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ON $t_n(p)$ AND $T_{p^2}(m)$ IN THE HECKE RING

By

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1. INTRODUCTION. We call the ring $\mathfrak{R}$ introduced in 2 the Hecke ring. G. Shimura[1] gave his conjecture on the Euler product of formal Dirichlet series attached to the ring $\mathfrak{R}$. We find there essentially an extended theory of Hecke operators to the automorphic functions of several variables, and that there is no recurrence formula of Hecke operators on the contrary with the automorphic functions of one variable. We find that some problems on the elementary divisors are so complicated and probably difficult. In this paper we investigate the rule of multiplication in $\mathfrak{R}$, the multiplicity. Principally $t_n(p)$ and $T_{p^2}(m)$ are considered, because $t_n(p)$ and $T_{p^2}(m)$ are generators of $\mathfrak{R}$ over the ring of rational integers, and the calculation of multiplicities are comparatively easy. In 3 and 4 we calculate the multiplicities, which are wanted to represent more explicitly. In 5 we find some interest polynomials which appear in the calculations of multiplicities.

2. Let $\boldsymbol{Z}$ be the ring of rational integers, and let $\boldsymbol{Z}_l$ be the set of all square matrices of size l with entries in $\boldsymbol{Z}$. We denote the unit matrix of size n by $E^{(n)}$, and $\begin{pmatrix} & E^{(n)} \\ -E^{(n)} & \end{pmatrix}$ by J_n.

Let $G_n^{(m)}=\{M \mid {}^tMJ_nM=mJ_n,\ M\in\boldsymbol{Z}_{2n}\}$, where $m\in\boldsymbol{Z}$, $m>0$, and ${}^tM$ denotes the transposed matrix of M. Therefore $G_n^{(1)}$ is identical with the symplectic modular group, and we denote it by G_n. Let $G_n\backslash G_n^{(m)}$ be the set of all left cosets of $G_n^{(m)}$ by G_n, and $G_n\backslash G_n^{(m)}/G_n$ be the set of all double cosets of $G_n^{(m)}$ by G_n. Let $\mathfrak{R}$ be the free $\boldsymbol{Z}$-module generated by all the double cosets for all m. By the well known law of the multiplication, $\mathfrak{R}$ becomes a commutative ring. Since the diagonal matrices $[e_1,\dots, e_n, f_1,\dots, f_n]$ such that $e_if_i=m$, e_{i+1} are divisible by e_i for $1\leq i\leq n-1$, are the complete system of representatives of $G_n\backslash G_n^{(m)}/G_n$, we especially denote the corresponding cosets by $T(e_1,\dots e_n\,;\, f_n,\dots, f_1)$ and denote the element $\sum T(e_1,\dots,e_n\,;\, f_n,\dots, f_1)$ in $\mathfrak{R}$ by $t_n(m)$, where the summation ranges over the above e_i and f_j for an integer m.

Throughout the paper, except for particular notations, we denote matrices with entries in $\boldsymbol{Z}$ by capital latain letters, for example $A, B, C, \ldots$ etc. The upper indices i and j of $A^{(i,j)}$ designate the numbers of rows and columns of the matrix A. particularly if $i=j$, then we denote $A^{(i,i)}$ by $A^{(i)}$. All small latain letters of the Gothic type denote columns, and the upper index of $\boldsymbol{a}^{(i)}$ designates the number of rows. The upper$\sim$of $\tilde{A}$ means that all the elements below the diagonal and on the diagonal of the matrix A are all 0. Let ${}^t\boldsymbol{a}^{(i)}=(a_1,\ldots, a_i)$ and ${}^t\boldsymbol{b}^{(i)}=(b_1,\ldots, b_i)$. If $a_j \equiv b_j$ *mod.* m, for every j, then we denote it by $\boldsymbol{a}^{(i)} \equiv \boldsymbol{b}^{(i)}$ *mod.*$\{m\}$.

Let $A^{(i,j)}=(\boldsymbol{a}_1^{(i)},\ldots, \boldsymbol{a}_j^{(i)})$ and $B^{(i,j)}=(\boldsymbol{b}_1^{(i)},\ldots, \boldsymbol{b}_j^{(i)})$. If $\boldsymbol{a}_k^{(i)} \equiv \boldsymbol{b}_k^{(i)}$ *mod.*$\{m_k\}$ for every k, then we denote it by$A^{(i,j)} \equiv B^{(i,j)}$ *mod.* $\{m_1,\ldots,m_j\}$. At last p denotes always a prime number in $\boldsymbol{Z}$.

In the following we write three lemmata, the first one of which is well-known. Their proof are obvious.

LEMMA 1. We can choose as the complete system of the representatives of cosets in $G_n \backslash G_n^{(m)}$, all matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in G_n^{(m)}$ such that

(1) $A, B, C \in \boldsymbol{Z}_n$, $A^t C = mE^{(n)}$, $A^t B = B^t A$,

(2) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} = (\boldsymbol{a}_1,\ldots, \boldsymbol{a}_n)$, where all the elements a_{ij} of $\boldsymbol{a}_j$ are 0

if $i>j$ and are $0 \leqq a_{ij} < a_{jj}$ if $i<j$.

(3) Let $C=\begin{pmatrix} c_{11} & & \\ & \ddots & \\ * & & c_{nn} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$.

Then all the elements b_{ij} are determined *mod.* c_{jj} if $i \leqq j$.

Now we state the law of multiplication in $\mathfrak{R}$ following Shimura [1]. Let G_nKG_n, G_nLG_n and G_nMG_n be three cosets in $\mathfrak{R}$, and let $G_nKG_n=\bigcup_i G_nK_i$, $G_nLG_n=\bigcup_j G_nL_j$ be their disjoint expressions. By $\mu(G_nKG_n \cdot G_nLG_n,\ G_nMG_n)$ we denote the number of (i, j) which satisfy $G_n\ K_iL_j=G_nM$, and we define the product $G_nKG_n \cdot G_nLG_n = \sum \mu(G_nKG_n \cdot G_nLG_n,\ G_nMG_n) G_nMG_n$.

$$G_nMG_n \subset G_nKG_nLG_n$$

LEMMA 2. Let $T(e_1,\ldots, e_n; f_n,\ldots, f_1) \in G_n \backslash G_n^{(g)}/G_n$ and let $T(k_1,\ldots, k_n; l_n,\ldots, l_1) \in G_n \backslash G_n^{(m)}/G_n$. If $(g, m)=1$, then $T(e_1,\ldots, e_n; f_n,\ldots, f_1)T(k_1,\ldots, k_n; l_n,\ldots, l_1)=T$

$T(e_1k_1,\dots, e_nk_n\,;\, f_nl_n,\dots, f_1l_1)\in G_n\backslash G_n^{(gm)}/G_n$.

