ON AN EXTENSION THEOREM OF n-SEPARABLE BOOLEAN ALGEBRAS

Mori, Tôru
Department of Mathematics, Saga University

https://doi.org/10.15017/1448919

出版情報：九州大学教養部数学雑誌. 4 (1), pp.1-10, 1966-05. General Education Department, Kyushu University
バージョン：
権利関係：

Math. Rep.
IV-1, 1966.

ON AN EXTENSION THEOREM OF $\mathfrak{n}$-SEPARABLE BOOLEAN ALGEBRAS

By

Tôru Mori

[1966. 1. 29.]

1. Introduction. In 1948, Alfred Horn and Alfred Tarski [2] has proved the following theorem.

Theorem 8′. *For every separable Boolean algebra* B *the following conditions are equivalent*:

(i) B *has a stricily positive* σ*-additive measure*;

(ii) B *is atomic*;

(iii) B *is weakly* σ*-distributive.*

The purpose of the present paper is to generalized this theorem to $\mathfrak{n}$-separable Boolean algebras.

2. Terminology and symbolism.

The symbol $\cup$ stands for the Boolean join. The symbol $\cap$, similarly, stands for the Boolean meet. The zero element and the unit element of a Boolean algebra are denoted by 0 and 1, respectively. If a is an element of a Boolean algebra, a' denotes the complement of a in it.

A Boolean algebra and the set of all its elements will be denoted by the same letter.

A subset A of a Boolean algebra B is said to be covering of B if $\bigcup_{a\in A} a=1$.

A covering of a Boolean algebra is said to be $\mathfrak{m}$-covering of B if $|A|\leqq\mathfrak{m}$, where $|A|$ represents the cardinal number of the set A. A covering or $\mathfrak{m}$-covering A is called partition, respectively, $\mathfrak{m}$-partition if elements of A are disjoint. In the sequel we assume that $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals.

If A and C are subsets of a Boolean algebra B, we say that A refines C, if for every $a\in A$ there exists $c\in C$ such that $a\subset c$; we say that A weakly refines C if for every $a\in A$ there exists a finite sequence

$$\{c_1, c_2,\dots, c_k\}\subset C$$

such that $a \subset \bigcup_{i=1}^{k} c_i$.

3. *T*. Traczyk's results.

If S and T are non-empty sets, then S^T will denote the set of all mappings of T into S. $F(S)$ will denote the class of all finite non-void subsets of S. According to the convention assumed S^T, $F(S)^T$ is the set of all functions φ defined on T with values in $F(S)$, i.e. such that, for every $t \in T$, $\varphi(t)$ is a finite non-void subset of S. If $\{a_{ts}: t \in T, s \in S\}$ is any set of elements of a Boolean algebra, and $\varphi \in F(S)^T$, then

$$a_{t\varphi(t)} = \bigcup_{s \in \varphi(t)} a_{ts}.$$

A Boolean algebra $\boldsymbol{B}$ is said to be weakly $(\mathfrak{m}, \mathfrak{n})$-distributive provided

$$\bigcap_{t \in T} \bigcup_{s \in S} a_{ts} = \bigcup_{\varphi \in F(S)^T} \bigcap_{t \in T} a_{t\varphi(t)}$$

for every set $\{a_{ts}: t \in T, s \in S, |T| \leqq \mathfrak{m}, |S| \leqq \mathfrak{n}\}$ of elements in $\boldsymbol{B}$, such that all the meets $\bigcap_{t \in T} a_{t\varphi(t)}$ exist and all the joins $\bigcup_{s \in S} a_{ts}$ exist and the meet $\bigcap_{t \in T} \bigcup_{s \in S} a_{ts}$ exist.

A Boolean algebra $\boldsymbol{B}$ is said to be weakly $\mathfrak{m}$-distributive if it is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive.

Now we intoduce the following results which were proved by T. Traczyk in [4].

