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Asymptotic Analysis of Laplace Integrals by use of Newton Polyhedra

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Preface

The purpose of this paper is to investigate the asymptotic behavior of *Laplace integrals* defined by

$$L(t;\varphi) := \int_{\mathbb{R}^n} e^{-tf(x)}\varphi(x)dx. \tag{1}$$

Here, f is a real-valued smooth function, φ is a complex-valued smooth function which has a sufficiently small support and t is a real parameter. Functions f and φ are called the *phase* and the *amplitude*, respectively. When $n = 1$, the Laplace integral can be regarded a generalization of the two-sided Laplace transform, which is defined by

$$(\mathcal{L}\varphi)(t) = \int_{-\infty}^{\infty} e^{-tx}\varphi(x)dx.$$

About the Laplace transform, see also §7.1.

Before investigating Laplace integrals, let us consider the asymptotic behavior of *oscillatory integrals* defined by

$$I(t;\varphi) := \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)dx. \tag{2}$$

Here, f is a smooth real-valued function, φ is a smooth complex-valued function and t is a real parameter. Also in this case, functions f and φ are called the phase and the amplitude, respectively. When $n = 1$, the oscillatory integral can be regarded as a kind of generalization of the Fourier transform of φ, which is expressed as

$$(\mathcal{F}\varphi)(t) = \int_{-\infty}^{\infty} e^{-itx}\varphi(x)dx.$$

We always assume f has some critical points (in particular, $\nabla f(0) = 0$), otherwise $I(t;\varphi) = \mathsf{O}\left(t^{-N}\right)$ for any $N \in \mathbb{N}$ (see Propositions 3.1.2 and 3.1.4). Suppose that the support of the amplitude φ is sufficiently small and contains the origin. If the Hessian matrix is invertible, one can easily get the asymptotic expansion of $I(t;\varphi)$ by calculating. In this case, Morse's lemma gives that the phase f can be $f(x) = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$ by exchanging the variables.

When the Hessian matrix is not invertible, by using Hironaka's resolution of singularities, the asymptotic expansion of $I(t;\varphi)$:

$$I(t;\varphi) \sim e^{itf(0)} \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha}(\log t)^{k-1}, \quad (\text{as } t \to \infty) \tag{3}$$

is obtained by P. Jeanquartier ([9], 1970) and B. Malgrange ([13], 1974) in the case when f is real-analytic. A. N. Varchenko achieved the oscillatory index and the multiplicity specifically, which are indices of t and $\log t$ in the leading term ([21], 1976).

Recently, J. Kamimoto and T. Nose [11] gave a generalization of the work of A. N. Varchenko. One of their themes is to extend the condition of the phase f. They innovated a class of $C^\infty(U)$ functions which is denoted by $\hat{\mathcal{E}}(U)$. Roughly speaking, every element of $\hat{\mathcal{E}}(U)$ can be expressed as a product of a polynomial and a smooth function. In fact, $\hat{\mathcal{E}}(U)$ includes all real-analytic functions (see §2.3). The asymptotic expansion is same as (3) in such a case. One of essential tool is the Newton polyhedron, which is defined from the Taylor expansion of a smooth function. Details for local zeta functions are in Chapter Chapter 8.

On the other hand, the asymptotic behavior of $L(t;\varphi)$ as $t \to \infty$ is obtained by Arnold, Gusein-Zade and Varchenko in the case that the phase function f is real-analytic and so on (see [1] and Theorem 5.1.2):

$$L(t;\varphi) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha} (\log t)^{k-1}, \tag{4}$$

as $t \to \infty$, where each $C_{\alpha,k}(\varphi)$ is a constant which depends on φ and $\{\alpha\}$ belongs to finitely many arithmetic progressions.

The main theme of this paper is to get the asymptotic expansions of the Laplace integrals for $f \in \hat{\mathcal{E}}(U)$, which is analogous to the work of Kamimoto and Nose [11]. Also in this case, we can get the same expansion as (4) and the indices of t and $\log t$ in the leading term of the expansion specifically (see Theorem 5.2.1).

This paper is organized as following. We define Newton polyhedra (in Chapter 1) and the function class $\hat{\mathcal{E}}(U)$ (in Chapter 2) and observe some properties for oscillatory integrals in Chapter 3. In Chapter 4, we show fundamental propositions along [19]. In Chapter 5, we introduce the prior research and the main theme of this paper. To obtain more precise results, we study toric variables (in Chapter 6) and local zeta functions (in Chapter 7 and Chapter 8). Finnaly, in Chapter 9, we show the main theorem.

Notations and Symbols

The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, and the set of non-negative integers and real numbers are denoted by $\mathbb{Z}_+$, $\mathbb{R}_+$, respectively. The complex space $\mathbb{C}$ can be regarded as the 2-dimensional real space $\mathbb{R}^2$. For $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ mean the real part and the imaginally part of z, respectively.

The standard inner product on $\mathbb{R}^n$ is expressed as $\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n$ and the standard norm of $\mathbb{R}^n$ is $\|x\| := \sqrt{\langle x, x\rangle} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$. The symbol $B(a, r)$ means an open ball in $\mathbb{R}^n$ whose center is $a \in \mathbb{R}^n$ and radius is $r > 0$, that is, $B(a, r) := \{x \in \mathbb{R}^n;\ \|x - a\| < r\}$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, define $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\langle\alpha\rangle := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$ and

$$\partial^\alpha f := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f.$$

Sometimes we write $\partial^\alpha = \partial_x^\alpha$.

Let A, B be subsets in $\mathbb{R}^n$ and c be a non-zero real constant. Then $A + B := \{a + b \in \mathbb{R}^n;\ a \in A,\ b \in B\}$, $c \cdot A := \{ca \in \mathbb{R}^n;\ a \in A\}$. In particular, when $A = \{a\}$, $A + B = a + B = \{a + b \in \mathbb{R}^n;\ b \in B\}$. For a finite set A, we write the number of elements of the set A with $\#A$.

For a C^∞ function f on $\mathbb{R}^n$, we call the set $\overline{\{x \in \mathbb{R}^n;\ f(x) \neq 0\}}$ the *support* of f, and write it with $Supp(f)$. Here $\overline{A}$ means the closure of the set A in the sense of the standard topology in $\mathbb{R}^n$. If the support of f is compact, we write $f \in C_0^\infty(\mathbb{R}^n)$.

Some symbols used in this paper will be found in P.69.

Contents

Chapter 1 Newton Polyhedra

In this chapter, we define the Newton polyhedra of C^∞ functions and see some examples. First, we recall inportant concepts about convex rational polyhedra. After that, we define the γ-part of f in §1.3.

§1.1 Polyhedra

For $(a, l) \in \mathbb{R}^n \times \mathbb{R}$, let $H(a, l)$ and $H^+(a, l)$ be a hyperplane and a closed half-space in $\mathbb{R}^n$ defined by

$$\begin{aligned} H(a,l) &:= \{x \in \mathbb{R}^n;\ \langle a, x\rangle = l\}, \\ H^+(a,l) &:= \{x \in \mathbb{R}^n;\ \langle a, x\rangle \geq l\}, \end{aligned}$$

respectively. It is clear that $H^+(a, l)$ is a convex set in $\mathbb{R}^n$ and $H(a, l)$ is the topological boundary of $H^+(a, l)$ unless $a = 0$.

Definition 1.1.1

$P \subset \mathbb{R}^n$ is called a *(convex rational) polyhedron* if P is expressed as an intersection of some closed half-spaces, that is,

$$P = \bigcap_{j=1}^{N} H^+(a^j, l_j), \tag{1.1}$$

for $(a^j, l_j) \in \mathbb{Z}^n \times \mathbb{Z}$ $(j = 1, \ldots, N)$.

Definition 1.1.2

A pair $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$ is *valid* for P if P is contained in $H^+(a, l)$. A set $\gamma \subset P$ is called a *face* if $\gamma = H^+(a, l) \cap P$ for some valid pairs $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$.

Remark 1.1.3

(i) Since $\mathbb{R}^n = H^+(0, 0)$, $\mathbb{R}^n$ is a polyhedron and valid for any polyhedron. Thus, P is a trivial face, and the other faces are called *proper faces.*

(ii) The pair $(0, -1)$ is valid for any polyhedron and $H(0, -1) \cap P = \varnothing$. This implies that the empty set is also a face of the polyhedron P.

(iii) Every pair (a^j, l_j) in (1.1) is valid for P.

From the definitions above, we can easily know every proper face γ is contained in

$$\bigcap_{j=1}^{M} H(a^j, l_j) \tag{1.2}$$

for some $\{(a^j, l_j) \in \mathbb{Z}^n \times \mathbb{Z}\}$.

Definition 1.1.4

The dimension of the face γ is defined by the intersection of all affine flats such as (1.2) and denoted by $dim(\gamma)$. The face whose dimension is 0, 1, $dim(P) - 1$ is called *vertex*, *edge*, *facet*, respectively.

Notice $dim(\gamma) \leq n - M$, where M is as in (1.2).

In this paper, we always consider tha case that the dimension of the polyhedron P equals to the dimension n. In this case, the dimension of a facet of P is $n - 1$. In particular, when $n = 2$, the only facet is P.

§1.2 Definition of Newton Polyhedra

Definition 1.2.1

Let U be an open neighborhood of the origin and f be a real-valued C^∞ function on U. Write the Taylor series of f around the origin by

$$\tilde{f}(x) := \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha, \tag{1.3}$$

where $c_\alpha = \partial^\alpha f(0)/\alpha!$. The *Newton polyhedron* $\Gamma_+(f)$*Newtonpolyhedron* of f is defined as the convex hull of the set

$$\mathfrak{D} := \bigcup_{c_\alpha \neq 0} (\alpha + \mathbb{R}_+^n),$$

where c_α is as in (1.3). In other words, the Newton polyhedron of f is the smallest set of all convex sets which contain the set $\mathfrak{D}$. The union of all compact faces of $\Gamma_+(f)$ is called *Newton diagram* $\Gamma(f)$ of f. We define the *principal part* of f by

$$f_*(x) := \sum_{\alpha \in \Gamma(f) \cap \mathbb{Z}_+^n} c_\alpha x^\alpha.$$

Immediately we can know $\Gamma_+(f) = \Gamma_+(f_*)$. It has been already known that the Newton polyhedron $\Gamma_+(f)$ of f is a polyhedron. (See [22].)