Let $G_nKG_n\in G_n\backslash G_n^{(g)}/G_n$ and let $G_nLG_n\in G_n\backslash G_n^{(m)}/G_n$. When their disjoint representations are $G_nKG_n=\bigcup\limits_i G_nK_i$ and $G_nLG_n=\bigcup\limits_j G_nL_j$, assume K_i and L_j be the representatives in Lemma 1. From the definition of multiplication in $\mathfrak{R}$ we obtain easily

LEMMA 3. If $G_n\backslash G_n^{(mg)}/G_n\ni G_nMG_n=T(x_1,\dots, x_n\,;\, y_n,\dots, y_1)$, and K,L,K_i and L_j are as the above, then $\mu(G_nKG_n\cdot G_nLG_n,\ G_nMG_n)$ is equal to the number of (i, j) of (K_i, L_j) such that

$$\begin{pmatrix} e_1 & & * \\ & \ddots & \\ & & e_n \end{pmatrix}\begin{pmatrix} k_1 & & * \\ & \ddots & \\ & & k_n \end{pmatrix}\equiv 0 \ mod.\ \{x_1,\dots,x_n\}$$

and

$$\begin{pmatrix} e_1 & & * \\ & \ddots & \\ & & e_n \end{pmatrix} G+B\begin{pmatrix} l_1 & & \\ & \ddots & \\ * & & l_n \end{pmatrix}\equiv 0 \ mod.\ \{y_1,\dots, y_n\},$$

where $K_i=\begin{pmatrix} A & B \\ 0 & C\end{pmatrix}$, $A=\begin{pmatrix} e_1 & & * \\ & \ddots & \\ & & e_n\end{pmatrix}$ and $L_j=\begin{pmatrix} F & G \\ 0 & H\end{pmatrix}$, $H=\begin{pmatrix} l_1 & & \\ & \ddots & \\ * & & l_n\end{pmatrix}$, $F=\begin{pmatrix} k_1 & & * \\ & \ddots & \\ & & k_n\end{pmatrix}$.

3. From Lemma 2, the product of $t_n(g)$ and $t_n(m)$ is essentially reduced to the product of $t_n(p^r)$ and $t_n(p^s)$. Therefore we now observe the product $t_n(p)t_n(p^l)$ which is most simple. Let $t_n(p)=G_n\begin{pmatrix} E^{(n)} & \\ & pE^{(n)}\end{pmatrix}G_n=\bigcup\limits_K G_nK$. We may assume $K=\begin{pmatrix} A & B \\ 0 & C\end{pmatrix}$ with the properties in Lemma 1.

Particularly let $A=\begin{pmatrix} E^{(i_1)} & & & \\ & pE^{(i_2)} & & \\ & & E^{(i_3)} & \\ & & & \ddots \end{pmatrix}+\begin{pmatrix} \tilde{A}_1 & A_{12} & A_{13} & \\ & \tilde{A}_2 & A_{23} & \\ & & \tilde{A}_3 & \\ & & & \ddots \end{pmatrix}$,

where $i_1\geqslant 0,\ i_2,\dots,\geqslant 1$ and $i_1+i_2+\dots=n$.

LEMMA 4. If $t_n(p)=\bigcup\limits_K G_nK,\ K=\begin{pmatrix} A & B \\ 0 & C\end{pmatrix}$, and A is as the above, then $\tilde{A}_\sigma=0$, $A_{\sigma 2\tau+1}=0$ and $A_{2\sigma 2\tau}=0$ for every σ and τ.

PROOF. Since the elementary divisors of K are $(1,\dots,1,p,\dots,p)$ and the $\tilde{A}_{2\sigma+1}=0$, therefore the proof is obvious.

Let L be the representative of coset in $G_n\backslash G_n^{(p^l)}$ and let $L=\begin{pmatrix} U & V \\ 0 & W\end{pmatrix}$. According with the splitting of A of K, we split the matrix U in such as

$$U=\begin{pmatrix} p^{s_1} & & & & & \\ & \ddots & & & & \\ & & p^{s_{i_1}} & & & \\ & & & p^{t_1} & & \\ & & & & \ddots & \\ & & & & & p^{t_{i_2}} \\ & & & & & & \ddots \end{pmatrix}+\begin{pmatrix} \tilde{U}_1 & U_{12} & \cdots \\ & \tilde{U}_2 & \cdots \\ & & \ddots \end{pmatrix}.$$

We denote the diagonal matrices $[p^{s_1},\dots, p^{s_{i_1}}]$, $[p^{t_1},\dots, p^{t_{i_2}}],\dots,$ by P_1, $P_2,\dots,$ and $[p^{l-s_1},\dots, p^{l-s_{i_1}}]$, $[p^{l-t_1},\dots, p^{l-t_{i_2}}],\dots,$ by Q_1, $Q_2,\dots$. Moreover we denote *mod.* $\{p^{s_1},\dots, p^{s_{i_1}}\}$, *mod.* $\{p^{t_1},\dots, p^{t_{i_2}}\}\dots,$ by *mod.* $\{P_1\}$, *mod.*$\{P_2\}$ and so on. It is same about the Q_1, $Q_2,\dots$. By simple calculation we come to

LEMMA 5. If $AU\equiv 0$ *mod.* $\{P_1, pP_2,\dots\}$, then $A_{2\sigma+1\,2\tau}=0$, $\tilde{U}_\lambda=0$ and $U_{\lambda\mu}=0$ except the case $\lambda\equiv 0$, *mod.* 2 and $\mu\equiv 1$ *mod.* 2, and $pU_{2\sigma\,2\tau+1}\equiv 0$ *mod.* $\{P_{2\tau+1}\}$.

From Lemma 4 and Lemma 5, we have

COROLLARY 1. The matrix A is diagonal under the assumption in Lemma 5.

COROLLARY 2. The Matrix B in K is symmetric under the assumption in Lemma 5, i.e. ${}^tB=B$, therefore ${}^tB_{\sigma\tau}=B_{\tau\sigma}$ and $B_{\sigma\tau}=0$ except σ and $\tau\equiv 1$ *mod.* 2 at the same time.

PROOF. From the above corollary and $A^tB=B^tA$ the Proof follows.

COROLLARY 3. Under the assumption in Lemma 5 the matrix W is of such a form

$$W=\begin{pmatrix} Q_1 & & \\ & Q_2 & \\ & W_{\lambda\mu} & \ddots \end{pmatrix},$$

where $W_{\lambda\mu}=0$ except $\lambda\equiv 1$ *mod.* 2 and $\mu\equiv 0$ *mod.* 2. Moreover

$$P_{2\tau}\ {}^tW_{2\sigma+1\,2\tau}+U_{2\tau\,2\sigma+1}Q_{2\sigma+1}=0.$$

PROOF. Since $U^tW=p^lE^{(n)}$ and U is of the form in Lemma 5, therefore the corollary follows.

Let $V=(V_{\sigma\tau})$. If $AV+BW=(R_{\sigma\tau})$, then under the assumption in Lemma 5, $R_{\sigma\tau}=V_{\sigma\tau}+B_{\sigma\tau}Q_\tau$ if $\sigma\equiv\tau\equiv 1$ *mod.* 2, $R_{\sigma\tau}=V_{\sigma\tau}+\sum_{\rho\geqslant\tau/2} B_{\sigma\,2\rho+1}W_{2\rho+1\,\tau}$ if $\sigma\equiv 1$ *mod.* 2 and $\tau\equiv 0$ *mod.* 2, and $R_{\sigma\tau}=pV_{\sigma\tau}$ if $\sigma\equiv 0$ *mod.* 2. Now we consider the congruence equation $(R_{\sigma\tau})\equiv 0$ *mod.* $\{pQ_1, Q_2,\dots\}$. When $\sigma\equiv\tau\equiv 1$ *mod.* 2 and $\sigma<\tau$, then $R_{\sigma\tau}\equiv 0$ *mod.* $\{pQ_\tau\}$. Since $V_{\sigma\tau}$ is determined *mod.* Q_τ and $B_{\sigma\tau}$ is determined *mod.* $\{pE^{(\tau)}\}$, therefore in the first case we obtain $V_{\sigma\tau}=0$, $B_{\sigma\tau}=0$.