Lemma 1. *A Boolean algebra $\boldsymbol{B}$ is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive if and only if for every class*

$$\{\boldsymbol{A}_t: t \in T, |T| \leqq \mathfrak{m}\}$$

of $\mathfrak{n}$-coverings of $\boldsymbol{B}$ there exists a covering $\boldsymbol{A}$ of $\boldsymbol{B}$ which weakly refines every $\boldsymbol{A}_t$.

Proof of necessity. Let

$$\boldsymbol{A}_t = \{a_{ts}: s \in S, |S| \leqq \mathfrak{n}\}$$

and

$$\boldsymbol{A} = \{a \in \boldsymbol{B}: \{a\} \text{ weakly refines every } \boldsymbol{A}_t\}$$

In fact, $\boldsymbol{A}$ is non-empty, for example $0 \in \boldsymbol{A}$ and obviously $\boldsymbol{A}$ weakly refines every $\boldsymbol{A}_t$. Suppose that $\boldsymbol{A}$ is not a covering of $\boldsymbol{B}$. Then $\bigcup_{a \in A} a$ does not exist or it exists but is not equal to 1. Therefore there is some $b \neq 0$, $b \in \boldsymbol{B}$, disjoint with every $a \in \boldsymbol{A}$.

By the weakly $(\mathfrak{m}, \mathfrak{n})$-distributivity of $\boldsymbol{B}$ there exists a mapping $\varphi \in F(S)^T$

such that (See [1], p. 128 (w_2))

$$b \cap \bigcap_{t\in T} a_{t\varphi(t)} \neq 0.$$

Where this inequality should be read as follows: the meet b and all elements $a_{t\varphi(t)} (t\in T)$ does not exist or it exists but is not equal to 0. Hence there is an element $c\neq 0$, $c\in \boldsymbol{B}$ such that $c\subset b$ and $c\in \boldsymbol{A}$. This leads to a contradiction because $b\cap c\neq 0$.

Proof of sufficiency. Let

$$\{b_{ts}: t\in T,\ s\in S.\ |T|\leqq \mathfrak{m},\ |S|\leqq \mathfrak{n}\}\subset \boldsymbol{B}$$

We assume now the existence of

$$\bigcup_{s\in S} b_{ts} \text{ for every } t\in T,\ \bigcap_{t\in T}\bigcup_{s\in S} b_{ts}=b\neq 0$$

and

$$\bigcap_{t\in T} b_{t\varphi(t)} \text{ for every } \varphi\in F(S)^T.$$

Let $s_0\notin S$, $S_0=S\cup\{s_0\}$, $b_{ts_0}=b'$ for every $t\in T$ and let

$$\boldsymbol{A}_t=\{b_{ts}: s\in S_0\}.$$

Then every $\boldsymbol{A}_t$ becomes a covering of $\boldsymbol{B}$ and, by the hypothesis, there exists a covering $\boldsymbol{A}$ which weakly refines every $\boldsymbol{A}_t$. Therefore there exists an element $a\in \boldsymbol{A}$ such that $b\cap a\neq 0$. By setting $d=b\cup a$, we obtain $d\subset a$ and $d\subset b$. It follows from the property of $\boldsymbol{A}$ that there exists a mapping $\varphi\in F(S_0)^T$ such that

$$d\subset \bigcap_{t\in T} b_{t\varphi(t)}.$$

Since $d\subset b$, we may assume with no loss of generality that $\varphi(t)$ does not contains s_0 for every $t\in T$. Therefore we can say that there exists a mapping $\varphi\in F(S)^T$ such that

$$\bigcap_{t\in T} b_{t\varphi(t)}\neq 0.$$

This means that the Boolean algebra $\boldsymbol{B}$ is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive (See [1], p. 127 (w_1)). Q. E. D.

A subset $\boldsymbol{A}$ of a Boolean algebra $\boldsymbol{B}$ is said to be dense (in $\boldsymbol{B}$) if, for every non-zero element $b\in \boldsymbol{B}$, there exists an element $a\in \boldsymbol{A}$ such that $0\neq a\subset b$.

