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Willox, Ralph
Graduate School of Mathematical Sciences, The University of Tokyo

Ramani, Alfred
Centre de Physique Théorique, Ecole Polytechnique, France

Grammaticos, Basil
IMNC, Université Paris VII & XI, CNRS, France

https://doi.org/10.15017/1448845

出版情報：応用力学研究所研究集会報告. 25AO-S2 (5), pp.29-34, 2014-03. Research Institute for Applied Mechanics, Kyushu University

応用力学研究所研究集会報告 No.25AO-S2
「非線形波動研究の拡がり」（研究代表者　増田　哲）

Reports of RIAM Symposium No.25AO-S2

The breadth and depth of nonlinear wave science

Proceedings of a symposium held at Chikushi Campus, Kyushu Universiy,
Kasuga, Fukuoka, Japan, October 31 - November 2, 2013

Article No. 05 (pp. 29 - 34)

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WILLOX　Ralph（WILLOX Ralph），RAMANI　Alfred（RAMANI Alfred），GRAMMATICOS　Basil（GRAMMATICOS Basil）

（Received 28 November 2013; accepted 25 February 2014）

Research Institute for Applied Mechanics
Kyushu University
March, 2014

The ultradiscrete sine-Gordon equation: introducing the oiston

Graduate School of Mathematical Sciences, the University of Tokyo WILLOX Ralph‡

Centre de Physique Théorique, Ecole Polytechnique, France RAMANI Alfred

IMNC, Université Paris VII & XI, CNRS, France GRAMMATICOS Basil

We study the interaction of solitons of the ultradiscrete sine-Gordon equation. We give an explicit algorithm which allows one to calculate the asymptotic state of the solitons after interaction, from their asymptotic state before interaction. This algorithm is formulated in terms of a new entity, which we call 'oiston', the origin of which can be traced back to an ultradiscrete version of the discrete KdV equation.

1. Introduction

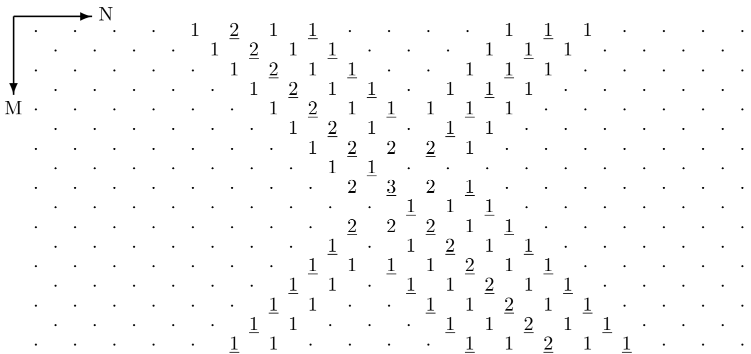
In this paper we shall study the interaction of soliton-like structures of the cellular automaton related to the sine-Gordon equation and interpret their dynamics in terms of a new entity which we have called "oiston" [1]. Starting from the discrete analogue of the sine-Gordon (sG) equation, introduced in [2] by Hirota,

$$z_{N+1}^{M+1} z_{N+1}^{M-1} = \frac{\delta + z_{N+2}^{M} z_{N}^{M}}{1 + \delta z_{N+2}^{M} z_{N}^{M}}\,, \tag{1}$$

we proceed to derive its ultradiscrete limit following the procedure proposed by Tokihiro and collaborators [3]. Namely, making the assumption that z is positive definite, we introduce the ansatz $z = e^{Z/\epsilon}$ (and $\delta = e^{-1/\epsilon}$, which implicitly assumes that $\delta < 1$) and take the limit $\epsilon \to +0$. Using the well-known identity $\lim_{\epsilon\to+0} \epsilon \log(e^{A/\epsilon} + e^{B/\epsilon}) = \max[A, B]$ we obtain

$$Z_{N+1}^{M+1} + Z_{N+1}^{M-1} = \max[-1, Z_{N+2}^{M} + Z_{N}^{M}] - \max[0, -1 + Z_{N+2}^{M} + Z_{N}^{M}]\,. \tag{2}$$

In the sections that follow we restrict the dependent variable to integer values and thus define a generalised cellular automaton. Moreover we consider the evolution along the M direction, which plays the role of "time", while N is the "space" variable. Given the form of (2), the evolution must take place on a checkerboard, i.e. on the sites (M, N) of the lattice where the sum $M + N$ has the same parity (which we shall take here to be even, without loss of generality).