In the case $\sigma=\tau\equiv 1$ *mod.* 2 also we have $V_\sigma=0$ and $B_\sigma=0$. In the second case $\sigma\equiv 1$ *mod.* 2 and $\tau\equiv 0$ *mod.* 2, and $\sigma<\tau$, we have $R_{\sigma\tau}=V_{\sigma\tau}$ from $B_{\sigma\ 2\rho+1}=0$ in the preceeding case. Therefore $R_{\sigma\tau}\equiv 0$ *mod.* $\{Q_\tau^1\}$ implies $V_{\sigma\tau}\equiv 0$ *mod.* $\{Q_\tau\}$ and $V_{\sigma\tau}=0$. Before the third case, we have to observe V more precisely. Since $L\in G_n^{(p^l)}$, $U^tV=V^tU$, it follows $P_{2\sigma+1}{}^tV_{2\tau+1\ 2\sigma+1}=V_{2\sigma+1\ 2\tau+1}P_{2\tau+1}$ for $\tau\leq\sigma$. As already observed, from the first case $V_{2\tau+1\ 2\sigma+1}$ and $V_{2\sigma+1\ 2\tau+1}=0$. Moreover from $V_{2\tau+1\ 2\sigma+1}=0$, the second case and $P_{2\sigma}{}^tV_{2\tau+1\ 2\sigma}=V_{2\sigma\ 2\tau+1}P_{2\tau+1}$ or $P_{2\tau+1}{}^tV_{2\sigma\ 2\tau+1}=V_{2\tau+1\ 2\sigma}P_{2\sigma}$, it follows $V_{2\sigma\ 2\tau+1}=0$. Thererore $V_{\sigma\tau}=0$ except $\sigma\equiv\tau\equiv 0$ *mod.* 2.

Now we may assume $R_{2\sigma\ 2\tau}=pV_{2\sigma\ 2\tau}\equiv 0$ *mod.* $\{Q_\tau\}$ in the third case. From $U^tV=V^tU$, we obtain $P_{2\sigma}{}^tV_{2\tau\ 2\sigma}=V_{2\sigma\ 2\tau}P_{2\tau}$ and this shows the elements a_{jk} of $V_{2\tau\ 2\sigma}$ are uniquelydetermined by a_{kj} of $V_{2\sigma\ 2\tau}$. Therefore the elements over the diagonal of V are uniquely determined, depending on the elements under the diagonal of V.

Now we obtain

THEOREM 1. Let $t_n(p)=\bigcup_K G_nK$, where $K=\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}$, and

$$A=\begin{pmatrix} E^{(i_1)} & & \\ & pE^{(i_2)} & \\ & & \ddots \end{pmatrix}+\begin{pmatrix} \tilde{A}_1 & A_{12} & \\ & A_2 & \\ & & \ddots \end{pmatrix}.$$ Let T be a member of $t_n(p^l)$ and let $T=\bigcup_L G_nL$,

where $L=\begin{pmatrix} U & V\\ 0 & W\end{pmatrix}$ and the splitting into the blocks of U and V according with the one of A and B are

$$U=\begin{pmatrix} p^{s_1} & & & & & \\ & \ddots & & & & \\ & & p^{s_{i_1}} & & & \\ & & & p^{t_1} & & \\ & & & & \ddots & \\ & & & & & p^{t_{i_2}} \\ & & & & & & \ddots \end{pmatrix}+\begin{pmatrix} \tilde{U}_1 & U_{12}\cdot & \cdots \\ & \tilde{U}_2 & \cdots \\ & & \ddots \end{pmatrix}$$

and

$$V=\begin{pmatrix} V_1 & V_{12} & \cdots \\ V_{21} & V_2 & \cdots \\ \vdots & & \end{pmatrix}.$$

Let $t_n(p^{l+1})=\sum T(p^{u_1},\dots,p^{u_n};\ p^{l-u_n+1},\dots,p^{l-n_1+1})=\bigcup_M G_nMG_n$ and let $G_nMG_n=T(p^{u_1},\dots,p^{u_n};\ p^{l-u_n+1},\dots,p^{l-u_1+1})$.

Then $G_nKL=G_nM$ if and only if the following conditions are fullfiled.

(1) The diagonal elements of AU are $p^{n_1},\dots,p^{u_n}$ from the upper left corner

and $u_1 \leqq u_2 \leqq \ldots \leqq u_n \leqq l-u_n+1 \leqq \ldots \leqq l-u_1+1$.

(2) A is diagonal and $B=0$, therefore K is diagonal.

(3) All $\tilde{U}_\sigma=0$, $U_{\lambda\mu}=0$ except $\lambda\equiv 0$ and $\mu\equiv 1$ *mod.* 2, $V_{\lambda\mu}=0$ except $\lambda\equiv\mu\equiv 0$ *mod.* 2, and $pU_{2\sigma\,2\tau+1}\equiv 0$ *mod.* $\{P_{2\tau+1}\}$, $pV_{2\sigma\,2\tau}\equiv 0$ *mod.* $\{Q_{2\tau}\}$, where $P_{2\tau+1}$ and $Q_{2\tau}$ are the already predicated diagonal matrices.

COROLLARY. The number of $\tilde{U}$ and V which satisfy the Theorem, when the diagonal of L is fixed, is

$$p^{\sum\{i_{2\sigma}(i_{2\sigma}+1)/2+i_{2\sigma}\sum_{\substack{\rho>\sigma\\ \tau>\sigma}}(i_{2\rho}+i_{2\tau+1})\}}$$

PROOF. It is easy to prove that $U_{2\sigma\,2\tau+1}$ and $V_{2\sigma\,2\tau}$ can be chosen independently from each other and that arbitrary $U_{2\sigma\,2\tau+1}$, $V_{2\sigma\,2\tau}$ which satisfy the condition (3) in Theorem 1, construct a matrix in $G_n(p^l)$ with P_λ, Q_μ and suitable $W_{2\tau+1\,2\sigma}$. Therefore we obtain the corollary.

Let m be a positive integer. We split m into some positive integers. Let $(m_1,\ldots, m_s)$ be one of the splittings, i. e.

$$m=m_1+\ldots+m_s,\ m_i>0.$$

Let x be an indeterminate and consider the following polynomial of x,

$$\sum_{(m_1,\cdots m_s)} x^{\sum_i\{m_{2i-1}(m_{2i-1}+1)/2+m_{2i-1}\sum_{\substack{j>i\\ k>i}}(m_{2k-1}+m_{2j})\}}$$

where the first summation ranges over all possible splittings. We denote the polynomial by $\alpha_x(m)$, and let $\alpha_x(0)=1$.

