On algebraic independence of periods in characteristic p

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On algebraic independence of periods in characteristic p

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Contents

Introduction

In this thesis, we study periods in characteristic p. In particular we treat positive characteristic multizeta values over function fields and the values of Carlitz multiple polylogarithms at algebraic points. We prove several results on algebraic independence of them.

Classical case

The multiple zeta values (MZVs) in characteristic 0 was defined by Euler (depth two) and Hoffman (higher depth). These are defined by

$$\zeta_{\mathbb{Z}}(\underline{n}) = \zeta_{\mathbb{Z}}(n_1, \dots, n_d) := \sum_{m_1 > \cdots > m_d \geq 1} \frac{1}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{R}^{\times}$$

for a d-tuple of positive integers $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ with $n_1 \geq 2$. The sum $\mathrm{wt}(\underline{n}) := \sum_i n_i$ is called the weight and $\mathrm{dep}(\underline{n}) := d$ is called the depth of $\zeta_{\mathbb{Z}}(\underline{n})$. One of the goals of this topic is to determine all algebraic relations over $\overline{\mathbb{Q}}$ among the MZVs. Although many relations among MZVs are known, very few linear/algebraic independence results on MZVs are known. For example, Euler proved that when $d = 1$, the ratio $\zeta_{\mathbb{Z}}(n)/(2\pi\sqrt{-1})^n$ is a rational number if and only if $n \geq 2$ is a positive even integer. However, we do not know whether $\zeta_{\mathbb{Z}}(n)/\pi^n$ is a transcendental number for each odd integer $n \geq 3$. It is conjectured that $\pi, \zeta_{\mathbb{Z}}(3), \zeta_{\mathbb{Z}}(5), \zeta_{\mathbb{Z}}(7), \dots$ are algebraically independent over $\overline{\mathbb{Q}}$.

For each integer $w \geq 2$, we denote by $\mathfrak{Z}_w$ the $\mathbb{Q}$-vector space spanned by the MZVs of weight w. We also define $\mathfrak{Z}_0 := \mathbb{Q}$, $\mathfrak{Z}_1 := \{0\}$ and $\mathfrak{Z} := \sum_w \mathfrak{Z}_w$. The harmonic product formula shows that the product of two MZVs of weights w_1 and w_2 is described as a sum of MZVs of weight $w_1 + w_2$. The simplest case is as follows:

$$\zeta_{\mathbb{Z}}(n_1)\zeta_{\mathbb{Z}}(n_2) = \zeta_{\mathbb{Z}}(n_1, n_2) + \zeta_{\mathbb{Z}}(n_2, n_1) + \zeta_{\mathbb{Z}}(n_1 + n_2).$$

Goncharov ([**G1**]) conjectured that MZVs of different weights are linearly independent over $\mathbb{Q}$; this means

$$\mathfrak{Z} = \bigoplus_{w \geq 0} \mathfrak{Z}_w.$$

Thus it is conjectured that $\mathfrak{Z}$ is a graded $\mathbb{Q}$-algebra graded by weights. Zagier ([**Z**]) conjectured that

$$\dim_{\mathbb{Q}} \mathfrak{Z}_w = d_w,$$

where $d_0 := 1$, $d_1 := 0$, $d_2 := 1$ and $d_w := d_{w-2} + d_{w-3}$ for $w \geq 3$. Goncharov ([**G2**]) and Terasoma ([**Te**]) showed that the inequality $\dim \mathfrak{Z}_w \leq d_w$ holds for each $w \geq 0$. To show the converse inequalities, we need linear/algebraic independence results of MZVs, and thus this seems to be very difficult. André ([**Andr**, p. 231]) asked whether there exists $\underline{n}$ such that $\zeta_{\mathbb{Z}}(\underline{n}) \notin \mathbb{Q}[\zeta_{\mathbb{Z}}(2), \zeta_{\mathbb{Z}}(3), \zeta_{\mathbb{Z}}(4), \zeta_{\mathbb{Z}}(5), \dots]$. Note that this comes from the above two conjectures because $d_8 = 4$ and the weight 8 monomials of single zeta values are only $\zeta_{\mathbb{Z}}(2)^4$, $\zeta_{\mathbb{Z}}(2)\zeta_{\mathbb{Z}}(3)^2$ and $\zeta_{\mathbb{Z}}(3)\zeta_{\mathbb{Z}}(5)$ up to non-zero rational factors. We do not also have an answer to this question. These conjectures are also formulated when we replace $\mathbb{Q}$ by $\overline{\mathbb{Q}}$. In this thesis, consequences of our main result are to give some lower bounds of the dimension of the vector space spanned by the positive characteristic MZVs of fixed weight and an affirmative answer to the function field analogue of a question of André.

Positive characteristic multizeta values

Let $K := \mathbb{F}_q(\theta)$ be the rational function field over the finite field of q elements with variable θ, p the characteristic of K, $K_\infty := \mathbb{F}_q(\!(\theta^{-1})\!)$ the ∞-adic completion of K, $\overline{K_\infty}$ a fixed algebraic closure of K_∞, $\mathbb{C}_\infty$ the ∞-adic completion of $\overline{K_\infty}$, and $\overline{K}$ the algebraic closure of K in $\mathbb{C}_\infty$. We fix a $(q-1)$-st root of $-\theta$ and let

$$\widetilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} \cdot K_\infty^\times$$

be the fundamental period of the Carlitz module. This is a generator of the kernel of the exponential map of the Carlitz module and a function field analogue of $2\pi\sqrt{-1}$ which is a generator of the kernel of the usual exponential map. Wade ([**W**]) proved that $\widetilde{\pi}$ is transcendental over K. As $\#\mathbb{F}_q[\theta]^\times = q-1$, we say that an integer n is "odd" if $q-1$ does not divide n, and "even" if $q-1$ divides n. In this thesis, an *index* means an element of $(\mathbb{Z}_{\geq 1})^d$ for some positive integer $d \geq 1$. Thakur ([**Th1**]) defined the *positive characteristic multizeta values* (also denoted by MZVs) by

$$\zeta(\underline{n}) = \zeta(n_1, \dots, n_d) := \sum \frac{1}{a_1^{n_1} \cdots a_d^{n_d}} \in K_\infty^\times$$

for indices $\underline{n} = (n_1, \dots, n_d)$, where the sum is over all monic polynomials a_i in $\mathbb{F}_q[\theta]$ such that $\deg a_1 > \cdots > \deg a_d \geq 0$. It is clear that $\zeta(p^e \underline{n}) = \zeta(\underline{n})^{p^e}$ for all $e \geq 0$, where we set $p^e\underline{n} := (p^e n_1, \dots, p^e n_d)$. These are called the *p-th power (Frobenius) relations*. As in the classical case, we want to determine all algebraic relations among the MZVs over $\overline{K}$.

The MZVs of depth one had been studied by Carlitz ([**Ca**]) and they are called the *Carlitz zeta values*. He showed that if $n \geq 1$ is a positive "even" integer, then we have the *Euler-Carlitz relation*

$$\frac{\zeta(n)}{\widetilde{\pi}^n} = \frac{B_n}{\Gamma_{n+1}} \in K^\times,$$

where $B_n \in \mathbb{F}_q[\theta]$ is the *Bernoulli-Carlitz number* and $\Gamma_{n+1} \in \mathbb{F}_q[\theta]$ is the *factorial of Carlitz* (see Chapter 1). This is an analogue of the relations of the special zeta values at positive even integers. Thus the Carlitz zeta values at positive "even" integers are transcendental over K. Anderson and Thakur ([**AT1**]) showed that the Carlitz zeta value $\zeta(n)$ appears as an integral point of the logarithm of the n-th tensor power of the Carlitz module for each $n \geq 1$. Yu ([**Y1**]) proved that $\zeta(n), \zeta(n)/\widetilde{\pi}^n \notin \overline{K}$ for each positive "odd" integer $n \geq 1$. In [**Y2**], he also determined all linear relations over $\overline{K}$ among the Carlitz zeta values and the powers of $\widetilde{\pi}$. Finally, Chang and Yu ([**CY**]) proved that all algebraic relations over $\overline{K}$ among the Carlitz zeta values come from the p-th power relations or the Euler-Carlitz relations. Note that Chang, Papanikolas and Yu ([**CPY**]) also showed the algebraic independence of MZVs when the constant field $\mathbb{F}_q$ varies.

Several results on the higher depth case were also proved. Thakur ([**Th2**]) showed that MZVs are non-zero. Anderson and Thakur ([**AT2**]) showed that the MZVs have an interpretation as periods of t-motives. For each $w \geq 1$, we denote by $\overline{\mathcal{Z}}_w$ the $\overline{K}$-vector space spanned by the MZVs of weight w in $\mathbb{C}_\infty$. We also define $\overline{\mathcal{Z}}_0 := \overline{K}$ and $\overline{\mathcal{Z}} := \sum_w \overline{\mathcal{Z}}_w$. In positive characteristic, the harmonic product formula does not hold in general. Thakur ([**Th1**, Theorem 5.10.6]) showed that if weight is not more than q, then MZVs satisfy the classical harmonic product formula. In particular, the harmonic product formula

$$\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2)$$

holds if $n_1 + n_2 \leq q$ (see Remark 1.2). In [**Th4**], he also showed that the product of two MZVs of weights w_1 and w_2 is described as a sum of MZVs of weight $w_1 + w_2$. Chang ([**Ch2**]) showed that

$$\overline{\mathcal{Z}} = \bigoplus_w \overline{\mathcal{Z}}_w.$$

Thus $\overline{\mathcal{Z}}$ is a graded $\overline{K}$-algebra graded by weights.

The above results do not give the algebraic independence of MZVs of higher weights. In this thesis, we study algebraic relations over $\overline{K}$ among the elements of the set

$$\{\widetilde{\pi}\} \cup \{\zeta(n_j, n_{j+1}, \ldots, n_i) | 1 \leq j \leq i \leq d\}$$

for a fixed index $\underline{n} = (n_1, \ldots, n_d)$ such that n_i is "odd" for each i. For a positive "odd" integer $n \geq 1$, we prove that $\widetilde{\pi}$, $\zeta(n)$ and $\zeta(n, n)$ are algebraically independent over $\overline{K}$ if $2n$ is "odd" (Theorem 2.7). If furthermore $3n$ is "odd", then $\widetilde{\pi}$, $\zeta(n)$, $\zeta(n, n)$ and $\zeta(n, n, n)$ are algebraically independent over $\overline{K}$ (Theorem 2.13). We also prove that the elements of the above set are algebraically independent over $\overline{K}$ if n_i is "odd" for each i and n_i/n_j is not an integral power of p for each $i \neq j$ (Theorem 2.17). We also treat some cases where n_i/n_j may be an integral power of p for some $i \neq j$. Then under some conditions, we prove that the elements of the above set have only the p-th power relations (Theorems 2.23 and 2.24). A consequence of our results is to give an affirmative answer to the function field analogue of a question in [**Andr**, p. 231]. By using these results, we also obtain non-trivial lower bounds of the dimension of $\overline{\mathcal{Z}}_w$. In particular, we determine the dimension of $\overline{\mathcal{Z}}_2$ in any p and $\overline{\mathcal{Z}}_3$ when $p \neq 2, 3$. These results are proved in [**M3**] and [**M4**].

Carlitz multiple polylogarithms

In [**Ch2**], Chang defined the *Carlitz multiple polylogarithms* (CMPLs) by

$$\mathrm{Li}_{\underline{n}}(z_1,\dots,z_d) := \sum_{i_1>\cdots>i_d\geq 0} \frac{z_1^{q^{i_1}}\cdots z_d^{q^{i_d}}}{((\theta-\theta^q)\cdots(\theta-\theta^{q^{i_1}}))^{n_1}\cdots((\theta-\theta^q)\cdots(\theta-\theta^{q^{i_d}}))^{n_d}}$$

for indices $\underline{n}$. It converges if $|z_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, where $|-|_\infty$ is an ∞-adic valuation on $\mathbb{C}_\infty$. In [**AT1**], Anderson and Thakur showed that $\zeta(n)$ is described as a K-linear combination of the values of CMPLs of weight n and depth one at rational points for each $n \geq 1$. Moreover, in [**Ch2**], Chang showed that for each index $\underline{n}$ with $\mathrm{wt}(\underline{n}) = w$ and $\mathrm{dep}(\underline{n}) = d$, $\zeta(\underline{n})$ is described as a K-linear combination of the values of CMPLs of weight w and depth d at rational points. He also proved that CMPLs take non-zero values when $z_i \neq 0$ for each i. We are interested in the algebraic independence of their values at algebraic points over $\overline{K}$. Let $n \geq 1$ be a positive integer, and let $\alpha_1,\dots,\alpha_r \in \overline{K}^\times$ be algebraic points such that $|\alpha_j|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for each j. Papanikolas ([**P**]), Chang and Yu ([**CY**]) proved that if $\widetilde{\pi}^n, \mathrm{Li}_n(\alpha_1),\dots,\mathrm{Li}_{\underline{n}}(\alpha_r)$ are linearly independent over K, then they are algebraically independent over $\overline{K}$. Let $n_1,\dots,n_d \geq 1$ be positive integers such that n_i/n_j is not an integral power of p for each $i \neq j$. For each i, let $\alpha_{i1},\dots,\alpha_{ir_i} \in \overline{K}^\times$ be algebraic points such that $|\alpha_{ij}|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each j. Chang and Yu ([**CY**]) also proved that if $\widetilde{\pi}^{n_i}, \mathrm{Li}_{n_i}(\alpha_{i1}),\dots,\mathrm{Li}_{n_i}(\alpha_{ir_i})$ are linearly independent over K for each i, then the elements of the set $\{\widetilde{\pi}\} \cup \{\mathrm{Li}_{n_i}(\alpha_{ij}) | i,j\}$ are algebraically independent over $\overline{K}$. As in the case of the MZVs, several results on the higher depth case were also proved. Chang ([**Ch2**]) showed that values of Carlitz multiple polylogarithms at algebraic points of different weights are linearly independent over $\overline{K}$.

In this thesis, we study algebraic relations over $\overline{K}$ among the elements of the set

$$\{\widetilde{\pi}\} \cup \{\mathrm{Li}_{n_j,n_{j+1},\dots,n_i}(\alpha_j,\alpha_{j+1},\dots,\alpha_i) | 1 \leq j \leq i \leq d\}$$

for a fixed index $\underline{n} = (n_1,\dots,n_d)$ and a d-tuple of algebraic points $\underline{\alpha} = (\alpha_1,\dots,\alpha_d) \in (\overline{K}^\times)^d$ such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. For a positive "odd" integer $n \geq 1$ and a rational point $\alpha \in K^\times$ such that $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, we prove that $\widetilde{\pi}$, $\mathrm{Li}_n(\alpha)$ and $\mathrm{Li}_{n,n}(\alpha,\alpha)$ are algebraically independent over $\overline{K}$ if $2n$ is "odd" (Theorem 2.30). If furthermore $3n$ is "odd", then $\widetilde{\pi}$, $\mathrm{Li}_n(\alpha)$, $\mathrm{Li}_{n,n}(\alpha,\alpha)$ and $\mathrm{Li}_{n,n,n}(\alpha,\alpha,\alpha)$ are algebraically independent over $\overline{K}$ (Theorem 2.31). We also prove that the elements of the above set are algebraically independent over $\overline{K}$ if $\widetilde{\pi}, \mathrm{Li}_{n_1}(\alpha_1),\dots,\mathrm{Li}_{n_d}(\alpha_d)$ are algebraically independent over $\overline{K}$ (Theorem 2.33). If n_i is "odd" and $\alpha_i \in K^\times$ for each i and n_i/n_j is not an integral power of p for each $i \neq j$, then the above assumption is satisfied. These results are proved in [**M3**] and [**M4**].

Outline of this thesis

In Chapter 1, we define notations which are used in this thesis. In Chapter 2, we state our results on algebraic independence of MZVs and values of CMPLs. In Chapter 3, at first we review the (pre-)t-motives which were originally defined by Anderson ([**Ande**]). We explain the way how we obtain periods from pre-t-motives following the work of Anderson and Thakur ([**AT1**], [**AT2**]). Then we recall Papanikolas' theory ([**P**]) which states that the transcendental degree of the field generated by periods in question over a base field coincides with the dimension of the "motivic Galois group" of a pre-t-motive. As an example (see Example 3.5), we see that MZVs and CMPLs at algebraic points appear as periods of some pre-t-motives. The primary tools of proving the main results are Papanikolas' theory. In Chapter 4, we give proofs of our theorems by using the arguments of Chapter 3.

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CHAPTER 1

Notations

We continue to use the notations of the Introduction. Let t be a variable independent of θ. Let $\mathbb{T} := \{f \in \mathbb{C}_\infty[\![t]\!] | f \text{ converges on } |t|_\infty \leq 1\}$ be the Tate algebra and $\mathbb{L}$ the fractional field of $\mathbb{T}$. We set

$$\mathbb{E} := \left\{ \sum a_i t^i \in \mathbb{C}_\infty[\![t]\!] \middle| \lim_{i \to \infty} \sqrt[i]{|a_i|_\infty} = 0,\ [K_\infty(a_0, a_1, \dots) : K_\infty] < \infty \right\}.$$

For any integer $n \in \mathbb{Z}$ and any formal Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty(\!(t)\!)$, let

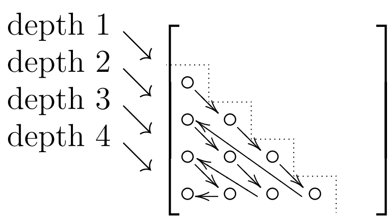
$$f^{(n)} := \sum_i a_i^{q^n} t^i$$

be the n-fold twist of f, and set $\sigma(f) := f^{(-1)}$. The fields $\mathbb{L}$ and $\overline{K}(t)$ are stable under the operation $f \mapsto f^{(n)}$ and we have $\mathbb{L}^{\sigma=1} = \mathbb{F}_q(t)$ where $(-)^{\sigma=1}$ is the σ-fixed part.

Definition 1.1. Let $d \geq 1$ be a positive integer. We set

$$I_d := \{(i,j) \in \mathbb{Z}^2 | 1 \leq j < i \leq d+1\}.$$

We define a depth of $(i,j) \in I_d$ by $\mathrm{dep}(i,j) := i - j$ and a total order on I_d by setting $(i,j) \leq (k,\ell)$ if either $\mathrm{dep}(i,j) = \mathrm{dep}(k,\ell)$ and $j \leq \ell$ (hence $i \leq k$), or $\mathrm{dep}(i,j) < \mathrm{dep}(k,\ell)$. For example, the order on I_4 is illustrated as the following diagram:



For each $(i,j) \in I_d$ and a d-tuple of symbol $\underline{y} = (y_1, \dots, y_d)$, we set

$$\underline{y}_{ij} := (y_j, y_{j+1}, \dots, y_{i-1}).$$

So, we have

$$\{\underline{y}_{ij} | (i,j) \in I_d\} = \{(y_j, y_{j+1}, \dots, y_i) | 1 \leq j \leq i \leq d\}.$$

Note that the MZV $\zeta(n_1, \dots, n_d)$ appears as a period of a t-motive of rank $d+1$ (Example 3.5). Moreover, the MZV $\zeta(n_j, \dots, n_{i-1})$ for $(i,j) \in I_d$ appears as an (i,j)-th component of a matrix of periods of that t-motive.