In this paper, we usually assume that f is *nonflat*, that is, $\Gamma_+(f) \neq \varnothing$. For example, $f_1(x) := 0$ and $f_2(x) := x_1 \cdot \exp(-x_2^2)$ are flat, but every non-zero polynomial is nonflat. In particular, the Newton polyhedron of $f \equiv 1$ is $\mathbb{R}_+^n$.

Definition 1.2.2

Define the *Newton distance* of $\Gamma_+(f)$ by $d(f) := \min\{t \geq 0;\ (t, \ldots, t) \in \Gamma_+(f)\}$, the *principal face* τ_* as the smallest face of $\Gamma_+(f)$ that contains $q_* := (d(f), \ldots, d(f))$ and the *Newton multiplicity* $m(f)$ of $\Gamma_+(f)$ as the codimension of τ_*.

A smooth function f is called *convenient* if the Newton polyhedron $\Gamma_+(f)$ of f intersects all coordinate axes.

For instance, when $n=2$, $f_1(x) := x_1^2 + x_2^4$ is convenient, however, $f_2(x) := x_1^2 + x_2^4 \cdot \exp(-1/x_1^2)$ is not convenient.

If q_* is a vertex of the Newton polyhedron $\Gamma_+(f)$, then $\tau_* = q_*$ and $m(f) = n$. From the definitions above, we can immediately get that τ_* is compact if f is convenient. However, the converse does not hold, that is, f is not always convenient even though the principal compact.

§1.3 The γ-part of f

The function $f \in C^\infty(U)$ is called *real-analytic* at $x=0$ if

$$f(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha$$

on a sufficiently small open neighborhood of the origin, where c_α is as in (1.3). If the same property is satisfied at every point in U, f is called real-analytic on U. Denote by $C^\omega(U)$ the set of all real-analytic functions on U.

Let U be an open neighborhood of the origin. When γ is compact or f is real-analytic on U, the γ-part of f is defined by

$$\tilde{f}_\gamma(x) := \sum_{\alpha \in \gamma} c_\alpha x^\alpha, \tag{1.4}$$

where each c_α is same as in (1.3). Notice that this is well-defined on U. However, this definition can't be generalized in the case when γ is noncompact or f is not real-analytic. Therefore, we must look for another definition of the γ-part.

From now, we always assume that U is an open neighborhood of the origin in $\mathbb{R}^n$, a function f is real-valued and C^∞ on U and a nonempty polyhedron P satisfies $P + \mathbb{R}_+^n \subset P \subset \mathbb{R}_+^n$, unless otherwise stated.

Definition 1.3.1

Let γ be a nonempty face of P. If for any $x \in U$, the limitation

$$\lim_{t \to 0} \frac{f(t^{a_1} x_1, \ldots, t^{a_n} x_n)}{t^l} \tag{1.5}$$

exists whenever $(a,l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ is a valid pair for P defining γ (that is, $H(a,l) \cap P = \gamma$), then we say that f *admits the γ-part* on U. For each $x \in U$, we denote the limitation (1.5) by $f_\gamma(x)$.

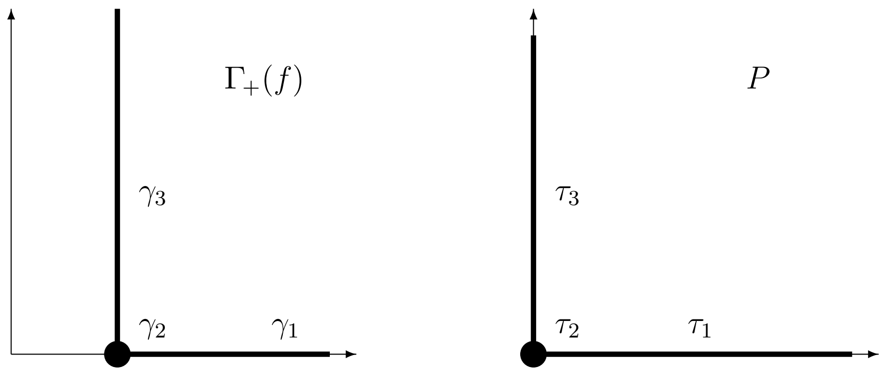

Figure 1: The Newton polyhedron $\Gamma_+(f)$ [left] and the polyhedron P [right]

Since $H(0,0) = \mathbb{R}^n$, for any C^∞ function f, every polyhedron P has the P-part and $f_P(x)$ is equal to $f(x)$. If γ is contained in some coordinate plane, then $l = 0$ and the limitation (1.5) exists.

In Proposition 2.2.5, we show that f always admits the γ-part and $f_\gamma(x)$ is equal to (1.4) if γ is compact. On the other hand, for a noncompact face γ, f does not always admit the γ-part (see the example below).

Example 1.3.2

Assume $n = 2$ and define $f_k(x_1, x_2) := x_1 + x_1^k \cdot \exp(-1/x_2^2)$ for $k \in \mathbb{Z}_+$. Then the Newton polyhedron of f_k is $\Gamma_+(f_k) = (1,0) + \mathbb{R}_+^2$ for each k and has three proper faces

$$
\begin{aligned}
\gamma_1 &:= \{(\alpha_1, 0);\ \alpha_1 \geq 1\}, \\
\gamma_2 &:= \{(1, 0)\} \text{ and} \\
\gamma_3 &:= \{(1, \alpha_2);\ \alpha_2 \geq 0\}.
\end{aligned}
$$

On the other hand, $P = \mathbb{R}_+^2$ also has three proper faces

$$
\begin{aligned}
\tau_1 &:= \{(\alpha_1, 0);\ \alpha_1 \geq 0\}, \\
\tau_2 &:= \{(0, 0)\} \text{ and} \\
\tau_3 &:= \{(0, \alpha_2);\ \alpha_2 \geq 0\}.
\end{aligned}
$$

Shapes of these polyhedra and their faces are seen in Figure 1. The γ_j-parts and τ_j-parts of f_k ($j = 1, 2, 3$ and $k \in \mathbb{Z}_+$) are as below.

<table>
<tr><th></th><th>$(f_k)_{\gamma_1}(x)$</th><th>$(f_k)_{\tau_1}(x)$</th><th>$(f_k)_{\gamma_2}(x)$</th><th>$(f_k)_{\tau_2}(x)$</th><th>$(f_k)_{\gamma_3}(x)$</th><th>$(f_k)_{\tau_3}(x)$</th></tr>
<tr><td>$k = 0$</td><td colspan="2" rowspan="3">x_1</td><td rowspan="3">x_1</td><td rowspan="3">0</td><td>**Not Defined!**</td><td>$\exp(-1/x_2^2)$</td></tr>
<tr><td>$k = 1$</td><td>$f_1(x_1, x_2)$</td><td rowspan="2">0</td></tr>
<tr><td>$k \geq 2$</td><td>x_1</td></tr>
</table>

Chapter 2 The class $\hat{\mathcal{E}}(U)$

In this chapter, we always assume that U is an open neighborhood of the origin in $\mathbb{R}^n$, a function f is smooth on U and $P \subset \mathbb{R}^n_+$ is a convex rational polyhedron which contains the Newton polyhedron $\Gamma_+(f)$.

§2.1 The definition of $\hat{\mathcal{E}}(U)$

Definition 2.1.1

Let U be an open neighborhood of the origin in $\mathbb{R}^n$ and P be a convex rational polyhedron satisfying $P + \mathbb{R}^n_+ \subset P \subset \mathbb{R}^n_+$. Then

$\mathcal{E}[P](U) := \{f \in C^\infty(U);\ \Gamma_+(f) \subset P\}$,

$\hat{\mathcal{E}}[P](U) := \{f \in \mathcal{E}[P](U);\ f$ admits the γ-part on U for any face $\gamma \subset P\}$ $(P \neq \varnothing)$,

$\hat{\mathcal{E}}[\varnothing](U) := \{0\}$,

$\hat{\mathcal{E}}(U) := \{f \in C^\infty(U);\ f \in \hat{\mathcal{E}}[\Gamma_+(f)](U)\}$.

In this paper, we call $\hat{\mathcal{E}}(U)$ the *Kamimoto-Nose class* and sometimes write it by "K-N class". The K-N class $\hat{\mathcal{E}}(U)$ contains $C^\omega(U)$ (see Proposition 2.2.6). The Denjoy-Carleman quasianalytic class, which is referred in [2], [20], [11] and so on, contains all real-analytic functions and is contained in the K-N class $\hat{\mathcal{E}}(U)$ (see [11]).

Remark 2.1.2

If $P = \mathbb{R}^n_+$, the polyhedron P always contains the Newton polyhedron of any $f \in C^\infty(U)$. Accordingly, $\mathcal{E}[\mathbb{R}^n_+](U) = \hat{\mathcal{E}}[\mathbb{R}^n_+](U) = C^\infty(U)$.

Remark 2.1.3

Consider the case when $n = 1$ and $P = [p, \infty)$ for $p \in \mathbb{Z}_+$. Since $\Gamma_+(f)$ is contained in P, every function in $\mathcal{E}[P](U)$ has a non-zero k-th derivative for some $k \geq p$ unless $f \equiv 0$, that is, $\mathcal{E}[P](U) = \hat{\mathcal{E}}[P](U) = \{x^\alpha \psi(x);\ \alpha \in \mathbb{Z}_+$ with $\alpha \geq p,\ \psi \in C^\infty(U)\} \cup \{0\}$.

From this fact, we can easily specify the K-N class as $\hat{\mathcal{E}}(U) = \{x^\alpha \psi(x);\ \alpha \in \mathbb{Z}_+,\ \psi \in C^\infty(U)$ with $\psi(0) \neq 0\} \cup \{0\}$. In other words, the K-N class is the set of all nonflat smooth functions and $f \equiv 0$.

Example 2.1.4

Let $n = 2$.