THEOREM 2. $t_n(p)t_n(p^l)=\sum\alpha_p(m)T(p^{s_1},\ldots, p^{s_{n-m}}, p^{s_{n-m+1}+1},\ldots, p^{s_n+1}; p^{l-s_n},\ldots, p^{l-s_{n-m+1}},\ldots, p^{l-s_1+1})$, here the summation ranges all over

$$s_1\leqq s_2 \leqq\ldots\leqq s_{n-m}\leqq s_{n-m+1}+1\leqq\ldots\leqq s_n+1\leqq l-s_n\leqq\ldots\leqq l-s_{n-m+1}\leqq l-s_{n-m}+1 \leqq\ldots\leqq l-s_1+1\,.$$

We shall find the explicit form of $\alpha_x(m)$ in 5.

4. Let $t_n(p^2)=\sum_{r=0}^{n} T(\overbrace{1,\ldots, 1}^{r}, \overbrace{p,\ldots, p}^{n-r}; p,\ldots, p, p^2,\ldots, p^2)$. Throughout the section we denote by $t_n(p^2)\ni T(\overbrace{1,\ldots, 1}^{r}, \overbrace{p,\ldots, p}^{n-r}; p,\ldots, p, p^2,\ldots, p^2)$ that the latter is one of the member of the summand representing the former, and denote the latter by $T_{p^2}(r)$ for simplicity. Let $T_{p^2}(r)=\bigcup_K G_nK$ and let K be chosen as in Lemma 1. For a while we consider the following particular K,

$$K=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},\quad A=\begin{pmatrix} E^{(i_1)} & & \\ & pE^{(i_2)} & \\ & & p^2E^{(i_3)} \end{pmatrix}+\begin{pmatrix} \tilde{A}_1 & A_{12} & A_{13} \\ & \tilde{A}_2 & A_{23} \\ & & \tilde{A}_3 \end{pmatrix},$$

$B=(B_{\sigma\tau})$, where $i_1+i_2+i_3=n$ and $i_2\geqq n-r$.

As in the case of $t_n(p^2)$, we take a representative of $t_n(p^l)$ for $l\geqq 2$, and let the representative be L,

$$L=\begin{pmatrix} U & V \\ 0 & W \end{pmatrix},\quad U=\begin{pmatrix} p^{r_1} & & & & & & & & \\ & \ddots & & & & & & & \\ & & p^{r_{i_1}} & & & & & & \\ & & & p^{s_1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & p^{s_{i_2}} & & & \\ & & & & & & p^{t_1} & & \\ & & & & & & & \ddots & \\ & & & & & & & & p^{t_{i_3}} \end{pmatrix}+\begin{pmatrix} \tilde{U}_1 & U_{12} & U_{13} \\ & \tilde{U}_2 & U_{23} \\ & & \tilde{U}_3 \end{pmatrix},$$

$V=(V_{\sigma\tau})$, like the above we split L according with the splitting of K.

LEMMA 6. If $T_{p^2}(r)=\bigcup_K G_nK$ and K is the above matrix, then $\tilde{A}_1=0$, $\tilde{A}_3=0$ and $A_{23}\equiv 0$ *mod.* $\{pE^{(i_3)}\}$.

PROOF. From the elementary divisors of K and $A^tC=p^2E^{(n)}$ the proof is obvious.

We denote tne diagonal matrices $[p^{r_1},\dots, p^{r_{i_1}}]$, $[p^{s_1},\dots, p^{s_{i_2}}]$ and $[p^{t_1},\dots, p^{t_{i_3}}]$ by P_1, P_2, P_3 respectively, and $[p^{l-r_1},\dots, p^{l-r_{i_1}}]$, $[p^{l-s_1},\dots, p^{l-s_{i_2}}]$ and $[p^{l-t_1},\dots, p^{l-t_{i_3}}]$ by Q_1, Q_2, Q_3 respectively.

LEMMA 7. If $AU\equiv 0$ *mod.* $\{P_1,\ pP_2,\ p^2P_3\}$, then $A_{12}=0$, $A_{13}=0$, $A_{23}=0$, and consequently $\tilde{U}_1=0$, $\tilde{U}_3=0$ and $U_{\sigma\tau}=0$ for every σ, τ.

PROOF. It is from simple calculation,

COROLLARY 1. Under the assumption of Lemma 7, if the j-th column of $\tilde{A}$ is not 0, and this is the first one, then s_j is not 0, i.e. $s_j\geqq 1$.

COROLLARY 2. Under the assumption of Lemma 7,

$$W=\begin{pmatrix} Q_1 & & \\ & Q_2 & \\ & & Q_3 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{\tilde{W}}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

all the elements above the diagonal and on the diagonal are 0, and

$$(P_2+\tilde{U}_2)^t(Q_2+\tilde{\tilde{W}}_2)=p^lE^{(i_2)},$$

COROLLARY 3. Under the assumption of Lemma 7,

$$V_{11}=0,\ V_{12}=0,\ V_{13}=0,\ V_{21}=0,\ V_{31}=0.$$

LEMMA 8. The solution $\tilde{U}_2$ *mod.* $\{P_2\}$ of the congruence equation

$$(pE^{(i_2)}+\tilde{A}_2)(P_2+\tilde{U}_2)\equiv 0 \text{ mod. } \{pP_2\}$$

is uniquely determined.

PROOF. Let $\tilde{A}_2=(a_{ij})$ and let $\tilde{U}_2=(u_{ij})$. If $i_2=1$, then it is obvious that the lemma holds. Assume for all $i_2<m$ the lemma is true. We exclude the first rows of $P_2+\tilde{U}_2$ and $pE^{(m)}+\tilde{A}_2$, and assume that the lemma holds for the rest matrices. We consider the product of the first row of $pE^{(m)}+\tilde{A}_2$ and the j-th column of $P_2+\tilde{U}_2$. Then it holds

$$pb_{1j}+a_{12}b_{2j}+\ldots+a_{1j-1}b_{j-1j}+p^{s_j}a_{1j}\equiv 0 \text{ mod. } p^{s_j+1}.$$

Since $b_{2j},\ldots, b_{j-1j}$ are uniquely determined *mod.* p^{s_j}, b_{1j} is uniquely determined too. Therefore the lemma is true for all integers i_2.

Let $D^{(s,t)}$ have all entries in $\boldsymbol{Z}$. If we canonically map all entries to the corresponding elements in the residue class field of $\boldsymbol{Z}$ by p, we obtain the matrix D^*. Then we call the rank of D^* the rank of D *mod.* p and denote it by $\rho_p(D)$. The following Lemma is almost clear from the elementary divisors of M.

LEMMA 9. Let a matrix M be in $G_m^{(p^2)}$: If $M=\begin{pmatrix} F & G \\ 0 & H\end{pmatrix}$ and $F=pE^{(m)}+\tilde{F}$, then $\rho_p(\tilde{F})=\rho_p(F)\leqq[m/2]$.

PROOF. If $m=1$, then $\tilde{F}=0$. Therefore $\rho_p(\tilde{F})=0$ and the lemma holds truely. Let assume for $m<t$ the lemma is true. If the first row of $\tilde{F}$ is 0, then from the assumption on Induction $\rho_p(\tilde{F})\leqq[m-1/2]\leqq[m/2]$. When the first row is not 0, let a_{1j} be the first element not 0.
Therefore the matrix F is transformed by suitable unimodular matrices to

$$pE^{(t)}+\begin{pmatrix} 0 & a_{1j} & 0 & \ldots \\ & 0 & 0 & \ldots \\ & & \boxed{\tilde{F}_1} & \end{pmatrix}.$$

Since the matrix $\tilde{F}_1$ is of degree $t-2$ and the assumption on the Inducticn holds, therefore

$$\rho_p(F)=1+\rho_p(\tilde{F}_1)\leqq 1+[t-2/2]=[t/2].$$

Therefore for all positive integers m, $\rho_p(F)\leqq[m/2]$ holds.