We set

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \overline{K_\infty}[\![t]\!],$$

which is in fact an element of $\mathbb{E}$. Since Ω has a simple zero at θ^{q^i} for each $i = 1, 2, \ldots$, it is transcendental over $\overline{K}(t)$. It satisfies the equation

$$\Omega^{(-1)} = (t-\theta)\Omega$$

and we have

$$\Omega(\theta) = \frac{1}{\widetilde{\pi}}.$$

We set $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j})$ for $i \geq 1$. For each integer $n \geq 0$ with q-adic expansion $n = \sum_i n_i q^i$ $(0 \leq n_i < q)$, the Carlitz factorial is defined by

$$\Gamma_{n+1} := \prod_i D_i^{n_i}.$$

Let $\underline{n} = (n_1, \ldots, n_d)$ be an index and $\underline{u} = (u_1, \ldots, u_d) \in (\overline{K}[t])^d$ a d-tuple of polynomials. For a polynomial $u = \sum_j \alpha_j t^j \in \overline{K}[t]$, we set $\|u\|_\infty := \max_j |\alpha_j|_\infty$. When $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, we set

$$L_{\underline{u},\underline{n}}(t) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{u_1^{(i_1)} \cdots u_d^{(i_d)}}{((t-\theta^q)\cdots(t-\theta^{q^{i_1}}))^{n_1} \cdots ((t-\theta^q)\cdots(t-\theta^{q^{i_d}}))^{n_d}} \in \overline{K_\infty}[\![t]\!],$$

which converges on $|t|_\infty < |\theta|_\infty^q$ and satisfies the equation

$$L_{\underline{u},\underline{n}}^{(-1)} = \frac{u_d^{(-1)}}{(t-\theta)^{n_1+\cdots+n_{d-1}}} L_{\underline{u}_{d1},\underline{n}_{d1}} + \frac{L_{\underline{u},\underline{n}}}{(t-\theta)^{n_1+\cdots+n_d}},$$

where we set $L_{\underline{u}_{11},\underline{n}_{11}} = L_{\emptyset,\emptyset} := 1$. When $\underline{u} = \underline{\alpha} \in \overline{K}^d$ with $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, we have $L_{\underline{\alpha},\underline{n}}(\theta) = \mathrm{Li}_{\underline{n}}(\underline{\alpha})$. Anderson and Thakur ([**AT1**], [**AT2**]) showed that there exists a polynomial $H_{n-1} \in \mathbb{F}_q[\theta, t]$ for each $n \geq 1$ such that $\|H_{n-1}\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ and $L_{H(\underline{n}),\underline{n}}(\theta) = \Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(\underline{n})$ where $H(\underline{n}) := (H_{n_1-1}, \ldots, H_{n_d-1})$.

Remark 1.2. We can easily show that

$$L_{\alpha_1,n_1} L_{\alpha_2,n_2} = L_{\alpha_1,\alpha_2,n_1,n_2} + L_{\alpha_2,\alpha_1,n_2,n_1} + L_{\alpha_1\alpha_2,n_1+n_2}$$

for each α_i and n_i (for more general cases, see [**Ch2**]. He treated $L_{\underline{\alpha},\underline{n}}(\theta)$, but the arguments are the same). By definition, $\Gamma_n = 1$ for each $1 \leq n \leq q$, and by the construction in [**AT1**], we know that $H_{n-1} = 1$ for $1 \leq n \leq q$. Thus if $n_1 + n_2 \leq q$, we have

$$L_{H_{n_1-1}H_{n_2-1},n_1+n_2} = L_{1,n_1+n_2} = L_{H_{n_1+n_2-1},n_1+n_2}.$$

Therefore, we obtain the harmonic shuffle product formula in Remark 2.9.

CHAPTER 2

Algebraic independence

In this chapter, we state linear/algebraic independence results on MZVs (Section 1) and values of CMPLs (Section 2). These are special cases of theorems in Chapter 4 and we will prove our theorems there in general settings.

1. Independence of multizeta values

Wade showed the following theorem:

THEOREM 2.1 ([**W**, Theorem 6.1]). *The Carlitz period $\widetilde{\pi}$ is transcendental over K.*

Thus, $\zeta(n)$ is transcendental over K for each positive "even" integer $n \geq 1$ by the Euler-Carlitz relation. Yu showed the transcendence of the Carlitz zeta values at the positive "odd" integers:

THEOREM 2.2 ([**Y1**, Theorem 3.1, Corollary 3.4]). *For each "odd" positive integer $n \geq 1$, the elements $\zeta(n)$ and $\zeta(n)/\widetilde{\pi}^n$ are both transcendental over K.*

Therefore all Carlitz zeta values are transcendental over K. He also determined all $\overline{K}$-linear relations among the powers of $\widetilde{\pi}$ and the Carlitz zeta values:

THEOREM 2.3 ([**Y2**, Theorem 4.1]). *Let $m_1, \ldots, m_r \geq 0$ be distinct non-negative integers and $n_1, \ldots, n_d \geq 1$ distinct positive "odd" integers. Then $\widetilde{\pi}^{m_1}$, ..., $\widetilde{\pi}^{m_r}$, $\zeta(n_1)$, ..., $\zeta(n_d)$ are linearly independent over $\overline{K}$.*

Finally, Chang and Yu determined all algebraic relations over $\overline{K}$ among the Carlitz zeta values:

THEOREM 2.4 ([**CY**, Corollary 4.6]). *Let $n_1, \ldots, n_d \geq 1$ be positive "odd" integers such that n_i/n_j is not an integral power of p for each $i \neq j$. Then $\widetilde{\pi}$, $\zeta(n_1)$, ..., $\zeta(n_d)$ are algebraically independent over $\overline{K}$.*

Thus all algebraic relations over $\overline{K}$ among the Carlitz zeta values come from the Euler-Carlitz relations and the p-th power relations.

For the higher depth case, Thakur showed that any MZVs are non-zero:

THEOREM 2.5 ([**Th2**, Theorem 4]). *For any index $\underline{n}$, we have $\zeta(\underline{n}) \neq 0$.*

Note that although the same statement in the classical case is trivial, this theorem is non-trivial.

The following theorem gives an affirmative answer to the function field analogue of Goncharov's conjecture:

THEOREM 2.6 ([**Ch2**, Theorem 2.2.1]). *We have*

$$\overline{\mathcal{Z}} = \bigoplus_{w \geq 0} \overline{\mathcal{Z}}_w.$$

Next, we state algebraic independence results of MZVs of higher depth. These are proved in Chapter 4. We treat the set

$$\{\widetilde{\pi}, \zeta(\underline{n}_{ij}) | (i,j) \in I_d\} = \{\widetilde{\pi}\} \cup \{\zeta(n_j, n_{j+1}, \ldots, n_i) | 1 \leq j \leq i \leq d\} \tag{2.1}$$

for a fixed index $\underline{n} = (n_1, \ldots, n_d)$ such that n_i is "odd" for each i.

First, we consider cases where $n_1 = n_2 = \cdots = n_d$.

THEOREM 2.7. *Let $n \geq 1$ be a positive "odd" integer. Then $\widetilde{\pi}$, $\zeta(n)$ and $\zeta(n,n)$ are algebraically independent over $\overline{K}$, or $\zeta(n)^2 - 2\zeta(n,n) \in \widetilde{\pi}^{2n} \cdot K^\times$. If $2n$ is "odd", then we have the former case.*

REMARK 2.8. If $p = 2$, then $2n$ is "odd" if and only if n is "odd". Thus $\widetilde{\pi}$, $\zeta(n)$ and $\zeta(n,n)$ are algebraically independent over $\overline{K}$ for each positive "odd" integer n.

On the other hand, in characteristic zero, $2n$ is always even. Thus the second part of Theorem 2.7 does not occur in this case. In fact, we have the relation $\zeta_{\mathbb{Z}}(n)^2 - 2\zeta_{\mathbb{Z}}(n,n) = \zeta_{\mathbb{Z}}(2n) \in \pi^{2n} \cdot \mathbb{Q}^\times$.

REMARK 2.9. If p^e divides n_1 and n_2 and $n_1/p^e + n_2/p^e \leq q$ for some $e \geq 0$, then we have the harmonic shuffle product $\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2)$ ([**Th2**, Theorem 1], or see Remark 1.2). In particular, if $2n = p^e(q-1)$ then we have the relation $\zeta(n)^2 - 2\zeta(n,n) = \zeta(2n) \in \widetilde{\pi}^{2n} \cdot K^\times$ (when $p = 2$, this follows directly, but in this case n is "even"). Thus, the latter case of the first part of Theorem 2.7 actually occurs when $p \geq 3$. We do not know what happens in the case where $2n = m(q-1)$ for general m (including the case where n is "even").

Since $\widetilde{\pi}$ and $\zeta(n)$ are algebraically independent over $\overline{K}$ for each "odd" integer n ([**CY**]), we have the following corollary:

COROLLARY 2.10. *Let $n \geq 1$ be an "odd" integer. Then any two elements of $\widetilde{\pi}$, $\zeta(n)$ and $\zeta(n,n)$ are algebraically independent over $\overline{K}$.*

COROLLARY 2.11. *We have*

$$\dim_{\overline{K}} \overline{\mathcal{Z}}_2 = \begin{cases} 2 & (q > 2) \\ 1 & (q = 2). \end{cases}$$

PROOF. Note that by Remark 2.9, we have $\zeta(1)^2 = 2\zeta(1,1) + \zeta(2) \in \overline{\mathcal{Z}}_2$ for each q. If $q \geq 4$ then 2 is "odd". Thus $\zeta(1)$ and $\zeta(1,1)$ (and $\widetilde{\pi}$) are algebraically independent over $\overline{K}$ by Theorem 2.7. Thus $\zeta(1)^2$ and $\zeta(1,1)$ form a basis of $\overline{\mathcal{Z}}_2$. If $q = 3$ then 2 is "even", and hence we have $\zeta(2) \in \widetilde{\pi}^2 \cdot K^\times$. However $\widetilde{\pi}$ and $\zeta(1)$ are algebraically independent over $\overline{K}$ ([**CY**]). Thus $\zeta(1)^2$ and $\zeta(2)$ form a basis of $\overline{\mathcal{Z}}_2$. When $q = 2$, we have the relation $\zeta(1,1) = \zeta(2)/(\theta^2 + \theta)$ ([**Th1**, Theorem 5.10.13]). □

REMARK 2.12. If $p \neq 2$ then $\zeta(1)$ and $\zeta(2)$ are algebraically independent over $\overline{K}$ ([**CY**]). Thus a new result in Corollary 2.11 is the characteristic 2 case with $q \neq 2$.

THEOREM 2.13. *Let $n \geq 1$ be a positive "odd" integer and set*

$$s := \mathrm{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \zeta(n), \zeta(n,n), \zeta(n,n,n)).$$

Then one and only one of the following holds:
(*i*) $s = 4$,
(*ii*) $s = 3$ *and* $\zeta(n)^2 - 2\zeta(n,n) \in \widetilde{\pi}^{2n} \cdot K^\times$,
(*iii*) $s = 3$ *and* $\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n) \in \widetilde{\pi}^{3n} \cdot K^\times$.
If $2n$ is "odd", then we have (i) or (iii). If $3n$ is "odd", then we have (i) or (ii).

REMARK 2.14. If $p = 3$, then $3n$ is "odd" if and only if n is "odd". Thus (*i*) or (*ii*) holds for each positive "odd" integer $n \geq 1$.

In characteristic zero, $3n$ is always odd if n is odd. Thus it is conjectured that the condition (*ii*) always occurs in this case.

REMARK 2.15. In Theorems 2.7 and 2.13, we do not know about the $K^\times$-factors of the relations when MZVs satisfies the relations as in the theorems. In these cases, we expect that the harmonic product formulas

$$\zeta(n)^2 - 2\zeta(n,n) = \zeta(2n) = \widetilde{\pi}^{2n}\frac{B_{2n}}{\Gamma_{2n+1}}$$

and

$$\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n) = \zeta(3n) = \widetilde{\pi}^{3n}\frac{B_{3n}}{\Gamma_{3n+1}}$$

hold.

We also have the following corollary:

COROLLARY 2.16. *Let $n \geq 1$ be a positive "odd" integer. Then $\zeta(n,n,n)$ and any two elements of the set $\{\widetilde{\pi}, \zeta(n), \zeta(n,n)\}$ are algebraically independent over $\overline{K}$.*

The algebraic independence of $\widetilde{\pi}$ and $\zeta(n,n)$ (resp $\zeta(n,n,n)$) in Corollary 2.10 (resp. Corollary 2.16) also follows from the "Eulerian" criterion ([**CPY**]) and the fact that if a multizeta value is not "Eulerian" then it is algebraically independent from $\widetilde{\pi}$ over $\overline{K}$ ([**Ch2**]).

Next, we consider the case where the depth one MZVs do not have relations.

THEOREM 2.17. *Let $d \geq 1$ be a positive integer, and let $n_1, \dots, n_d \geq 1$ be d distinct positive integers. If n_i is "odd" for each i and n_i/n_j is not an integral power of p for each $i \neq j$, then the the following $1 + \frac{d(d+1)}{2}$ elements*

$$\begin{gathered}
\widetilde{\pi}, \quad \zeta(n_1), \quad \zeta(n_2), \quad \zeta(n_3), \quad \zeta(n_4), \ \dots\dots\dots, \ \zeta(n_d), \\
\zeta(n_1,n_2), \ \zeta(n_2,n_3), \ \zeta(n_3,n_4), \ \dots\dots, \ \zeta(n_{d-1},n_d), \\
\zeta(n_1,n_2,n_3), \zeta(n_2,n_3,n_4), \dots, \zeta(n_{d-2},n_{d-1},n_d), \\
\vdots \\
\zeta(n_1,n_2,\dots,n_{d-1}), \zeta(n_2,n_3,\dots,n_d), \\
\zeta(n_1,n_2,\dots,n_d)
\end{gathered}$$

are algebraically independent over $\overline{K}$.

Theorem 2.17 provides many MZVs which are algebraically independent over $\overline{K}$. The next theorem gives a positive answer to the function field analogue of a question in [**Andr**, p. 231].

THEOREM 2.18. *For each positive integer $d \geq 1$, we set K_d to be the field generated by the MZVs of depth 1 or d over K. When $q \neq 2$, we have*

$$\operatorname{tr.deg}_{K_1} K_d = \infty$$

for each $d \geq 2$.

PROOF. Since $q \neq 2$, the set $\mathbb{Z}_{\geq 1} \smallsetminus ((q-1)\mathbb{Z}_{\geq 1} \cup p\mathbb{Z}_{\geq 1})$ is an infinite set. We denote the elements of this set by $n_1, n_2, n_3, \ldots$. Hence we have

$$K_1 = K(\widetilde{\pi}^{q-1}, \zeta(n_1), \zeta(n_2), \zeta(n_3), \ldots).$$

By Theorem 2.17, the elements $\zeta(n_1, \ldots, n_d), \zeta(n_{d+1}, \ldots, n_{2d}), \zeta(n_{2d+1}, \ldots, n_{3d}), \ldots$ are algebraically independent over K_1. □

REMARK 2.19. (1) Similarly, we can prove that for any integers $d_1, d_2, d_3, \cdots \geq 2$, there exist indices $\underline{n}_1, \underline{n}_2, \underline{n}_3, \ldots$ such that $\operatorname{dep}(\underline{n}_j) = d_j$ for each j and $\zeta(\underline{n}_1)$, $\zeta(\underline{n}_2)$, $\zeta(\underline{n}_3)$, $\ldots$ are algebraically independent over K_1.

(2) When $q = 2$, Chang ([**Ch2**]) showed that either $\zeta(1,2)$ or $\zeta(2,1)$ is transcendental over K_1. However we do not know whether there exist infinitely many MZVs which are algebraically independent over K_1 when $q = 2$.

By Theorem 2.17, we may obtain some lower bounds of the dimension of the vector space over K (or $\overline{K}$) spanned by the MZVs of fixed weight. We do not pursue this problem in this thesis and content ourselves with stating the following easily obtained lower bounds of the transcendental degree of the field generated by the MZVs of bounded weights and the dimension of $\overline{\mathcal{Z}}_3$:

COROLLARY 2.20. *Let $w \geq 1$ be a positive integer. If there exist positive integers $d_1, \ldots, d_r \geq 1$ and an "odd" positive integer $n_{ij} \geq 1$ for each $1 \leq i \leq r$ and $1 \leq j \leq d_i$ such that $n_{ij}/n_{i'j'}$ is not an integral power of p for each $(i,j) \neq (i', j')$ and $\sum_j n_{ij} \leq w$ for each i, then we have*

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, \zeta(\underline{n}) | \operatorname{wt}(\underline{n}) \leq w) \geq 1 + \sum_{i=1}^{r} \frac{d_i(d_i+1)}{2}.$$

COROLLARY 2.21. *We have*

$$\dim_{\overline{K}} \overline{\mathcal{Z}}_3 \begin{cases} = 4 & (p \neq 2, 3) \\ \geq 3 & (p = 2 \text{ or } 3,\ q \neq 2, 3) \\ = 3 & (q = 3) \\ \geq 2 & (q = 2). \end{cases}$$

PROOF. Note that $\dim_{\overline{K}} \overline{\mathcal{Z}}_3 \leq 4$. Assume that $p \neq 2, 3$. By Theorem 2.17, $\zeta(1)$, $\zeta(2)$, $\zeta(3)$ and $\zeta(1,2)$ are algebraically independent over $\overline{K}$. Thus $\zeta(1)^3$, $\zeta(1)\zeta(2)$, $\zeta(3)$ and $\zeta(1,2)$ form a $\overline{K}$-basis of $\overline{\mathcal{Z}}_3$. Next assume that $q \neq 2$. By Corollary 2.16, $\zeta(1)$, $\zeta(1,1)$ and $\zeta(1,1,1)$ are algebraically independent over $\overline{K}$. Thus $\zeta(1)^3$, $\zeta(1)\zeta(1,1)$ and $\zeta(1,1,1)$ are linearly independent over $\overline{K}$. When $q = 3$, we have

$\zeta(1,2) = \zeta(3)/(\theta - \theta^3)$ ([**Th3**, Theorem 5]). When $q = 2$, since $\zeta(1)$ and either $\zeta(1,2)$ or $\zeta(2,1)$ are algebraically independent over $\overline{K}$ (see Theorem 2.1 and Remark 2.19), $\zeta(1)^3$ and either $\zeta(1,2)$ or $\zeta(2,1)$ are linearly independent over $\overline{K}$. □

Next, we consider cases where indices may have p-th power relations. We fix an index $\underline{n} = (n_1, \ldots, n_d)$ such that n_i's are "odd" and distinct from each other.