(i) Suppose $P = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2_+;\ \alpha_1 + \alpha_2 \geq 2\}$. Define functions f and g as $f(x_1, x_2) := x_1^2 + x_2^2$, $g(x_1, x_2) := x_1^2 + x_1 + x_2^2$. Then f belongs to $\hat{\mathcal{E}}[P](U)$. But g does not belong to $\hat{\mathcal{E}}[P](U)$, because

$$\frac{\partial g}{\partial x_1}(0) = 1 \neq 0.$$

One can prove that those functions f and g above belong to $\hat{\mathcal{E}}(U)$. In fact, all real-analytic functions belong to $\hat{\mathcal{E}}(U)$. (See the next section.)
(ii) The function f_k, which is defined in Example 1.3.2, belongs to $\hat{\mathcal{E}}(U)$ if $k \geq 1$. This is because the Newton polyhedron of f_k is expressed as $\Gamma_+(f_k) = \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_+^2;\ \alpha_1 \geq 1\}$.

From the definition of $\hat{\mathcal{E}}(U)$, it is trivial

$$\hat{\mathcal{E}}(U) \subset \bigcup_{P \subset \mathbb{R}_+^n:\text{polyhedron}} \hat{\mathcal{E}}[P](U).$$

But, in general, $\hat{\mathcal{E}}(U)$ is a proper subset of $\cup \hat{\mathcal{E}}[P](U)$. To know the properties of $\hat{\mathcal{E}}(U)$, we define another function class $\check{\mathcal{E}}[P](U)$ and show

$$\hat{\mathcal{E}}(U) = \bigcup_{\varnothing \neq P \subset \mathbb{R}_+^n:\text{polyhedron}} \check{\mathcal{E}}[P](U) \tag{2.1}$$

in §2.3. In order to achieve the purpose, we will analyze the phase function and its γ-part in the next section.

§2.2 Analysis of the phase function

Definition 2.2.1

We denote by $\mathfrak{P}(\{1, \dots, n\})$ the set of all subsets in $\{1, \dots, n\}$. For $I \in \mathfrak{P}(\{1, \dots, n\})$, define the mapping T_I^r as

$$(y_1, \dots, y_n) = T_I^r(x_1, \dots, x_n),$$

where y_j equals to r for $j \in I$ and otherwise $y_j = x_j$.

In particular, we denote by T_I when $r = 0$.

For example, let $n = 3$, $I = \{3\}$ and $U = \{x \in \mathbb{R}^3;\ x_1^2 + x_2^2 + x_3^2 < 1\}$, then $T_I(U) = \{y \in \mathbb{R}^3;\ y_1^2 + y_2^2 < 1 \text{ and } y_3 = 0\}$. This is the image of the projection from $\mathbb{R}^3$ to the x_1x_2-plain.

Remark 2.2.2

Let V be an element of $\mathfrak{P}(\{1, \dots, n\})$, $W := \{1, \dots, n\} \setminus V$. Then $T_V(x) + T_W(x) = x$ for all $x \in \mathbb{R}^n$.

The definition above gives the proof of this fact.

Lemma 2.2.3

Let $V \in \mathfrak{P}(\{1,\ldots,n\})$, $W := \{1,\ldots,n\} \setminus V$, N be a natural number, $A_V(N) := \{\alpha \in T_V(\mathbb{Z}_+^n);\ \langle\alpha\rangle < N\}$ and $B_V(N) := \{\alpha \in T_V(\mathbb{Z}_+^n);\ \langle\alpha\rangle = N\}$. Then one can get

$$f(x) = \sum_{\alpha \in A_V(N)} \frac{1}{\alpha!}(\partial^\alpha f)(T_W(x))x^\alpha + \sum_{\alpha \in B_V(N)} R_\alpha(x)x^\alpha,$$

for any $x \in U$, where

$$R_\alpha(x) = \frac{N}{\alpha!}\int_0^1 (1-t)^{N-1}(\partial^\alpha f)(tT_V(x) + T_W(x))dt.$$

Proof.

Let δ be a positive constant and ψ be a smooth function on the interval $(-\delta, 1+\delta)$. Integrating by parts, we have

$$\begin{aligned}
\psi(1) &= \psi(0) + \int_0^1 \psi'(t)dt \\
&= \cdots \\
&= \sum_{k=0}^{N-1} \frac{\psi^{(k)}(0)}{k!} + \frac{1}{(N-1)!}\int_0^1 (1-t)^{N-1}\psi^{(N)}(t)dt.
\end{aligned}$$

Putting $\psi(t) = f(tT_V(x) + T_W(x))$, one can get the conclusion. □

From now, we consider the γ-part of f. First, we prepare the following sets of integers $V(\gamma)$ and $W(\gamma)$.

Definition 2.2.4

Let $e_j = (e_1^{[j]}, \ldots, e_n^{[j]})$ be a unit vector with $e_k^{[j]} = \delta_{j,k}$, where $\delta_{j,k}$ is Kronecker's delta. Suppose that γ be a proper face of P. Then we define the sets $V(\gamma)$ and $W(\gamma)$ by $V(\gamma) := \{j \in \{1,\ldots,n\};\ \gamma + \mathbb{R}_+ e_j \subset \gamma\}$ and $W(\gamma) := \{1,\ldots,n\} \setminus V$.

By the definition above, it is obvious that $V(\gamma) = \varnothing$ if and only if γ is compact.

For an arbitrary valid pair (a,l) defining γ, $V(\gamma)$ is the set of j with $a_j = 0$. This fact implies that $W(\gamma) = \{j \in \{1,\ldots,n\};\ a_j \neq 0\}$, and thus, sets $\{j \in \{1,\ldots,n\};\ a_j = 0\}$ and $\{j \in \{1,\ldots,n\};\ a_j \neq 0\}$ are independent of the vector $a \in \mathbb{R}^n$, only depend on γ.

Hereafter in this section, $V(\gamma)$ and $W(\gamma)$ are simply denoted by V and W, respectively. Then $T_W(t^{a_1}x_1, \ldots, t^{a_n}x_n) = T_W(x)$ for $t \in [0,1]$ and $x \in U$, where (a,l) is a valid pair defining γ and Lemma 2.2.3 yields that

$$f(t^{a_1}x_1, \ldots, t^{a_n}x_n) = \sum_{\alpha \in A_V(N)} \frac{1}{\alpha!}(\partial^\alpha f)(T_W(x))x^\alpha t^{\langle a,\alpha\rangle}$$

$$+\sum_{\alpha\in B_V(N)} R_\alpha(t^{a_1}x_1,\dots,t^{a_n}x_n)x^\alpha t^{\langle a,\alpha\rangle}, \tag{2.2}$$

where N is an integer such that $H(a,l)\cap T_V(\mathbb{Z}_+^n)$ is contained in $A_V(N)$.

Proposition 2.2.5

For any compact face $\gamma\subset P$, f always admit the γ-part and

$$f_\gamma(x)=\sum_{\alpha\in\gamma\cap\mathbb{Z}_+^n} c_\alpha x^\alpha, \tag{2.3}$$

where each c_α is a coefficient of the Taylor expansion of f as in (1.3).

Proof.

Calculating the first term in the right-hand side of (2.2), we get

$$\begin{aligned}\sum_{\alpha\in A_V(N)}\frac{1}{\alpha!}(\partial^\alpha f)(T_W(x))x^\alpha t^{\langle a,\alpha\rangle} &= \sum_{\langle\alpha\rangle<N}\frac{1}{\alpha!}(\partial^\alpha f)(0)x^\alpha t^{\langle a,\alpha\rangle}\\ &= \sum_{\langle\alpha\rangle<N} c_\alpha x^\alpha t^{\langle a,\alpha\rangle}.\end{aligned}$$

Since the polyhedron P contains the Newton polyhedron of f, the coefficient c_α is equal to 0 if $\langle a,\alpha\rangle<l$ and $\langle a,\alpha\rangle-l$ is positive if $\langle\alpha\rangle=N$. Thus we obtain

$$\begin{aligned}\frac{f(t^{a_1}x_1,\dots,t^{a_n}x_n)}{t^l} &= \sum_{\langle\alpha\rangle<N} c_\alpha x^\alpha t^{\langle a,\alpha\rangle-l}+\sum_{\langle\alpha\rangle=N} R_\alpha(t^{a_1}x_1,\dots,t^{a_n}x_n)x^\alpha t^{\langle a,\alpha\rangle-l}\\ &\to \sum_{\langle\alpha\rangle<N \text{ with } \langle a,\alpha\rangle=l} c_\alpha x^\alpha,\end{aligned}$$

as $t\to 0$. □

From the proof above, we can immediately get that, if f admits the γ-part, then

$$f_\gamma(x)=\sum_{\alpha\in\gamma\cap T_V(\mathbb{Z}_+^n)} c_\alpha x^\alpha, \tag{2.4}$$

because $\langle a,\alpha\rangle>l$ for all $\alpha\in B_V(N)$ and $(\partial^\alpha f)(T_W(x))=0$ if $\langle a,\alpha\rangle<l$. Since $V=V(\gamma)=\varnothing$, the expression (2.4) holds for every compact face γ.

The γ-part f_γ is smooth on U. In addition, if f is real-analytic on U, the γ-part is also real-analytic on U. This fact is shown from the equation (2.4).

Proposition 2.2.6

If f is real-analytic on U, then f always admits the γ-part even if γ is not compact.

Proof.

Since f is real-analytic, $\partial^\alpha f$ is also real-analytic on U. If $\alpha \in A_V(N)$ satisfies $\langle a, \alpha \rangle < l$, then $(\partial^\alpha f)(T_W(x)) \equiv 0$ on U, because the Newton polyhedron of f is contained in P and (a, l) is valid. Similarly as above, we get the conclusion. □

This proposition implies that every real-analytic function f belongs to the class $\hat{\mathcal{E}}[P](U)$ while P contains $\Gamma_+(f)$. In particular, all real-analytic functions are elements of the class $\hat{\mathcal{E}}(U)$.

§2.3 Properties of $\hat{\mathcal{E}}(U)$

It is time-consuming to check whether f belongs to the K-N class. Hence, we consider an equivalent condition in this section. For that goal, we define a class $\check{\mathcal{E}}[P](U)$ and prove the relation (2.1).

