SMOOTHNESS OF THE GENERALIZED SOLUTIONS OF LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Ikebe, Nobunori
General Education Department, Kyushu University

https://doi.org/10.15017/1448917

出版情報：九州大学教養部数学雑誌. 3 (1), pp.19-26, 1965-12. General Education Department, Kyushu University

Math. Rep.
III-1, 1965.

SMOOTHNESS OF THE GENERALIZED SOLUTIONS OF LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

By

Nobunori IKEBE

[1965. 8. 23]

Introduction

In the smoothness problem of the generalized solutions in Hilbert space of linear elliptic partial differential equations, it has been necessary to assume considerable differentiability for the coefficients of equations.

For an elliptic equation

$$Lu=f \qquad\qquad \text{(order of } L=2m)$$

in a domain Ω, assuming the coefficients and f belong to $C^{\alpha}(\Omega)$, S Agmon A. Douglis and L. Nirenberg ([1]) have shown that the generalized solution in $H_{2m},\,_{L_2}$ belongs to C^{2m+d} in any compact subdomain of Ω.

The purpose of this note is to extend the result up to the boundary for general boundary value problems.

We shall divide the theory into three parts. In § 1, it will be shown $u \in C^{2m-\varepsilon}(\Omega)$. In § 2, assuming a hyper-plane $t=0$ as a part of boundary, making difference quotient for translation tangent to this hyper-plane, we shall show that $u \in C^{2m}$ and $D^{2m}u$ satisfies the Hölder condition with respect to x.

§ 1. Lp-estimates.

Let $u(P)$ be a function defined in a semi-sphere $\sum_R$: $|x|^2+t^2 \leqq R^2$, $t \geqq 0$, where $x=(x_1, \ldots, x_n)$, and $u(P)$ be in $H_{2m},\,_{L_2}$ (see p. 69 in [1]) on $\sum_R$.

Now, we consider the following boundary value problem

$$(1.1)\qquad \begin{aligned} &L(P, D)\ u(P)=F(P) \text{ in } \textstyle\sum_R, \\ &B_j(x, D)\ u(P)=\phi_j(x) \text{ on } \sigma_R(x^2 \leqq R^2,\ t=0),\ j=1, \ldots, m, \end{aligned}$$

(an order of $L=2m$, an order of $B_j=m_j(<2m)$), under the conditions of (i) (condition on L), (ii) (complementing condition) and (iii) (see [1] p. 663).

(i) *Condition on L. We assume that L is uniformly elliptic. In addition in the case of two variables* ($n=1$) *we assume that for each P in* $\sum_R$ *and real* $\xi=0$ *exactly half of the roots of the polynomial* $L'(P, \xi, \tau)$ *lie in the upper half of the complex* τ*-plane, where* $L'(P, \xi, \tau)$ *is the characteristic form associated with* L' *which is the principal part of L.*

(ii) *Complementing Condition of the boundary operator relative to L. For every fixed* $P=(x, t)$ $(\in \sum_R)$ *and the operator* $L'(P, D)$, $B'_j(x, D)$, $j=1, \dots, m$, *which is the principal part of* B_j *satisfies the Complementing Condition uniformly for all P in* $\sum_R$.

(iii) *We shall assume that F and the coefficients of L belong to* $C^{\alpha}(\sum_R)$[1], $(0<\alpha<1)$ *and have* $|\ |_{\alpha}$ *norms bounded by k, while* ϕ_j *and the coefficients of* B_j *belong to* $C^{2m-m_j+\alpha}(\sigma_R)$[2] *and have their* $|\ |_{2m-m_j+\alpha}$ *norms bounded by k.*

The main purpose of this note is to prove

Theorem 1. *Let* $u(P)$ *be a generalized solution of* (1, 1) *in* H_{2m,L_2} *on* $\sum_R$ *under the conditions of* (i), (ii) *and* (iii), *then* $u(P)$ *belongs to* $C^{2m}(\sum_{R'})$ *and* $D_{2m}u$ *satisfies the Hölder condition with exponent* α' *with respect to x for any* α', R' *such that* $0<R'<R$, $0<\alpha'<\alpha$.