Definition 2.22. We say that elements $(i,j), (k,\ell) \in I_d$ are equivalent and denote by $(i,j) \sim (k,\ell)$ if $\mathrm{dep}(i,j) = \mathrm{dep}(k,\ell)$ and there exists an integer $e \in \mathbb{Z}$ such that $\underline{n}_{ij} = p^e \underline{n}_{k\ell} := (p^e n_\ell, \ldots, p^e n_{k-1})$. When $(i,j) \sim (k,\ell)$ and $\mathrm{dep}(i,j) = \mathrm{dep}(k,\ell) = 1$, we write $j \sim \ell$ instead.

Of course, this equivalence relation depends on the fixed index $\underline{n}$. When (i,j) and (k,ℓ) are equivalent, we have the p-th power relation $\zeta(\underline{n}_{ij}) = \zeta(\underline{n}_{k\ell})^{p^e}$ where $e \in \mathbb{Z}$ is an integer such that $\underline{n}_{ij} = p^e \underline{n}_{k\ell}$. We expect that when $\underline{n}$ satisfies some "good" condition, the p-th power relations are the only relations among the elements of the set (2.1). This means that the equality

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{n}_{ij}) | (i,j) \in I_d) = 1 + \#(I_d/\sim) \tag{2.2}$$

holds for certain $\underline{n}$. The equality (2.2) does not hold in general. For example, set $\underline{n} = (n_1, n_2, p^e n_2, p^e n_1, n_1 + n_2)$ for $n_1 + n_2 \le q$ and $e \ge 1$. Then the harmonic product formula for $\zeta(n_1)\zeta(n_2)$ holds and we have the relation

$$(\zeta(n_1)\zeta(n_2) - \zeta(n_1,n_2) - \zeta(n_1+n_2))^{p^e} = \zeta(p^e n_2, p^e n_1).$$

We show that the equality (2.2) holds in some cases.

Theorem 2.23. *Let $\underline{n} = (n_1, \ldots, n_d)$ be an index such that n_i's are "odd" and distinct from each other. If there exists exactly one pair $j_1 \neq j_2$ such that $j_1 \sim j_2$ in I_d, then the equality (2.2) holds. This means that we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{n}_{ij}) | (i,j) \in I_d) = \#I_d = \frac{d(d+1)}{2}.$$

Theorem 2.24. *If $d \le 3$, then the equality (2.2) holds.*

2. Independence of values of Carlitz multiple polylogarithms

The following lemma is used as a criterion whether values of CMPLs satisfy assumptions of our theorems.

Lemma 2.25. *Let $m \ge 1$ be a positive "odd" integer, $\underline{n} = (n_1, \ldots, n_d)$ an index and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (K^\times)^d$ a d-tuple of non-zero rational points such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. Then $\widetilde{\pi}^m$ and $\mathrm{Li}_{\underline{n}}(\underline{\alpha})$ are linearly independent over K_∞.*

Proof. We have $\widetilde{\pi}^m \notin K_\infty$ and $\mathrm{Li}_{\underline{n}}(\underline{\alpha}) \in K_\infty^\times$ for such m, $\underline{n}$ and $\underline{\alpha}$ (see Theorem 2.28). □

First we state algebraic independence results in depth one cases. Papanikolas ($n = 1$), Chang and Yu ($n \ge 1$) proved the following theorem. This gives a criterion of the algebraic independence of the values of CMPLs at algebraic points of depth one.

THEOREM 2.26 ([**P**, Theorem 6.3.2], [**CY**, Theorem 3.1]). *Let $n \geq 1$ be a positive integer, and let $\alpha_1, \ldots, \alpha_r \in \overline{K}$ be algebraic points such that $|\alpha_j|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for each j. If $\widetilde{\pi}^n, \mathrm{Li}_n(\alpha_1), \ldots, \mathrm{Li}_n(\alpha_r)$ are linearly independent over K, then they are algebraically independent over $\overline{K}$.*

By Lemma 2.25 and Theorem 2.26, $\widetilde{\pi}$ and $\mathrm{Li}_n(\alpha)$ are algebraically independent over $\overline{K}$ if $n \geq 1$ is a positive "odd" integer and $\alpha \in K^\times$ is a non-zero rational point such that $|\alpha|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$.

Chang and Yu studied the algebraic independence of values of CMPLs of depth one when weights vary.

THEOREM 2.27 ([**CY**, Theorem 4.5]). *Let $n_1, \ldots, n_d \geq 1$ be positive integers such that n_i/n_j is not an integral power of p for each $i \neq j$. For each i, we take algebraic points $\alpha_{i1}, \ldots, \alpha_{ir_i} \in \overline{K}$ with $|\alpha_{ij}|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for $j = 1, \ldots, r_i$. If $\widetilde{\pi}^{n_i}, \mathrm{Li}_{n_i}(\alpha_{i1}), \ldots, \mathrm{Li}_{n_i}(\alpha_{ir_i})$ are linearly independent over K for each i, then the $1 + \sum_{i=1}^d r_i$ elements $\{\widetilde{\pi}, \mathrm{Li}_{n_i}(\alpha_{ij}) | 1 \leq i \leq d,\ 1 \leq j \leq r_i\}$ are algebraically independent over $\overline{K}$.*

For the higher depth case, Chang showed that any values of the CMPLs at non-trivial points are non-zero:

THEOREM 2.28 ([**Ch2**, Proposition 6.1.1]). *For any index $\underline{n} = (n_1, \ldots, n_d)$ and a d-tuple of non-zero points $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{C}_\infty^\times)^d$ such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$, we have $\mathrm{Li}_{\underline{n}}(\underline{\alpha}) \neq 0$.*

The following theorem gives an affirmative answer to the CMPLs analogue of Goncharov's conjecture:

THEOREM 2.29 ([**Ch2**, Theorem 6.4.3]). *Values of CMPLs at non-trivial algebraic points of different weights are linearly independent over $\overline{K}$.*

Next, we state algebraic independence results of the values of CMPLs at algebraic points of higher depth. These are proved in Chapter 4. We treat the set

$$\{\widetilde{\pi}, \mathrm{Li}_{\underline{n}_{ij}}(\underline{\alpha}_{ij}) | (i,j) \in I_d\} = \{\widetilde{\pi}\} \cup \{\mathrm{Li}_{n_j, n_{j+1}, \ldots, n_i}(\alpha_j, \alpha_{j+1}, \ldots, \alpha_i) | 1 \leq j \leq i \leq d\}$$

for a fixed index $\underline{n} = (n_1, \ldots, n_d)$ and a d-tuple of algebraic points $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \overline{K}^d$ such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i.

First, we consider cases where $n_1 = n_2 = \cdots = n_d$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_d$.

THEOREM 2.30. *Let $n \geq 1$ be a positive integer and $\alpha \in \overline{K}$ an algebraic point such that $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Assume that $\widetilde{\pi}^n$ and $\mathrm{Li}_n(\alpha)$ are linearly independent over K. Then $\widetilde{\pi}$, $\mathrm{Li}_n(\alpha)$ and $\mathrm{Li}_{n,n}(\alpha, \alpha)$ are algebraically independent over $\overline{K}$, or $\mathrm{Li}_n(\alpha)^2 - 2\,\mathrm{Li}_{n,n}(\alpha, \alpha) = \mathrm{Li}_{2n}(\alpha^2) \in \widetilde{\pi}^{2n} \cdot K^\times$. If $\widetilde{\pi}^{2n}$ and $\mathrm{Li}_{2n}(\alpha^2)$ are linearly independent over K, then we have the former case.*

Note that by Lemma 2.25, the assumption of Theorem 2.30 is satisfied if n is "odd" and $\alpha \in K^\times$. Similarly, the assumption of the second part of Theorem 2.30 is satisfied if $2n$ is "odd" and $\alpha^2 \in K^\times$.

THEOREM 2.31. *Let $n \geq 1$ be a positive integer and $\alpha \in \overline{K}$ an algebraic point such that $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Assume that $\widetilde{\pi}^n$ and $\mathrm{Li}_n(\alpha)$ are linearly independent over K. Set*

$$s := \mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \mathrm{Li}_n(\alpha), \mathrm{Li}_{n,n}(\alpha, \alpha), \mathrm{Li}_{n,n,n}(\alpha, \alpha, \alpha)).$$

Then one and only one of the following holds:

(*i*) $s = 4$,

(*ii*) $s = 3$ *and* $\mathrm{Li}_n(\alpha)^2 - 2\,\mathrm{Li}_{n,n}(\alpha, \alpha) = \mathrm{Li}_{2n}(\alpha^2) \in \widetilde{\pi}^{2n} \cdot K^\times$,

(*iii*) $s = 3$ *and* $\mathrm{Li}_n(\alpha)^3 - 3\,\mathrm{Li}_n(\alpha)\,\mathrm{Li}_{n,n}(\alpha, \alpha) + 3\,\mathrm{Li}_{n,n,n}(\alpha, \alpha, \alpha) = \mathrm{Li}_{3n}(\alpha^3) \in \widetilde{\pi}^{3n} \cdot K^\times$,

(*iv*) $s = 2$ *and the above two relations are satisfied.*

If $\widetilde{\pi}^{2n}$ and $\mathrm{Li}_{2n}(\alpha^2)$ are linearly independent over K, then we have (i) or (iii). If $\widetilde{\pi}^{3n}$ and $\mathrm{Li}_{3n}(\alpha^3)$ are linearly independent over K, then we have (i) or (ii).

Note that the assumption of the second (resp. third) part of Theorem 2.31 is satisfied if $2n$ (resp. $3n$) is "odd" and $\alpha^2 \in K^\times$ (resp. $\alpha^3 \in K^\times$).

We have the following corollary:

COROLLARY 2.32. *Let $n \geq 1$ be a positive "odd" integer and $\alpha \in K^\times$ a non-zero rational point such that $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Then $\mathrm{Li}_{n,n,n}(\alpha, \alpha, \alpha)$ and any two elements of the set $\{\widetilde{\pi}, \mathrm{Li}_n(\alpha), \mathrm{Li}_{n,n}(\alpha, \alpha)\}$ are algebraically independent over $\overline{K}$.*

The next theorem gives many values of CMPLs of higher depth which are algebraically independent over $\overline{K}$.

THEOREM 2.33. *Let $n_1, \ldots, n_d \geq 1$ be positive integers. For each i, we take $\alpha_i \in \overline{K}^\times$ such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. If $\widetilde{\pi}, \mathrm{Li}_{n_1}(\alpha_1), \ldots, \mathrm{Li}_{n_d}(\alpha_d)$ are algebraically independent over $\overline{K}$, then the cardinality of the set*

$$\{\pi\} \cup \{\mathrm{Li}_{n_j, n_{j+1}, \ldots, n_i}(\alpha_j, \alpha_{j+1}, \ldots, \alpha_i) | 1 \leq j \leq i \leq d\}$$

is $1 + \frac{d(d+1)}{2}$ and all elements of this set are algebraically independent over $\overline{K}$.

By Lemma 2.25 and Theorem 2.27, the assumption of Theorem 2.33 is satisfied when n_i is "odd" and $\alpha_i \in K^\times$ for each i and n_i/n_j is not an integral power of p for each $i \neq j$.

CHAPTER 3

Review of pre-t-motives

In this chapter, we review the notions of pre-t-motives and Papanikolas' theory for pre-t-motives. For more details, see [**P**]. A *pre-t-motive* is an étale φ-module over $(\overline{K}(t), \sigma)$; this means a finite-dimensional $\overline{K}(t)$-vector space M equipped with a σ-semilinear bijective map $\varphi\colon M \to M$. A morphism of pre-t-motives is a $\overline{K}(t)$-linear map which is compatible with the φ's. A tensor product of two pre-t-motives are defined naturally. For any pre-t-motive M, the *Betti realization* of M is defined by

$$M^B := \Big(\mathbb{L} \otimes_{\overline{K}(t)} M\Big)^{\sigma\otimes\varphi=1},$$

where $(-)^{\sigma\otimes\varphi=1}$ is the $\sigma\otimes\varphi$-fixed part. A pre-t-motive M is called *rigid analytically trivial* if the natural injection $\mathbb{L} \otimes_{\mathbb{F}_q(t)} M^B \hookrightarrow \mathbb{L} \otimes_{\overline{K}(t)} M$ is an isomorphism. The category of rigid analytically trivial pre-t-motives forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor $M \mapsto M^B$. For any such M, we denote by G_M the fundamental group of the Tannakian subcategory generated by M with respect to the Betti realization. By definition, G_M is an $\mathbb{F}_q(t)$-subgroup scheme of $\mathrm{GL}(M^B)$.

Let $\Phi \in \mathrm{GL}_r(\overline{K}(t))$ be a matrix. We consider the system of Frobenius difference equations

$$\Psi^{(-1)} = \Phi\Psi \tag{3.1}$$

with solution entries of $\Psi = (\Psi_{ij})$ in $\mathbb{L}$. The matrix Φ defines the pre-t-motive $M_\Phi := \overline{K}(t)^r$ with

$$\varphi(x_1, \ldots, x_r) = (x_1^{(-1)}, \ldots, x_r^{(-1)})\Phi.$$

The pre-t-motive M_Φ is rigid analytically trivial if and only if the system of Frobenius difference equations (3.1) has a solution matrix Ψ in $\mathrm{GL}_r(\mathbb{L})$, and in this case $\Psi^{-1}\mathbf{m}$ forms an $\mathbb{F}_q(t)$-basis of $(M_\Phi)^B$, where $\mathbf{m}$ is the standard basis of $\overline{K}(t)^r$ on which the action of φ is presented as Φ. Such matrix Ψ is called a *rigid analytic trivialization* of Φ, and the values $\Psi_{ij}(\theta)$ of its components at $t = \theta$ (if they converge) are called *periods* of M_Φ. For such Ψ, we set $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$, where Ψ_1 (resp. Ψ_2) is the matrix in $\mathrm{GL}_r(\mathbb{L}\otimes_{\overline{K}(t)}\mathbb{L})$ such that $(\Psi_1)_{ij} = \Psi_{ij}\otimes 1$ (resp. $(\Psi_2)_{ij} = 1\otimes\Psi_{ij}$). Let $X = (X_{ij})$ be the $r \times r$ matrix of independent variables X_{ij}. We define an $\mathbb{F}_q(t)$-algebra homomorphism ν by

$$\nu\colon \mathbb{F}_q(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L};\ X_{ij} \mapsto \widetilde{\Psi}_{ij}$$

and set

$$G_\Psi := \mathrm{Spec}(\mathbb{F}_q(t)[X, 1/\det X]/\ker\nu) \subset \mathrm{GL}_{r/\mathbb{F}_q(t)}.$$

For each $\mathbb{F}_q(t)$-algebra R, we have the map given by

$$G_\Psi(R) \to G_{M_\Phi}(R);\ g \mapsto (\mathbf{f}\cdot\Psi^{-1}\mathbf{m} \mapsto \mathbf{f}g^{-1}\cdot\Psi^{-1}\mathbf{m}) \tag{3.2}$$

where $\mathbf{f}$ runs over all elements of $\mathrm{Mat}_{1\times r}(R)$.

THEOREM 3.1 ([**P**, Theorems 4.3.1, 4.5.10, 5.2.2]). *Let Φ and Ψ be matrices satisfying (3.1), and let G_{M_Φ} and G_Ψ be as above.*

(1) *The scheme G_Ψ is a smooth subgroup scheme of $\mathrm{GL}_{r/\mathbb{F}_q(t)}$ and the above map $G_\Psi \to G_{M_\Phi}$ is an isomorphism of group schemes over $\mathbb{F}_q(t)$.*

(2) *Let $\overline{K}(t)(\Psi)$ be the field generated by the entries of Ψ over $\overline{K}(t)$. Then we have*

$$\dim G_\Psi = \mathrm{tr.deg}_{\overline{K}(t)}\,\overline{K}(t)(\Psi).$$

(3) *Assume that $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$, $\Psi \in \mathrm{GL}_r(\mathbb{T})\cap\mathrm{Mat}_r(\mathbb{E})$, and $\det\Phi = c(t-\theta)^d$ for some $c \in \overline{K}^\times$ and $d \geq 0$. Let $\overline{K}(\Psi(\theta))$ be the field generated by the entries of $\Psi(\theta)$ over $\overline{K}$. Then we have*

$$\mathrm{tr.deg}_{\overline{K}(t)}\,\overline{K}(t)(\Psi) = \mathrm{tr.deg}_{\overline{K}}\,\overline{K}(\Psi(\theta)).$$

REMARK 3.2. The result (3) in Theorem 3.1 is rooted in the deep result in [**ABP**], which is addressed as ABP-criterion. However, the restriction of the condition on $\det\Phi$ originated from Anderson t-motives but such restriction indeed can be relaxed (see [**Ch1**]). But for our purpose, the above is sufficient and so we do not state the refined version given in [**Ch1**].

REMARK 3.3. Let $v \in \mathbb{F}_q[t]$ be an irreducible monic polynomial. Then we can consider *v-adic realizations* and *v-adic periods* of t-motives. In [**M1**] and [**M2**], we proved a v-adic analogue of (1) and (2) of Theorem 3.1. However, we do not know whether a v-adic analogue of ABP-criterion holds.

EXAMPLE 3.4. The Carlitz pre-t-motive C is the pre-t-motive defined by the 1×1-matrix $\begin{bmatrix} t-\theta \end{bmatrix}$. Since $\Omega^{(-1)} = (t-\theta)\Omega$, the Carlitz pre-$t$-motive is rigid analytically trivial. Since Ω is transcendental over $\overline{K}(t)$, we have $\dim G_{[\Omega]} = 1$, and thus $G_C = G_{[\Omega]} = \mathbb{G}_m$.

EXAMPLE 3.5. Let $\underline{n} = (n_1,\dots,n_d)$ be an index and $\underline{u} = (u_1,\dots,u_d) \in (\overline{K}[t])^d$ be a d-tuple of polynomials such that $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. We consider $(d+1)\times(d+1)$-matrices

$$\Phi := \begin{bmatrix} (t-\theta)^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ u_1^{(-1)}(t-\theta)^{n_1+\cdots+n_d} & (t-\theta)^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ 0 & u_2^{(-1)}(t-\theta)^{n_2+\cdots+n_d} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{n_d} & 0 \\ 0 & \cdots & 0 & u_d^{(-1)}(t-\theta)^{n_d} & 1 \end{bmatrix}$$

and

$$\Psi := \begin{bmatrix} \Omega^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{u}_{21},\underline{n}_{21}} & \Omega^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{u}_{31},\underline{n}_{31}} & \Omega^{n_2+\cdots+n_d} L_{\underline{u}_{32},\underline{n}_{32}} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \Omega^{n_d} & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{u}_{d+1,1},\underline{n}_{d+1,1}} & \Omega^{n_2+\cdots+n_d} L_{\underline{u}_{d+1,2},\underline{n}_{d+1,2}} & \cdots & \Omega^{n_d} L_{\underline{u}_{d+1,d},\underline{n}_{d+1,d}} & 1 \end{bmatrix},$$

where the notations $\underline{n}_{ij}$ and $\underline{u}_{ij}$ are defined in Definition 1.1. These satisfy the Frobenius difference equations (3.1). Hence Ψ is a rigid analytic trivialization of Φ. Let M be the pre-t-motive defined by Φ. By Theorem 3.1, we have an isomorphism $G_\Psi \to G_M$ and

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, L_{\underline{u}_{ij},\underline{n}_{ij}}(\theta) | (i,j) \in I_d) = \dim G_\Psi.$$

Thus when $\underline{u} = H(\underline{n})$ (resp. $\underline{u} = \underline{\alpha} \in \overline{K}^d$ with $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i), the multizeta values $\zeta(\underline{n}_{ij})$ (resp. the Carlitz multiple polylogarithms $\mathrm{Li}_{\underline{n}_{ij}}(\underline{\alpha}_{ij})$) appear as periods of the pre-t-motive M. By the definition of G_Ψ, we also have the inclusion

$$G_\Psi \subset \left\{ \begin{bmatrix} a^{n_1+\cdots+n_d} & & & \\ x_{21} & a^{n_2+\cdots+n_d} & & \\ \vdots & \ddots & \ddots & \\ x_{d+1,1} & \cdots & x_{d+1,d} & 1 \end{bmatrix} \right\}.$$

We can calculate $\widetilde{\Psi}$ explicitly as

$$\widetilde{\Psi}_{ij} = (\Omega^{-1} \otimes \Omega)^{n_i+\cdots+n_d} \sum_{s=j}^{i} \sum_{r=0}^{i-s} (-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=i}} L_{i_1 i_0} \cdots L_{i_r i_{r-1}} \otimes \Omega^{n_j+\cdots+n_{i-1}} L_{sj}$$

for each $(i,j) \in I_d$, where we write $L_{k\ell} := L_{\underline{u}_{k\ell},\underline{n}_{k\ell}}$.

