|u(x+h_m)-u(x)|}{|h_m|^{\alpha+\beta}}=A(x).$$

1. When $A(x)<\infty$, from (A.2) we have

$$\text{(A.3)} \qquad \begin{aligned} K &\geqq \left|2\frac{u(x+h_m)-u(x)}{|h_m|^{\alpha+\beta}}-2^{\alpha+\beta}\frac{u(x+2h_m)-u(x)}{|2h_m|^{\alpha+\beta}}\right| \\ &\geqq 2\frac{|u(x+h_m)-u(x)|}{|h_m|^{\alpha+\beta}}-2^{\alpha+\beta}\frac{|u(x+2h_m)-u(x)|}{|2h_m|^{\alpha+\beta}}. \end{aligned}$$

Set $\sup\limits_{m} \dfrac{|u(x+2h_m)-u(x)|}{|2h_m|^{\alpha+\beta}}=B(x)$, then

$$\text{(A.4)} \qquad 0 \leqq B(x) \leqq A(x).$$

From (A.3), (A.4) we see that

$$K \geqq 2A(x)-2^{\alpha+\beta}B(x) \geqq (2-2^{\alpha+\beta})\ A(x).$$

Hence we conclude that $A(x) \leqq \dfrac{K}{2-2^{\alpha+\beta}}$.

2. To complete the proof we have to show that $A(x) \neq \infty$.

If $A(x)=\infty$ for some $x$, for the fixed $x$ we can find the sequence $\{h'_m\}$ such that

$$\frac{|u(x+h'_m)-u(x)|}{|h'_m|^{\alpha+\beta}} \geqq \frac{|u(x+2h'_m)-u(x)|}{|2h'_m|^{\alpha+\beta}}, \quad \lim_{m\to\infty} \frac{|u(x+h'_m)-u(x)|}{|h'_m|^{\alpha+\beta}}=\infty.$$

This sequence $\{h'_m\}$ is made as follows:
there exists the integer $n_m$ such that

$$\frac{|u(x+h_m)-u(x)|}{|h_m|^{\alpha+\beta}} < \frac{|u(x+2h_m)-u(x)|}{|2h_m|^{\alpha+\beta}} < \cdots\cdots$$

$$< \frac{|u(x+2^{n_m}h_m)-u(x)|}{|2^{n_m}h_m|^{\alpha+\beta}} \geqq \frac{|u(x+2^{n_m+1}h_m)-u(x)|}{|2^{n_m+1}h_m|^{\alpha+\beta}},$$

so we may set $h'_m=2^{n_m}h_m$. Because for sufficiently large integer, $\frac{|u(x+2^{n_m+1}h_m)-u(x)|}{|2^{n_m+1}h_m|}$ are bounded.

Thus we have

$$K \geqq (2-2^{\alpha+\beta}) \frac{|u(x+h'_m)-u(x)|}{|h'_m|^{\alpha+\beta}} \tag{A.5}$$

similarly to (A.3). The right hand side of (A.5) tends to infinity. This is the contradiction, and the proof is complete.

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