Proof. Without loss of generality we may assume that u, F, ϕ_j vanish for $|P|>R''(R'<R''<R)$. For any fixed point $P_0=(x_0, t_0)\in\sum_R$, we shall write the operators L and B_j as follows:

$$L(P, D)\ u(P)=L'(P_0, D)\ u(P)+(L'(P, D)-L'(P_0, D))\ u(P)+L''(P, D)\ u(P)$$

$$B_j(x, D)\ u(P)=B_j(x_0, D)\ u(P)+(B_j(x, D)-B_j(x_0, D))\ u(P)+B_j''(x, D)\ u(P).$$

By the use of Representation formulas (4, 7) for $H_{2m,Lp}$ in [1] (See p. 653 and p. 702), we have

$$\begin{aligned}D^{2m}u(x, t)=D^{2m}&\int\Gamma(P_0, P-Q)\{F(Q)-(L'(Q, D)u(Q)-L'(P_0, D))u(Q)\\ &-L''(Q, D)\ u(Q)\}_N{}^{3)}dQ\\ &+\sum_{j=1}^{m}\int D^{2m}K_j(P_0, x-y, t)\{\phi_j(y)+(B_j'(x_0, D)-B_j'(y, D))\\ &\times u(y, 0)-B_j''(y, D)\ u(y, 0)-B_j'(x_0, D)\ v(y, 0)\}dy,\end{aligned}\tag{1.2}$$

1), 2) See p 657 in [1].
3) See (4. 4) in p. 652 in [1].

where $v(x, t)$ denotes the first intengral before operating D^{2m}, $\Gamma(P_0, P-Q)$ is the fundamental solution of $L'(P_0, D)$ with a singularity at $Q=P$, and $K_j(P_0, x-y, t)$[4] are the Poisson kernels for the operators $L'(P_0, D)$ and $B_j(x_0, D)$ with singularities at $x=y$, $t=0$. It holds for $P_0=P$ also, where D^{2m} should not be operated for P_0. Validity of (1, 2) will be easily seen by (1, 4), (1, 5)

For convenience in estimating (1, 2), we shall denote the first term of (1, 2) by A_1, and set

$$A_2=\sum_{j=1}^{m} D^{2m}K_j * \phi_j$$

$$A_3=\sum_{j=1}^{m} D^{2m}K_j * (B_j'(x_0, D)-B_j'(y, D))\ u(y, 0),$$

$$A_4=\sum_{j=1}^{m} D^{2m}K_j * B_j''(y, D)\ u(y, 0),$$

$$A_5=\sum_{j=1}^{m} D^{2m}K_j * B_j(x_0, D)\ v(y, 0).$$

Consider D^{2m} as D_iD^{2m-1}, where D_i denotes differential operator with respect to x_i and D denotes differential operator with respect to x_i or t. In view of the properties of the kernels K_j and $K_{j,q}$[5], we can write $A_i (i=2, 3, 4, 5)$ in the form:

$$\sum_k D_i\{D_kD^{2m-1}\Delta^{(n+q-2m+m_j)/2}K_{j,q} * D_k\Delta^{(2m-m_j-2)/2}f\}^{6)},$$

when $2m-m_j$=even. For $2m-m_j$=odd, similar representations are possible. Denote by $g(x)$ the quantity

$$g(x)=D_k\Delta^{(2m-m_j-2)/2}f.$$

If $g\in H_{1-\frac{1}{q}}$[7] $(q>1)$, we can find $w(x, t)$ such that $w(x, 0)=g(x)$, $\|w\|_{1,\ L_q}$[8] $<\infty$, $\|w\|_q$[9] (x-space) $\leq Ct^{1-\frac{1}{q}}$ (for $t\to\infty$).

Let $D_kD^{2m-1}\Delta^{(n+q-2m+m_j)/2}K_{j,q}=K(x_0, t_0, x, t)$ and $J=D_i(K(x_0, t_0, x, t) * g(x))$, then by the argument used in [1] p. 712, we obtain

$$-J=\iint_{s>0} D_sW(y, s)\ D_{y_i}K(x_0, t_0, x-y, t+s)\ dyds$$

$$+\iint_{s>0} D_{y_i}W(y, s)\ D_sK(x_0, t_0, x-y, t+s)\ dyds.$$

4) See p. 635 in [1].
5) See p. 636 in [1].
6) See p. 701 in [1].
7) We need to take note of that this assumption for g is used for A_2 only
8), 9) See p. 689 in [1].