Definition 2.3.1

Let $P \subset \mathbb{R}^n_+$ be a polyhedron.
(i) Denote by $\mathcal{S}[P]$ and $\mathcal{V}(P)$ the set of finite set in $P \cap \mathbb{Z}^n_+$ and the set of vertices of P, respectively.
(ii) Let U be a neighborhood of the origin. Denote by $\check{\mathcal{E}}[P](U)$ the set of functions $f \in \hat{\mathcal{E}}[P](U)$ which are expressed as

$$f(x) = \sum_{p \in S} x^p \psi_p(x), \tag{2.5}$$

where $\mathcal{V}(P) \subset S \in \mathcal{S}[P]$, $\psi_p \in C^\infty(U)$ and $\psi_p(0) \neq 0$ for $p \in \mathcal{V}(P)$.
(iii) Define the class $\tilde{\mathcal{E}}(U)$ by

$$\tilde{\mathcal{E}}(U) := \left\{ \sum_{p \in S} x^p \psi_p(x);\ S \in \mathcal{S}[\mathbb{R}^n_+], \psi_p \in C^\infty(U) \text{ with } \psi_p(0) \neq 0 \text{ for } p \in S \right\}.$$

If the set S in (ii) or (iii) is empty, we regard the summation

$$f(x) = \sum_{p \in S} x^p \psi_p(x)$$

as $f \equiv 0$. Indeed, the function $f \equiv 0$ is real-analytic on U, and thus, it belongs to the K-N class. However, if the phase function constantly equals to 0, then the Laplace integral is

$$L(t; \varphi) = \int_{Supp(\varphi)} \varphi(x) dx.$$

This is a complex constant which is independent of the parameter t.

Proposition 2.3.2

If f belongs to the K-N class, then f can be expressed as (2.5), where $P = \Gamma_+(f)$.

Proof.

Since $P = \Gamma_+(f)$ is a polyhedron, there exists a family $\{(a^j, l_j)\}_{j=1}^M$ such that

$$P = \bigcap_{j=1}^{M} H^+(a^j, l_j)$$

and $H(a^j, l_j) \cap P$ is a facet of P for every $j = 1, \ldots, M$. This fact is referred in [22].

Put $P_0 := \mathbb{R}_+^n$ and

$$P_k := P_0 \cap \left(\bigcap_{j=1}^{k} H^+(a^j, l_j) \right),$$

for $k = 1, \ldots, M$. Then $P_M = P$. Now we start to prove the lemma by induction with respect to k.

It is trivial for $k = 0$.

Suppose that the lemma holds for $k-1$ and show that it holds for k ($k = 1, \ldots, M$). From the assumption, f can be expressed as

$$f(x) = \sum_{p \in S_{k-1}} x^p \psi_{k-1,p}(x),$$

where $S_{k-1} \in \mathcal{S}[P_{k-1}]$ with $\mathcal{V}(P_{k-1}) \subset S_{k-1}$ and each $\psi_{k-1,p}$ is smooth on U. Let $\gamma_k := P \cap H(a^k, l_k)$, $V = V(\gamma_k)$ and $W = W(\gamma_k)$. Applying Lemma 2.2.3 to functions $\psi_{k-1,p}$, we get

$$\psi_{k-1,p}(x) = \sum_{\alpha \in A_V(N)} C_{p,\alpha}^{[k]}(T_W(x)) x^\alpha + \sum_{\alpha \in B_V(N)} R_{p,\alpha}^{[k]}(x) x^\alpha, \tag{2.6}$$

for each p, where $C_{p,\alpha}^{[k]}$ and $R_{p,\alpha}^{[k]}$ are smooth on U and N is an integer such that $H(a^k, l_k) \cap T_V(\mathbb{Z}_+^n) \subset A_V(N)$. Substituting (2.6) to (2.5), one can find

$$\begin{aligned} f(x) &= \sum_{p \in S_{k-1},\, \alpha \in A_V(N)} C_{p,\alpha}^{[k]}(T_W(x)) x^{p+\alpha} + \sum_{p \in S_{k-1},\, \alpha \in B_V(N)} R_{p,\alpha}^{[k]}(x) x^{p+\alpha} \\ &=: f_1(x) + f_2(x). \end{aligned}$$

Write $a^k = (a_1^{[k]}, \ldots, a_n^{[k]})$. The assumption $f \in \hat{\mathcal{E}}[P](U)$ follows that the function f admits the γ_k-part, that is, the following limitation exists:

$$\lim_{t \to 0} \frac{f(t^{a_1^{[k]}} x_1, \ldots, t^{a_n^{[k]}} x_n)}{t^{l_k}}.$$

Since $H(a^k, l_k) \cap T_V(\mathbb{Z}_+^n) \subset A_V(N)$, the set $B_V(N)$ is contained in $H^+(a^k, l_k) \cap T_V(\mathbb{Z}_+^n)$, and thus, $\langle a, p+\alpha\rangle = \langle a, p\rangle + \langle a, \alpha\rangle \geq 0 + l_k = l_k$ for $\alpha \in B_V(N)$. This inequality yields that the following limitation exists:

$$\lim_{t\to 0} \frac{f_2(t^{a_1^{[k]}}x_1, \ldots, t^{a_n^{[k]}}x_n)}{t^{l_k}} = \lim_{t\to 0} \sum_{p\in S_{k-1},\, \alpha\in B_V(N)} R_{p,\alpha}^{[k]}(t^{a_1^{[k]}}x_1, \ldots, t^{a_n^{[k]}}x_n)t^{\langle a, p+\alpha\rangle - l_k}x^{p+\alpha}.$$

Consequently,

$$\lim_{t\to 0} \frac{f_1(t^{a_1^{[k]}}x_1, \ldots, t^{a_n^{[k]}}x_n)}{t^{l_k}} = \lim_{t\to 0} \frac{f(t^{a_1^{[k]}}x_1, \ldots, t^{a_n^{[k]}}x_n)}{t^{l_k}} - \lim_{t\to 0} \frac{f_2(t^{a_1^{[k]}}x_1, \ldots, t^{a_n^{[k]}}x_n)}{t^{l_k}} \tag{2.7}$$

must have a limitation, because the limitations in the right-hand side exist. Since the left-hand side of (2.7) is expressed as

$$\lim_{t\to 0} \frac{f_1(t^{a_1^{[k]}}x_1, \ldots, t^{a_n^{[k]}}x_n)}{t^{l_k}} = \lim_{t\to 0} \sum_{p\in S_{k-1},\, \alpha\in A_V(N)} C_{p,\alpha}^{[k]}(T_W(x))t^{\langle a, p+\alpha\rangle - l_k}x^{p+\alpha},$$

the function $C_{p,\alpha}^{[k]} \equiv 0$ for each pair (p, α) with $\langle a, p+\alpha\rangle < l_k$, otherwise, the limitation does not exist. Now we get the expression (2.5).

Assume that there exists $p \in \mathcal{V}(P)$ such that $\psi_p(0) = 0$ or $\mathcal{V}(P)$ is not contained in S. Then $\Gamma_+(\sum x^p\psi_p) \subsetneq \Gamma_+(f)$, this is a contradiction. □

The proposition above holds even if $P \neq \Gamma_+(f)$. It is suffice that we suppose $f \in \hat{\mathcal{E}}[P](U)$. When we prove for $P \supset \Gamma_+(f)$, the condition "$\psi_p(0) \neq 0$ for $p \in \mathcal{V}(P)$" is not necessary. Conversely, if f has an expression as (2.5), then f belongs to the class $\hat{\mathcal{E}}[P](U)$.

Corollary 2.3.3

The set $\hat{\mathcal{E}}[P](U)$ can be written as follows:

$$\hat{\mathcal{E}}[P](U) = \left\{ \sum_{p\in S} x^p\psi_p(x);\ S \in \mathcal{S}[P] \text{ and } \psi_p \in C^\infty(U) \text{ for } p \in S \right\}.$$

Corollary 2.3.4

The relationship (2.1) holds.

These facts imply that every function which belongs to $\hat{\mathcal{E}}[P](U)$ or $\hat{\mathcal{E}}(U)$ is expressed as a product of a polynomial and a smooth function. Indeed, all polynomials are contained in such a function class, but some smooth function are not contained.

Remark 2.3.5

Applying the proposition above, we get the expression

$$f_\gamma(x) = \sum_{p\in\gamma\cap S} x^p\psi_p(T_{W(\gamma)}(x)), \tag{2.8}$$

where S and ψ_p are as in (2.5) and $W(\gamma)$ is same as in Definition 2.2.4.

Next, we consider the relationship between $\hat{\mathcal{E}}(U)$ and $\tilde{\mathcal{E}}(U)$. Definitions of these classes lead the containment relationship $\tilde{\mathcal{E}}(U) \subset \hat{\mathcal{E}}(U)$. When $n = 1$, it is obvious $\tilde{\mathcal{E}}(U) = \hat{\mathcal{E}}(U)$. Hence, from now, we consider the case when $n \geq 2$.

Proposition 2.3.6

When $n = 2$, every function $f \in \hat{\mathcal{E}}(U)$ belongs to the class $\tilde{\mathcal{E}}(U)$.

Proof.

Applying Proposition 2.3.2 to f, we get that there exist a set $S \in \mathcal{S}[\Gamma_+(f)]$ and functions $\psi_p \in C^\infty(U)$ for $p \in S$ such that $\mathcal{V}(\Gamma_+(f)) \subset S$, $\psi_p(0) \neq 0$ for $p \in \mathcal{V}(\Gamma_+(f))$ and

$$f(x) = \sum_{p\in S} x^p\psi_p(x). \tag{2.9}$$

Applying Taylor's theorem to $x^q\psi_q(x)$ for $q \in S \setminus \mathcal{V}(\Gamma_+(f))$, we have

$$x^q\psi_q(x) = P_q(x) + \sum_{\alpha\in\mathcal{V}(\Gamma_+(f))} x^\alpha\psi_{q,\alpha}(x), \tag{2.10}$$

where P_q is a polynomial and every $\psi_{q,\alpha}$ is smooth on U. Substituting (2.10) to the equation (2.9), one can obtain

$$f(x) = \sum_{p\in S\setminus\mathcal{V}(\Gamma_+(f))} P_q(x) + \sum_{p\in\mathcal{V}(\Gamma_+(f))} x^p\tilde{\psi}_p(x), \tag{2.11}$$

where every $\tilde{\psi}_p \in C^\infty(U)$ satisfies $\tilde{\psi}_p(0) \neq 0$. This equation implies that the function f belongs to $\tilde{\mathcal{E}}(U)$. □

At the last part of the proof above, the reason that $\tilde{\psi}_p(0)$ does not equal to 0 is as following. Each term in the last summation in the right-hand side of (2.11) is made from some $\psi_{q,\alpha}$ and ψ_p. Since $\psi_{q,\alpha}(0) = 0$ for any $q \in S \setminus \mathcal{V}(\Gamma_+(f))$, the value $\tilde{\psi}_p(0)$ equals to $\psi_p(0)$, which is not equal to 0.