The dynamics we shall be interested in concern the interaction of solutions with finite support. As shown in [1], the ultradiscrete sG has soliton-like solutions which satisfy $\left|Z_N^M + Z_{N+2}^M\right| \le 1$ and which can move in either direction at the "speed of light", i.e. with speed ± 1. The soliton collisions of the ultradiscrete sG were studied in [4] where three different types of behaviour were identified. First, the solitons can be mutually transparent. Second, the solitons can interact elastically, emerging from the collision with just a phase-shift. The unusual interaction is the third one. Indeed the solitons can interact *inelastically*, emerging from the collision

‡ R.W. would like to acknowledge support by JSPS KAKENHI, Grant-in-Aid for Scientific Research, Grant Number 24540204.

having acquired (or lost) a head or a tail. The figure above shows such an inelastic collision. Note that throughout the paper, dots denote zero entries and underscores denote negative values.

Moreover, this dressing/undressing of solitons has a finite capacity, i.e. for every soliton there exists a finite set of obtainable states – obtainable through successive dressing and undressing – amongst which there are unique states (up to an overall change of sign) with minimal and maximal length. Practically, in the example at hand this means: starting with a soliton of the form $[1, -2, 1, -1]$ we can undress it to the minimal state $[1, -2, 1]$ which, in turn, can be dressed to $[-1, 1, -2, 1]$. Both 'dressed' states can be dressed once more to the maximal state $[-1, 1, -2, 1, -1]$. Similarly, the second soliton $[1, -1, 1]$ cannot be dressed further and has a unique undressed state $[-1, 1]$.

The difficulty in interpreting the dynamics of the ultradiscrete sG solitons stems from the fact that these solitons are not the ultradiscrete analogues of the solitons of the discrete sG equation (1), which are not positive definite and therefore not ultradiscretisable. Still, as we shall explain in the following sections, it is possible to obtain a complete description of the dynamics, introducing a new entity which we have dubbed the "oiston".

2. The oiston

In order to implement the algorithm we start by introducing the rotated axes $m = \frac{M+N}{2}, n = \frac{M-N}{2}$, which are aligned with the light-cones of the solitons. We now have solitons that propagate either along the m direction, or along the n direction. For the solitons that propagate along the n-axis we introduce a convenient auxiliary variable

$$Y_n^m = (-1)^m Z_N^M \,, \tag{3}$$

which has the advantage of rectifying the alternating sign in the solitons. Similarly, for the solitons that propagate along the m-axis we introduce

$$\widetilde{Y}_n^m = (-1)^n Z_N^M \,. \tag{4}$$

Furthermore, for both types of soliton we define new quantities

$$V^m = \max\left[Y_\nu^{m+1}, Y_\nu^{m-1}\right] - Y_\nu^m, \qquad W^m = Y_\nu^m - \min\left[Y_\nu^{m+1}, Y_\nu^{m-1}\right] \tag{5}$$

$$\widetilde{V}_n = \max\left[Y_{n+1}^\mu, Y_{n-1}^\mu\right] - Y_n^\mu, \qquad \widetilde{W}_n = Y_n^\mu - \min\left[Y_{n+1}^\mu, Y_{n-1}^\mu\right] \tag{6}$$

which are computed from the asymptotic state of the solitons before interaction. Since an asymptotically free soliton retains its shape during propagation these quantities are independent of the positions ν and μ, provided the latter are situated sufficiently far in the past. It is the vector $\begin{pmatrix} W \\ V \end{pmatrix}$ which defines the oistons. Let us give a few examples of the oiston construction.