Put $x_0=x,\ t_0=t$, then

$$(1,4)\qquad \begin{aligned}-J=&\iint_{s>0} D_s W(y,s)\ D_{y_i}K(x,t,x-y,t+s)\ dyds\\ &+\iint_{s>0} D_{y_i}W(y,s)\ D_sK(x,t,x-y,t+s)\ dyds.\end{aligned}$$

For simplicity, denote the kernels D_iK and D_sK by $G(x, t, x-y, t+s)$, and define a kernel $N(x, t, x-y, t+s)$ as follows:

$$N(x,t,x-y,t+s)=G(x,t,x-y,t+s) \text{ in } t+s\geqq 0$$

and N is odd in $t+s<0$. And it is easily seen that the kernel N satisfies the conditions of Calderon-Zygmund theorem 2 in [3]. In virtue of the theorem, we conclude

$$(1,5)\qquad \|J\|_{Lq}\leqq C\ \|W\|_{1,Lq}.$$

Now, let us estimate $A_k(k=1,\dots,5)$. Put $P_0=P$ in all A_k.

Estimate of A_1.

The terms which correspond to the first and the third intengrands belong to $L_q\left(q\leqq\dfrac{(n+1)2}{n+1-2}\right)$ by Sobolev's theorem and Calderon-Zygmund theorem 2[10].

The second one does so, too $\left(q\leqq\dfrac{(n+1)2}{n+1-2\alpha}\right)$ by Sobolev's theorem, because

$$|(L'(Q,D)-L'(P,D))\ u(Q)|\leqq C|P-Q|^\alpha|D^{2m}u(Q)|.$$

Estimates of A_2, A_4 and A_5.

Applying the inequality (1,5) and using the above estimates, we see that A_2, A_4 and A_5 belong to L_q for all $q>1$, $q\leqq\dfrac{(n+1)2}{n+1-2}$ and $q\leqq\dfrac{(n+1)2}{n+1-2\alpha}$, respectively.

Estimate of A_3.

Setting

$$D_k\Delta^{(2m-m_j-2)/2}(B_j'(x,D)-B_j'(y,D))\ u(y,s)=W$$

and put it into the quantity (1,4). Then the absolute value of A_3 is bounded by the sum of

10) $D^{2m}\Gamma$ satisfies the conditions of Calderon-Zygmund theorem 2, See [1].

$$\left|\int \frac{C|x-y|^{\alpha} D^{2m} u(y, s)}{(|x-y|+|t+s|)^{n+1}}\, dyds\right| \text{ and } C|DK * D^{2m-1}u|.$$

Consequently $A_8 \in L_q\left(q \leqq \frac{(n+1)2}{n+1-2\alpha}\right)$ by Sobolev's theorem.

By the above estimates, we have $D_i D^{2m-1}u \in L_q\left(q \leqq \frac{(n+1)2}{n+1-2\alpha}\right)$, and in virtue of ellipticity we may conclude $D^{2m}u \in L_q\left(q \leqq \frac{(n+1)2}{n+1-2\alpha}\right)$.

Next, take q in place of 2, then $A_k \in L_{\frac{(n+1)q}{n+1-q\alpha}}$, $k=1, \ldots, 5$.
Repeating this process, q will be taken arbitrarily large. Applying the inequality (11, 15) in [11], we have

$$|D^{2m-1}u(P+\Delta P)-D^{2m-1}u(P)| < K|\Delta P|^{\beta},\ P,\ P+\Delta P \in \textstyle\sum_R, \beta = \min\left(1, 1-\frac{n+1}{q}\right).$$

Since q is choosen arbitrarily large, u belongs to $C^{2m-\varepsilon}(\sum_R)$ for any $\varepsilon > 0$.

Now, we summarize the above results as follows:

Theorem 2. *Assume that $u(P)$ satisfies the all conditions of theorem* 1 *except for F and ϕ_j, F belongs to L_p on $\sum_R$ for all $p \geqq 2$, ϕ_j belongs to C_{2m-m_j-1} on σ_R and $D^{2m-m_j-1}\,\phi_j$ belongs to $H_{1-1/p, L_p}$ for all $p \geqq 2$. Then $u(P)$ belongs to $C^{2m-\varepsilon}(\sum_{R'})$ for any R', ε, such that $0 < R' < R$, $\varepsilon > 0$.*

§ 2. Differentiability and the Hölder condition

We shall now concern ourselves with a more detailed smoothness of the solution u.