At the end of this section, ket us consider the case when $n \geq 3$.

Proposition 2.3.7

If $n \geq 3$, then $\tilde{\mathcal{E}}(U) \subsetneq \hat{\mathcal{E}}(U)$. In other words, there exists a function $f \in \hat{\mathcal{E}}(U)$ such that $f \notin \tilde{\mathcal{E}}(U)$.

Proof.

Define the function f by

$$f(x) := \sum_{j=1}^{n-1} x_j^4 + x_1^2 x_2^2 \cdot e^{-1/x_n^2}.$$

Then the Newton polyhedron of f is explicitly written as $\Gamma_+(f) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n;\ \alpha_1 + \cdots + \alpha_{n-1} \geq 4 \text{ and } \alpha_n \geq 0\}$ and the set of vertices is $\mathcal{V}(\Gamma_+(f)) = \{4e_j;\ j = 1, \ldots, n-1\}$, where each e_j means same as in Definition 2.2.4. From the definition above, it is clear that f can be expressed as (2.5). This implies f belongs to $\check{\mathcal{E}}[P](U)$ for $P = \Gamma_+(f)$, and thus, f belongs to the K-N class. Nevertheless, f is not an element of $\tilde{\mathcal{E}}(U)$, because $\exp(-1/x_n^2)$ can't be rewritten as a product of a monomial (or polynomial) and a smooth function which does not vanish on a neighborhood of the origin. $\square$

One of differences between $n \leq 2$ and $n \geq 3$ is the convenience of f. When $n = 1$ or 2, every nonflat function is expressed as a product of a monomial and a convenient function. On the other hand, though the function f defined in the proposition above is nonflat, it can't be rewritten in such a product, because of the shape of the Newton polyhedron.

Chapter 3 Known results for oscillatory integrals

In this chapter, we study some properties about the oscillatory integrals. See also [21], [12], [19], [11] and so on.

§3.1 Analysis of oscillatory integrals

Definition 3.1.1

Let F be a function defined on $(0, \infty)$. We call F *rapidly decreasing* as $u \to \infty$ if

$$\lim_{u \to \infty} u^N F(u) = 0,$$

for any $N \in \mathbb{Z}_+$.

Generally, F is called a rapidly decreasing function if $F \in C^\infty((0, \infty))$ satisfies

$$\lim_{u \to \infty} u^N F^{(M)}(u) = 0,$$

for arbitrary N, $M \in \mathbb{Z}_+$. However, since we do not need to consider the differentiability in this paper, we adopt Definition 3.1.1.

Proposition 3.1.2

Let $n = 1$ and φ be a smooth function satisfying Supp$(\varphi) \subset (a, b)$. If $f'(x) \neq 0$ for all $x \in [a, b]$, then

$$I(t; \varphi) = \int_a^b e^{itf(x)} \varphi(x) dx = \mathsf{O}\left(t^{-N}\right),$$

as $t \to \infty$, for any non-negative integer N. Here, $\mathsf{O}(\cdot)$ is the Landau notation, that is, $t^N I(t; \varphi)$ has a limitation as t tends to infinity.

Proof.

Let D_f be an operator defined by

$$D_f := \frac{1}{itf'(x)} \cdot \frac{d}{dx}$$

and let ${}^tD_f$ denote the transpose of D_f, that is,

$${}^tD_f = -\frac{d}{dx}\left(\frac{\cdot}{itf'(x)}\right).$$

Since $D_f^N(\exp(itf(x))) = \exp(itf(x))$ for any non-negative integer N, integrating by parts, we get

$$I(t; \varphi) = \int_a^b D_f^N (e^{itf(x)}) \varphi(x) dx = \int_a^b e^{itf(x)} \left({}^tD_f\right)^N \varphi(x) dx.$$

Thus, $|I(t;\varphi)| \leq Ct^{-N}$ for some $C > 0$ which is independent of t. □

It is clear that $I(t;\varphi)$ in the proposition above is rapidly decreasing as $t \to \infty$ in the sense of Definition 3.1.1. In fact, $I(t;\varphi)$ is differenciable with respect to t and

$$\lim_{t\to\infty} t^N \frac{\partial^M}{\partial t^M} I(t;\varphi) = 0$$

in such a case, where N and M are (arbitrary) natural numbers.

To discuss about the cases when $n \geq 2$, we define a critical point as below.

Definition 3.1.3

Suppose that f is a partial differentiable function near $x_0 \in \mathbb{R}^n$. The point x_0 is called a *critical point* if

$$(\nabla f)(x_0) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right)\Bigg|_{x=x_0} = 0.$$

Proposition 3.1.4

Suppose that φ has a sufficiently small support and f has no critical points on Supp(φ). Then

$$I(t;\varphi) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)dx = \mathsf{O}\left(t^{-N}\right),$$

as $t \to \infty$, for any non-negative integer N.

Applying Proposition 3.1.2 (if necessary, choosing a coordinate system), one can prove this proposition. This fact shows that the behavior of an oscillatory integral depends on critical points of f. Indeed, in the theorems in the next section, f is required to have some critical points. To simplify, we usually assume that $f(0) = 0$, $\nabla f(0) = (0,\ldots,0)$ and the origin is the only critical point in the support of the amplitude φ.

Proposition 3.1.5 ([19])

Suppose that f has a critical zero point at the origin and the Hessian matrix

$$\mathrm{Hess}\, f(0) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1}(0) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}(0) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_n}(0) \end{pmatrix} \tag{3.1}$$

is invertible. Then the asymptotic expansion is

$$I(t;\varphi) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{-j}, \tag{3.2}$$

as $t \to \infty$, where each a_j is a constant depending f and φ. In particular,

$$a_0 = \frac{(2\pi i)^{n/2} \cdot \varphi(0)}{\sqrt{|\mathrm{Hess}\, f(0)|}}.$$

§3.2 Recent results for oscillatory integrals

Theorem 3.2.1 (Varchenko, [21])

Let U be a sufficiently small open neighborhood of the origin in $\mathbb{R}^n$. Suppose that f is real-analytic on U, the Newton polyhedron of f is not an empty set, the support of φ is contained in U and f has only one critical zero point at 0 in Supp(φ). If f satisfies that

$$\nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \ldots, \frac{\partial f_\gamma}{\partial x_n}\right) \neq (0, \ldots, 0)$$

on $U \cap (\mathbb{R} \setminus \{0\})^n$ for every compact face γ of $\Gamma_+(f)$, then one can obtain the n asymptotic expansion:

$$I(t;\varphi) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha} (\log t)^{k-1}, \tag{3.3}$$

as $t \to \infty$, where each $C_{\alpha,k}(\varphi)$ is a constant which depends on φ and $\{\alpha\}$ belongs to finitely many arithmetic progressions which are determined by the Newton polyhedron of f. Moreover, following facts hold.

(i) *The value $\beta_0(f) := \max\{\alpha; (\alpha, k) \in S_0(f)\}$ is not larger than $-1/d(f)$, where $S_0(f) := \{(\alpha,k) \in S_0(f);$ the value $C_{\alpha,k}$ in (3.3) is not equal to $0\}$.*

(ii) *If at least one condition of the followings (a)-(c) is satisfied, then $\beta_0(f)$ is equal to $-1/d(f)$ and the value $\eta_0(f)$, which is defined by $\eta_0(f) := \max\{k; (\beta_0(f), k) \in S_0(f)\}$, is equal to $m(f)$.*

(a) $d(f) > 1$.

(b) *f is non-negative or non-positive on U.*

(c) *$1/d(f)$ is not an odd integer and f_{τ_*} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$.*

(iii) *The constants $\beta_0(f)$ and $\eta_0(f)$ defined as above are called the oscillation-index and the multiplicity of f, respectively. They depend functions f and φ and satisfy*

$$|I(t;\varphi)| = \left| \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx \right| \leq C(\varphi)\, t^{\beta_0(f)} (\log t)^{\eta_0(f)},$$

for some $C(\varphi) > 0$ and sufficiently large $t > 0$.

(iv) *The limitation*

$$\lim_{t\to\infty} t^{-\beta_0(f)} (\log t)^{-\eta_0(f)} I(t;\varphi)$$

exists. Indeed, the limitation is equal to $C_{\beta_0(f),\eta_0(f)}(\varphi)$.

Theorem 3.2.2 (Kamimoto, Nose, [11])

Suppose that $f \in \hat{\mathcal{E}}(U)$ and the other assumptions are same as Theorem 3.2.1. Then we can get the asymptotic expansion

$$I(t;\varphi) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha} (\log t)^{k-1} \tag{3.4}$$

as $t \to \infty$, where $C_{\alpha,k}(\varphi)$ and $\{\alpha\}$ are also same as above. Furthermore, properties (i)-(iv) in Theorem 3.2.1 also hold.

Chapter 4 Elementary properties of Laplace integrals

In this chapter, we study fundamental properties of Laplace integrals. First, we assume $n = 1$ in §4.1. Next, we deal with some simple cases in §4.2.

Now we define the *Laplace-index* $\beta(f)$ and the *multiplicity* (of the Laplace integral) $\eta(f)$ as following. If we can obtain the symptotic expansion

$$L(t;\varphi) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha} (\log t)^{k-1}, \tag{4.1}$$

as $t \to \infty$, then $\beta(f) := \max\{\alpha;\, (\alpha, k) \in S(f)\}$ and $\eta(f) := \max\{k;\, (\beta_0(f), k) \in S(f)\}$, where $S(f) := \{(\alpha, k) \in S(f);$ there exists $\varphi \in C_0^{\infty}(U)$ such that $C_{\alpha,k}(\varphi)$ in (5.2) is not equal to $0\}$.

§4.1 One variable cases

When we consider Laplace integrals, we should suppose $f(x) \geq 0$. If f has no zero point, the integral $L(t;\varphi)$ is rapidly decreasing (see the proposition below).

Proposition 4.1.1

Let f be a smooth function which satisfies $f(x) > 0$ for all $x \in [a, b]$ and φ be a complex-valued smooth function whose support is a relatively compact set of an finite open interval (a, b). Then

$$L(t;\varphi) = \int_a^b e^{-tf(x)} \varphi(x) dx = \mathsf{O}\left(t^{-N}\right),$$

as $t \to \infty$, for any non-negative integer N.