LEMMA 10. Let $A_2=pE^{(i_2)}+\tilde{A}_2$, $\rho_p(\tilde{A}_2)=g$, and let $U_2=P_2+\tilde{U}_2$. Under the assumption of Lemma 7, for suitable unimodular matrices X, Y, Z in $\boldsymbol{Z}_{i_2}$ it holds that

$$A'_2=XA_2Y=\begin{pmatrix} p & & A^* \\ & \ddots & \\ & & p \end{pmatrix},\quad U'_2=Y^{-1}U_2Z=\begin{pmatrix} p^{s'_1} & & U^* \\ & \ddots & \\ & & p^{s'_{i_2}} \end{pmatrix},$$

where A^* and U^* are in $\boldsymbol{Z}_g$, and lie at the right upper corner of A'_2 and U'_2 respectively. Moreover $\rho_p(A^*)=g$, and $(s'_1,\dots, s'_{i_2})$ is a permutation of $(s_1,\dots, s_{i_2})$.

PROOF. Assume $X'A_2Y'=pE^{(i_2)}+\tilde{A}'$, $Y'^{-1}U_2Z'=P_2+\tilde{U}'$, where $\tilde{A}'=(0,\dots, 0, \boldsymbol{a}_{j_1}, 0,\dots,\dots, \boldsymbol{a}_{j_\sigma}, \boldsymbol{a}_{j_\sigma+1},\dots)$ and $\boldsymbol{a}_{j_1},\dots, \boldsymbol{a}_{j_\sigma}$ are linearly independent with respect to *mod.* p, and $\tilde{U}'=(0,\dots, 0, \boldsymbol{u}_{j_1}, 0,\dots,\dots, \boldsymbol{u}_{j_\sigma}, \boldsymbol{u}_{j_\sigma+1},\dots)$, and the matrices X', Y', Z' are triangular unimodular matrices whose elements under the diagonal are 0. If $\boldsymbol{a}_{j_1},\dots, \boldsymbol{a}_{j_\sigma}, \boldsymbol{a}_{j_\sigma+1}$ are linearly dependent, then there exist integers λ_r, not all are 0, such that

$$\lambda_1\begin{pmatrix}\boldsymbol{a}_{j_1}\\ p\end{pmatrix}+\dots+\lambda_\sigma\begin{pmatrix}\boldsymbol{a}_{j_\sigma}\\ p\end{pmatrix}+\begin{pmatrix}\boldsymbol{a}_{j_\sigma+1}\\ p\end{pmatrix}=p\begin{pmatrix}\boldsymbol{b}\\ 1\end{pmatrix}.$$

Therefore

$$j_\sigma+1\left\{\begin{pmatrix} 1 & -p\boldsymbol{b} & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}\right. X'A_2Y' \begin{pmatrix} 1 & & \vdots & & \\ & \ddots & \lambda_{j_\sigma} & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} p & & \boldsymbol{a}_{j_1} & \boldsymbol{a}_{j_\sigma} & 0 & \\ & \ddots & & & & \\ & & p & & & \\ & & & \ddots & & \\ & & & & p & \\ & & & & & p & \ddots \end{pmatrix}.$$

If we suitably determine $\{b'_i\}$ considering $A_2U_2\equiv 0$ *mod.* $\{pP_2\}$ in

$$\begin{pmatrix} 1 & & \vdots & & \\ & \ddots & -\lambda_{j_1} & & \\ & & \vdots & & \\ & & \ddots\ -\lambda_{j_\sigma} & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} Y'^{-1}U_2Z' \begin{pmatrix} 1 & & p^{s_{j_\sigma+1}-s_1}b_1' & & \\ & \ddots & \vdots & & \\ & & \vdots & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix},$$

then we obtain

$$\begin{pmatrix} p^{s_1} & u_{j_1} & u_{j_\sigma} & 0 & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & p^{s_{j_\sigma}} & & \\ & & & & p^{s_{j_\sigma}+1} & \\ & & & & & \ddots \end{pmatrix}.$$

After these transformations, we obtain g linearly independent columns w. r. t. *mod.* p, $a_{j_1}, \ldots, a_{j_g}$. Now consider the matrix

$$\begin{pmatrix} p & a_{j_1} & a_{j_g} & & \\ & \ddots & & & \\ & & p & & \\ & & & \ddots & \\ & & & & p \\ & & & & & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} p^{s_1} & u_{j_1} & u_{j_g} & & \\ & \ddots & & & \\ & & p^{s_{j_2}} & & \\ & & & \ddots & \\ & & & & p^{s_{j_g}} \\ & & & & & \ddots \end{pmatrix}.$$

About the rows of these matrices we do the same transformations from the bottom to the top, and we obtain the matrices of degree g if 0 rows and 0 columns are excluded. These transformations are also done by unimodular matrices whose elements under the diagonal are 0. At last we gather the last matrices by the matrices

$$\begin{pmatrix} E^{(j_k-1)} & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & 1 & & 0 & \\ & & & & & E^{(n-k)} \end{pmatrix}, \quad (k=1, \ldots, g),$$

at the right upper corner. It is possible because $g \leqq [i_2/2]$. Then of course $|A^*| \not\equiv 0$ *mod.* p, and the diagonal elements are transformed to the diagonal elements.

COROLLARY 1, Under the assumption in Lemma 10, Lemma 8 is once more proved.

PROOF. Since A^* and U^* are in Z_g and $g \leqq [i_2/2]$, therefore the proof is seen easily.

COROLLARY 2. $A_2U_2 \equiv 0$ *mod.* $\{p^{s_1+1}, \ldots, p^{s_{i_2}+1}\}$ if and only if $XA_2U_2Z \equiv 0$ *mod.* $\{p^{s'_1+1}, \ldots, p^{s'_{i_2}+1}\}$, where $s_1, \ldots, s_{i_2}$ and $s'_1, \ldots, s'_{i_2}$ are given in Lemma 10.

LEMMA 11. Let $K \in G_{m+h}^{(p^2)}$ and let $L \in G_{m+h}^{(pl)}$, and they are

$$K=\begin{pmatrix} pE^{(g)} & & A^{*(g)} & & B_{11} & B_{12} & B_{13} & \\ & pE^{(m-2g)} & & & B_{21} & B_{22} & B_{23} & \\ & & pE^{(g)} & & B_{31} & B_{32} & B_{33} & \\ & & & p^2E^{(h)} & & & & \\ & & & & pE^{(g)} & & & \\ & & & & & pE^{(m-2g)} & & \\ & & & & C^{*(g)} & & pE^{(g)} & \\ & & & & & & & E^{(h)} \end{pmatrix},\quad L=\begin{pmatrix} P_1 & U^{*(g)} & & & V_{11} & V_{12} & V_{13} & V_{14} \\ & P_2 & & & \cdot & \cdot & \cdot & \cdot \\ & & P_3 & & \cdot & \cdot & \cdot & \cdot \\ & & & P_4 & V_{41} & V_{42} & V_{43} & V_{44} \\ & & & & Q_1 & & & \\ & & & & & Q_2 & & \\ & & & & W^{*(g)} & & Q_3 & \\ & & & & & & & Q_4 \end{pmatrix}$$

where P_i and Q_j are diagonal matrices whose elements are p-powers. Then for a given K, there exist $phm+h(h+1)$ different $L's$ such that

$$KL\equiv 0 \ mod.\ \{pP_1,\ pP_2,\ pP_3,\ p^2P_4,\ pQ_1,\ pQ_2,\ pQ_3,\ Q_4\}.$$