CHAPTER 4

Proofs

In this chapter, we prove our results in Chapter2. For square matrices A and B, we denote by $A \oplus B$ the diagonal block matrix made of A and B. We use the letters a and x_{ij}'s as coordinate variables of algebraic groups. We use also these variables to define various algebraic groups. For example, we denote by

$$\left\{ \begin{bmatrix} a & 0 \\ x & 1 \end{bmatrix} \right\}$$

the subgroup of GL_2 over $\mathbb{F}_q(t)$ whose valued points are of the form

$$\begin{bmatrix} a & 0 \\ x & 1 \end{bmatrix}.$$

To prove our theorems, we use the following lemma. This lemma is clear, but very useful.

LEMMA 4.1. *Let $V \subset {\mathbb{G}_a}^r$ be an algebraic subgroup of dimension zero. Let $m_1, \ldots, m_r \in \mathbb{Z}$ be non-zero integers. Assume that V is stable under the $\mathbb{G}_m$-action on ${\mathbb{G}_a}^r$ defined by*

$$a.(x_1, \ldots, x_r) = (a^{m_1} x_1, \ldots, a^{m_r} x_r) \quad (a \in \mathbb{G}_m, (x_i) \in {\mathbb{G}_a}^r).$$

Then $V(\overline{\mathbb{F}_q(t)})$ is trivial.

1. Depth one case

We state several algebraic independence results concerning the case of depth one. Papanikolas, Chang and Yu proved the following theorem which states a criterion of the algebraic independence of MZVs and CMPLs at algebraic points of depth one. Note that they discussed only the case where $u_j = \alpha_j \in \overline{K}$ with $|\alpha_j|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ (see Theorem 2.26), but their arguments work also for any $u_j \in \overline{K}[t]$ with $\|u_j\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$.

THEOREM 4.2 ([**P**, Theorem 6.3.2], [**CY**, Theorem 3.1]). *Let $n \geq 1$ be a positive integer and $u_1, \ldots, u_r \in \overline{K}[t]$ polynomials with $\|u_j\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for each j. If $\widetilde{\pi}^n, L_{u_1,n}(\theta), \ldots, L_{u_r,n}(\theta)$ are linearly independent over K, then they are algebraically independent over $\overline{K}$.*

Thus $\widetilde{\pi}$ and $\zeta(n)$ (or $\mathrm{Li}_n(\alpha)$) are algebraically independent over $\overline{K}$ for each "odd" integer $n \geq 1$ and $\alpha \in K^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, because $\widetilde{\pi}^n \notin K_\infty$ and $\zeta(n), \mathrm{Li}_n(\alpha) \in K_\infty^\times$ for such n and α (see also Lemma 2.25).

Theorem 4.3. *Let $n_1, \dots, n_d \geq 1$ be positive integers such that n_i/n_j is not an integral power of p for each $i \neq j$. For each i, we take polynomials $u_{i1}, \dots, u_{ir_i} \in \overline{K}[t]$ with $\|u_{ij}\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for $j = 1, \dots, r_i$. If $\widetilde{\pi}^{n_i}, L_{u_{i1},n_i}(\theta), \dots, L_{u_{ir_i},n_i}(\theta)$ are linearly independent over K for each i, then the $1 + \sum_{i=1}^d r_i$ elements $\{\widetilde{\pi}, L_{u_{ij},n_i}(\theta) | 1 \leq i \leq d,\ 1 \leq j \leq r_i\}$ are algebraically independent over $\overline{K}$.*

This is almost proved in [**CY**]. In [**CY**], they treated some special case, but their proof works also for any n_i not divisible by p and any $\alpha_{ij} \in \overline{K}[t]$ with $\|\alpha_{ij}\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$. By a slight modification of their proof, we can weaken the condition on n_i's as in our statement. Note that when $\alpha_{ij} \in \overline{K}$ or $\alpha_{ij} = H_{n_i-1}$, we can reduce Theorem 4.3 to the case where n_i is not divisible by p, and we do not need the following proof in such cases.

In our proofs, our purpose is to show that the dimension of the algebraic group in question is maximal as large as possible, and so we always work on the $\overline{\mathbb{F}_q(t)}$-valued points without studying the reduced/non-reduced structures, where $\overline{\mathbb{F}_q(t)}$ is a fixed algebraic closure of $\mathbb{F}_q(t)$. So for an algebraic group G over $\mathbb{F}_q(t)$, when it is clear from the context, without confusion we still denote by G the $\overline{\mathbb{F}_q(t)}$-valued points of G.

Proof of Theorem 4.3. We set $I := \{(i,j) \in \mathbb{Z}^2 | 1 \leq i \leq d,\ 1 \leq j \leq r_i\}$. In this proof, (i,j) and (k,ℓ) are always assumed to be elements of I. We define an order on I by the lexicographic order; this means $(i,j) \leq (k,\ell)$ if and only if $i = k$ and $j \leq \ell$, or $i < k$. For $(k,\ell) \in I$, we define 2×2-matrices

$$\Phi[k,\ell] := \begin{bmatrix} (t-\theta)^{n_k} & 0 \\ \alpha_{k\ell}^{(-1)}(t-\theta)^{n_k} & 1 \end{bmatrix} \quad \text{and} \quad \Psi[k,\ell] := \begin{bmatrix} \Omega^{n_k} & 0 \\ \Omega^{n_k} L_{\alpha_{k\ell},n_k} & 1 \end{bmatrix}.$$

Then they satisfy the Frobenius difference equations $\Psi[k,\ell]^{(-1)} = \Phi[k,\ell]\Psi[k,\ell]$. Let $M[k,\ell]$ be the pre-t-motive defined by $\Phi[k,\ell]$ and $G(k,\ell)$ (resp. $G_k(\ell)$) the fundamental group of the pre-t-motive

$$M(k,\ell) := C \oplus \bigoplus_{(i,j) \leq (k,\ell)} M[i,j] \quad \left(\text{resp. } M_k(\ell) := C \oplus \bigoplus_{j \leq \ell} M[k,j] \right).$$

We identify $G(k,\ell)$ (resp. $G_k(\ell)$) with the algebraic group defined by

$$[\Omega] \oplus \bigoplus_{(i,j) \leq (k,\ell)} \Psi[i,j] \quad \left(\text{resp. } [\Omega] \oplus \bigoplus_{j \leq \ell} \Psi[k,j] \right)$$

as in Theorem 3.1. Then we have the inclusion (resp. equality)

$$G(k,\ell) \subset \left\{ [a] \oplus \bigoplus_{(i,j) \leq (k,\ell)} \begin{bmatrix} a^{n_i} & 0 \\ x_{ij} & 1 \end{bmatrix} \right\} \quad \left(\text{resp. } G_k(\ell) = \left\{ [a] \oplus \bigoplus_{j \leq \ell} \begin{bmatrix} a^{n_k} & 0 \\ x_{kj} & 1 \end{bmatrix} \right\} \right)$$

for each (k,ℓ). By Theorem 3.1, it suffices to show that the above inclusion is actually an equality for each (k,ℓ). We prove this by induction on $(k,\ell) \in I$ with respect to the total order "$\leq$".

By the assumption, this is true for $(1,1) \le (k,\ell) \le (1,r_1)$. Let $(k,\ell) \ge (2,1)$ and assume that the inclusion is an equality for the greatest element (k',ℓ') of $\{(i,j) \in I | (i,j) < (k,\ell)\}$. Thus $(k',\ell') = (k,\ell-1)$ if $\ell \ne 1$ and $(k',\ell') = (k-1,r_{k-1})$ if $\ell = 1$. By definition, $M(k',\ell')$ and $M_k(\ell)$ are subobjects of $M(k,\ell)$ and C is a subobject of $M(k,\ell)$, $M(k',\ell')$ and $M_k(\ell)$. By the Tannakian duality, we have surjections $\psi\colon G(k,\ell) \to G(k',\ell')$, $\psi_k\colon G(k,\ell) \to G_k(\ell)$, $\pi\colon G(k,\ell) \to \mathbb{G}_m$, $\pi'\colon G(k',\ell') \to \mathbb{G}_m$ and $\pi''\colon G_k(\ell) \to \mathbb{G}_m$, where we identify G_C with $\mathbb{G}_m$. The projections π, π' and π'' map the matrices of the above forms to a and ψ (resp. ψ_k) maps them to the same matrices with the (k,ℓ)-th component matrices (resp. all (i,j)-th component matrices $(i \ne k)$) removed. We set $V := \operatorname{Ker}\pi$, $V' := \operatorname{Ker}\pi'$ and $V'' := \operatorname{Ker}\pi''$ to be the unipotent radicals of $G(k,\ell)$, $G(k',\ell')$ and $G_k(\ell)$. Then we have the following diagram

$$\begin{tikzcd}
1 \arrow[r] & V'' \arrow[r] & G_k(\ell) \arrow[r, "\pi''"] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & V \arrow[r] \arrow[u, "\psi_k|_V"] \arrow[d, "\psi|_V"] & G(k,\ell) \arrow[r, "\pi"] \arrow[u, two heads, "\psi_k"] \arrow[d, two heads, "\psi"] & \mathbb{G}_m \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & V' \arrow[r] & G(k',\ell') \arrow[r, "\pi'"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

which is commutative and whose rows are exact.

It is clear that $\psi|_V$ is surjective. Note that the coordinate variable $x_{k\ell}$ of $G(k,\ell)$ is the only coordinate variable which does not appear as a coordinate variable of $G(k',\ell')$. Thus we know that $\dim G(k',\ell') \le \dim G(k,\ell) \le \dim G(k',\ell') + 1$. This also follows from Theorem 3.1 (2). It suffices to show that the second inequality is an equality.

Now, assume that $\dim G(k,\ell) = \dim G(k',\ell')$. Then $\dim \operatorname{Ker}(\psi|_V) = 0$. We identify $V \subset \prod_{(i,j)\le(k,\ell)} \mathbb{G}_a$, $V' = \prod_{(i,j)<(k,\ell)} \mathbb{G}_a$ and $V'' = \prod_{j \le \ell} \mathbb{G}_a$ by means of the coordinates x_{ij}. The $\mathbb{G}_m$-action on V (resp. V', resp. V'') defined by $a.X := \widetilde{a}^{-1} X \widetilde{a}$, where $\widetilde{a} \in G(k,\ell)$ (resp. $G(k',\ell')$, resp. $G_k(\ell)$) is a lift of $a \in \mathbb{G}_m$, is described as $x_{ij} \mapsto a^{n_i} x_{ij}$ on each coordinate. By Lemma 4.1 we have $\operatorname{Ker}(\psi|_V) = 1$. Thus the morphism $\psi|_V$ is bijective (but not necessary an isomorphism of varieties) and we have the surjective map

$$\psi_k|_V \circ \psi|_V^{-1}\colon V' \xleftarrow{\sim} V \twoheadrightarrow V''.$$

For each $(i,j) \ne (k,\ell)$, we set V_{ij} (resp. V'_{ij}) to be the subvariety of V (resp. V') defined by $x_{i'j'} = 0$ for each $(i',j') \ne (i,j),(k,\ell)$. Then $\psi|_{V_{ij}}\colon V_{ij} \to V'_{ij} = \mathbb{G}_a$ is a bijective $\mathbb{G}_m$-homomorphism. Thus we have $\dim V_{ij} = 1$. Hence the algebraic set[1] V_{ij} is defined by a separable polynomial of the form $x_{k\ell}^{p^e} - \sum_{n=0}^m b_n x_{ij}^{p^n}$ for some $e, m \ge 0$ and $b_n \in \overline{\mathbb{F}_q(t)}$ (See [**Co**, Corollary 1.8]). Now we take $i \ne k$ and assume that the $\mathbb{G}_m$-homomorphism $\psi_k|_{V_{ij}} \circ \psi|_{V_{ij}}^{-1}$ is non-zero. Then we can take $b_m \ne 0$ and we have $(\sum_n b_n (a^{n_i} x_{ij})^{p^n})^{p^{-e}} = a^{n_k} (\sum_n b_n x_{ij}^{p^n})^{p^{-e}}$ for each $a \in \mathbb{G}_m$. By comparing

[1]More precisely, the smooth algebraic group $(V_{ij} \otimes \overline{\mathbb{F}_q(t)})_{\mathrm{red}}$ is defined by such a polynomial.

the coefficients of $x_{ij}^{p^{m-e}}$, we have $n_i p^{m-e} = n_k$, which is a contradiction. Thus we conclude that $\psi_d|_{V_{ij}} \circ \psi|_{V_{ij}}^{-1} = 0$. Therefore we have $\psi_k|_V(\psi|_V^{-1}({\mathbb{G}_a}^{\ell-1})) = V''$, whence a contradiction since $\dim V'' = \ell$. $\square$

2. Proofs of Theorems 2.7, 2.13, 2.30 and 2.31

Let $\overline{F}$ be a fixed algebraic closure of $F := \mathbb{F}_q(t)$ and F^{sep} the separable closure of F in $\overline{F}$. For a scheme S over F, we write $S_{\overline{F}}$ for its base extension to $\overline{F}$. The following lemma is proved by using the same argument as in [**Ch2**, Lemma 5.3.1] and we omit the proof.

LEMMA 4.4. *Let $\underline{n} = (n_1, \ldots, n_d)$ be an index and $\underline{u} = (u_1, \ldots, u_d) \in (\overline{K}[t])^d$ a d-tuple of polynomials such that $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. For each non-negative integer $N \geq 0$, we have*

$$(\Omega^{n_1+\cdots+n_d} L_{\underline{u},\underline{n}})(\theta^{q^N}) = \left(\frac{L_{\underline{u},\underline{n}}(\theta)}{\widetilde{\pi}^{n_1+\cdots+n_d}} \right)^{q^N}.$$

THEOREM 4.5. *Let $n \geq 1$ be a positive integer and $u \in \overline{K}[t]$ a polynomial such that $\|u\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Then the following conditions are equivalent:*
(1) $\widetilde{\pi}$ and $L_{u,n}(\theta)$ are algebraically independent over $\overline{K}$,
(2) Ω and $L_{u,n}$ are algebraically independent over $\overline{K}(t)$,
(3) $\widetilde{\pi}^n$ and $L_{u,n}(\theta)$ are linearly independent over K,
(4) $\Omega^n L_{u,n} - c_1$ and Ω^n are linearly independent over $\overline{K}(t)$ for each $c_1 \in \mathbb{F}_q(t)$.

PROOF. Let $L_1 := L_{u,n}$. We set

$$\Phi_1 := \begin{bmatrix} (t-\theta)^n & \\ u^{(-1)}(t-\theta)^n & 1 \end{bmatrix} \in \mathrm{GL}_2(\overline{K}(t)) \text{ and } \Psi_1 := \begin{bmatrix} \Omega^n & \\ \Omega^n L_1 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{L}).$$

Let M_1 be the pre-t-motive defined by Φ_1 and G_1 its fundamental group. Then the n-th tensor power $C^{\otimes n}$ of the Carlitz pre-t-motive (Example 3.4) is a subobject of M_1 by

$$C^{\otimes n} \hookrightarrow M_1;\ x \mapsto (x, 0).$$

By Tannakian duality and Theorem 3.1, we have a diagram of smooth group schemes over F

$$\begin{array}{ccccc} G_1 & \xrightarrow{\pi_1} & G_{C^{\otimes n}} & & \\ \simeq \uparrow & & \simeq \uparrow & & \\ G_{\Psi_1} & & G_{\Omega^n} & = & \mathbb{G}_m . \end{array}$$

In the following, we identify the upper group schemes with the lower group schemes in the above diagram. At first, we describe the morphism π_1 in the above diagram explicitly. By definition, we have

$$\widetilde{\Psi_1} = \begin{bmatrix} \Omega^{-n} \otimes \Omega^n & \\ -L_1 \otimes \Omega^n + 1 \otimes \Omega^n L_1 & 1 \end{bmatrix}.$$

Thus we have the inclusion

$$G_1 \cong G_{\Psi_1} \subset \overline{G_1} := \left\{ \begin{bmatrix} a & \\ x & 1 \end{bmatrix} \right\} \subset \mathrm{GL}_2 .$$

By using the above identification, we can write

$$\pi_1 \colon G_1 \twoheadrightarrow \mathbb{G}_m; \ \begin{bmatrix} a & \\ x & 1 \end{bmatrix} \mapsto a.$$

This follows from the description of the map (3.2). The arguments are the same as in [**P**, §6.2.2], [**CY**, §4.3] and [**CPY**, Remark 2.3.2]. We set $V_1 := \operatorname{Ker} \pi_1$ to be the unipotent radical of G_1. Then we have

$$V_1 \subset \overline{V_1} := \left\{ \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} \right\}$$

and obtain the following diagram

$$\begin{tikzcd}
1 \arrow[r] & \overline{V_1} \arrow[r] & \overline{G_1} \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & V_1 \arrow[r] \arrow[u, hook] & G_1 \arrow[r, "\pi_1"] \arrow[u, hook] & \mathbb{G}_m \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

which is commutative and whose rows are exact.

The group scheme V_1 is smooth over F. Indeed, take $a_0 \in \mathbb{G}_m(\overline{F}) \smallsetminus \mathbb{G}_m(\overline{\mathbb{F}_q})$ and its lift $\widetilde{a_0} \in G_1(\overline{F})$. Let $T \subset G_{1,\overline{F}}$ be the Zariski closure of the group generated by $\widetilde{a_0}$. Then T is a rank one torus and isomorphic to $\mathbb{G}_{m,\overline{F}}$ via $\pi_{1,\overline{F}}$. In particular, $\mathrm{Lie}(\pi_{1,\overline{F}}) \colon \mathrm{Lie}(G_{1,\overline{F}}) \to \mathrm{Lie}(\mathbb{G}_{m,\overline{F}})$ is nonzero. Since $G_{1,\overline{F}}$ and $\mathbb{G}_{m,\overline{F}}$ are smooth over $\overline{F}$, we have $\dim V_{1,\overline{F}} = \dim \mathrm{Lie}(V_{1,\overline{F}})$, and hence V_1 is smooth over F.