Y	0	0	1	0	0
W	0	0	1	0	0
V	0	1	$\underline{1}$	1	0

Y	0	0	1	1	1	0	0
W	0	0	1	0	1	0	0
V	0	1	0	0	0	1	0

We remark, reading from left to right, that in the arrangement of vectors $\begin{pmatrix} W \\ V \end{pmatrix}$ in the picture on the left, we have an ascending slope of 1's followed by a descending one, enclosing a -1, which constitute an elementary trapezoidal oiston. In the picture on the right we have first an ascending elementary oiston followed by a descending one.

Y	0	0	1	0	1	0	0
W	0	0	1	$\underline{1}$	1	0	0
V	0	1	$\underline{1}$	1	$\underline{1}$	1	0

Y	0	0	1	0	1	1	1	0	0
W	0	0	1	$\underline{1}$	1	0	1	0	0
V	0	1	$\underline{1}$	1	0	0	0	1	0

In the picture on the left we still have a single trapezoidal oiston, although with more structure than the elementary one above. On the right we have, first, an ascending parallelogram-shaped oiston which is again more complicated than the elementary one that had width one, followed by an elementary descending oiston. These examples show how to derive the oistons from the solitons of the ultradiscrete sG. The reverse construction is also possible. Given an arrangement of oistons, one can obtain the corresponding soliton in the Y variable by a very simple procedure. We illustrate this procedure on the last example of the decomposition into oistons given above. Starting from

$$\begin{array}{c|ccccccccc} W & 0 & 0 & 1 & \underline{1} & 1 & 0 & 1 & 0 & 0 \\ V & 0 & 1 & \underline{1} & 1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

we discard the negative 1's,

$$\begin{array}{c|ccccccccc} W & 0 & 0 & 1 & & 1 & 0 & 1 & 0 & 0 \\ V & 0 & 1 & & 1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

and follow from left-to-right (or from right-to-left) the slopes in the remaining pattern of 1's. We first assign the value 0 to Y and, sweeping for example from left-to-right, we increase the value of Y by 1 when we encounter an ascending slope within an oiston, or decrease Y by 1 in case of a descending slope.

$$\begin{array}{c|ccccccccccccccccc} W & 0 & & 0 & & 1 & & & & 1 & & 0 & & 1 & & 0 & & 0 \\ V & 0 & & 1 & & & & 1 & & 0 & & 0 & & 0 & & 1 & & 0 \\ & & & & + & & - & & + & & & & & & - & & & \\ Y & 0 & & 0 & & 1 & & 0 & & 1 & & 1 & & 1 & & 0 & & 0 \end{array}$$

Hence we see that the oistons offer a faithful representation of the (asymptotically free) solitons for the ultradiscrete sG, expressed in the Y variable.

3. The interaction rules

In order to explain the interaction rules for the oistons, we start by decomposing the initial condition $[\ldots,0,1,-2,1,-1,0,0,0,0,1,-1,1,0,\ldots]$ whose evolution is given in the introduction. Following the procedure given in the previous section, we find

$$\begin{array}{ccccccccccccccccc} & \cdot & \cdot & 1 & \underline{2} & 1 & \underline{1} & \cdot & \cdot & \cdot & \cdot & 1 & \underline{1} & 1 & \cdot & \cdot & \\ \widetilde{Y} & & & 1 & 2 & 1 & 1 & & & & & 1 & 1 & 1 & & & Y \\ \widetilde{W} & & \cdot & 1 & 1 & \cdot & 1 & \cdot & & & \cdot & 1 & \cdot & 1 & \cdot & \cdot & W \\ \widetilde{V} & \cdot & 1 & 1 & \underline{1} & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & V \end{array}$$

as decompositions for the solitons moving along the m-axis (with tildes) and n-axis (without tildes). Sweeping from left to right, for the contribution stemming from $\begin{pmatrix} \widetilde{W} \\ \widetilde{V} \end{pmatrix}$ we encounter an elementary upwards pointing oiston of width 1, an elementary trapezoidal oiston touching it, followed by an elementary downwards pointing oiston of width 1 at distance 1. These three oistons will interact with the two oistons stemming from $\begin{pmatrix} W \\ V \end{pmatrix}$ which are at distance 1 relative to each other.