PROOF. At first from Corollary 2 of Lemma 10 U^* is uniquely determined and therefore W^* is also unique. If $p^2V_{44}\equiv 0$ *mod.* $\{Q_4\}$, then $V_{44}=p^{-2}X_{44}Q_4$, where all elements of X_{44} are determined *mod.* p^2. From $P_4{}^tV_{44}=V_{44}{}^tP_4$ it follows $X_{44}={}^tX_{44}$. Hence we can take $p^{h(h+1)}$ different V_{44} as the solutions. From similar investigation we obtain $V_{24}=p^{-1}X_{24}\,Q_4$, $V_{42}=p^{-1}{}^tX_{24}\,Q_2$, all elements of X_{24} are determined *mod.* p, and $V_{34}=p^{-1}X_{34}\,Q_4$, $V_{43}=p^{-1}{}^tX_{34}\,Q_3$, all elements of X_{34} are determined *mod.* p. Since $pU^*+A^*P_3\equiv 0$ *mod.* $\{pP_3\}$, therefore $U^*+p^{-1}A^*P_3=XP_3$ for suitable X determined *mod.* p. If $V_{43}=p^{-1}X_{43}\,Q_3$, then $V_{14}=p^{-1}({}^tX_{41}+XX_{34})-p^{-2}A^*X_{34}\,Q_4$. Hence we obtain p^{hm} different solutions V_{14}, V_{24}, V_{34}, and the multiplicity of $(V_{14},\ V_{24},\ V_{34},\ V_{44},\ V_{43},\ V_{42},\ V_{41})$ is $p^{hm+h(h+1)}$. If we investigate V_{11}, V_{12}, V_{13}, V_{21}, V_{31}, V_{22}, V_{23}, V_{32}, V_{33}, then we know it is also determined uniquely, and the multiplicity of these parts is 1.

COROLLARY. B_{23}, B_{32} and B_{33} are all 0.

PROOF. It is from $|A^*|\not\equiv 0$ *mod.* p.

LEMMA 12. Let $K\in G_m^{(p^2)}$, $K=\begin{pmatrix} A & B \\ 0 & C\end{pmatrix}$. If $\rho_p(B_{22})=h$ and $\rho_p(A^*)=g$, then the elementary divisors of K are

$$(\underbrace{1,\ \dots,\ 1}_{h+2g},\ \underbrace{p,\ \dots,\ p}_{m-h-2g};\ p,\ \dots,\ p,\ p^2,\ \dots,\ p^2),$$

where B_{22} and A^* are the matrices in Lemma 10 and in Lemma 11, and

$$A=\begin{pmatrix} pE^{(g)} & & A^* \\ & pE^{(m-2g)} & \\ & & pE^{(g)} \end{pmatrix}.$$

Let K be the matrices in Lemma 7, with $\rho(A^*)=g$ and $\rho(B_{22})=h$. We denote the number of the above matrices K by $\delta_p(i_2;\ g,\ h,\ f)$ such that the

f-th column of $\tilde{A}_2$ is not 0 *mod. p* and it is first one, and $\rho_p(A^*)=g$, $\rho_p(B_{22})=h$. Particularly if $\rho_p(A^*)=0$, then we denote the number by $\varepsilon_p(i_2\,;\,h)$. Let $T(p^{r_1},\ldots,p^{r_{i_1}},p^{s_1},\ldots,p^{s_{i_2}},p^{t_1},\ldots,p^{t_{i_3}})$ be in $t_n(p^l)$. Then we use the convention in this notation that if $i_1=0$, $i_2=0$ or $i_3=0$, then the above $T(\ldots)$ does not contain $(p^{r_1},\ldots,p^{r_{i_1}})$, $(p^{s_1},\ldots,p^{s_{i_2}})$ or $(p^{t_1},\ldots,p^{t_{i_3}})$.

LEMMA 13. $T_{p^2}(r)t_n(p^l)\ \ni$

$$\sum_{\substack{i_2\geqslant 2g+h+(i_1+i_3)=n-r\\ i_2\geqslant f\geqslant 1}} p^{i_3(i_3+1+i_2)}\delta_p(i_2\,;\,g,\,h,\,f)T(p^{r_1},\ldots,p^{r_{i_1}},\,p^{s_1+1},\ldots,p^{s_{f-1}+1},\,p^{s_f+2},\ldots,$$

$p^{s_{i_2}+2},\,p^{t_1+2},\ldots,p^{t_w+2}\,;\,\ldots\ldots)$, where $i_1+i_2+i_3=n$.

COROLLARY. $T_{p^2}(r)t_n(p^l)\ \ni$

$$\sum_{i_2\geqq n-r=h+(i_1+i_3)} \varepsilon_p(i_2\,;\,h)T(p^{r_1},\ldots,p^{r_{i_1}},\,p^{s_1+1},\ldots,p^{s_{i_2}+1}).$$

Now we consider more general type of the representative matrices of $T_{p^2}(m)$. They are transformed to the firstly treated matrices by interchanges between columns or rows, and the transformations are done by symplectic modular matrices. Therefore our problem is to calculate the multiplicities which are brought on by these transformations. The canonical forms of cosets in the product $T_{p^2}(m)t_n(p^l)$ are $T(p^{r_1},\ldots,p^{r_u},p^{s_1+1},\ldots,p^{s_v+1},p^{t_1+2},\ldots,p^{t_w+2}\,;\ldots)$, where $u+v+w=n$. For convenience we classfy them into three classes, (1) $w=0$, (2) $v=0$, $w\neq 0$, (3) $vw\neq 0$.

Let $u\geqq 0$, $e\geqq n-m$, $f\geqq 0$ and $e+f=n-u$, where $u, e, f, \in \boldsymbol{Z}$. When $e>0$, let $e=e_1+\ldots+e_d$ be a representation of e as the sum of any number of integral parts and let $f=f_1+\ldots+f_d$ be a representation of f as the sum of integral parts of same number with e's such that the last part is non-negative and the other are positive. We consider two representations of e or f are different if their arrangements of integral parts are different. Then we define a polynomial of an indeterminate x

$$\beta_x(e,f)=\sum_{(e_i),(f_j)} x^{\sum_i e_i(\sum_{j\geqq i} f_j)},$$

where the first summation is taken all over the above representaions of e and f. Particularly we define $\beta_x(0,f)=1$ if $f=0$ and $=0$ if $f\neq 0$. For the case of $w=0$ we obtain

THEOREM 3. $T_{p^2}(m)t_n(p^l)\ \ni$

$\sum_{u=0}^{m} \sum_{\substack{e+f=n-u \\ e \geqq n-m}} \beta_p(e, f)\varepsilon_p(e, m-n+e)T(p^{r_1}, \ldots, p^{r_u}, p^{s_1+1}, \ldots, p^{s_{n-u}+1}; \ldots\ldots)$, where $r_1 \leqq \ldots \leqq r_u \leqq s_1+1 \leqq \ldots \leqq s_{n-u}+1 \leqq l+2-(s_{n-u}+1) \leqq \ldots \leqq l+2-(s_1+1) \leqq l+2-r_u \leqq \ldots \leqq l+2-r_1$.