In view of the above short exact sequence, we let $\mathbb{G}_m(\overline{F})$ act on $V_1(\overline{F})$ by $a.X := \widetilde{a}^{-1} X \widetilde{a}$ for $a \in \mathbb{G}_m(\overline{F})$ and $X \in V_1(\overline{F})$, where $\widetilde{a} \in G_1(\overline{F})$ is a lift of a. In term of matrices, this action is given by

$$a. \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ ax & 1 \end{bmatrix}.$$

By Theorem 3.1, we have

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, L_1(\theta)) = \mathrm{tr.deg}_{\overline{K}(t)}\, \overline{K}(t)(\Omega, L_1) = \dim G_1$$

and this value is one or two. Thus we have

$$(3) \Leftarrow (1) \Leftrightarrow (2) \Rightarrow (4).$$

Assume that the condition (4) does not hold. Then there exists $c_1 \in \mathbb{F}_q(t)$ and $f \in \overline{K}(t)$ such that

$$\Omega^n L_1 - c_1 = f \Omega^n.$$

For some positive integer $N \geq 1$, the rational function f has no pole at $t = \theta^N$. Then we have

$$\left(\frac{L_1(\theta)}{\widetilde{\pi}^n} \right)^{q^N} - c_1(\theta)^{q^N} = 0$$

by Lemma 4.4. Thus the condition (3) does not hold. This means that (3) $\Rightarrow$ (4).

From now on, we suppose that the condition (1) does not hold. Thus we have $\dim G_1 = 1$ and hence $\dim V_1 = 0$. By Lemma 4.1, we conclude that $V_1 = \{1\}$. Next, we determine the group scheme G_1. Fix an element $a_0 \in \mathbb{G}_m(F)$ which has infinite order and set $\overline{a_0} \in \mathbb{G}_m(\overline{F})$ to be the geometric point above a_0. Since the geometric fiber $G_{1,\overline{a_0}}$ of π_1 over $\overline{a_0}$ is a $V_{1,\overline{F}}$-torsor, it is isomorphic to $V_{1,\overline{F}}$ which is smooth over $\overline{F}$. Thus the fiber G_{1,a_0} is smooth over F. By [**L**, Chapter 3, Proposition 2.20], we have $G_{1,a_0}(F^{\mathrm{sep}}) \neq \emptyset$, and hence we can take a lift $\widetilde{a_0} = \begin{bmatrix} a_0 & \\ x_0 & 1 \end{bmatrix}$ of a_0 in $G_1(F^{\mathrm{sep}})$. Then for each integer $r \in \mathbb{Z}$, we have

$$\widetilde{a_0}^r = \begin{bmatrix} a_0^r & \\ \frac{x_0}{1-a_0}(1-a_0^r) & 1 \end{bmatrix}.$$

Since a_0 has infinite order, we have

$$\left\{ \begin{bmatrix} a & \\ c_1(1-a) & 1 \end{bmatrix} \right\} \subset G_{1,\overline{F}},$$

where $c_1 := \frac{x_0}{1-a_0} \in F^{\mathrm{sep}}$. Since $G_{1,\overline{F}}$ is a one-dimensional irreducible reduced group scheme, we conclude that the above inclusion is actually an equality.

We set a polynomial

$$Q := X_{21} - c_1(1 - X_{11}) \in F^{\mathrm{sep}}[X_{11}, X_{21}] \subset F^{\mathrm{sep}}[X_{11}, \dots, X_{22}, 1/\det X].$$

Then $G_{1,\overline{F}} = \operatorname{Spec} \overline{F}[X_{11}, X_{21}, X_{11}^{-1}]/(Q)$. Since G_1 is defined over F, the ideal (Q) is stable under the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F) = \mathrm{Aut}(\overline{F}/F)$. Therefore for each $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$, we can write $\sigma(Q) = P_\sigma Q$ for some $P_\sigma \in \overline{F}[X_{11}, X_{21}, X_{11}^{-1}]$. By comparing the degree of each variables, P_σ must be a constant. Comparing the both sides again, we have $P_\sigma = 1$ and $\sigma(c_1) = c_1$ for each $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Hence we have $c_1 \in F$ and $Q \in F[X, 1/\det X]$. Since $Q \equiv 0$ on the reduced scheme $G_{1,\overline{F}}$, we have $Q(\widetilde{\Psi_1}) = 0$. By the definition of $\widetilde{\Psi_1}$, this is equivalent to the equality

$$(\Omega^n L_1 - c_1) \otimes \Omega^n = \Omega^n \otimes (\Omega^n L_1 - c_1)$$

in $\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L}$. Thus the condition (4) does not hold. This means (4) $\Rightarrow$ (1). $\square$

Remark 4.6. By the proof of Theorem 4.5, when the equivalent conditions are satisfied, we have

$$G_1 = \left\{ \begin{bmatrix} a & \\ x & 1 \end{bmatrix} \right\}.$$

When the equivalent conditions are not satisfied, we have

$$G_1 = \left\{ \begin{bmatrix} a & \\ c_1(1-a) & 1 \end{bmatrix} \right\}$$

for some $c_1 \in \mathbb{F}_q(t)$. Such c_1 gives the linear dependence of Ω^n and $\Omega^n L_{u,n} - c_1$ over $\overline{K}(t)$, and c_1 is uniquely determined by

$$c_1(\theta) = \frac{L_{u,n}(\theta)}{\widetilde{\pi}^n}.$$

THEOREM 4.7. *Let $n \geq 1$ be a positive integer and $u \in \overline{K}[t]$ a polynomial such that $\|u\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Assume that the equivalent conditions of Theorem 4.5 are satisfied. Then the following conditions are equivalent:*

(1) *$\widetilde{\pi}$, $L_{u,n}(\theta)$ and $L_{u,u,n,n}(\theta)$ are algebraically independent over $\overline{K}$,*
(2) *Ω, $L_{u,n}$ and $L_{u,u,n,n}$ are algebraically independent over $\overline{K}(t)$,*
(3) *$\widetilde{\pi}^{2n}$ and $L_{u,n}(\theta)^2 - 2L_{u,u,n,n}(\theta)$ are linearly independent over K,*
(4) *$\Omega^{2n}L_{u,n}^2 - 2\Omega^{2n}L_{u,u,n,n} - c_2$ and Ω^{2n} are linearly independent over $\overline{K}(t)$ for each $c_2 \in \mathbb{F}_q(t)$.*

REMARK 4.8. The equivalent conditions of Theorem 4.7 are satisfied if $p = 2$. For example, we can easily check the condition (3).

PROOF OF THEOREM 4.7. We continue to use the notations in the proof of Theorem 4.5. Let $L_2 := L_{u,u,n,n}$. We set

$$\Phi_2 := \begin{bmatrix} (t-\theta)^{2n} & & \\ u^{(-1)}(t-\theta)^{2n} & (t-\theta)^n & \\ & u^{(-1)}(t-\theta)^n & 1 \end{bmatrix} \in \mathrm{GL}_3(\overline{K}(t))$$

and

$$\Psi_2 := \begin{bmatrix} \Omega^{2n} & & \\ \Omega^{2n}L_1 & \Omega^n & \\ \Omega^{2n}L_2 & \Omega^n L_1 & 1 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{L}),$$

Let M_2 be the pre-t-motive defined by Φ_2 and G_2 its fundamental group. Then we have a homomorphism of pre-t-motives

$$M_2 \twoheadrightarrow M_1;\ (x_1, x_2, x_3) \mapsto (x_2, x_3).$$

By Tannakian duality and Theorem 3.1, we have a diagram of smooth group schemes over F

$$\begin{array}{ccccc} G_2 & \xrightarrow{\psi_{12}} & G_1 & \xrightarrow{\pi_1} & G_{C^{\otimes n}} \\ \simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\ G_{\Psi_2} & & G_{\Psi_1} & & G_{\Omega^n} = \mathbb{G}_m\,. \end{array}$$

In the following, we identify the upper group schemes with the lower group schemes in the above diagram. At first, we describe the morphism ψ_{12} in the above diagram explicitly. By Remark 4.6, we have

$$G_1 = \left\{ \begin{bmatrix} a & \\ x & 1 \end{bmatrix} \right\}.$$

By the definition of $\widetilde{\Psi_2} = ((\widetilde{\Psi_2})_{ij})$, we have the relations

$$(\widetilde{\Psi_2})_{11} = (\widetilde{\Psi_2})_{22}^2, \quad (\widetilde{\Psi_2})_{22} = \Omega^{-n} \otimes \Omega^n, \quad (\widetilde{\Psi_2})_{33} = 1,$$

$$(\widetilde{\Psi_2})_{21} = (\widetilde{\Psi_2})_{22}(\widetilde{\Psi_2})_{32}, \quad (\widetilde{\Psi_2})_{32} = 1 \otimes \Omega^n L_1 - L_1 \otimes \Omega^n,$$

$$(\widetilde{\Psi_2})_{31} = (L_1^2 - L_2) \otimes \Omega^{2n} - L_1 \otimes \Omega^{2n}L_1 + 1 \otimes \Omega^{2n}L_2,$$

and $(\widetilde{\Psi_2})_{ij} = 0$ if $i < j$. Thus we have the inclusion

$$G_2 \cong G_{\Psi_2} \subset \overline{G_2} := \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ y & x & 1 \end{bmatrix} \right\} \subset \mathrm{GL}_3 .$$

By using the above identifications, we can write

$$\psi_{12} \colon G_2 \twoheadrightarrow G_1; \quad \begin{bmatrix} a^2 & & \\ ax & a & \\ y & x & 1 \end{bmatrix} \mapsto \begin{bmatrix} a & \\ x & 1 \end{bmatrix}$$

as in the proof of Theorem 4.5. We set $\pi_2 := \pi_1 \circ \psi_{12}$ and $V_2 := \operatorname{Ker} \pi_2$ to be the unipotent radical of G_2. Then we have

$$V_2 \subset \overline{V_2} := \left\{ \begin{bmatrix} 1 & & \\ x & 1 & \\ y & x & 1 \end{bmatrix} \right\}$$

and obtain the following diagram

$$\begin{tikzcd}
1 \arrow[r] & \overline{V_2} \arrow[r] & \overline{G_2} \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & V_2 \arrow[r] \arrow[u, hook] \arrow[d, "\psi_{12}|_{V_2}"] & G_2 \arrow[r, "\pi_2"] \arrow[u, hook] \arrow[d, two heads, "\psi_{12}"] & \mathbb{G}_m \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & V_1 \arrow[r] & G_1 \arrow[r, "\pi_1"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

which is commutative and whose rows are exact. Clearly $\psi_{12}|_{V_2}$ is surjective. The group scheme V_2 is smooth over F as in the proof of Theorem 4.5. A $\mathbb{G}_m(\overline{F})$-action on $V_2(\overline{F})$ is given by

$$a. \begin{bmatrix} 1 & & \\ x & 1 & \\ y & x & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ ax & 1 & \\ a^2 y & ax & 1 \end{bmatrix}.$$

By Theorem 3.1, we have

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, L_1(\theta), L_2(\theta)) = \operatorname{tr.deg}_{\overline{K(t)}} \overline{K}(t)(\Omega, L_1, L_2) = \dim G_2$$

and this value is two or three. Thus we have

$$(3) \Leftarrow (1) \Leftrightarrow (2) \Rightarrow (4)$$

and $(3) \Rightarrow (4)$ as in the proof of Theorem 4.5.

From now on, we suppose that the condition (1) does not hold. We have $\dim G_2 = 2$ and hence $\dim V_2 = 1$. Then $\operatorname{Ker}(\psi_{12}|_{V_2} \colon V_2(\overline{F}) \to V_1(\overline{F})) = V_2(\overline{F}) \cap \overline{V_{2,0}}(\overline{F})$ has dimension zero, where we set

$$\overline{V_{2,0}} := \left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ y & & 1 \end{bmatrix} \right\}.$$

Since $V_2(\overline{F}) \cap \overline{V_{2,0}}(\overline{F})$ is closed under the $\mathbb{G}_m(\overline{F})$-action on $V_2(\overline{F})$, we conclude that $V_2(\overline{F}) \cap \overline{V_{2,0}}(\overline{F}) = \{1\}$ by Lemma 4.1. For a matrix

$$X = \begin{bmatrix} 1 & & \\ x & 1 & \\ y & x & 1 \end{bmatrix} \in V_2(\overline{F}),$$

we have

$$X^r = \begin{bmatrix} 1 & & \\ rx & 1 & \\ \frac{(r-1)r}{2}x^2 + ry & rx & 1 \end{bmatrix} \in V_2(\overline{F})$$

for each integer $r \in \mathbb{Z}$. Thus if r is not divisible by p, we obtain

$$V_2(\overline{F}) \ni (r.X)X^{-r} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ \frac{r(r-1)}{2}(2y - x^2) & 0 & 1 \end{bmatrix} \in \overline{V_{2,0}}(\overline{F}).$$

Since $V_2(\overline{F}) \cap \overline{V_{2,0}}(\overline{F}) = \{1\}$, we have the relation

$$\frac{r-1}{2}(2y - x^2) = 0.$$

We take $r \not\equiv 1 \mod 2p$. Then we have the relation

$$2y - x^2 = 0.$$

When $p = 2$, we have $x = 0$. This contradicts the surjectivity of $\psi_{12}|_{V_2}$. Thus when $p = 2$, we always have $\dim G_2 = 3$. In the following, we assume that $p \neq 2$. Since $\dim V_2 = 1$, we conclude that

$$V_2 = \left\{ \begin{bmatrix} 1 & & \\ x & 1 & \\ \frac{x^2}{2} & x & 1 \end{bmatrix} \right\}.$$

Next, we determine the group scheme G_2. Fix an element $a_0 \in \mathbb{G}_m(F)$ which has infinite order. As in the proof of Theorem 4.5, the fiber G_{2,a_0} is smooth over F and we have $G_{2,a_0}(F^{\mathrm{sep}}) \neq \emptyset$. Let $\widetilde{a_0}$ be a lift of a_0 in $G_2(F^{\mathrm{sep}})$. Since $G_2(F^{\mathrm{sep}})$ contains $V_2(F^{\mathrm{sep}})$, we can eliminate the x-coordinate of $\widetilde{a_0}$. Thus we may assume that

$$\widetilde{a_0} = \begin{bmatrix} a_0^2 & & \\ & a_0 & \\ y_0 & & 1 \end{bmatrix} \in G_2(F^{\mathrm{sep}}).$$

Then for each integer $r \in \mathbb{Z}$, we have

$$\widetilde{a_0}^r = \begin{bmatrix} a_0^{2r} & & \\ & a_0^r & \\ \frac{y_0}{1-a_0^2}(1 - a_0^{2r}) & & 1 \end{bmatrix} \in G_2(F^{\mathrm{sep}}).$$

Since a_0 has infinite order, we have

$$\left\{ \begin{bmatrix} a^2 & & \\ & a & \\ -\frac{c_2}{2}(1 - a^2) & & 1 \end{bmatrix} \right\} \subset G_{2,\overline{F}},$$

where $c_2 := -\frac{2y_0}{1-a_0^2} \in F^{\mathrm{sep}}$. Since $G_{2,\overline{F}}$ is a two-dimensional irreducible reduced group scheme which also contains $V_{2,\overline{F}}$, we conclude that

$$G_{2,\overline{F}} = \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}.$$

As in the proof of Theorem 4.5, we have $c_2 \in F$. We set a polynomial

$$Q := 2X_{31} - X_{32}^2 + c_2(1 - X_{22}^2) \in F[X_{22}, X_{31}, X_{32}] \subset F[X_{11}, \ldots, X_{33}, 1/\det X].$$

Then we have $Q(\widetilde{\Psi_2}) = 0$. By the definition of $\widetilde{\Psi_2}$, this is equivalent to the equality

$$(\Omega^{2n} L_1^2 - 2\Omega^{2n} L_2 - c_2) \otimes \Omega^{2n} = \Omega^{2n} \otimes (\Omega^{2n} L_1^2 - 2\Omega^{2n} L_2 - c_2)$$

in $\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L}$. Thus the condition (4) does not hold. This means (4) $\Rightarrow$ (1). $\square$

Remark 4.9. By the proof of Theorem 4.7, when the equivalent conditions are satisfied, we have

$$G_2 = \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ y & x & 1 \end{bmatrix} \right\}.$$

When the equivalent conditions are not satisfied, we have

$$G_2 = \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}$$

for some $c_2 \in \mathbb{F}_q(t)$. Such c_2 gives the linear dependence of $\Omega^{2n} L_{u,n}^2 - 2\Omega^{2n} L_{u,u,n,n} - c_2$ and Ω^{2n} over $\overline{K}(t)$, and c_2 is uniquely determined by

$$c_2(\theta) = \frac{L_{u,n}(\theta)^2 - 2L_{u,u,n,n}(\theta)}{\widetilde{\pi}^{2n}} = \frac{L_{u^2,2n}(\theta)}{\widetilde{\pi}^{2n}}.$$

Proofs of Theorems 2.7 and 2.30. First we prove Theorem 2.7. We fix a positive "odd" integer $n \geq 1$ and set $u := H_{n-1}$. Since $\widetilde{\pi}^n \notin K_\infty$ and $L_{\alpha,n}(\theta) = \Gamma_n \zeta(n) \in K_\infty^\times$, they are linearly independent over K. Thus the equivalent conditions of Theorem 4.5 are satisfied. By Theorem 4.7, the elements $\widetilde{\pi}$, $\Gamma_n\zeta(n)$ and $\Gamma_n^2\zeta(n,n)$ are algebraically independent over $\overline{K}$, or $\widetilde{\pi}^{2n}$ and $\Gamma_n^2(\zeta(n)^2 - 2\zeta(n,n))$ are linearly dependent over K.

Now we assume that $2n$ is "odd". Then $\widetilde{\pi}^{2n} \notin K_\infty$ and $\Gamma_n^2(\zeta(n)^2 - 2\zeta(n,n)) \in K_\infty$. Since $|\zeta(n)|_\infty = 1$ and $|\zeta(n,n)|_\infty < 1$, we have $\zeta(n)^2 - 2\zeta(n,n) \neq 0$. Thus the condition (3) of Theorem 4.7 holds.

Theorem 2.30 is proved similarly. $\square$

Next, we consider depth three cases. To show the classifications of Theorems 2.13 and 2.31, we prove Theorems 4.10 and 4.13.