<u>Proof</u>.

Since f is continuous and $Supp(\varphi)$ is compact in $\mathbb{R}$, there exists a constant $c > 0$ such that $f(x) \geq c$ on $Supp(\varphi)$. Then

$$\left| \int_a^b e^{-tf(x)} \varphi(x) dx \right| \;\leq\; \int_a^b \left| e^{-tf(x)} \right| |\varphi(x)| dx \;\leq\; e^{-ct} \int_a^b |\varphi(x)| dx \;=\; e^{-ct} \cdot \|\varphi\|_{L^{\infty}(\mathbb{R})}.$$

This estimate yields the conclusion, because the last integral is a constant which is independent of t. □

Of course, we can prove this proposition with operators

$$D_f := -\frac{1}{tf'(x)} \cdot \frac{d}{dx}, \quad {}^{t}D_f = \frac{d}{dx}\left(\frac{\cdot}{tf'(x)}\right),$$

as with the proof of Proposition 3.1.2. It is clear that $L(t;\varphi)$ is smooth and rapidly decreasing with respect to t in the sense of Definition 3.1.1.

Remark 4.1.2

Notice that the Newton polyhedron $\Gamma_+(f)$ is $\mathbb{R}_+$ and $d(f)=0$ in this case.

Hereafter, we always assume that f has some zeros on the support of the amplitude φ. In particular, we often assume $f(0)=0$. The assumption $f(x)\geq 0$ shows that every zero point of f satisfies $f'(x)=0$, otherwise, f is negative near the point.

Proposition 4.1.3

Suppose that there exists an integer k such that $|f^{(k)}(x)|\geq 1$ for any $x\in(a,b)$. Then there exists a positive c_k which independent of f and t such that

$$\left|\int_a^b e^{-tf(x)}dx\right| \leq c_k\, t^{-1/k}. \tag{4.2}$$

Proof.

First we show in the case $k=1$. Integrating by parts, we can immediately get

$$\int_a^b e^{-tf(x)}dx = \left[-\frac{e^{-tf(x)}}{tf'(x)}\right]_{x=a}^{b} + \int_a^b e^{-tf(x)}\frac{d}{dx}\left(\frac{1}{tf'(x)}\right)dx. \tag{4.3}$$

Since $|f'(x)|\geq 1$ for all $x\in(a,b)$,

$$\left|\left[-\frac{e^{-tf(x)}}{tf'(x)}\right]_{x=a}^{b}\right| = \left|-\frac{e^{-tf(b)}}{tf'(b)}+\frac{e^{-tf(a)}}{tf'(a)}\right| \leq \frac{1}{t}\left(\frac{e^{-tf(b)}}{|f'(b)|}+\frac{e^{-tf(a)}}{|f'(a)|}\right) \leq \frac{2}{t}.$$

The absolute value of the second term in (4.3) is less than (or equal to)

$$\frac{1}{t}\int_a^b \left|e^{-tf(x)}\frac{d}{dx}\left(\frac{1}{f'(x)}\right)\right|dx \leq \frac{1}{t}\left|\int_a^b \frac{d}{dx}\left(\frac{1}{f'(x)}\right)dx\right| \leq \frac{2}{t}.$$

Next, let us assume the case k is proved and start to show the case $k+1$. There exists only one constant $c\in[a,b]$ such that $|f^{(k)}(x)|\geq|f^{(k)}(c)|$, because $|f^{(k+1)}(x)|\geq 1$ on $[a,b]$.

If $f^{(k)}(c)$ is equal to 0, then $|f^{(k)}(x)|\geq\delta$ on $[a,b]\backslash(c-\delta,c+\delta)$ for any $\delta>0$. Thus, the integrals on $[a,c-\delta]$ and $[c+\delta,b]$ are estimated as

$$\begin{aligned}\left|\int_a^{c-\delta} e^{-tf(x)}dx\right| &= \left|\int_a^{c-\delta} e^{-t\delta F(x)}dx\right| \leq c_k(t\delta)^{-1/k},\\ \left|\int_{c+\delta}^{b} e^{-tf(x)}dx\right| &= \left|\int_{c+\delta}^{b} e^{-t\delta F(x)}dx\right| \leq c_k(t\delta)^{-1/k},\end{aligned}$$

where $f(x) = \delta F(x)$. On the other hand,

$$\left| \int_{c-\delta}^{c+\delta} e^{-tf(x)} dx \right| \leq \int_{c-\delta}^{c+\delta} |e^{-tf(x)}| dx \leq 2\delta.$$

Therefore, the following estimate holds:

$$\left| \int_a^b e^{-tf(x)} dx \right| \leq 2c_k (t\delta)^{-1/k} + 2\delta.$$

If $c = a$ or b, then we similarly get

$$\left| \int_a^b e^{-tf(x)} dx \right| \leq c_k (t\delta)^{-1/k} + \delta \leq 2c_k (t\delta)^{-1/k} + 2\delta.$$

Since δ is arbitrary, substituting $\delta = t^{-1/(k+1)}$, we get

$$\left| \int_a^b e^{-tf(x)} dx \right| \leq 2c_k \left(t^{-k/(k+1)} \right)^{1/k} + 2t^{-1/(k+1)} = 2(c_k + 1) t^{-1/(k+1)}.$$

Therefore, we had better take $c_{k+1} = 2(c_k + 1)$. □

From the recurrence relation in the proof above, we get $c_k = 2^{k+1} - 2 = 2\left(2^k - 1\right)$.

Corollary 4.1.4

Under the same assumption as Proposition 4.1.3, we get

$$\left| \int_a^b e^{-tf(x)} \varphi(x) dx \right| \leq c_k \, t^{-1/k} \cdot \|\varphi\|_{L^\infty(\mathbb{R})}. \tag{4.4}$$

Proof.

From the definition of $\| \cdot \|_{L^\infty(\mathbb{R})}$, we can immediately get

$$\begin{aligned} \left| \int_a^b e^{-tf(x)} \varphi(x) dx \right| &\leq \int_a^b \left| e^{-tf(x)} \varphi(x) \right| dx \\ &\leq \|\varphi\|_{L^\infty(\mathbb{R})} \int_a^b e^{-tf(x)} dx. \end{aligned}$$

The previous proposition gives the conclusion. □

Consider the case that $k = 1$, $f'(x) \geq 1$, $a = 0$ and $b = 1$. Since f' is always positive, f is strictly monotone increasing function. Accordingly, $f(0) = 0$ or $f(x) > 0$ on $[0, 1]$. If $f(0) > 0$, by Proposition 4.1.1, the integral in (4.2) is rapidly decreasing as $t \to \infty$. If $f(0) = 0$, from the assumption of the proposition above, $f(x) \geq x$ for $x \geq 0$. Thus we have

$$\int_0^1 e^{-tf(x)} dx \leq \int_0^\infty e^{-tx} dx = \frac{1}{t}.$$

For odd integer $k \geq 3$, we get

$$\int_0^\infty e^{-tx^k} dx = \frac{1}{k} t^{-1/k} \cdot \Gamma\left(\frac{1}{k}\right) = t^{-1/k} \cdot \Gamma\left(1 + \frac{1}{k}\right),$$

where Γ is the Gamma function defined by

$$\Gamma(z) := \int_0^\infty u^{z-1} e^{-u} du$$

for $\Re(z) > 0$. Of course, this equation is true for $k = 1$. But, unfortunately we can't estimate like $f(x) \geq x^k/k!$ under the assumption $k \geq 3$.

Indeed, neither (4.2) nor (4.4) is always the best estimate. Roughly speaking, it is because the proof of Proposition 4.1.3 does not require the information of $f, f', \ldots, f^{(k-1)}$.

For example, $f(x) := x(x-3)^2$ satisfies $f'''(x) \equiv 6$. By the proposition, we have

$$\left| \int_0^4 e^{-tf(x)} dx \right| \leq Ct^{-1/3}, \quad \left| \int_2^4 e^{-tf(x)} dx \right| \leq Ct^{-1/3},$$

where C is a positive constant (independent of f and t). However, we can get that the second integral is $\mathsf{O}\left(t^{-1/2}\right)$ from the proposition below.

Proposition 4.1.5

Let $k \geq 2$ be an even integer and suppose that f satisfies $f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0$ and $f^{(k)}(x_0) \neq 0$ for a point $x_0 \in \mathbb{R}$. If the support of φ is contained in a sufficiently small neighborhood of x_0, then

$$L(t;\varphi) \sim t^{-1/k} \sum_{j=0}^\infty a_j t^{-j/k}, \tag{4.5}$$

where each a_j is a complex constant. Here the symbol $\sim$ means

$$\left(\frac{d}{dt}\right)^M \left(L(t;\varphi) - t^{-1/k} \sum_{j=0}^N a_j t^{-j/k} \right) = \mathsf{O}\left(t^{-M-(N+1)/k}\right),$$

as $t \to \infty$, for any non-negative integers N and M.

<u>Proof</u>.