Proof. Let K be representatives of cosets in $T_{p^2}(m)$ such that

$$K=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \quad A=\begin{pmatrix} E^{(u)} & & & \\ & pE^{(e_1)} & & \\ & & E^{(f_1)} & \\ & & & \ddots \end{pmatrix},$$

only such K bring the case $w=0$. Let e be the number of p and let $f+u$ be the number of 1 on the diagonal of A. For given K from similar argument in Lemma 7, we obtain the multiplicity $\beta_p(e,f)$ of U-part. If all K are considered, though V is uniquely determined by B, then the number of choice of B is $\varepsilon_p(e, m-n+e)$. Hence the multiplicity in the case $w=0$ is $\beta_p(e,f)\varepsilon_p(e, m-n+e)$, from this the proof is obvious.

Secondly we consider the case $v=0$ and $w \neq 0$.
Let $u \geqq 0$, $h \geqq n-m$, $g>0$ and $h+g=n-u$. Let $g=g_1+\ldots+g_d$, and $h=h_1+\ldots+h_d$ be the same notations which are treated in the first case. Then we define a polynomial of an indeterminate x

$$\gamma_x(g,h)=\sum_{(g_i),(h_j)} x^{\sum_i \{g_i(\sum_{j\geqq i} h_j+\sum_{k>i} g_k)+g_i(g_i+1)/2\}},$$

particularly $\gamma_x(0,h)=1$ if $h=0$ and $=0$ if $h \neq 0$.
For the second case we obtain

Theorem 4. $T_{p^2}(m)t_n(p^l) \ni$

$\sum_{u=0}^{m-1} \sum_{\substack{g+h=n-u \\ m-u\geqq g\geqq 1}} \sum_{\substack{0\leqq k\leqq h-e \\ e+f=h-k \\ n-m\leqq e\leqq h}} \sum_{\substack{2a+b=e+m-n \\ 2\leqq c\leqq e}} \gamma_{p^2}(g,h)\beta_p(e,f)\delta_p(e; a,b,c)T(p^{r_1}, \ldots, p^{r_u}, p^{t_1+2}, \ldots, p^{t_w+2}; \ldots\ldots)$, where $r_1 \leqq \ldots \leqq r_u \leqq t_1+2 \leqq \ldots \leqq t_w+2 \leqq l-t_w \leqq \ldots \leqq l-t_1 \leqq l+2-r_u \leqq \ldots \leqq l+2-r_1$.

Proof. Let K be representatives of cosets in $T_{p^2}(m)$ which bring $T(p^{r_1}, \ldots, p^{r_u}, p^{t_1+2}, \ldots, p^{t_w+2}; \ldots\ldots)$ in $T_{p^2}(m)t_n(p^l)$. Then they are of only such forms

$$K=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},\quad A=\begin{pmatrix} E^{(u)} & & & & & \\ & p^2E^{(g_1)} & & & & \\ & & E^{(k)} & & & \\ & & & pE^{(e_1)} & & \\ & & & & E^{(f_1)} & \\ & & & & & \ddots \end{pmatrix}+\tilde{A}.$$

Let g be the number of p^2, e be of p and $u+k+f$ be of 1 on the diagonal of A. About U-part from Theorem 3 the multiplicity is

$$p^{g_1\{\sum_{j\geqq 1}(2f_j+e_j)+2k\}+2\sum_{i\gg 2}g_i\{\sum_{j\geqq i}(f_j+e_j)\}}\beta_p(e,f),$$

wher $h+g=n-u$, $h=e+f+k$, $e=e_1+\dots+e_d$ and $f=f_1+\dots+f_d$ which is the representaion of f corresponding to e's. About V-part, from similar argument with Lemma 11, the multiplicity is

$$p^{\sum_i g_i(g_i+1)+\sum_i g_i(\sum_{j\gg i}e_j)+2\sum_i(g_i\sum_{j>i}g_j)}.$$

Therefore when all K are considered, then the multiplicity is

$$\sum_{\substack{g+h=n-u\\ m-u\geqq g\geqq 1}}\ \sum_{\substack{0\leqq k\leqq h-e\\ e+f=h-k\\ n-m\leqq e\leqq h}}\ \sum_{\substack{2a+b=e+m-n\\ 2\leqq c\leqq e}}\gamma_{p^2}(g,h)\beta_p(e,f)\delta_p(e\,;a,b,c).$$

Hence we obtain the theorem.

For the third case we have

THEOREM 5. $T_{p^2}(m)t_n(p^l)\ni$

$$\sum_{u,v}\{(\sum_{\substack{0\leqq g+h\leqq w-1\\ \min\{m-u,n-2-u,w-1\}\geqq g\geqq 0}}\ \sum_{\substack{e+f=v+v-g\\ e\geqq \max\{n-m,2\}}}\ \sum_{\substack{2a+b=e+m-n\\ \max\{2,e-w+1\}\leqq c\leqq e}}\gamma_{p^2}(g,h)\beta_p(e,f)\delta_p(e\,;a,b,c))+$$

$$(\sum_{\min\{m-n,n-1-u\}\geqq g\geqq 1}\ \sum_{\substack{e+f=v+w-g\\ e\geqq \max\{1,n-m\}}}\ \sum_{\substack{2a+b=e+m-n\\ \max\{2,e-w+2\}\leqq c\leqq e}}\gamma_{p^2}(g,w-g)\beta_p(e,f)\delta_p(e\,;a,b,c))\}\,T(p^{r_1},$$

$\dots, p^{r_u}, p^{s_1+1},\dots, p^{s_v+1}, p^{t_1+2},\dots, p^{t_w+2}\,;\dots\dots)$, where $r_1\leqq\dots\leqq r_u\leqq s_1+1\dots\leqq s_v+1\leqq t_1+2\leqq\dots\leqq t_w+2\leqq l-t_w\leqq\dots l-t_1\leqq l+1-s_v\leqq\dots\leqq l+1-s_1\leqq l+2-r_u\leqq\dots\leqq l+2-r_1$.

PROOF. Let K be representatives of cosets in $T_{p^2}(m)$ which bring $T(p^{r_1},\dots, p^{r_u}, p^{s_1+1},\dots, p^{s_v+1}, p^{t_1+2},\dots, p^{t_m+2})$ in $T_{p^2}(m)t_n(p^l)$, where $vw\neq 0$. Therefore if

$$K=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\text{ and } A=\begin{pmatrix} E^{(u)} & & & \\ & pE^{(e_1)} & & \\ & & E^{(f_1)} & \\ & & & \ddots \end{pmatrix}+\tilde{A},$$

then we have two cases. In one case the $u+v+1-$th element on the diagonal of A is p, and in another case the element is p^2. Let g be the number of p^2, e be of p and $u+f$ be of 1 on the diagonal of A. In the first case we have at least two p on the diagonal, and all elements p^2 belong to w-part. Let h be the number of 1 belonging to w-part and they lie at the lower part of the first p^2. Then from Theorem 3 and 4 we have the multiplicity

$$\sum_{\substack{0\leqq g+h\leqq w-1 \\ \min\{m-u,n-2-u,w-1\}\geqq g\geqq 0}} \sum_{\substack{e+f=v+w-g \\ e\geqq \max\{n-m,2\}}} \sum_{\substack{2a+b=e+m-n \\ \max\{2,e-w+1\}\leqq c\leqq e}} \gamma_{p^2}(g, h)\cdot\beta_p(e, f)\delta_p(e\,;a, b, c).$$

In the second case we have at least one p on the diagonal. Therefore we have the multiplicity

$$\sum_{\min\{m-u,n+u\}\geqq g\geqq 1} \sum_{\substack{e+f=v+w-g \\ e\geqq \max\{1,n-m\}}} \sum_{\substack{2a+b=e+m-n \\ \max\{2,e-w+2\}\leqq c\leqq e}} \gamma_{p^2}(g, w-g)\beta_p(e, f)\delta_p(e\,;a, b, c).$$

Hence we have the theorem.