The asymptotic consequence of the interaction is such that each set of oistons is conserved, separately, but that the relative distances within each oiston train change according to a very particular rule. Moreover, what really matters are not the distances themselves but their parities. If we denote the inter-oiston distances in each oiston train by $\widetilde{D}_i$ $(i = 1, \ldots, p-1)$ and D_i $(i = 1, \ldots, q-1)$ for the general case of a train consisting of p oistons for the solitons moving along the m-axis and q oistons for the solitons that move along the n-axis, we define

$$\widetilde{\theta}_i = \widetilde{D}_i \bmod 2 \;\; (i = 1, \ldots, p-1)\,, \qquad \theta_i = D_i \bmod 2 \;\; (i = 1, \ldots, q-1)\,. \tag{7}$$

In fact, the changes of the inter-oiston distances due to the interaction are all of the form

$$D_i = 2L_i + \theta_i \;\rightarrow\; D'_i = 2L_i + \theta'_i \tag{8}$$

(and similarly for the tildes) where the new parities θ'_i will be given by the procedure described below. Practically speaking this means that the inter-oiston distances can change at most by one unit.

The remaining quantity that plays a role in the interaction will be denoted as κ and encodes the relative signs of the 'heads' of the first two solitons that will interact: $\kappa = 0$ if these heads have the same sign and $\kappa = 1$ if they have opposite signs. For the initial condition analysed above, we find

$$\begin{array}{cccccccccccccccccc}
 & \cdot & \cdot & 1 & \underline{2} & 1 & \underline{1} & \cdot & \cdot & \cdot & \cdot & 1 & \underline{1} & 1 & \cdot & \cdot & \\
\widetilde{Y} & & & 1 & 2 & 1 & 1 & & & & & 1 & 1 & 1 & & & Y \\
\widetilde{W} & & \cdot & 1 & 1 & \cdot & 1 & \cdot & & & \cdot & 1 & \cdot & 1 & \cdot & & W \\
\widetilde{V} & \cdot & 1 & 1 & \underline{1} & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & V \\
 & & \uparrow & & & \uparrow & & & & & & & \uparrow & & & & \\
 & & \widetilde{\theta}_2{=}0 & & & \widetilde{\theta}_1{=}1 & & & (\kappa{=}1) & & & & \theta_1{=}1 & & & &
\end{array}$$

Using these data we construct a master array

$$A = \big[\, \widetilde{\theta}_{p-1}, \widetilde{\theta}_{p-2}, \cdots, \widetilde{\theta}_1, \kappa\,, \theta_1, \theta_2, \cdots, \theta_{q-1} \,\big]\,, \tag{9}$$

which in the present case is $A = \big[0, 1, 1, 1\big]$.

In general, for a master array $A = [a_1, \ldots, a_{p+q-1}]$, the asymptotic consequence of the oiston interaction is to 'exchange' in some sense inter-oiston gaps between the two oiston trains. More precisely, from this master array we can recover the gaps in the oiston trains after interaction, by reinterpreting the entries a_i in the master array as

$$A = \big[\, \theta'_1, \theta'_2, \cdots, \theta'_{q-1}, \kappa'\,, \widetilde{\theta}'_{p-1}, \widetilde{\theta}'_{p-2}, \cdots, \widetilde{\theta}'_1 \,\big]\,, \tag{10}$$

i.e.,

$$\theta'_i = a_i \;\; (i = 1, \ldots, q-1)\,, \qquad \kappa' = a_q\,, \qquad \widetilde{\theta}'_i = a_{q+i} \;\; (i = 1, \ldots, p-1) \tag{11}$$