By minimal extension of $\boldsymbol{B}$ we mean a complete Boolean algebra $\boldsymbol{B}^\infty$

which contains $\boldsymbol{B}$ as a dense subalgebra.

A Boolean algebra $\boldsymbol{B}$ is said to satisfy the $\mathfrak{m}$-chain condition provided every set of disjoint elements in $\boldsymbol{B}$ has power $\leqq \mathfrak{m}$.

Theorem 1. *If a Boolean algebra* $\boldsymbol{B}$ *satisfies the* $\mathfrak{n}$*-chain condition, then the following conditions are equivalent*:

(i) $\boldsymbol{B}$ *is weakly* $(\mathfrak{m}, \mathfrak{n})$*-distributive*;

(ii) *for every family* $\{\boldsymbol{A}_t\colon t \in T, |T| \leqq \mathfrak{m}\}$ *of* $\mathfrak{n}$*-partitions of* $\boldsymbol{B}$ *there exists a covering* $\boldsymbol{A}$ *of* $\boldsymbol{B}$ *which weakly refines every* $\boldsymbol{A}_t$.

Proof. Evidently (i) $\rightarrow$ (ii) by Lemma 1. The proof (ii) $\rightarrow$ (i) is based on the well-ordering principle.

Let $\boldsymbol{C}$ be any $\mathfrak{n}$-covering of $\boldsymbol{B}$. Let

$$c_1,\ c_2, \ldots,\ c_\alpha, \ldots\ (\alpha < \beta)$$

be a transfinite sequence of all elements of $\boldsymbol{C}$, where β is the least ordinal number of power . We denote by $\boldsymbol{B}^\infty$ the minimal extension of $\boldsymbol{B}$. Let us define in $\boldsymbol{B}^\infty$.

$$b_1 = c_1,\ b_2 = c_2 \cap c'_1, \ldots,\ b_\alpha = c_\alpha \cap \bigcap_{\gamma<\alpha} c'_\gamma, \ldots$$

The set

$$\{b_\alpha\colon \alpha < \beta\}$$

is an $\mathfrak{n}$-partition of $\boldsymbol{B}^\infty$ such that $b_\alpha \subset c_\alpha$ for every $\alpha < \beta$.

$\boldsymbol{B}^\infty$ satisfying the $\mathfrak{n}$-chain condition and $\boldsymbol{B}$ being a dense subset of $\boldsymbol{B}^\infty$, every b_α is a join of at most $\mathfrak{n}$ disjoint elements of $\boldsymbol{B}$. (see [1], p. 95 F). Thus there exists in $\boldsymbol{B}$ an $\mathfrak{n}$-partition which refines $\boldsymbol{C}$.

Now let

$$\{\boldsymbol{A}_t\colon t \in T,\ |T| \leqq \mathfrak{m}\}$$

be a family of $\mathfrak{n}$-coverings of $\boldsymbol{B}$. Let $\boldsymbol{C}_t$ be an $\mathfrak{n}$-partition which refines $\boldsymbol{A}_t$. By (ii) there exists a covering $\boldsymbol{A}$ which weakly refines every $\boldsymbol{C}_t$; obviously, it weakly refines every $\boldsymbol{A}_t$, too. Therefore $\boldsymbol{B}$ is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive by Lemma 1. Q. E. D.

4. $\mathfrak{n}$-separable Boolean algebra.

A Boolean algebra $\boldsymbol{B}$ is said to be $\mathfrak{n}$-separable provided it contains a dense subset $\boldsymbol{A}$ such that $|\boldsymbol{A}| \leqq \mathfrak{n}$. If $\boldsymbol{B}$ is σ-separable, then $\boldsymbol{B}$ is said to be a separable Boolean algebra.