Let L, B_j be in the forms:

$$(2,1)\qquad \begin{aligned} &\sum_{\beta} a^{(\beta)}(x, t)\ D^{\beta}u(x, t) = F(x, t) \\ &\sum_{\gamma} b_{j\gamma}(x)\ D^{\gamma}u(x, 0) = \phi_j(x). \end{aligned}$$

With the aid of the well-known difference quotient method, we have

$$\sum a^{\beta}(x+h, t)\frac{D^{\beta}u(x+h, t)-D^{\beta}u(x, t)}{|h|^{\alpha''}} + \sum \frac{a^{\beta}(x+h, t)-a^{\beta}(x, t)}{|h|^{\alpha''}} D^{\beta}u(x, t)$$
$$= \frac{F(x+h, t)-F(x, t)}{|h|^{\alpha''}},$$

$$\sum b_{j},\,_{\gamma}(x+h)\frac{D^{\gamma}u(x+h,0)-D^{\gamma}u(x,0)}{|h|^{\alpha''}}+\sum\frac{b_{j},\,_{\gamma}(x+h)-b_{j\gamma}(x)}{|h|^{\alpha''}}D^{\gamma}u(x,0)$$
$$=\frac{\phi_{j}(x+h)-\phi_{j}(x)}{|h|^{\alpha''}},$$

where $\alpha'<\alpha''<\alpha$, (x, t), $(x+h, t) \in \sum_R$, Taking $\alpha-\alpha''$ in place of α, these equations with respect to $U(x, t)=(u(x+h, t)-u(x, t))/|h|^{\alpha''}$ satisfy the conditions of theorem 2. Then $U \in C^{2m-\varepsilon}(\sum_R)$ in virtue of Theorem 2. And the norm $|U|_{2m-\varepsilon}$ is bounded by constant independent of h.

To complete the proof of theorem 1 we need

Lemma (Ono and Ikebe [6]). *Let P be any point $(x_1, \ldots, x_n)$ in $S_R(|x|\leq R)$ and h, k denote any vectors such that $P+h$, $P+h+k \in S_R$.* If *$V(P)$ $(\in C(S_R))$ vanishes near the boundary and satisfies that*

$$\left|\frac{V(P+h)-V(P)-V(P+h+k)+V(P+k)}{|h|^{\beta}|k|^{\alpha}}\right|\leq K \tag{A.1}$$

for all P, h, k and some α, β such that $0<\alpha, \beta<1$, $\alpha+\beta<1$, then $DV \in C^{\alpha+\beta-1}(S_R)$.

Proof will be given in appendix.

By the lemma, we have that $D_iD^{2m-1}u$ satisfies Hölder condition with exponent $\alpha''-\varepsilon$ with respect to x_i. For the reason that u is the solution of elliptic equation, $D^{2m}u$ satisfies it also.

Appendix

Proof of lemma. Let us restate the condition

$$\left|\frac{v(P+h)-v(P)-(v(P+h+k)-v(P+k))}{|h|^{\beta}|k|^{\alpha}}\right|\leq K. \tag{A. 1}$$

First we shall show total differentiability of v. For this purpose, use the following inequality repeatedly, which is the special case of (A. 1),

$$\left|\frac{v(P+h)-v(P)}{|h|}-\frac{v(P+2h)-v(P)}{2|h|}\right|\leq\frac{K}{2}|h|^{\alpha+\beta-1}. \tag{A. 2}$$

That is

$$\frac{|v(P+h)-v(P)|}{|h|}\leq\frac{|v(P+2h)-v(P)|}{2|h|}+\frac{K}{2}|h|^{\alpha+\beta-1}$$
$$\leq\frac{|v(P+2^{2}h)-v(P)|}{2^{2}|h|}+\frac{K}{2}(|2h|^{\alpha+\beta-1}+|h|^{\alpha+\beta-1})\leq\cdots$$

$$\leqq \frac{|v(P+2^{\mu+1}h)-v(P)|}{2^{\mu+1}|h|}+\frac{K}{2}(|2^{\mu}h|^{\alpha+\beta-1}+\cdots+|2h|^{\alpha+\beta-1}+|h|^{\alpha+\beta-1}).$$

Now, if $\frac{|h_0|}{2^{\mu+1}}\leqq|h|\leqq\frac{|h_0|}{2^{\mu}}\to 0$, so that $P+h_0 \varepsilon S_R$, then

$$\leqq\frac{|v(P+h)-v(P)|}{|h|}\leqq\frac{2\max|v|}{|h_0|}+\frac{K}{2}(|2^{\mu}h|^{\alpha+\beta-1}+\cdots$$

$$+|2h|^{\alpha+\beta-1}+|h|^{\alpha+\beta-1})$$

$$<\frac{2\max|v|}{|h_0|}+\frac{K|h_0|^{\alpha+\beta-1}/2}{1-2^{-(\alpha+\beta-1)}}\quad .$$

Hence $\overline{lim}_{|h|\to 0}\left|\frac{v(P+h)-v(P)}{|h|}\right|<\infty$, and by a well-known theorem v is totally differentiable almost everywhere.