Theorem 4.10. *Let $n \geq 1$ be a positive integer and $u \in \overline{K}[t]$ a polynomial such that $\|u\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Assume that the equivalent conditions of Theorem 4.7 are satisfied. Then the following conditions are equivalent:*

(1) $\widetilde{\pi}$, $L_{u,n}(\theta)$, $L_{u,u,n,n}(\theta)$ *and* $L_{u,u,u,n,n,n}(\theta)$ *are algebraically independent over* $\overline{K}$,
(2) Ω, $L_{u,n}$, $L_{u,u,n,n}$ *and* $L_{u,u,u,n,n,n}$ *are algebraically independent over* $\overline{K}(t)$,
(3) $\widetilde{\pi}^{3n}$ *and* $L_{u,n}(\theta)^3 - 3L_{u,n}(\theta)L_{u,u,n,n}(\theta) + 3L_{u,u,u,n,n,n}(\theta)$ *are linearly independent over* K,
(4) $\Omega^{3n}L_{u,n}^3 - 3\Omega^{3n}L_{u,n}L_{u,u,n,n} + 3\Omega^{3n}L_{u,u,u,n,n,n} - c_3$ *and* Ω^{3n} *are linearly independent over* $\overline{K}(t)$ *for each* $c_3 \in \mathbb{F}_q(t)$.

REMARK 4.11. The equivalent conditions of Theorem 4.10 are satisfied if $p = 3$. For example, we can easily check the condition (3).

PROOF OF THEOREM 4.10. We continue to use the notations in the proofs of Theorems 4.5 and 4.7. Let $L_3 := L_{\alpha,\alpha,\alpha,n,n,n}$. We set

$$\Phi_3 := \begin{bmatrix} (t-\theta)^{3n} & & & \\ u^{(-1)}(t-\theta)^{3n} & (t-\theta)^{2n} & & \\ & u^{(-1)}(t-\theta)^{2n} & (t-\theta)^n & \\ & & u^{(-1)}(t-\theta)^n & 1 \end{bmatrix} \in \mathrm{GL}_4(\overline{K}(t))$$

and

$$\Psi_3 := \begin{bmatrix} \Omega^{3n} & & & \\ \Omega^{3n}L_1 & \Omega^{2n} & & \\ \Omega^{3n}L_2 & \Omega^{2n}L_1 & \Omega^n & \\ \Omega^{3n}L_3 & \Omega^{2n}L_2 & \Omega^n L_1 & 1 \end{bmatrix} \in \mathrm{GL}_4(\mathbb{L}).$$

Let M_3 be the pre-t-motive defined by Φ_3 and G_3 its fundamental group. Then we have a homomorphism of pre-t-motives

$$M_3 \twoheadrightarrow M_2;\ (x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4).$$

By the Tannakian duality and Theorem 3.1, we have a diagram of smooth group schemes over F

$$\begin{tikzcd} G_3 \arrow[r, two heads, "\psi_{23}"] & G_2 \arrow[r, two heads, "\psi_{12}"] & G_1 \arrow[r, two heads, "\pi_1"] & G_{C^{\otimes n}} & \\ G_{\Psi_3} \arrow[u, "\simeq"] & G_{\Psi_2} \arrow[u, "\simeq"] & G_{\Psi_2} \arrow[u, "\simeq"] & G_{\Omega^n} \arrow[u, "\simeq"] \arrow[r, equal] & \mathbb{G}_m . \end{tikzcd}$$

In the following, we identify the upper group schemes with the lower group schemes in the above diagram. At first, we describe the morphism ψ_{23} in the above diagram explicitly. By Remark 4.9, we have

$$G_2 = \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ y & x & 1 \end{bmatrix} \right\}.$$

By the definition of $\widetilde{\Psi_3} = ((\widetilde{\Psi_3})_{ij})$, we have the relations

$$(\widetilde{\Psi_3})_{11} = (\widetilde{\Psi_3})_{33}^3,\quad (\widetilde{\Psi_3})_{22} = (\widetilde{\Psi_3})_{33}^2,\quad (\widetilde{\Psi_3})_{33} = \Omega^{-n} \otimes \Omega^n,\quad (\widetilde{\Psi_3})_{44} = 1,$$
$$(\widetilde{\Psi_3})_{21} = (\widetilde{\Psi_3})_{33}^2(\widetilde{\Psi_3})_{43},\quad (\widetilde{\Psi_3})_{32} = (\widetilde{\Psi_3})_{33}(\widetilde{\Psi_3})_{43},\quad \widetilde{\Psi}_{43} = -L_1 \otimes \Omega^n + 1 \otimes \Omega^n L_1,$$
$$(\widetilde{\Psi_3})_{31} = (\widetilde{\Psi_3})_{33}(\widetilde{\Psi_3})_{42},\quad (\widetilde{\Psi_3})_{42} = (L_1^2 - L_2) \otimes \Omega^{2n} - L_1 \otimes \Omega^{2n}L_1 + 1 \otimes \Omega^{2n}L_2,$$

$(\widetilde{\Psi_3})_{41} = (-L_1^3 + 2L_1L_2 - L_3)\otimes\Omega^{3n} + (L_1^2 - L_2)\otimes\Omega^{3n}L_1 - L_1\otimes\Omega^{3n}L_2 + 1\otimes\Omega^{3n}L_3$,
and $(\widetilde{\Psi_3})_{ij} = 0$ if $i < j$. Thus we have the inclusion

$$G_3 \cong G_{\Psi_3} \subset \overline{G_3} := \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ ay & ax & a & \\ z & y & x & 1 \end{bmatrix} \right\} \subset \mathrm{GL}_4 .$$

By using the above identifications, we can write

$$\psi_{23}\colon G_3 \twoheadrightarrow G_2; \quad \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ ay & ax & a & \\ z & y & x & 1 \end{bmatrix} \mapsto \begin{bmatrix} a^2 & & \\ ax & a & \\ y & x & 1 \end{bmatrix}$$

as in the proof of Theorem 4.5. We set $\pi_3 := \pi_2\circ\psi_{23} = \pi_1\circ\psi_{12}\circ\psi_{23}$ and $V_3 := \operatorname{Ker}\pi_3$ to be the unipotent radical of G_3. Then we have

$$V_3 \subset \overline{V_3} := \left\{ \begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & x & 1 & \\ z & y & x & 1 \end{bmatrix} \right\}$$

and obtain the following diagram

$$\begin{tikzcd}
1 \arrow[r] & \overline{V_3} \arrow[r] & \overline{G_3} \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & V_3 \arrow[r] \arrow[u, hook] \arrow[d, "\psi_{23}|_{V_3}"] & G_3 \arrow[r, "\pi_3"] \arrow[u, hook] \arrow[d, "\psi_{23}"] & \mathbb{G}_m \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & V_2 \arrow[r] & G_2 \arrow[r, "\pi_2"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

which is commutative and whose rows are exact. Clearly $\psi_{23}|_{V_3}$ is surjective. The group scheme V_3 is smooth over F as in the proof of Theorem 4.5. A $\mathbb{G}_m(\overline{F})$-action on $V_3(\overline{F})$ is given by

$$a. \begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & x & 1 & \\ z & y & x & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ ax & 1 & & \\ a^2y & ax & 1 & \\ a^3z & a^2y & ax & 1 \end{bmatrix}.$$

By Theorem 3.1, we have

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, L_1(\theta), L_2(\theta), L_3(\theta)) = \operatorname{tr.deg}_{\overline{K}(t)} \overline{K}(t)(\Omega, L_1, L_2, L_3) = \dim G_3$$

and this value is three or four. Thus we have

$$(3) \Leftarrow (1) \Leftrightarrow (2) \Rightarrow (4)$$

and $(3) \Rightarrow (4)$ as in the proof of Theorem 4.5.

From now on, we suppose that the condition (1) does not hold. We have $\dim G_3 = 3$ and hence $\dim V_3 = 2$. Then $\mathrm{Ker}(\psi_{23}|_{V_3} \colon V_3(\overline{F}) \to V_2(\overline{F})) = V_3(\overline{F}) \cap \overline{V_{3,0}}(\overline{F})$ has dimension zero, where we set

$$\overline{V_{3,0}} := \left\{ \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ z & & & 1 \end{bmatrix} \right\}.$$

Since $V_3(\overline{F}) \cap \overline{V_{3,0}}(\overline{F})$ is closed under the $\mathbb{G}_m(\overline{F})$-action on $V_3(\overline{F})$, we conclude that $V_3(\overline{F}) \cap \overline{V_{3,0}}(\overline{F}) = \{1\}$ by Lemma 4.1. For a matrix

$$X = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & x & 1 & \\ z & y & x & 1 \end{bmatrix} \in V_3(\overline{F}),$$

we have

$$X^r = \begin{bmatrix} 1 & & & \\ rx & 1 & & \\ \frac{r(r-1)}{2}x^2 + ry & rx & 1 & \\ \frac{r(r-1)(r-2)}{6}x^3 + r(r-1)xy + rz & \frac{r(r-1)}{2}x^2 + ry & rx & 1 \end{bmatrix} \in V_3(\overline{F})$$

for each integer $r \in \mathbb{Z}$. Thus if r is not divisible by p, we obtain

$$(r.X)X^{-r} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \frac{r(r-1)}{2}(2y - x^2) & 0 & 1 & \\ \frac{r(r-1)(r+1)}{3}(3z - 3xy + x^3) & \frac{r(r-1)}{2}(2y - x^2) & 0 & 1 \end{bmatrix}.$$

If s is not divisible by p, we obtain

$$(\sqrt{s}.((r.X)X^{-r}))((r.X)X^{-r})^{-s} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ s(\sqrt{s} - 1)\frac{r(r-1)(r+1)}{3}(3z - 3xy + x^3) & 0 & 0 & 1 \end{bmatrix}$$

which is contained in $V_3(\overline{F}) \cap \overline{V_{3,0}}(\overline{F}) = \{1\}$. Thus we have the relation

$$(\sqrt{s} - 1)\frac{(r-1)(r+1)}{3}(3z - 3xy + x^3) = 0.$$

When $p \neq 2$, we take $r \not\equiv \pm 1 \mod 3p$ and $s \neq 1$. Then we have the relation

$$3z - 3xy + x^3 = 0.$$

Assume $p = 2$. We denote by $Y_{(v,w)}$ the inverse image of

$$\begin{bmatrix} 1 & & \\ v & 1 & \\ w & v & 1 \end{bmatrix}$$

via the group isomorphism $\psi_{23}|_{V_3} \colon V_3(\overline{F}) \to V_2(\overline{F})$. We take $c_0 \in \overline{F}$ such that

$$Y_{(1,0)} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ c_0 & 0 & 1 & 1 \end{bmatrix}.$$

Then if $x, y \neq 0$, we have

$$V_3(\overline{F}) \ni (x.Y_{(1,0)})(\sqrt{y}.(Y_{(1,0)}^2)) = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & x & 1 & \\ xy + c_0x^3 & y & x & 1 \end{bmatrix} = Y_{(x,y)} = X.$$

If $x \neq 0$ and $y = 0$, we have

$$V_3(\overline{F}) \ni (x.Y_{(1,0)}) = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ 0 & x & 1 & \\ c_0x^3 & 0 & x & 1 \end{bmatrix} = Y_{(x,0)} = X.$$

If $x = 0$ and $y \neq 0$, we have

$$V_3(\overline{F}) \ni (\sqrt{y}.Y_{(1,0)}^2) = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ y & 0 & 1 & \\ 0 & y & 0 & 1 \end{bmatrix} = Y_{(0,y)} = X.$$

Thus in any case, we have $z = xy + c_0x^3$. We can compute $Y(v_1, w_1)Y(v_2, w_2)$ as follows:

$$\begin{bmatrix} 1 & & & \\ v_1 + v_2 & 1 & & \\ w_1 + v_1v_2 + w_2 & v_1 + v_2 & 1 & \\ (v_1 + v_2)(w_1 + v_1v_2 + w_2) + c_0(v_1 + v_2)^3 + \beta & w_1 + v_1v_2 + w_2 & v_1 + v_2 & 1 \end{bmatrix}$$

for each v_1, w_1, v_2 and w_2, where $\beta := -v_1v_2(v_1 + v_2)(3c_0 + 1)$. Since $V_3(\overline{F})$ is a group, the matrix $Y(v_1, w_1)Y(v_2, w_2)$ is contained in $V_3(\overline{F})$ and β must be zero. Thus we have $c_0 = -1/3$. Therefore in any characteristic case, we have the relation

$$3z - 3xy + x^3 = 0.$$

When $p = 3$, we have $x = 0$. This contradicts the surjectivity of $\psi_{23}|_{V_3}$. Thus when $p = 3$, we always have $\dim G_3 = 4$. In the following, we assume that $p \neq 3$. Since $\dim V_3 = 2$, we conclude that

$$V_3 = \left\{ \begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & x & 1 & \\ xy - \frac{x^3}{3} & y & x & 1 \end{bmatrix} \right\}.$$

Next, we determine the group scheme G_3. Fix an element $a_0 \in \mathbb{G}_m(F)$ which has infinite order. As in the proof of Theorem 4.5, the fiber G_{3,a_0} is smooth over F and we have $G_{3,a_0}(F^{\mathrm{sep}}) \neq \emptyset$. Let $\widetilde{a_0}$ be a lift of a_0 in $G_3(F^{\mathrm{sep}})$. Since $G_3(F^{\mathrm{sep}})$

contains $V_3(F^{\mathrm{sep}})$, we can eliminate the x- and y-coordinates of $\widetilde{a_0}$. Thus we may assume that

$$\widetilde{a_0} = \begin{bmatrix} a_0^3 & & & \\ & a_0^2 & & \\ & & a_0 & \\ z_0 & & & 1 \end{bmatrix} \in G_3(F^{\mathrm{sep}}).$$

Then for each integer $r \in \mathbb{Z}$, we have

$$\widetilde{a_0}^{\,r} = \begin{bmatrix} a_0^{3r} & & & \\ & a_0^{2r} & & \\ & & a_0^r & \\ \frac{z_0}{1-a_0^3}(1-a_0^{3r}) & & & 1 \end{bmatrix} \in G_3(F^{\mathrm{sep}}).$$

Since a_0 has infinite order, we have

$$\left\{ \begin{bmatrix} a^3 & & & \\ & a^2 & & \\ & & a & \\ \frac{c_3}{3}(1-a^3) & & & 1 \end{bmatrix} \right\} \subset G_{3,\overline{F}},$$

where $c_3 := \frac{3z_0}{1-a_0^3} \in F^{\mathrm{sep}}$. Since $G_{3,\overline{F}}$ is a three-dimensional irreducible reduced group scheme which also contains $V_{3,\overline{F}}$, we conclude that

$$G_{3,\overline{F}} = \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ ay & ax & a & \\ xy - \frac{x^3}{3} + \frac{c_3}{3}(1-a^3) & y & x & 1 \end{bmatrix} \right\}.$$

As in the proof of Theorem 4.5, we have $c_3 \in F$. We set a polynomial

$$Q := 3X_{41} - 3X_{43}X_{42} + X_{43}^3 - c_3(1 - X_{33}^3) \in F[X_{11}, \ldots, X_{44}, 1/\det X].$$

Then we have $Q(\widetilde{\Psi_3}) = 0$. By the definition of $\widetilde{\Psi_3}$, this is equivalent to the equality

$$(\Omega^{3n}L_1^3 - 3\Omega^{3n}L_1L_2 + 3\Omega^{3n}L_3 - c_3) \otimes \Omega^{3n} = \Omega^{3n} \otimes (\Omega^{3n}L_1^3 - 3\Omega^{3n}L_1L_2 + 3\Omega^{3n}L_3 - c_3)$$

in $\mathbb{L} \otimes_{\overline{K(t)}} \mathbb{L}$. Thus the condition (4) does not hold. This means (4) $\Rightarrow$ (1). □

Remark 4.12. By the proof of Theorem 4.10, when the equivalent conditions are satisfied, we have

$$G_3 = \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ ay & ax & a & \\ z & y & x & 1 \end{bmatrix} \right\}.$$

When the equivalent conditions are not satisfied, we have

$$G_3 = \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ ay & ax & a & \\ xy - \frac{x^3}{3} + \frac{c_3}{3}(1-a^3) & y & x & 1 \end{bmatrix} \right\}$$

for some $c_3 \in \mathbb{F}_q(t)$. Such c_3 gives the linear dependence of

$$\Omega^{3n} L_{u,n}^3 - 3\Omega^{3n} L_{u,n} L_{u,u,n,n} + 3\Omega^{3n} L_{u,u,u,n,n,n} - c_3 \quad \text{and} \quad \Omega^{3n}$$

over $\overline{K}(t)$, and c_3 is uniquely determined by

$$c_3(\theta) = \frac{L_{u,n}(\theta)^3 - 3L_{u,n}(\theta)L_{u,u,n,n}(\theta) + 3L_{u,u,u,n,n,n}(\theta)}{\widetilde{\pi}^{3n}} = \frac{L_{u^3,3n}(\theta)}{\widetilde{\pi}^{3n}}.$$

THEOREM 4.13. *Let $n \geq 1$ be a positive integer and $u \in \overline{K}[t]$ a polynomial such that $\|u\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Assume that the equivalent conditions of Theorem 4.5 are satisfied but the equivalent conditions of Theorem 4.7 are not satisfied. Then the following conditions are equivalent:*

(1) *$\widetilde{\pi}$, $L_{u,n}(\theta)$ and $L_{u,u,u,n,n,n}(\theta)$ are algebraically independent over $\overline{K}$,*
(2) *Ω, $L_{u,n}$ and $L_{u,u,u,n,n,n}$ are algebraically independent over $\overline{K}(t)$,*
(3) *$\widetilde{\pi}^{3n}$ and $L_{u,n}(\theta)^3 - 3L_{u,n}(\theta)L_{u,u,n,n}(\theta) + 3L_{u,u,u,n,n,n}(\theta)$ are linearly independent over K,*
(4) *$\Omega^{3n} L_{u,n}^3 - 3\Omega^{3n} L_{u,n} L_{u,u,n,n} + 3\Omega^{3n} L_{u,u,u,n,n,n} - c_3$ and Ω^{3n} are linearly independent over $\overline{K}(t)$ for each $c_3 \in \mathbb{F}_q(t)$.*

REMARK 4.14. The equivalent conditions of Theorem 4.13 are satisfied if $p = 3$. For example, we can easily check the condition (3).

PROOF OF THEOREM 4.13. We continue to use the notations in the proofs of Theorems 4.5, 4.7 and 4.10. By Remark 4.8, we have $p \neq 2$. By Remark 4.9, there exists $c_2 \in F$ such that $\Omega^{2n} L_1^2 - 2\Omega^{2n} L_2 - c_2$ and Ω^{2n} are linearly dependent over $\overline{K}(t)$ and

$$G_2 = \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}.$$

Thus we have the inclusions

$$G_3 \cong G_{\Psi_3} \subset \overline{G_3}' := \left\{ \begin{bmatrix} a^3 & & & \\ a^2 x & a^2 & & \\ a\left(\frac{x^2}{2} - \frac{c_2}{2}(1-a^2)\right) & ax & a & \\ z & \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\} \subset \mathrm{GL}_4$$

and

$$V_3 \subset \overline{V_3}' := \left\{ \begin{bmatrix} 1 & & & \\ x & 1 & & \\ \frac{x^2}{2} & x & 1 & \\ z & \frac{x^2}{2} & x & 1 \end{bmatrix} \right\}.$$

By Theorem 3.1, we have

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, L_1(\theta), L_2(\theta), L_3(\theta)) = \operatorname{tr.deg}_{\overline{K(t)}} \overline{K}(t)(\Omega, L_1, L_2, L_3) = \dim G_3$$

and this value is two or three. Thus we have

$$(3) \Leftarrow (1) \Leftrightarrow (2) \Rightarrow (4)$$

and (3) $\Rightarrow$ (4) as in the proof of Theorem 4.5. From now on, we suppose that the condition (1) does not hold. We have $\dim G_3 = 2$ and hence $\dim V_3 = 1$. Then we have $V_3(\overline{F}) \cap \overline{V_{3,0}}(\overline{F}) = 1$, and for a matrix

$$X = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ \frac{x^2}{2} & x & 1 & \\ z & \frac{x^2}{2} & x & 1 \end{bmatrix} \in V_3(\overline{F}),$$

we have the relation

$$3z - \frac{x^3}{2} = 0$$

as in the proof of Theorem 4.10.