Lemma 2. *If a Boolean algebra* $\boldsymbol{B}$ *satisfying the* $\mathfrak{n}$*-chain condition contains a non-zero element d which does not include any atom, then there exists an infinite disjoint subset*

$$\{a_s:\ s\in S,\ |S|\leqq \mathfrak{n}\}$$

of $\boldsymbol{B}$ *such that*

$$d=\bigcup_{s\in S} a_s\ \ (d\neq a_s,\ a_s\neq 0)$$

Proof. Since d does not contain any atom, there exists an infinite disjoint set composed of non-zero proper subelements of d. Let $\mathfrak{F}$ be the class of all such infinite disjoint sets. $\mathfrak{F}$ is partially ordered by defining $E\leqq F$ to mean that $E\subseteqq F$, where $\subseteqq$ is set-theoretical inclusion and E, F are elements of $\mathfrak{F}$. Let G be the set-theoretical union of all elements of a complete ordered subset[1] $\mathfrak{F}_0$ of $\mathfrak{F}$. Then G is a member of $\mathfrak{F}$ and moreover G is an upper bound of $\mathfrak{F}$.

We now know that F satisfies the conditions of Zorn's lemma, and, must, therefore, contain at least one maximal element, say M. Since B satisfies the $\mathfrak{n}$-chain condition, we can represent $M=\{a_s:\ s\in S,\ |S|\leqq \mathfrak{n}\}$.

It remains to be shown that

$$d=\bigcup_{s\in S} a_s\ (d\neq a_s,\ a_s\neq 0).$$

Suppose that d_0 is any upper bound of M and $d\not\subset d_0$. Then there exists an element s such that $0\neq s\subset d$, $s\cap d_0=0$ (Note that $d\not\subset d_0$ is equivalent to $d\cap d'_0\neq 0$). Thus we obtain an element $M\cup\{s\}\in\mathfrak{F}$ with $M\cup\{s\}\neq M$, $M\leqq M\cup\{s\}$, contrary to the maximality of $\mathfrak{F}$. Hence d is the least upper bound of M.

Theorem 2. *If an* $\mathfrak{n}$*-separable Boolean algebra* $\boldsymbol{B}$ *is weakly* $\mathfrak{n}$*-distributive, then* $\boldsymbol{B}$ *is atomic.*

Proof. Suppose that B is not atomic. By definition, there is an $a\neq 0$ in B which does not include any atom. Let D be a dense subset in B, and let D_0 be the set of all non-zero elements d_t of D for which $d_t\subset a$.

Now suppose that $D_0=\{d_t\neq 0;\ t\in T,\ |T|\leqq \mathfrak{n}\}$. By Lemma 2, for every d_t, there is an infinite disjoint set $\{a_{ts}:\ s\in S,\ |S|\leqq \mathfrak{n}\}$ such that

$$d_t=\bigcup_{s\in S} a_{ts}\ (d_t\neq a_{ts},\ a_{ts}\neq 0)$$

1) If P is partially ordered and if, moreover, for every pair a, b in P either $a\leqq b$ or $b\leqq a$, then P is said to be completely ordered.

Let $s_0 \notin S$, $S_0 = S \cup \{s_0\}$, $a_{ts_0} = d_t'$ for every $t \in T$
and let

$$A_t = \{a_{ts} : s \in S_0\}.$$

In this manner every A_t becomes a $\mathfrak{n}$-partition of B and then, we have
(1) $\quad d_t \cap a'_{t\varphi(t)} \neq 0$ for every $\varphi \in F(S)^T$, $t \in T$.

On account of the $\mathfrak{n}$-separability of B, B satisfies the $\mathfrak{n}$-chain condition. Hence, by Theorem 1, there exists a coveriong A of B which weakly refines every A_t.

Let $A = \{a_u : u \in U\}$. Then there is an a_{u_0} such that $a \cap a_{u_0} \neq 0$. This follows from that $\bigcup_{u \in U} a_u = 1$. Thus, by setting $b = a \cap a_{u_0}$, we obtain $b \subset a_{u_0}$ and $b \subset a$. Therefore there exists a mapping $\varphi \in F(S_0)^T$ such that $b \subset a_{t\varphi(t)}$ for every $t \in T$. Since $0 \neq q \subset a$, there is a t_0 such that $d_{t_0} \subset b$ and $d_{t_0} \in D_0$. Consequently, for this t_0, there is a mapping $\varphi \in F(S_0)^T$ such that $d_{t_0} \subset a_{t_0\varphi(t_0)}$.