First, we assume that $x_0 = 0$, $f(x) = x^k$ and $Supp(\varphi) \subset (-\delta, \delta)$ for a sufficiently small constant $\delta > 0$. Write the Taylor expansion of φ:

$$\varphi(x) = \sum_{j=0}^M b_j x^j + x^{M+1} \varphi_M(x),$$

where φ_M is a smooth function. Then

$$\int_{-\infty}^{\infty} e^{-tx^k}\varphi(x)dx = \sum_{j=0}^{M}\int_{-\infty}^{\infty} b_j e^{-tx^k}x^j dx + \int_{-\infty}^{\infty} e^{-tx^k}x^{M+1}\varphi_M(x)dx. \tag{4.6}$$

When j is even, the integrand is an even function. Putting $y = tx^k$, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-tx^k}x^j dx &= 2\int_0^{\infty} e^{-tx^k}x^j dx \\
&= 2\cdot\frac{t^{-(j+1)/k}}{k}\int_0^{\infty} e^{-y}y^{(j+1)/k-1}dy \\
&= \frac{2}{k}\cdot\Gamma\left(\frac{j+1}{k}\right)t^{-(j+1)/k}.
\end{aligned} \tag{4.7}$$

In particular, when $k = 2$ and $j \geq 2$, one can know $\Gamma((j+1)/2) = (j-1)!!\sqrt{\pi/2^j}$. On the other hand, if j is odd, then

$$\int_{-\infty}^{\infty} e^{-tx^k}x^j dx = 0. \tag{4.8}$$

Next let us estimate the second term of (4.6). Assume that $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 0$ for $|x| \geq 2$ and $\chi(x) = 1$ for $|x| \leq 1$. Then, for small $\varepsilon > 0$, one has

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-tx^k}x^{M+1}\varphi_M(x)dx \\
=\ &\int_{-\infty}^{\infty} e^{-tx^k}x^{M+1}\varphi_M(x)\,\chi\left(\frac{x}{\varepsilon}\right)dx + \int_{-\infty}^{\infty} e^{-tx^k}x^{M+1}\varphi_M(x)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)dx \\
=:\ &L_1(t) + L_2(t).
\end{aligned}$$

Since $\chi(x/\varepsilon) = 0$ for $|x| \geq 2\varepsilon$, $L_1(t)$ is estimated as

$$\begin{aligned}
|L_1(t)| &\leq \int_{-2\varepsilon}^{2\varepsilon}\left|e^{-tx^k}x^{M+1}\varphi_M(x)\,\chi\left(\frac{x}{\varepsilon}\right)\right|dx \\
&\leq (2\varepsilon)^{M+1}\sup_{x\in[-2\varepsilon,2\varepsilon]}\left|\varphi_M(x)\,\chi\left(\frac{x}{\varepsilon}\right)\right|\cdot 4\varepsilon \\
&= C_1\varepsilon^{M+2},
\end{aligned}$$

for some $C_1 > 0$. To estimate $L_2(t)$, define an operator D by

$$Df := -\frac{1}{ktx^{k-1}}\cdot\frac{d}{dx}f,$$

then its transpose ${}^tD$ is expressed as

$${}^tDF = \frac{1}{kt}\cdot\frac{d}{dx}\left(\frac{F}{x^{k-1}}\right).$$

Since there exists $C'' > 0$ such that φ_M, φ'_M, χ and χ' are less than C''/x, we get

$$\begin{aligned}
&\left|{}^tD\left(x^{M+1}\varphi_M(x)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)\right)\right| \\
\leq\ & \frac{1}{kt}\left(\left|(M-k+2)x^{M-k+1}\varphi_M(x)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)\right|\right. \\
&\quad \left.+\left|x^{M-k+2}\varphi'_M(x)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)\right|+\frac{1}{\varepsilon}\left|x^{M-k+2}\varphi_M(x)\,\chi\left(\frac{x}{\varepsilon}\right)\right|\right) \\
\leq\ & \frac{C'}{t}x^{M-k+1},
\end{aligned}$$

for some $C' > 0$. Thus, $L_2(t)$ is estimated as

$$\begin{aligned}
|L_2(t)| &= \left|\int_{-\infty}^{\infty} D^N(e^{-tx^k})x^{M+1}\varphi_M(x)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)dx\right| \\
&= \left|\int_{-\infty}^{\infty} e^{-tx^k}\left({}^tD\right)^N\left(x^{M+1}\varphi_M(x)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)\right)dx\right| \\
&\leq \frac{\tilde{C}}{t^N}\int_{|x|\geq\varepsilon}|x|^{M-kN+1}dx \\
&\leq \frac{C_2}{t^N}\,\varepsilon^{M-kN+2}, \qquad (4.9)
\end{aligned}$$

for some $\tilde{C}$, $C_2 > 0$ if $M - kN + 2 < -1$.

Since $\varepsilon > 0$ is arbitrary, putting $\varepsilon = 1/\sqrt[k]{t}$, we obtain

$$\begin{aligned}
|L_1(t)| &\leq C_1 t^{-(M+2)/k}, \\
|L_2(t)| &\leq C_2 t^{-(M+2)/k}.
\end{aligned}$$

M is also arbitrary, thus we get the conclusion in a special case.

From now, we prove in general cases. The assumption gives the expression $f(x) = a(x-x_0)^k + \mathsf{O}\left(|x-x_0|^{k+1}\right)$ with $a \neq 0$. Let η be a smooth function satisfying $f(x) = a(x-x_0)^k(1+\eta(x))$. By the definition, $\eta(x) = \mathsf{O}\left(|x-x_0|\right)$ as $x \to x_0$ and $|\eta(x)| < 1$ on a sufficiently small neighborhood $U = U(x_0)$. Taking smaller U (if necessary) and setting $y = (x-x_0)(1+\eta(x))^{1/k}$, the mapping $x \mapsto y$ is a diffeomorphism from U to $\tilde{U}$, where $\tilde{U}$ is a small open neighborhood of the origin. If $Supp(\varphi) \subset U$, then

$$\int_{-\infty}^{\infty} e^{-tf(x)}\varphi(x)dx = \int_{-\infty}^{\infty} e^{-aty^k}\tilde{\varphi}(y)dy,$$

where $\tilde{\varphi} \in C_0^{\infty}(\mathbb{R})$. By the proof of a simple case, we can similarly prove. □

Remark 4.1.6

Together with (4.6), (4.7) and (4.8), estimates of L_1 and L_2 yield

$$a_0 = \frac{2}{k}\cdot\tilde{\varphi}(0)a^{-1/k}\Gamma\left(\frac{1}{k}\right).$$

Since $\tilde{\varphi}(0) = \varphi(x_0)$ and $f^{(k)}(x_0) = a \cdot k!$, the first coefficient is rewritten as

$$a_0 = \frac{2}{k} \cdot \varphi(x_0) \left(\frac{k!}{f^{(k)}(x_0)} \right)^{1/k} \Gamma\left(\frac{1}{k}\right). \tag{4.10}$$

In particular, when $k = 2$,

$$a_0 = \sqrt{\frac{2\pi}{f''(x_0)}} \cdot \varphi(x_0). \tag{4.11}$$

Remark 4.1.7

It is clear that a_0 is not equal to 0 unless $\varphi(x_0) = 0$. In this case, the value $\beta_0(f)$ equals to $-1/2$ and the value $\eta_0(f)$ equals to 1.

§4.2 Several variables cases

In this section, we discuss about some Laplace integrals in the case when $n \geq 2$.

Proposition 4.2.1

If $f(x) > 0$ on $Supp(\varphi)$, then

$$L(t; \varphi) = \int_{\mathbb{R}^n} e^{-tf(x)} \varphi(x) dx = \mathsf{O}\left(t^{-N}\right)$$

as $t \to \infty$, for any non-negative integer N.

Outline of the proof.

The way to prove this fact is same as Proposition 4.1.1. □

It is clear that $L(t; \varphi)$ is smooth with respect to t and rapidly decreasing as $t \to \infty$.

From the proposition above, one can know that the behavior of the Laplace integral $L(t; \varphi)$ is independent of critical points $x_1, \ldots, x_\ell$ with $f(x_j) \neq 0$, though all critical points affects to the asymptotic behavior of the oscillatory integral.

Remark 4.2.2

Similarly as Remark 4.1.2, $\Gamma_+(f) = \mathbb{R}^n_+$ and $d(f) = 0$ in this case.

Hereafter, unless otherwise stated, we always assume the conditions (A) and (B):

(A) The support of φ is contained in a sufficiently small neighborhood of the origin.

(B) $f(x) \geq 0$ on $Supp(\varphi)$, $f(0) = 0$ and f has a critical point at $x = 0$.

Proposition 4.2.3

Suppose that k is a natural number, $Supp(\varphi) \subset B(0,1)$ and there exists a multi-index β with $\langle\beta\rangle = k$ such that $|\partial_x^\beta f| \geq 1$ on $B(0,1)$. Then there exists $\tilde{c}_k > 0$ which depends on f and is independent of φ and t such that

$$\left| \int_{B(0,1)} e^{-tf(x)} \varphi(x) dx \right| \leq \tilde{c}_k \, t^{-1/k} \|\varphi\|_{L^\infty(B(0,1))}.$$

Define $\mathcal{P}_k := \{\sum c_\alpha x^\alpha;\ \langle\alpha\rangle = k\}$, the set of homogeneous polynomials of degree k. If every coefficient c_α is equals to 0, then $P \equiv 0$ (we assume $\mathcal{P}_k$ contains this monomial). Then $\{x^\alpha;\ \langle\alpha\rangle = k\}$ is a basis for this linear space. Write the dimension of $\mathcal{P}_k$ as d_k. Of course, d_k depends on not only k but also n.

To prove the proposition, we need the lemma below (see also [19]).

Lemma 4.2.4

There are unit vectors of $\mathbb{R}^n$ $\xi_1, \ldots, \xi_{d_k}$ such that $\{\langle\xi_j, x\rangle^k;\ 1 \leq j \leq d_k\}$ is a basis of $\mathcal{P}_k$, where $\langle\cdot,\cdot\rangle$ means the standard inner product in $\mathbb{R}^n$ (not the inner product in the L^2-space).

Proof.