Next, to prove the Hölder continuity of Dv, by the above result, it is sufficient to verify the following inequality

$$\text{(A. 3)}\qquad \left|\frac{v(P+h+k)-v(P+k)}{|h|}-\frac{v(P+h)-v(P)}{|h|}\right|\leqq K'|k|^{\alpha+\beta-1}.$$

Now, $$\frac{v(P+h+k)-v(P+k)}{|h|}-\frac{v(P+h)-v(P)}{|h|}$$

$$=\sum_{\nu=1}^{\mu}\left\{\frac{v(P+k+2^{\nu-1}h)-v(P+k)}{2^{\nu-1}|h|}-\frac{v(P+k+2^{\nu}h)-v(P+k)}{2^{\nu}|h|}\right\}$$

$$-\sum_{\nu=1}^{\mu}\left\{\frac{v(P+2^{\nu-1}h)-v(P)}{2^{\nu-1}|h|}-\frac{v(P+2^{\nu}h)-v(P)}{2^{\nu}|h|}\right\}$$

$$+\left\{\frac{v(P+k+2^{\mu}h)-v(P+k)}{2^{\mu}|h|}-\frac{v(P+2^{\mu}h)-v(P)}{2^{\mu}|h|}\right\}.$$

Thus, with the aid of (A. 2) we have the following inequality

$$\left|\frac{v(P+h+k)-v(P+k)}{|h|}-\frac{v(P+h)-v(P)}{|h|}\right|$$

$$\leqq K\sum_{\nu=1}^{\mu}|2^{\nu-1}h|^{\alpha+\beta-1}+\left|\frac{v(P+k+2^{\mu}h)-v(P+k)}{2^{\mu}|h|}-\frac{v(P+2^{\mu}h)-v(P)}{2^{\mu}|h|}\right|$$

$$\text{(A. 4)}\qquad \leqq K|2^{\mu-1}h|^{\alpha+\beta-1}\sum_{\nu=1}^{\mu}\left(\frac{1}{2^{\alpha+\beta-1}}\right)^{\mu-\nu}+(\text{second term})$$

$$\leqq K|2^{\mu-1}h|^{\alpha+\beta-1}\frac{1}{1-\frac{1}{2^{\alpha+\beta-1}}}+(\text{second term}).$$

Choose μ, $\frac{1}{2^{\mu}}\leqq\frac{|h|}{|k|}<\frac{1}{2^{\mu-1}}$, then with the aid of (A. 1), the right hand side of (A.4) is majorized by $\dfrac{K|k|^{\alpha+\beta-1}}{1-\frac{1}{2^{\alpha+\beta-1}}}+K|k|^{\alpha+\beta-1}$. This completes the proof.

General Education Department of Kyushu University.

Bibliography

[1] AGMON, S., DOUGLIS, A., NIRENBERG, L., *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions 1,* Comm. Pure Appl. Math., **12** (1959), 623–729.

[2] CALDERON, A. P. and ZYGMUND, A., *On the existence of certain singular integrals,* Acta Math., **88** (1952) 85–139.

[3] CALDERON, A. P. and ZYGMUND, A., *On singular integrals,* Amer. J. Math., **78** (1956), 289–309.

[4] DOUGLIS, A. and NIRENBERG, L., *Interior estimates for elliptic system of partial differential equations,* Comm. Pure Appl. Math., **8** (1955), 503–538.

[5] HOPF, E., *Über den Funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung,* Math. Zeit., **34** (1931), 194–233.

[6] IKEBE, N., ONO, A., *Note on the local estimates for generalized solutions of elliptic partial differential eqnations,* Memoir Fac. Sci. Kyushu Univ. Ser. A, **18**, (1964), 109–113.

[7] MORREY, C. B., *Secoud order elliptic systems of differential equations,* Contributions to the theory of partial differential equations, Ann. of Math. Studies **No. 33**, Prinston Univ. Press (1954), 101–159.

[8] SCHECHTER, M., *General boundary value problems for elliptic partial differential equations,* Comm. Pure Appl. Math., **12** (1959), 457–486.

[9] SCHECHTER, M., *Remarks on elliptic boundary value problems,* Comm. Pure Appl. Math., **12** (1959), 561–578.

[10] SOBOLEV, S. L., *On a theorem of functional analysis,* Mat. Sbornik, **4** (1938), 471–497.

[11] SOBOLEV, S. L., *Application of functional analysis in mathematical physics,* Amer. Math. Soc. 1963).