It follows from this fact that we can find a mapping $\psi \in F(S)^T$ such that

$$\psi(t_0) = \begin{cases} \varphi(t_0) - \{s_0\} & \text{if } s_0 \in \varphi(t_0) \\ \varphi(t_0) & \text{if } s_0 \notin \varphi(t_0) \end{cases}$$

and $\psi(t) \in F(S)$ for every $t \in T - \{t_0\}$. For this ψ, we have $d_{t_0} \subset a_{t_0\psi(t_0)}$.

This means that

$$d_{t_0} \cap a'_{t_0\psi(t_0)} = 0 \text{ for } \psi \in F(S).$$

This contradicts (1). The proof is complete.

5. Some known results.

A subalgebra A of a Boolean algebra B is said to be an $\mathfrak{m}$-regular subalgebra of B, when for every set $C \subset A$, $|C| \leqq \mathfrak{m}$, if the join $\bigcup_{a \in C} a$ exists in A, it is also the join of this set in B. The following result is given by Smith and Tarski (See [3]).

Theorem 3. An atomic Boolean algebra B is complete distributive.

Proof. Fix $\mathfrak{m}$ arbitrarily. Let

$$\{a_{ts} : t \in T,\ s \in S,\ |T| \leqq \mathfrak{m},\ |S| \leqq \mathfrak{m}\} \subset B.$$

We suppose now that the existence of

$$\bigcup_{s \in S} a_{ts} \text{ for every } t \in T,\ \bigcap_{t \in T} \bigcup_{s \in S} a_{ts} = b \neq 0.$$

Let c be an atom of B such that $c \subset b$. Then $c \subset \bigcup_{s \in S} a_{ts}$ holds for every $t \in T$. Hence it follows that there exists a $s \in S$ such that $c \subset a_{ts}$ for $t \in T$. Thus we can choose a mapping $\varphi \in S^T$ such that $c \subset a_{t\varphi(t)}$ for every $t \in T$. Accordingly, $\bigcap_{t \in T} a_{t\varphi(t)}$ does not exist or is not zero. This implies B is $\mathfrak{m}$-distributive (See [1], p. 63 (d_1)). Since $\mathfrak{m}$ is arbitrary, B is complete distributive.

The following result is also given by Smith and Tarski (See [3]). For any subset A of a Boolean algebra, we shall let $\delta(A)$ denote the least cardinal number such that the power of every disjoint subset of A is at most $\mathfrak{n}$.

Theorem 4. *If a Boolean algebra B is isomorphic to an $\mathfrak{n}$-regular subfield $\mathfrak{F}_0$ of an $\mathfrak{n}$-field $\mathfrak{F}$ and $\delta(B) \leq \mathfrak{n}$ then B is atomic.*

Proof. Let X be the underlying set of $\mathfrak{F}_0$ and let h be an isomorphism of B onto $\mathfrak{F}_0$. For any element $x \in X$, there exists an element a such that $x \notin h(a)$. Therefore, by Zorn's lemma, let A be a maximal disjoint set of elements a of B satisfying $x \notin h(a)$.