in which the θ'_i are the parities of the new gaps, as defined in (8), for the oiston train that moves along the n-axis, and the $\widetilde{\theta}'_i$ are the parities of the new gaps for the oiston train that moves along the m-axis. The relative sign of the tails of the last two solitons to exit from the interaction is given by $(-1)^{\kappa'}$. Moreover, the front of each outgoing soliton train will incur a phase-shift. Namely, the leading oiston for the soliton moving along the m-axis is shifted back by δ_m steps in this direction and, analogously, that for the soliton moving along the n-axis is shifted back by δ_n steps, where

$$\delta_m = q - \kappa - \sum_{j=1}^{q-1} \theta_j\,, \qquad \delta_n = p - \kappa - \sum_{j=1}^{p-1} \widetilde{\theta}_j\,. \tag{12}$$

Note that these phase-shifts will also induce overall sign changes in the original Z variable, given by relations (3) and (4).

Going back to the example given in the beginning of this section, starting from the master array $A = \big[0, 1, 1, 1\big]$, we find the new arrangement of oistons given below. As $\theta'_1 = 0$, the two oistons that evolve along the n-axis and that were at distance $D_1 = 1$ are now at distance 0. Similarly, as $\widetilde{\theta}'_2 = \widetilde{\theta}'_1 = 1$, the three oistons evolving along the m-axis are now completely separated. The reconstruction of the corresponding soliton trains, expressed in the Y variable, is easily performed along the lines of the procedure explained at the end of section 2. From these values for Y and using the information on the phase-shifts obtained from (12), one can easily reconstruct the asymptotic state of the outgoing solitons in the variable Z using (3) and (4). This state corresponds exactly to what we obtained by a direct simulation of the initial condition.

$$
\begin{array}{cccccccl}
\cdot & \cdot & \underline{1} & 1 & \cdot & \cdot & & \\
 & & 1 & 1 & & & & Y' \\
 & \cdot & 1 & 1 & \cdot & & & W' \\
\cdot & 1 & \cdot & \cdot & 1 & \cdot & & V' \\
 & & \uparrow & & & & & \\
 & & \theta'_1{=}0 & & & & &
\end{array}
\qquad
\begin{array}{lccccccccc}
 & \cdot & \cdot & \underline{1} & 1 & \underline{2} & 1 & \underline{1} & \cdot & \cdot \\
\widetilde{Y}' & & & 1 & 1 & 2 & 1 & 1 & & \\
\widetilde{W}' & & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \\
\widetilde{V}' & \cdot & 1 & \cdot & 1 & \underline{1} & 1 & \cdot & 1 & \cdot \\
 & & & \uparrow & & & \uparrow & & & \\
 & & & \widetilde{\theta}'_2{=}1 & & & \widetilde{\theta}'_1{=}1 & & &
\end{array}
$$

Note that the value of $\kappa' = 1$ corresponds exactly to the relative sign of the tails of the reconstructed outgoing solitons. Notice also that for the soliton on the left, the oiston train is the maximally undressed one, while on the right we have a maximally dressed soliton. In fact, the dressing/undressing process is completely explained by the above interaction rules – and in particular by relation (8) – which show that for a given configuration of p oistons there are only 2^{p-1} different arrangements of inter-oiston gaps.

4. Where do the oistons come from?

While the introduction of the oistons for the construction of the asymptotic soliton states may appear somewhat magical, it has a quite rational origin. As we have shown in [5] the discrete sG equation is related to the discrete modified KdV equation

$$y_{n+1}^{m+1} = y_n^m \frac{y_n^{m+1} + \delta y_{n+1}^m}{\delta y_n^{m+1} + y_{n+1}^m} \tag{13}$$

by the transformation

$$y_n^m = (z_N^M)^{(-1)^m} . \tag{14}$$

At the ultradiscrete limit this relation leads to (3) and (4). Next, we use the Miura transformations [6]