When $p = 3$, we have $x = 0$. This contradicts the surjectivity of $\psi_{23}|_{V_3}$. Thus when $p = 3$, we always have $\dim G_3 = 3$. In the following, we assume that $p \neq 2, 3$. Since $\dim V_3 = 1$, we conclude that

$$V_3 = \left\{ \begin{bmatrix} 1 & & & \\ x & 1 & & \\ \frac{x^2}{2} & x & 1 & \\ \frac{x^3}{6} & \frac{x^2}{2} & x & 1 \end{bmatrix} \right\}.$$

Next, we determine the group scheme G_3. Fix an element $a_0 \in \mathbb{G}_m(F)$ which has infinite order. As in the proof of Theorem 4.5, the fiber G_{3,a_0} is smooth over F and we have $G_{3,a_0}(F^{\mathrm{sep}}) \neq \emptyset$. Let $\widetilde{a_0}$ be a lift of a_0 in $G_3(F^{\mathrm{sep}})$. Since $G_3(F^{\mathrm{sep}})$ contains $V_3(F^{\mathrm{sep}})$, we can eliminate the x-coordinates of $\widetilde{a_0}$. Thus we may assume that

$$\widetilde{a_0} = \begin{bmatrix} a_0^3 & & & \\ 0 & a_0^2 & & \\ -a_0\frac{c_2}{2}(1-a_0^2) & 0 & a_0 & \\ z_0 & -\frac{c_2}{2}(1-a_0^2) & 0 & 1 \end{bmatrix} \in G_3(F^{\mathrm{sep}}).$$

Then for each integer $r \in \mathbb{Z}$, we have

$$\widetilde{a_0}^r = \begin{bmatrix} a_0^{3r} & & & \\ 0 & a_0^{2r} & & \\ -a_0^r\frac{c_2}{2}(1-a_0^{2r}) & 0 & a_0^r & \\ \frac{z_0}{1-a_0^3}(1-a_0^{3r}) & -\frac{c_2}{2}(1-a_0^{2r}) & 0 & 1 \end{bmatrix} \in G_3(F^{\mathrm{sep}}).$$

Since a_0 has infinite order, we have

$$\left\{ \begin{bmatrix} a^3 & & & \\ 0 & a^2 & & \\ -a\frac{c_2}{2}(1-a^2) & 0 & a & \\ \frac{c_3}{3}(1-a^3) & -\frac{c_2}{2}(1-a^2) & 0 & 1 \end{bmatrix} \right\} \subset G_{3,\overline{F}},$$

where $c_3 := \frac{3z_0}{1-a_0^3} \in F^{\mathrm{sep}}$. Since $G_{3,\overline{F}}$ is a two-dimensional irreducible reduced group scheme which also contains $V_{3,\overline{F}}$, we conclude that

$$G_{3,\overline{F}} = \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ a(\frac{x^2}{2} - \frac{c_2}{2}(1-a^2)) & ax & a & \\ \frac{x^3}{6} - \frac{c_2}{2}(1-a^2)x + \frac{c_3}{3}(1-a^3) & \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}.$$

As in the proof of Theorem 4.5, we have $c_3 \in F$. We set a polynomial

$$Q := 6X_{41} - X_{43}^3 + 3c_2(1 - X_{33}^2)X_{43} - 2c_3(1 - X_{33}^3) \in F[X_{11}, \ldots, X_{44}, 1/\det X].$$

Then we have $Q(\widetilde{\Psi_3}) = 0$. By the definition of $\widetilde{\Psi_3}$ and the assumption $L_1^2 - 2L_2 - c_2\Omega^{-2n} \in \overline{K}(t)$, this is equivalent to the equality

$$(\Omega^{3n}L_1^3 - 3\Omega^{3n}L_1L_2 + 3\Omega^{3n}L_3 - c_3) \otimes \Omega^{3n} = \Omega^{3n} \otimes (\Omega^{3n}L_1^3 - 3\Omega^{3n}L_1L_2 + 3\Omega^{3n}L_3 - c_3)$$

in $\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L}$. Thus the condition (4) does not hold. This means (4) $\Rightarrow$ (1). $\square$

Remark 4.15. By the proof of Theorem 4.13, when the equivalent conditions are satisfied, we have

$$G_3 = \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ a\left(\frac{x^2}{2} - \frac{c_2}{2}(1-a^2)\right) & ax & a & \\ z & \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}.$$

When the equivalent conditions are not satisfied, we have

$$G_3 = \left\{ \begin{bmatrix} a^3 & & & \\ a^2x & a^2 & & \\ a(\frac{x^2}{2} - \frac{c_2}{2}(1-a^2)) & ax & a & \\ \frac{x^3}{6} - \frac{c_2}{2}(1-a^2)x + \frac{c_3}{3}(1-a^3) & \frac{x^2}{2} - \frac{c_2}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}.$$

for some $c_3 \in \mathbb{F}_q(t)$. Such c_3 gives the linear dependence of

$$\Omega^{3n}L_{u,n}^3 - 3\Omega^{3n}L_{u,n}L_{u,u,n,n} + 3\Omega^{3n}L_{u,u,u,n,n,n} - c_3 \quad \text{and} \quad \Omega^{3n}$$

over $\overline{K}(t)$, and c_3 is uniquely determined by

$$c_3(\theta) = \frac{L_{u,n}(\theta)^3 - 3L_{u,n}(\theta)L_{u,u,n,n}(\theta) + 3L_{u,u,u,n,n,n}(\theta)}{\widetilde{\pi}^{3n}} = \frac{L_{u^3,3n}(\theta)}{\widetilde{\pi}^{3n}}.$$

Proof of Theorems 2.13 and 2.31. First we prove Theorem 2.13. We fix a positive "odd" integer $n \geq 1$ and set $u := H_{n-1}$. Since $\widetilde{\pi}^n \notin K_\infty$ and $L_{\alpha,n}(\theta) = \Gamma_n\zeta(n) \in K_\infty^\times$, they are linearly independent over K. Thus the equivalent conditions of Theorem 4.5 are satisfied.

By Theorem 4.10, when the equivalent conditions of Theorem 4.7 are satisfied (thus $\widetilde{\pi}$, $\zeta(n)$ and $\zeta(n,n)$ are algebraically independent over $\overline{K}$), one and only one of the following holds:

- *(i)* $\widetilde{\pi}$, $\Gamma_n\zeta(n)$, $\Gamma_n^2\zeta(n,n)$ and $\Gamma_n^3\zeta(n,n,n)$ are algebraically independent over $\overline{K}$,
- *(iii)* $\widetilde{\pi}^{3n}$ and $\Gamma_n^3(\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n))$ are linearly dependent over K.

By Theorem 4.13, when the equivalent conditions of Theorem 4.7 are not satisfied (thus $\widetilde{\pi}^{2n}$ and $\Gamma_n^2(\zeta(n)^2 - 2\zeta(n,n))$ are linearly dependent over K), one and only one of the following holds:

(*ii*) $\widetilde{\pi}$, $\Gamma_n\zeta(n)$, and $\Gamma_n^3\zeta(n,n,n)$ are algebraically independent over $\overline{K}$,

(*iv*) $\widetilde{\pi}^{3n}$ and $\Gamma_n^3(\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n))$ are linearly dependent over K.

Now we assume that $2n$ is "odd". Then $\widetilde{\pi}^{2n} \notin K_\infty$ and $\Gamma_n^2(\zeta(n)^2 - 2\zeta(n,n)) \in K_\infty$. Since $|\zeta(n)|_\infty = 1$ and $|\zeta(n,n)|_\infty < 1$, we have $\zeta(n) - 2\zeta(n,n) \neq 0$. Thus $\widetilde{\pi}^{2n}$ and $\Gamma_n^2(\zeta(n)^2 - 2\zeta(n,n))$ are linearly independent over K, and hence we have (*i*) or (*iii*). Assume that $3n$ is "odd". Then $\widetilde{\pi}^{3n} \notin K_\infty$ and $\Gamma_n^3(\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n)) \in K_\infty$. Since $|\zeta(n,n,n)|_\infty < 1$, we have

$$\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n) \neq 0.$$

Thus $\widetilde{\pi}^{3n}$ and $\Gamma_n^3(\zeta(n)^3 - 3\zeta(n)\zeta(n,n) + 3\zeta(n,n,n))$ are linearly independent over K, and hence we have (*i*) or (*ii*). Since n is "odd", either $2n$ or $3n$ is "odd". Therefore, the condition (*iv*) does not occur.

Theorem 2.31 is proved similarly. □

3. Proofs of Theorems 2.17 and 2.33

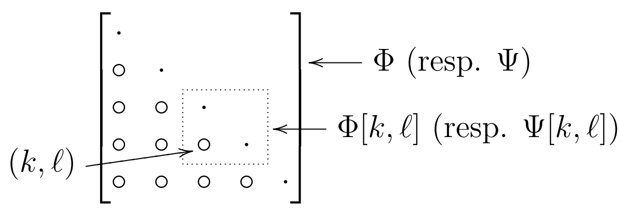
Next, we prove Theorems 2.17 and 2.33. As in the proof of Theorem 4.3, for an algebraic group G over $\mathbb{F}_q(t)$, when it is clear from the context, without confusion we still denote by G the $\overline{\mathbb{F}_q(t)}$-valued points of G.

Recall that I_d is the set defined in Definition 1.1. The notations $\underline{n}_{ij}$ and $\underline{u}_{ij}$ are also defined there. Clearly, Theorems 2.17 and 2.33 follow from Theorems 4.2, 4.3 and 4.16.

THEOREM 4.16. *Let* $\underline{n} = (n_1, \ldots, n_d)$ *be an index and* $\underline{u} = (u_1, \ldots, u_d) \in (\overline{K}[t])^d$ *a* d*-tuple of polynomials such that* $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ *for each* i*. If* $\widetilde{\pi}$, $L_{u_1,n_1}(\theta)$, $\ldots$, $L_{u_d,n_d}(\theta)$ *are algebraically independent over* $\overline{K}$*, then we have*

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, L_{\underline{u}_{ij},\underline{n}_{ij}}(\theta) | (i,j) \in I_d) = 1 + \#I_d = 1 + \frac{d(d+1)}{2}.$$

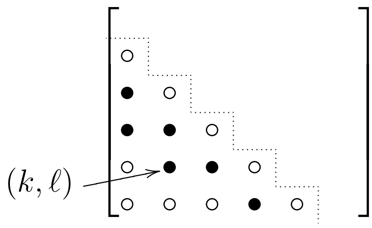
PROOF. In this proof, (i,j) and (k,ℓ) are always assumed to be elements of the totally ordered set I_d. Let Φ and Ψ be the $(d+1) \times (d+1)$-matrices defined in Example 3.5. These satisfy the Frobenius difference equations (3.1). For $(k,\ell) \in I_d$, we define $(\mathrm{dep}(k,\ell)+1) \times (\mathrm{dep}(k,\ell)+1)$-matrices $\Phi[k,\ell] = (\Phi[k,\ell]_{ij})$ and $\Psi[k,\ell] = (\Psi[k,\ell]_{ij})$ which are sub-matrices of Φ and Ψ, where $\Phi[k,\ell]_{ij} = \Phi_{i+\ell-1,j+\ell-1}$ and $\Psi[k,\ell]_{ij} = \Psi_{i+\ell-1,j+\ell-1}$. In particular, the lower left corner of $\Phi[k,\ell]$ (resp. $\Psi[k,\ell]$) is the (k,ℓ)-th entry of Φ (resp. Ψ). The following is an illustration of the relative positions of the matrices:



Let $M[k,\ell]$ be the pre-t-motive defined by $\Phi[k,\ell]$ and $G(k,\ell)$ the fundamental group of the pre-t-motive

$$M(k,\ell) := C \oplus \bigoplus_{\substack{\mathrm{dep}(i,j)\geq \mathrm{dep}(k,\ell)-1 \\ (i,j)\leq (k,\ell)}} M[i,j],$$

where C is the Carlitz pre-t-motive (see Example 3.4). The $\bullet$'s below illustrate the range in which (i,j) runs in the above direct sum:



We identify $G(k,\ell)$ with the algebraic group defined by $\left[\Omega\right] \oplus \bigoplus_{(i,j)} \Psi[i,j]$ as in Theorem 3.1. Then we have the inclusion

$$G(k,\ell) \subset \left\{ \left[a\right] \oplus \bigoplus_{\substack{\mathrm{dep}(i,j)\geq \mathrm{dep}(k,\ell)-1 \\ (i,j)\leq (k,\ell)}} \begin{bmatrix} a^{n_j+\cdots+n_d} & & & \\ x_{j+1,j} & a^{n_{j+1}+\cdots+n_d} & & \\ \vdots & \ddots & \ddots & \\ x_{ij} & \cdots & x_{i,i-1} & a^{n_i+\cdots+n_d} \end{bmatrix} \right\}$$

for each (k,ℓ). Note that some different entries/coordinates of different block matrices may be the same and denoted by same letters; this means that for $(i,j),(i',j')$ and r,r',s,s' with $1 \leq s < r \leq \mathrm{dep}(i,j)+1$ and $1 \leq s' < r' \leq \mathrm{dep}(i',j')+1$, if $(r+j-1, s+j-1) = (r'+j'-1, s'+j'-1)$, then the (r,s)-th entry of the (i,j)-th component matrix and the (r',s')-th entry of the (i',j')-th component matrix are the same and they are denoted by $x_{r+j-1,s+j-1}$. In fact, since Ψ is a lower triangular matrix, the (r,s)-th entry of $\widetilde{\Psi[i,j]}$ is equal to the $(r+j-1,s+j-1)$-th entry of $\widetilde{\Psi}$ (for the explicit description of $\widetilde{\Psi}$, see Example 3.5). Thus if $(r+j-1,s+j-1) = (r'+j'-1, s'+j'-1)$, then the (r,s)-th entry of $\widetilde{\Psi[i,j]}$ and the (r',s')-th entry of $\widetilde{\Psi[i',j']}$ coincide. Therefore the values of these entries in the algebraic group $G(k,\ell)$ are the same.

By Theorem 3.1, it suffices to show that the above inclusion is actually an equality for each (k,ℓ). We prove this by induction on $(k,\ell) \in I_d$ with respect to the total order "$\leq$".

By the assumption, this is true for $(2,1) \leq (k,\ell) \leq (d+1,d)$, the depth one cases. Let $(k,\ell) \geq (3,1)$ (this means $\mathrm{dep}(k,\ell) \geq 2$) and assume that the inclusion is an equality for (k',ℓ') the greatest element of $\{(i,j) \in I_d | (i,j) < (k,\ell)\}$, which means that $(k',\ell') = (k-1,\ell-1)$ if $\ell \neq 1$ and $(k',\ell') = (d+1, d+3-k)$ if $\ell = 1$. By definition, $M(k',\ell')$ is a subobject of $M(k,\ell)$ and C is a subobject of $M(k,\ell)$ and $M(k',\ell')$. By Tannakian duality, we have surjections $\psi\colon G(k,\ell) \to G(k',\ell')$, $\pi\colon G(k,\ell) \to \mathbb{G}_m$ and $\pi'\colon G(k',\ell') \to \mathbb{G}_m$, where we identify G_C with $\mathbb{G}_m$. These are projection maps. More precisely, π and π' map the matrices of the above forms to a and ψ maps them to the same matrices with the (k,ℓ)-th component matrices removed. This follows from the description of the map (3.2). The arguments are the same as in [**P**, §6.2.2], [**CY**, §4.3] and [**CPY**, Remark 2.3.2]. We set $V := \operatorname{Ker}\pi$ and $V' := \operatorname{Ker}\pi'$ to be the unipotent radicals of $G(k,\ell)$ and $G(k',\ell')$, respectively. Then we have the following diagram

$$\begin{tikzcd} 1 \arrow[r] & V \arrow[r] \arrow[d, "\psi|_V"] & G(k,\ell) \arrow[r, "\pi"] \arrow[d, "\psi"] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & V' \arrow[r] & G(k',\ell') \arrow[r, "\pi'"] & \mathbb{G}_m \arrow[r] & 1, \end{tikzcd}$$

which is commutative and whose rows are exact.

It is clear that $\psi|_V$ is surjective. Since V is non-commutative, the $G(k,\ell)$-action $A.X := A^{-1}XA$ on V ($X \in V$, $A \in G(k,\ell)$) depends not only on $\pi(A)$ but also on the other entries of A. Note that the coordinate variable $x_{k\ell}$ of $G(k,\ell)$ is the only coordinate variable which does not appear as a coordinate variable of $G(k',\ell')$. Thus we know that $\dim G(k',\ell') \le \dim G(k,\ell) \le \dim G(k',\ell')+1$. This also follows from Theorem 3.1 (2). It suffices to show that the second inequality is an equality.