First, we show that $\bigcup_{a \in A} a$ exists in B. Suppose that $\bigcup_{a \in A} a$ does not exist. Then, for any upper bound u of A, there is an element d such that $0 \neq d \subset u$ and $d \cap a = 0$ for every $a \in A$. Consequently, there would be elements b, $c \in B$ such that $b \cap c = 0$, $b \neq 0$, $c \neq 0$ and $a \cap b = a \cap c = 0$ for every $a \in A$. But the maximality of A would be imply that $x \in h(b)$ and $x \in h(c)$, so that $h(b) \cap h(c) \neq \phi$. On the other hand, since $b \cap c = 0$, we obtain $h(b) \cap h(c) = h(b \cap c) = \phi$. This leads to contradiction. Accordingly, $\bigcap_{a \in A} a$ must exist. Moreover, since A is disjoint, $|A| \leq \mathfrak{n}$ by $\delta(B) \leq \mathfrak{n}$. A mapping h being an isomorphism of B onto $\mathfrak{F}_0$, $\{h(a) : a \in A\}$ has a least upper bound in $\mathfrak{F}_0$. That is, $\bigcup_{a \in A} h(a)$ exists in $\mathfrak{F}_0$. Moreover, by the $\mathfrak{n}$-regularity of $\mathfrak{F}_0$, we have

$$\bigcup_{a \in A} h(a) = \sum_{a \in A} h(a),$$

where $\sum$ is the set-theoretical union in $\mathfrak{F}$.

Next, we show that if $x \in h(b)$ for some $b \in B$, then $h(b) - \bigcup_{a \in A} h(a) = h(b) - \sum_{a \in A} h(a) = D$ is an atom of $\mathfrak{F}_0$. Since x is contained in D, D is not empty. Suppose that $h(c)$ is a non-void subset of D, then we have

$$h(c) \cap (\sum_{a \in A} h(a)) = \phi$$

By the maximality of A, x is contained in $h(c)$. If $h(c) \neq D$, then a non-void subset $h(c)'\cap D$ of D contains x, similarly. This leads to contradiction because $h(c)'\cap D$ does not contain x.

Therefore, every element of X is contained in an atom. Thus $\mathfrak{F}_0$ is atomic, consequently, $\boldsymbol{B}$ is also atomic. Q. E. D.

A real function $\mathfrak{m}$ defined on a Boolean algebra $\boldsymbol{B}$ is said to belan $\mathfrak{m}$-additive measure provided:

(i) $0\leqq m(a)\leqq\infty$ for every $a\in\boldsymbol{B}$. There exists an element $b\in\boldsymbol{B}$ such that $m(b)<\infty$,

(ii) $m(\bigcup_{t\in T}a_t)=\sum_{t\in T}m(a_t)$

for every disjoint set $\{a_t: t\in T, |T|\leqq\mathfrak{m}\}\subset\boldsymbol{B}$, such that $\bigcup_{t\in T}a_t$ exists.

Where, the infinite sum $\sum_{t\in T}\mathfrak{m}(a_t)$ should be understood as follows: if $m(a_t)=0$ for all t except an finite or enumerable sequence $t_n(t_i\neq t_j$ for $i\neq j)$, and $\sum_n m(a_{t_n})=c<\infty$ then $\sum_{t\in T}m(a_t)=c$, otherwise $\sum_{t\in T}m(a_t)=\infty$.

An $\mathfrak{m}$-additive measure on a Boolean algebra $\boldsymbol{B}$ is said to be 2-valued provided it assumes exactly two values: the number 0 and the number 1.

We need the following result of Sikorski, but we does not prove here (See [1], p. 98).

Theorem 5. Let B be a Boolean algebra. The following conditions are equivalent:

(i) *for every element $a\neq 0$ $(a\in\boldsymbol{B})$ there exists a 2-valued $\mathfrak{m}$-additive mea sure such that $m(a)=1$.*

(ii) *$\boldsymbol{B}$ is isomorphic to an $\mathfrak{m}$-regular subalgebra of an $\mathfrak{m}$-field of sets.*

The following result is due to Horn and Tarski (See [2]).

Theorem 6. If $\boldsymbol{B}$ is an atomic Boolean algebra, then for every $a\neq 0$ $(a\in\boldsymbol{B})$ and for arbitrary power $\mathfrak{m}$ there exists a 2-valued $\mathfrak{m}$-additive measure such that $m(a)=1$.

Proof. Let b be an atom which is contained in a.
The formula

$$m(x)=\begin{cases}1 & if\ b\subset x\\ 0 & if\ b\subset x'\end{cases}$$

defines a 2-valued $\mathfrak{m}$-additive measure m. Q. E. D.