$$u_n^m = \frac{(1+\delta) y_n^m}{\delta y_n^{m-1} + y_{n-1}^m} , \tag{15a}$$

$$v_{n+1}^m = \frac{\delta}{1-\delta} \frac{y_n^{m+1} + y_n^{m-1}}{y_n^m} \tag{15b}$$

relating the discrete mKdV to the Yang-Baxter form [7] of the discrete KdV

$$u_n^{m+1} = \frac{1}{\Delta(u_n^m + v_n^m)} , \tag{16a}$$

$$v_{n+1}^m = \frac{u_n^{m+1}}{u_n^m} v_n^m , \tag{16b}$$

where $\Delta = (1-\delta)/(1+\delta)$.

Taking the ultradiscrete limit of (16) as $u_n^m = e^{U_n^m/\epsilon}$ and $v_n^m = \frac{2\delta}{1-\delta} e^{V_{n-1}^m/\epsilon}$ for $\delta = e^{-1/\epsilon}$ and $\epsilon \to +0$, we obtain

$$U_n^{m+1} = \min[-U_n^m, 1 - V_{n-1}^m] , \tag{17a}$$

$$V_n^m = V_{n-1}^m + U_n^{m+1} - U_n^m . \tag{17b}$$

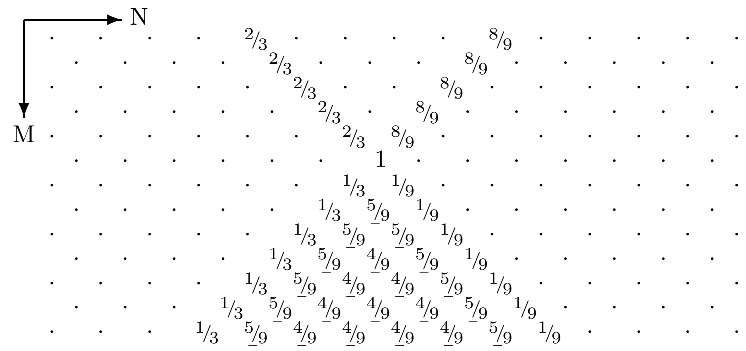
Note that eliminating the variable V from the system (17) yields an ultradiscrete version of the KdV equation which is not the standard Box&Ball one, since it corresponds to the limit $\Delta \to 1$. Note also that, at the ultradiscrete limit, the Miura transformation (15) for the variable v yields

$$V_n^m = \max[Y_n^{m+1}, Y_n^{m-1}] - Y_n^m\,, \tag{18}$$

which is nothing but the defining relation for the V part of the oistons given in section 2. In fact, the variable W, which together with V allows the construction of an oiston, is nothing but the value obtained from (18) by changing $Y \to -Y$, which corresponds to the fundamental symmetry $y \to 1/y$ of the discrete mKdV (13).

5. Conclusion

In this paper we have introduced an important new notion, that of the oiston. These new entities, which compose the solitons of the ultradiscrete sG, are conserved during a soliton collision and thus we may consider them as more fundamental than the soliton themselves. Be that as it may, it is the very fact that the ultradiscrete sG solitons are composed of oistons that explains the inelastic character of the scattering of these solitons.



Given the inelastic behaviour observed in the soliton collisions, one may wonder whether the term soliton is still appropriate in this case. While one can argue that it should be possible to trace the conservation of the oistons back to the conserved quantities of the sG (something that remains to be proven) the situation becomes more complicated when one strays from pure integers. (In fact, as we have shown in [8], if one wishes to have an as complete as possible description of the dynamics of an ultradiscrete system, one must not limit oneself to integer values.) An interaction of solitons that take rational values is presented in the figure above. We remark that the result of the collision is the creation of a bridge of constant height linking the two outgoing solitons. The figure, however, hints at a more physical interpretation of this phenomenon. One could interpret the above situation as that of two incoming waves which are partly reflected on the light-cone, but with constructive interference between the waves inside the light-cone. In fact, in a forthcoming paper we shall explain how such interactions can be understood in relation to dispersive waves for the discrete sG equation.

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