Now, assume that $\dim G(k,\ell) = \dim G(k',\ell')$. Then $\dim \operatorname{Ker}(\psi|_V) = 0$. It is clear that $\operatorname{Ker}(\psi|_V)$ is a normal subgroup of $G(k,\ell)$ and $A.x_{k\ell} = \pi(A)^{n_\ell+\cdots+n_d}x_{k\ell}$ for each $x_{k\ell} \in \operatorname{Ker}(\psi|_V)$ and $A \in G(k,\ell)$, where we identify $\operatorname{Ker}(\psi|_V) \subset \mathbb{G}_a$ by means of the coordinate $x_{k\ell}$. By Lemma 4.1 we have that $\operatorname{Ker}(\psi|_V)$ is trivial. We take any elements

$$X = \begin{bmatrix}1\end{bmatrix} \oplus \bigoplus_{\substack{\operatorname{dep}(i,j)\ge \operatorname{dep}(k,\ell)-1 \\ (i,j)\le(k,\ell)}} \begin{bmatrix} 1 & & & \\ x_{j+1,j} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{ij} & \cdots & x_{i,i-1} & 1 \end{bmatrix} \in V$$

and

$$A = \begin{bmatrix}1\end{bmatrix} \oplus \bigoplus_{\operatorname{dep}(i,j)=\operatorname{dep}(k,\ell)-1} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 0 & \ddots & \\ a_{ij} & & & 1 \end{bmatrix} \oplus \bigoplus_{\substack{\operatorname{dep}(i,j)=\operatorname{dep}(k,\ell) \\ (i,j)\le(k,\ell)}} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ a_{i-1,j} & 0 & & \ddots & \\ a_{ij} & a_{i,j+1} & & & 1 \end{bmatrix} \in V,$$

where we can take any $x_{ij} \in \overline{\mathbb{F}_q(t)}$ (resp. any $a_{ij} \in \overline{\mathbb{F}_q(t)}$) for each $(i,j) \in I_d$ such that $(i,j) \ne (k,\ell)$ (resp. $\operatorname{dep}(i,j) \ge \operatorname{dep}(k,\ell)-1$ and $(i,j) < (k,\ell)$) by the assumption on (k',ℓ') and the surjectivity of $\psi|_V$. Then $X^{-1}(A^{-1}XA)$ is equal to

$$
\begin{aligned}
[1] \quad &\oplus \bigoplus_{\mathrm{dep}(i,j)=\mathrm{dep}(k,\ell)-1} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \\
&\oplus \bigoplus_{\substack{\mathrm{dep}(i,j)=\mathrm{dep}(k,\ell)\\ (i,j)\le(k,\ell)}} \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & 0 & & \ddots \\ a_{i-1,j}x_{i,i-1}-a_{i,j+1}x_{j+1,j} & & & & 1 \end{bmatrix}.
\end{aligned}
$$

Now we take $a_{ij}=0$ for $(k-\ell,1)\le(i,j)<(k,\ell+1)$ and $a_{k,\ell+1}=1$. Then we see that $X^{-1}(A^{-1}XA)\in\mathrm{Ker}(\psi|_V)=\{0\}$ and so we have $x_{\ell+1,\ell}=0$. Since $(\ell+1,\ell)\neq(k,\ell)$, this is a contradiction. Therefore we have $\dim G(k,\ell)=\dim G(k',\ell')+1$. $\square$

4. Proofs of Theorems 2.23 and 2.24

We prove Theorems 2.23 and 2.24. For an index $\underline{n}=(n_1,\dots,n_d)$, we set

$$L_{\underline{n}} := L_{H(\underline{n}),\underline{n}},\quad \Omega^{\underline{n}} := \Omega^{n_1+\cdots+n_d} \quad\text{and}\quad \underline{n}' := (n_1,\dots,n_{d-1}).$$

First, we determine the relation between $L_{\underline{n}}$ and $L_{p^e\underline{n}}$ for a non-negative integer $e\ge 0$. For a positive integer $n\ge 1$, we set $\Gamma_n(t)$ to be the inverse image of Γ_n via the $\mathbb{F}_q$-isomorphism $\mathbb{F}_q(t)\to\mathbb{F}_q(\theta); t\mapsto\theta$.

Lemma 4.17. *For each positive integer $n\ge 1$ and each non-negative integer $e\ge 0$, we have*

$$\frac{H_{p^en-1}}{\Gamma_{p^en}(t)} = \left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}.$$

Proof. By definition, we have

$$(H_{s-1}\Omega^s)^{(i)}(\theta) = \frac{\Gamma_s S_i(s)}{\widetilde{\pi}^s}$$

for each $s\ge 1$ and $i\ge 0$, where

$$S_i(s) := \sum_{\substack{a\in\mathbb{F}_q[\theta]:\text{monic}\\ \deg(a)=i}} \frac{1}{a^s}$$

(see [**AT1**, 3.7.4], [**AT2**, 2.4.1]). Thus we have

$$\left(\frac{H_{s-1}}{\Gamma_s(t)}(\theta^{q^{-i}})\right)^{q^i} = \left(\frac{H_{s-1}}{\Gamma_s(t)}\right)^{(i)}(\theta) = \frac{S_i(s)}{\widetilde{\pi}^s(\Omega^s)^{(i)}(\theta)}.$$

Therefore

$$\left(\frac{H_{p^en-1}}{\Gamma_{p^en}(t)}(\theta^{q^{-i}})\right)^{q^i} = \frac{S_i(p^en)}{\widetilde{\pi}^{p^en}(\Omega^{p^en})^{(i)}(\theta)} = \left(\frac{S_i(n)}{\widetilde{\pi}^n(\Omega^n)^{(i)}(\theta)}\right)^{p^e}$$
$$= \left(\left(\frac{H_{n-1}}{\Gamma_n(t)}(\theta^{q^{-i}})\right)^{q^i}\right)^{p^e} = \left(\left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}(\theta^{q^{-i}})\right)^{q^i}.$$

for each $i \geq 0$. Thus we have

$$\frac{H_{p^en-1}}{\Gamma_{p^en}(t)}(\theta^{q^{-i}}) = \left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}(\theta^{q^{-i}}).$$

for each $i \geq 0$. □

We set

$$\gamma_{e,n} := \frac{H_{p^en-1}}{H_{n-1}^{p^e}} = \frac{\Gamma_{p^en}(t)}{\Gamma_n(t)^{p^e}} \in \mathbb{F}_q(t)^\times$$

and

$$\gamma_{e,\underline{n}} := \prod_{i=1}^{d} \gamma_{e,n_i}$$

for any index $\underline{n} = (n_1, \ldots, n_d)$.

LEMMA 4.18. *For each index $\underline{n}$ and each non-negative integer $e \geq 0$, we have*

$$L_{p^e\underline{n}} = \gamma_{e,\underline{n}} L_{\underline{n}}^{p^e}.$$

PROOF. We prove this equality by induction on d. When $d = 0$, it is clear. We take $d \geq 1$ and assume that the above equality holds for any indices whose depths are lower than d. Then we have

$$\begin{aligned}
&(\Omega^{p^e\underline{n}}L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e})^{(-1)} \\
&= H_{p^en_d-1}^{(-1)}(t-\theta)^{p^en_d}\Omega^{p^e\underline{n}}L_{p^e\underline{n}'} + \Omega^{p^e\underline{n}}L_{p^e\underline{n}} - \gamma_{e,\underline{n}}\left(H_{n_d-1}^{(-1)}(t-\theta)^{n_d}\Omega^{\underline{n}}L_{\underline{n}'} + \Omega^{\underline{n}}L_{\underline{n}}\right)^{p^e} \\
&= \Omega^{p^e\underline{n}}L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e} + (t-\theta)^{p^en_d}\Omega^{p^e\underline{n}}\left(H_{p^en_d-1}^{(-1)}L_{p^e\underline{n}'} - \gamma_{e,\underline{n}'}\gamma_{e,n_d}(H_{n_d-1}^{(-1)})^{p^e}L_{\underline{n}'}^{p^e}\right) \\
&= \Omega^{p^e\underline{n}}L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e} + (t-\theta)^{p^en_d}\Omega^{p^e\underline{n}}L_{p^e\underline{n}'}\left(H_{p^en_d-1}^{(-1)} - \gamma_{e,n_d}(H_{n_d-1}^{(-1)})^{p^e}\right) \\
&= \Omega^{p^e\underline{n}}L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e} + (t-\theta)^{p^en_d}\Omega^{p^e\underline{n}}L_{p^e\underline{n}'}\left(H_{p^en_d-1}^{(-1)} - \gamma_{e,n_d}^{(-1)}(H_{n_d-1}^{(-1)})^{p^e}\right) \\
&= \Omega^{p^e\underline{n}}L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e}.
\end{aligned}$$

Thus $\Omega^{p^e\underline{n}}L_{p^e\underline{n}} = \gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e} + c$ for some $c \in \mathbb{F}_q(t)$. Then we have

$$\begin{aligned}
(\gamma_{e,\underline{n}}(\Omega^{\underline{n}}L_{\underline{n}})^{p^e})(\theta) &= \frac{\Gamma_{p^en_1}}{\Gamma_{n_1}^{p^e}}\cdots\frac{\Gamma_{p^en_d}}{\Gamma_{n_d}^{p^e}}\left(\frac{\Gamma_{n_1}\cdots\Gamma_{n_d}\zeta(\underline{n})}{\widetilde{\pi}^{n_1+\cdots+n_d}}\right)^{p^e} \\
&= \frac{\Gamma_{p^en_1}\cdots\Gamma_{p^en_d}\zeta(p^e\underline{n})}{\widetilde{\pi}^{p^en_1+\cdots+p^en_d}} \\
&= (\Omega^{p^e\underline{n}}L_{p^e\underline{n}})(\theta).
\end{aligned}$$

Therefore we have $c(\theta) = 0$, and hence $c = 0$. □

Lemma 4.19. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index and $\Psi \in \mathrm{GL}_{d+1}(\mathbb{L})$ a matrix defined in Example 3.5 for $\underline{u} = H(\underline{n})$. Take $(i,j),(k,\ell) \in I_d$ such that $\underline{n}_{ij} = p^e \underline{n}_{k\ell}$ for some non-negative integer $e \geq 0$. Then we have*

$$\widetilde{\Psi}_{ij}/\widetilde{\Psi}_{ii} = \gamma_{e,\underline{n}_{k\ell}}(\widetilde{\Psi}_{k\ell}/\widetilde{\Psi}_{kk})^{p^e}.$$

Proof. By Example 3.5, we have

$$\begin{aligned}
\widetilde{\Psi}_{ij}/\widetilde{\Psi}_{ii} &= \sum_{s=j}^{i}\sum_{r=0}^{i-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=i}} L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{ij}} L_{\underline{n}_{sj}} \\
&= \sum_{s=\ell}^{k}\sum_{r=0}^{k-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=k}} L_{p^e\underline{n}_{i_1 i_0}} \cdots L_{p^e\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{p^e\underline{n}_{k\ell}} L_{p^e\underline{n}_{s\ell}} \\
&= \sum_{s=\ell}^{k}\sum_{r=0}^{k-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=k}} \gamma_{e,\underline{n}_{i_1 i_0}} \cdots \gamma_{e,\underline{n}_{i_r i_{r-1}}} \gamma_{e,\underline{n}_{s\ell}} (L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{k\ell}} L_{\underline{n}_{s\ell}})^{p^e} \\
&= \gamma_{e,\underline{n}_{k\ell}} \left(\sum_{s=\ell}^{k}\sum_{r=0}^{k-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=k}} L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{k\ell}} L_{\underline{n}_{s\ell}} \right)^{p^e} \\
&= \gamma_{e,\underline{n}_{k\ell}}(\widetilde{\Psi}_{k\ell}/\widetilde{\Psi}_{kk})^{p^e}.
\end{aligned}$$

□

Proof of Theorem 2.23. We use the notations of the proof of Theorem 4.16. By Lemma 4.19, $G(k,\ell)$ is an algebraic subgroup of

$$\left\{ [a] \oplus \bigoplus_{\substack{\mathrm{dep}(i,j)\geq \mathrm{dep}(k,\ell)-1 \\ (i,j)\leq(k,\ell)}} \begin{bmatrix} a^{n_j+\cdots+n_d} & & & \\ x_{j+1,j} & \ddots & & \\ \vdots & \ddots & \ddots & \\ x_{ij} & \cdots & x_{i,i-1} & a^{n_i+\cdots+n_d} \end{bmatrix} \;\middle|\; \begin{array}{c} x_{rs}/x_{rr} = \gamma_{e,\underline{n}_{vw}}(x_{vw}/x_{vv})^{p^e} \\ \text{for } \ (r,s),(v,w) \in I_d, e \geq 1 \\ \text{such that } \ \underline{n}_{rs} = p^e \underline{n}_{vw} \end{array} \right\}$$

for each (k,ℓ). By Theorem 3.1, it suffices to show that this inclusion is actually an equality for each (k,ℓ). We prove this by induction on $(k,\ell) \in I_d$ with respect to the total order "$\leq$".

By Theorem 2.4, this is true for $(2,1) \leq (k,\ell) \leq (d+1,d)$, the depth one cases. Let $(k,\ell) \geq (3,1)$ (this means $\mathrm{dep}(k,\ell) \geq 2$) and assume that the inclusion is an equality for (k',ℓ') the greatest element of $\{(i,j) \in I_d | (i,j) < (k,\ell)\}$. We know that $\dim G(k',\ell') \leq \dim G(k,\ell) \leq \dim G(k',\ell') + 1$. It suffices to show that the second inequality is an equality.

Now, assume that $\dim G(k,\ell) = \dim G(k',\ell')$. We shall induce a contradiction. The strategy is the same as the proof of Theorem 4.16 except that a_{ij}'s and x_{ij}'s may not be independent. By the same arguments, the $\ell-1$ equalities

$$\left\{\begin{array}{c} a_{k-\ell,1}x_{k-\ell+1,k-\ell}-a_{k-\ell+1,2}x_{21}=0 \\ a_{k-\ell+1,2}x_{k-\ell+2,k-\ell+1}-a_{k-\ell+2,3}x_{32}=0 \\ a_{k-\ell+2,3}x_{k-\ell+3,k-\ell+2}-a_{k-\ell+3,4}x_{43}=0 \\ \vdots \\ a_{k-2,\ell-1}x_{k-1,k-2}-a_{k-1,\ell}x_{\ell,\ell-1}=0 \end{array}\right. \tag{4.1}$$

imply the equality

$$a_{k-1,\ell}x_{k,k-1}-a_{k,\ell+1}x_{\ell+1,\ell}=0. \tag{4.2}$$

When the equivalence class of $(k,\ell+1)$ has one element, we can take $a_{k-\ell,1}=\cdots=a_{k-1,\ell}=0$ and $a_{k,\ell+1}x_{\ell+1,\ell}\neq 0$ by the induction hypothesis. Then the equalities (4.1) hold, and hence we have the equality (4.2) $a_{k,\ell+1}x_{\ell+1,\ell}=0$. This is a contradiction. When the equivalence class of $(k,\ell+1)$ has two elements, then $\mathrm{dep}(k,\ell+1)=1$ by the assumption. We set $a_j:=a_{j+1,j}$ and $x_j:=x_{j+1,j}$. Then the equalities (4.1) become

$$\left\{\begin{array}{c} a_1x_2-a_2x_1=0 \\ a_2x_3-a_3x_2=0 \\ a_3x_4-a_4x_3=0 \\ \vdots \\ a_{\ell-1}x_\ell-a_\ell x_{\ell-1}=0 \end{array}\right. \tag{4.3}$$

and the equality (4.2) becomes

$$a_\ell x_{\ell+1}-a_{\ell+1}x_\ell=0. \tag{4.4}$$

There exists $1\leq w\leq \ell$ such that $w\sim \ell+1$. This means that there exists a non-zero integer $e\neq 0$ such that $n_w=p^e n_{\ell+1}$ and hence we have the relations

$$a_w=\gamma_{e,n_{\ell+1}}a_{\ell+1}^{p^e} \quad\text{and}\quad x_w=\gamma_{e,n_{\ell+1}}x_{\ell+1}^{p^e},$$

where we set $\gamma_{e,n}:=(\gamma_{-e,p^e n})^{-p^e}$ if $e<0$ and n is divisible by p^{-e}.

If $1\leq w\leq \ell-2$, then we can take

$$a_1=\cdots=a_{w-1}=a_{w+1}=\cdots=a_\ell=0,\quad x_{w-1}=x_{w+1}=0 \quad\text{and}\quad a_{\ell+1}x_\ell\neq 0$$

by the induction hypothesis (if $w=1$, then we ignore x_{w-1}). Then the equalities (4.3) hold and the equality (4.4) does not hold. This is a contradiction.

If $w=\ell-1$, then the last equality of the equalities (4.3) becomes

$$\gamma_{e,n_{\ell+1}}(a_{\ell+1}^{p^e}x_\ell-a_\ell x_{\ell+1}^{p^e})=0$$

and we can take

$$a_1=\cdots=a_{\ell-2}=0,\quad x_{\ell-2}=0,\quad x_\ell,x_{\ell+1}\neq 0 \quad\text{and}\quad \left(\frac{a_{\ell+1}}{x_{\ell+1}}\right)^{p^e}=\frac{a_\ell}{x_\ell}\notin\mathbb{F}_{p^{|e|}}$$

by the induction hypothesis. Then the equalities (4.3) hold and the equality (4.4) does not hold. This is a contradiction.

If $w=\ell$, then the equality (4.4) becomes

$$\gamma_{e,n_{\ell+1}}(a_{\ell+1}^{p^e}x_{\ell+1}-a_{\ell+1}x_{\ell+1}^{p^e})=0$$

and we can take

$$a_1 = \cdots = a_{\ell-1} = 0, \quad x_{\ell-1} = 0, \quad x_{\ell+1} \neq 0 \quad \text{and} \quad \frac{a_{\ell+1}}{x_{\ell+1}} \notin \mathbb{F}_{p^{|e|}}$$

by the induction hypothesis. Then the equalities (4.3) hold and the equality (4.4) does not hold. This is a contradiction. $\square$

Proof of Theorem 2.24. We may assume that $d = 3$. We use the notations of the proofs of Theorems 4.16 and 2.23. By Theorem 3.1, it suffices to show that the inclusion from $G(k, \ell)$ to the algebraic group defined in the beginning of the proof of Theorem 2.23 is actually an equality for each (k, ℓ).

By Theorem 2.4, this is true for $(2, 1) \leq (k, \ell) \leq (4, 3)$, the depth one cases. Let $(k, \ell) = (3, 1)$ (resp. $(4, 1)$) and assume that the inclusion is an equality for $(4, 3)$ (resp. $(4, 2)$). Assume that $\dim G(k, \ell) = \dim G(4, 3)$ (resp. $\dim G(4, 2)$). Since $\ell = 1$, the equality (4.2) always holds. We can check easily that this is a contradiction even if $(k - 1, \ell) \sim (k, \ell + 1)$ (and hence $(\ell + 1, \ell) \sim (k, k - 1)$).

Let $(k, \ell) = (4, 2)$, and assume that $(4, 2) \not\sim (3, 1)$ and $\dim G(4, 2) = \dim G(3, 1)$. In this case, the equality $a_1x_2 - a_2x_1 = 0$ implies the equality $a_2x_3 - a_3x_2 = 0$. We may assume that $1 \sim 2 \sim 3$, otherwise we obtain a contradiction from Theorems 4.16 and 2.23. For each j, we have $n_j = p^{e_j}n$ for some $n \geq 1$ and $e_j \geq 0$ with $\min\{e_j\} = 0$. We set $a := a_{j_0}$ and $x := x_{j_0}$ for some j_0 such that $e_{j_0} = 0$. Thus we have $a_j = \gamma_{e_j,n}a^{p^{e_j}}$ and $x_j = \gamma_{e_j,n}x^{p^{e_j}}$ for each j. Then

$$a^{p^{e_1}}x^{p^{e_2}} - a^{p^{e_2}}x^{p^{e_1}} = 0 \text{ implies } a^{p^{e_2}}x^{p^{e_3}} - a^{p^{e_3}}x^{p^{e_2}} = 0$$

for any $a, x \in \overline{\mathbb{F}_q(t)}$. Since $e_1 \neq e_2$, we conclude that $e_1 - e_2$ divide $e_2 - e_3$. By symmetric arguments, since $e_2 \neq e_3$, we conclude that $e_3 - e_2$ divide $e_2 - e_1$ This means that $e_1 - e_2 = \pm(e_2 - e_3)$. However this is a contradiction because we assume that $(4, 2) \not\sim (3, 1)$ and $e_1 \neq e_3$. $\square$

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