5. In this section we seek explicit forms of the polynomials defined in **4.**

LEMMA 14. $\beta_x(1, f)=x^f$, and if $e\geqq 2$, then

$$\beta_x(e, f)=\frac{x^f(x^{f+1}-1)\dots(x^{f+e-1}-1)}{(x-1)\dots(x^{e-1}-1)}.$$

PROOF. If $e=1$, then from the definition directly $\beta_x(1, f)=x^f$. If $e\geqq 2$, then from the definition

$$\beta_x(e, f)=\sum_{e=1}+\sum_{e\neq 1}=x^f\sum_{f_1=0}^{f}\beta_x(e-1, f_1).$$

Therefore

$$\beta_x(e, f)=x^f\frac{(x^{f+1}-1)\dots(x^{f+e-1}-1)}{(x-1)\dots(x^{e-1}-1)}.$$

LEMMA 15. $\gamma_x(1, f)=x^{1+f}$, and if $e\geqq 2$, then

$$\gamma_x(e, f)=x^{f+e(e+1)/2}\frac{(x^{f+1}-1)\dots(x^{f+e-1}-1)}{(x-1)\dots(x^{e-1}-1)}.$$

PROOF. If $e=1$, then from the definition directly we obtain $\gamma_x(e, f)=x^{1+f}$. If $e\geqq 2$, then by the same manner

$$\gamma_x(e, f)=x^{f+e}\sum_{f_1=0}^{f}\gamma_x(e-1, f_1),$$

from this we obtain the formula.

COROLLARY. $\gamma_x(e, f)=x^{e(e+1)/2}\beta_x(e, f)$.

We define the polynomial $(x+1)\dots(x^m+1)=\sigma_x(m)$. About this polynomial it is well-known that

$$\sigma_x(m)=1+\sum_{e=1}^{m}x^{e(e+1)/2}\frac{(x^m-1)\dots(x^{m-e+1}-1)}{(x-1)\dots(x^e-1)}.$$

Then we obtain

LEMMA 16. If $m\geqq 1$, then $\alpha_x(m)=\sigma_x(m)-\sigma_x(m-1)$.

PROOF. From the definition $\alpha_x(m)=\sum_{\substack{e+f=m\\ m\geqslant e\geqslant 1}}\gamma_x(e,f)$. Therefore by the above well-known relation

$$\alpha_x(m)=x^m+\sum_{m\geqslant e\geqslant 2}x^{m+e(e-1)/2}\frac{(x^{m-1}-1)\dots(x^{m-e+1}-1)}{(x-1)\dots(x^{e-1}-1)}=x^m\sigma_x(m-1)=\sigma_x(m)-\sigma_x(m-1).$$

LEMMA 17. If $r\geqslant 2$, then $\varepsilon_p(n;r)=p^{[r/2]([r/2]+1)}\dfrac{(p^{n-r+1}-1)\dots(p^n-1)}{(p^2-1)\dots(p^{2[r/2]}-1)}$ and $\varepsilon_p(n;1)=p^n-1$, $\varepsilon_p(n,0)=1$.

PROOF. It is obvious that $\varepsilon_p(n;1)=p^n-1$, and $\varepsilon_p(n,0)=1$. Let (b_{ij}) be symmetric matrices of degree n and of $\rho_p(\!(b_{ij})\!)=r\geqq 2$. There are three cases in the following senses,

(1) $b_{11}\neq 0$, (2) $b_{11}=b_{12}=\dots=b_{1i-1}=0$, but $b_{1i}\neq 0$, and (3) $b_{11}=b_{12}=\dots=b_{1n}=0$. From these cases we obtain the equation

$$\varepsilon_p(n;r)=\varepsilon_p(n-1;r)+p^{n-1}(p-1)\varepsilon_p(n-1;r-1)+p^{n-1}(p^{n-1}-1)\varepsilon_p(n-2;r-2).$$

Let assume for $s<r$ $\varepsilon_p(n;s)=p^{[s/2]([s/2]+1)}(p^{n-s+1}-1)\dots(p^n-1)/(p^2-1)\dots(p^{2[s/2]}-1)$ for all n. Then $\varepsilon_p(n;\ \mathrm{r})-\varepsilon_p(n-1;r)$

$$=p^{n-1}p^{r(r-2)/4+1}\frac{(p^{n-r+1}-1)\dots(p^{n-1}-1)}{(p^2-1)\dots(p^{r-2}-1)},\ r\equiv 0\ (2),$$

$$=p^{n-1}p^{(r-3)(r-1)/4}\ \frac{(p^r-1)(p^{n-1}-1)\dots(p^{n-r+1}-1)}{(p^2-1)\dots(p^{r-3}-1)(p^{r-1}-1)},\ r\not\equiv 0\ (2).$$

Tnerefore $\varepsilon_p(n;r)=\sum_{m=r}^{n}p^{1+r(r-2)/4}\ \dfrac{p^{m-1}(p^{m-1}-1)\dots(p^{m-r+1}-1)}{(p^2-1)\dots\dots(p^{r-2}-1)},\ r\equiv 0\ (2),$

$$=\sum_{m=r}^{n}p^{(r-3)(r-1)/4}\ \frac{(p^r-1)p^{m-1}(p^{m-1}-1)\dots(p^{m-r+1}-1)}{(p^2-1)\dots(p^{r-3}-1)(p^{r-1}-1)},\ r\not\equiv 0\ (2).$$

Now if $n\geqslant r\geqslant 2$, then we can prove

$$\sum_{m=r}^{n}p^{m-1}(p^{m-1}-1)\dots(p^{m-r+1}-1)=\frac{p^{r-1}(p^n-1)\dots(p^{n-r+1}-1)}{p^r-1}.$$

Hence $\varepsilon_p(n;r)=\dfrac{p^{r(r/2+1)/2}(p^{n-r+1}-1)\dots(p^n-1)}{(p^2-1)\dots(p^{r-2}-1)(p^r-1)},\ r\equiv 0\ (2),$

$$=\frac{p^{(r-1)(1+(r-1)/2)/2}(p^n-1)\dots(p^{n-r+1}-1)}{(p^2-1)\dots(p^{r-1}-1)},\ r\not\equiv 0\ (2).$$

Therefore we obtain the lemma.

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Reference.

[1] SHIMURA, G., *On modular correspondences for Sp(N, Z) and their congruence relations,* Proc. N. A. S. **49** (1963), 824–828.