6. The extension theorem.

A real function m defined on a Boolean algebra $\boldsymbol{B}$ is said to be measure provided

(i) $0 \leqq m(a) \leqq \infty$ for every $a \in \boldsymbol{B}$, and there exists an element $b \in \boldsymbol{B}$ such that $m(b) < \infty$,

(ii) $m(a \cup b) = m(a) + m(b)$ whenever $a \cap b = 0$, a, $b \in \boldsymbol{B}$.

A measure m on a Boolean algebra $\boldsymbol{B}$ is said to be strictly positive if $\mathfrak{M}(a) > 0$ for all $a \neq 0$ $(a \in \boldsymbol{B})$.

Obviously every $\mathfrak{m}$-additive measure is a measure in the above sense.

Theorem 7. Let $\boldsymbol{B}$ be a separable Boolean algebra. The following conditions are equivalent:

(i) *$\boldsymbol{B}$ has a strictly positive σ-additive measure;*

(ii) *for every $a \neq 0$ $(a \in \boldsymbol{B})$ there exists a 2-valued σ-additive measure m such that $m(a) = 1$.*

Proof. (i)$\rightarrow$(ii). $\boldsymbol{B}$ is atomic by Theorem 8′. Hence $\boldsymbol{B}$ satisfies the condition (ii) by Theorem 6.

(ii)$\rightarrow$(i). By Theorem 5, $\boldsymbol{B}$ is isomorphic to a σ-regular subalgebra of an σ-field of sets. In addition, $\boldsymbol{B}$ being separable, we have $\delta(\boldsymbol{B}) \leqq \sigma$. Hence, by Theorem 4, $\boldsymbol{B}$ is atomic. Therefore $\boldsymbol{B}$ satisfies the condition (i) by Theorem 8′.

Theorem 8. (The extension theorem) Let $\boldsymbol{B}$ be an $\mathfrak{m}$-separable Boolean algebra. The following conditions are equivalent:

(i) *for every $a \neq 0$ $(a \in \boldsymbol{B})$ there exists a 2-valued $\mathfrak{m}$-additive measure m;*

(ii) *$\boldsymbol{B}$ is atomic;*

(iii) *$\boldsymbol{B}$ is weakly $\mathfrak{m}$-distributive.*

Proof. (i)$\rightarrow$(ii). By Theorem 5, $\boldsymbol{B}$ is isomorphic to an $\mathfrak{m}$-regular subalgebra of an $\mathfrak{m}$-field of sets. Moreover, by the $\mathfrak{m}$-separability of $\boldsymbol{B}$, it follows that $\delta(\boldsymbol{B}) \leqq \mathfrak{m}$. Therefore $\boldsymbol{B}$ is atomic by Theorem 4.

(ii)$\rightarrow$(iii). By Theorem 3, $\boldsymbol{B}$ is complete distributive and consequently $\mathfrak{m}$-distrbutive. This implies that $\boldsymbol{B}$ is weakly $\mathfrak{m}$-distributive (See [1], p. 129, 30.2).

(iii)$\rightarrow$(ii). By Theorem 2, $\boldsymbol{B}$ is atomic.

(ii)$\rightarrow$(i). This follows directly from Theorem 6. Q. E. D.

Department of Mathematics, Saga University, Saga.

Bibliography

[1] R. Sikorski, *Boolean algebras*, Berlin 1964.

[2] A. Horn and A. Tarski, *Measures in Boolean algebras*, Trans. Amer. Math. Soc. **64**, 1948 pp. 467∼497.

[3] E. C. SMITH and A. TARSKI, ***Higher degrees of distrbutiviiy and completeness in Boolean algebras***, Trans. Amer. Math. Soc. **84**. 1957, pp. 230~257.

[4] T. TRACZYK, *Minimal extension of weakly distrbutive Boolean algebras*, Coll. Math. XI_1, 1963, pp. 17~24.