Define $\mathcal{P}_\xi := \{\langle\xi, x\rangle^k = (\xi_1 x_1 + \cdots + \xi_n x_n)^k;\ \xi \in S^{n-1}\}$ and an inner product of $\mathcal{P}_k$ as

$$\langle P, Q\rangle_{\mathcal{P}_k} := \sum_{\langle\alpha\rangle = k} \alpha! a_\alpha b_\alpha,$$

where $P(x) = \sum a_\alpha x^\alpha$ and $Q(x) = \sum b_\alpha x^\alpha$. It is clear that $\mathcal{P}_\xi$ is a subset of $\mathcal{P}_k$. Since

$$Q\left(\frac{\partial}{\partial x}\right) = \sum_{\langle\alpha\rangle = k} b_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha = \sum_{\langle\alpha\rangle = k} b_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

then we obtain

$$\left(Q\left(\frac{\partial}{\partial x}\right)\right)(P) = \langle P, Q\rangle_{\mathcal{P}_k}.$$

If there exists a polynomial $P \in \mathcal{P}_k \setminus \{0\}$ such that $\left\langle P, \langle\xi, x\rangle^k \right\rangle_{\mathcal{P}_k} = 0$ for all $\xi \in S^{n-1}$ (that is, $\mathcal{P}_\xi \subsetneq \mathcal{P}_k$), then $\left\langle \langle\xi, \nabla\rangle^k, P \right\rangle_{\mathcal{P}_k} = 0$ for any $\xi \in \mathbb{R}^n$. For an arbitrary multi-index β,

$$\xi^\beta \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} x^\alpha = \begin{cases} 0 & (\text{if } \beta_j > \alpha_j \text{ for at least one } j), \\ \xi^\beta \cdot \prod_{j=1}^n \left(\dfrac{\alpha_j!}{(\alpha_j - \beta_j)!} x_j^{\alpha_j - \beta_j} \right) & (\text{otherwise}). \end{cases}$$

Now, we consider the case $\langle\beta\rangle = \langle\alpha\rangle = k$, thus we have

$$\begin{aligned} 0 = \langle P, \langle \xi, \nabla\rangle^k \rangle_{\mathcal{P}_k}(x) &= \sum_{\langle\alpha\rangle=k} a_\alpha \langle \xi, \nabla\rangle^k x^\alpha \\ &= \sum_{\langle\alpha\rangle=k} a_\alpha \xi^\alpha \alpha! \binom{k}{\alpha_1} \cdots \binom{k-\alpha_1-\cdots\alpha_{n-1}}{\alpha_n} \\ &= \sum_{\langle\alpha\rangle=k} a_\alpha \xi^\alpha k! \quad = k!\, P(\xi). \end{aligned}$$

The equation above shows $P \equiv 0$, this is a contradiction, that is, $\mathcal{P}_\xi = \mathcal{P}_k$.

Therefore, there are $\xi_1, \ldots, \xi_{d_k} \in S^{n-1}$ such that $\{\langle \xi, x\rangle^k\}$ is a basis of $\mathcal{P}_k$. □

Proof of Proposition 4.2.3.

By the assumption, there exists a unit vector $\xi = \xi(x_0)$ such that $|\langle \xi, \nabla\rangle^k f(x_0)| \geq \varrho_k$ for any $x_0 \in B(0,1)$, where $\langle\beta\rangle = k$ and ϱ_k is a constant depending on f (independent of x_0 and φ). Thus, there exists $r = r(x_0)$ such that $|\langle \xi, \nabla\rangle^k f(x_0)| \geq \varrho_k/2$ on $B(x_0, r(x_0))$. Since

$$Supp(\varphi) \subset \bigcup_{x \in Supp(\varphi)} B(x, r(x))$$

and the support of φ is compact, there exist $x_1, \ldots, x_N \in Supp(\varphi)$ such that

$$Supp(\varphi) \subset \bigcup_{j=1}^{N} B(x_j, r(x_j)).$$

We take a partition of unity such that each η_j satisfies $Supp(\eta_j) \subset B(x_j, r(x_j))$ and $0 \leq \eta_j(x) \leq 1$. It is clear that $\varphi_j := \varphi \cdot \eta_j$ belongs to $C_0^\infty(B(x_j, r(x_j)))$. For each j, choose a coordinate system which satisfies $\xi(x_j) = (1, 0, \ldots, 0)$. Then there exist constants u_j, v_j and a rectangle $D_j \subset \mathbb{R}^{n-1}$ such that $B(x_j, r(x_j)) \subset (u_j, v_j) \times D_j$. Therefore,

$$\begin{aligned} \left| \int_{B(x_j, r(x_j))} e^{-tf(x)} \varphi_j(x) dx \right| &= \left| \int_{D_j} \int_{u_j}^{v_j} e^{-tf(x)} \varphi_j(x) dx_1 dx' \right| \\ &\leq \int_{D_j} \left| \int_{u_j}^{v_j} e^{-tf(x)} \varphi_j(x) dx_1 \right| dx' \\ &\leq \int_{D_j} c_k \left(\frac{\varrho_k t}{2} \right)^{-1/k} \sup_{x_1 \in (u_j, v_j)} |\varphi_j(x_1, x_2, \ldots, x_n)| dx' \\ &\leq c_{j,k} t^{-1/k} \|\varphi\|_{L^\infty}, \end{aligned}$$

where c_k is same as in Proposition 4.1.3 (and Corollary 4.1.4) and $c_{j,k} = \sqrt[k]{2} \cdot c_k \varrho_k^{-1/k}$ $(2r(x_j))^{n-1}$. Defining $\tilde{c}_k := \sum_j c_{j,k}$, we get the conclusion. □

Since $r(x_j) \leq 1$ and $\sqrt[k]{2} \leq 2$, we can put $c_{j,k} = 2^n \cdot c_k \varrho_k^{-1/k}$ and $\tilde{c}_k = N c_{j,k}$.

Proposition 4.2.5

Suppose that $f(0) = 0$ *and* f *has a nondegenerate critical point at* 0, *that is,* $\nabla f(0) = 0$ *and the matrix* $\operatorname{Hess} f(0)$ *is invertible. If* $0 \in Supp(\varphi) \subset B(0,\delta)$ *for sufficiently small* $\delta > 0$, *then*

$$L(t;\varphi) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{-j}, \tag{4.12}$$

as $t \to \infty$, *where each* a_j *is a constant depending* f *and* φ *and the notation* $\sim$ *is same as in Proposition 4.1.5.*

Proof.

First we assume $f(x) = x_1^2 + \cdots + x_n^2$. Write the Taylor expansion of φ as

$$\varphi(x) = \sum_{\langle\alpha\rangle \leq M} c_\alpha x^\alpha + \sum_{\langle\alpha\rangle = M+1} x^\alpha \varphi_{M,\alpha}(x),$$

where each $\varphi_{M,\alpha}$ is smooth on $B(0,\delta)$. Decompose the Laplace integral in the same way as 4.6. Since

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-t(x_1^2+\cdots+x_n^2)} x^\alpha dx &= \prod_{\ell=1}^{n} \left(\int_{\mathbb{R}} e^{-tx_\ell^2} x_\ell^{\alpha_\ell} dx_\ell \right) \\ &\sim \prod_{\ell=1}^{n} \left(t^{-(1+\alpha_j)/2} \sum_{j_\ell=0}^{\infty} a_{\ell,j_\ell} t^{-j_\ell} \right) \\ &= t^{-(n+\langle\alpha\rangle)/2} \prod_{\ell=1}^{n} \left(\sum_{j_\ell=0}^{\infty} a_{\ell,j_\ell} t^{-j_\ell} \right) \end{aligned} \tag{4.13}$$

for $\langle\alpha\rangle \leq M$ and

$$\int_{\mathbb{R}^n} e^{-t(x_1^2+\cdots+x_n^2)} x^\alpha \varphi_{M,\alpha}(x) dx = \mathsf{O}\left(t^{-(M+1+n)/2} \right) \tag{4.14}$$

for $\langle\alpha\rangle = M+1$, one can obtain the expansion (4.12) for a special case.

When we show in general cases, we had better apply Morse's lemma (see [19] and Lemma 4.2.7). Then

$$L(t;\varphi) = \int e^{-t(y_1^2+\cdots+y_n^2)} (\varphi \circ \Psi)(y) |J\Psi(y)| dy,$$

where Ψ is a diffeomorphism and $J\Psi$ is its Jacobian. Since $(\varphi \circ \Psi) \cdot |J\Psi|$ belongs to $C_0^\infty(\mathbb{R}^n)$ and $Supp(\varphi)$ is sufficiently small, we get the conclusion. $\square$

Lemma 4.2.6

The estimate (4.14) holds.

Proof.

Fix a multi-index α such that $\langle\alpha\rangle = M+1$ and write $\alpha = (\alpha_1, \ldots, \alpha_n)$. Let χ be a cut-off function which satisfies $\chi(x) = 1$ for $\|x\| \leq 1$ and $\chi(x) = 0$ for $\|x\| \geq 2$. If necessary, we suppose $0 \leq \chi(x) \leq 1$ on $\mathbb{R}^n$. Then

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-t(x_1^2+\cdots+x_n^2)} x^\alpha \varphi_{M,\alpha}(x) dx \\ = \; & \int_{\mathbb{R}^n} e^{-t(x_1^2+\cdots+x_n^2)} x^\alpha \varphi_{M,\alpha}(x) \chi\left(\frac{x}{\varepsilon}\right) dx + \int_{\mathbb{R}^n} e^{-t(x_1^2+\cdots+x_n^2)} x^\alpha \varphi_{M,\alpha}(x) \left(1 - \chi\left(\frac{x}{\varepsilon}\right)\right) dx, \\ =: \; & L_1(t) + L_2(t) \end{aligned}$$

for an arbitrary $\varepsilon > 0$.

Since $\chi(x/\varepsilon) = 0$ for $\|x\| \geq 2$, we have

$$\begin{aligned} |L_1(t)| \; & \leq \; \int_{\|x\| \leq 2\varepsilon} \left| e^{-t(x_1^2+\cdots+x_n^2)} x^\alpha \varphi_{M,\alpha}(x) \chi\left(\frac{x}{\varepsilon}\right) \right| dx \\ & \leq \; (2\varepsilon)^{M+1} \cdot \sup_{\|x\| \leq 2\varepsilon} \left| \varphi_{M,\alpha}(x) \chi\left(\frac{x}{\varepsilon}\right) \right| \cdot (4\varepsilon)^n \; = \; C_1 \varepsilon^{M+1+n} \end{aligned}$$

for some $C_1 > 0$. From the same reason as (4.9), $L_2(t)$ can be estimated as

$$|L_2(t)| \leq \frac{C_2}{t^{\langle N \rangle}} \varepsilon^{M+1+n-2\langle N \rangle},$$

where $C_2 > 0$ and $N = (N_1, \ldots, N_n)$ such that $\alpha_j - 2N_j < -1$ for every $j = 1, \ldots, n$.

Substituting $\varepsilon = 1/\sqrt{t}$, we obtain (4.14). □

Lemma 4.2.7 (Morse's lemma, [19])

The assumption is same as Proposition 4.2.5. Then there exists a diffeomorphism $\Psi : U \to \tilde{U}$ such that $f \circ \Psi(y) = y_1^2 + \cdots + y_n^2$, where U and $\tilde{U}$ are open neighborhoods of the origin.

Proof.

Let U be a sufficiently small open neighborhood of the origin. We can express

$$f(x) = \sum_{j,k=1}^{n} x_j x_k f_{j,k}(x),$$

where each $f_{j,k}$ is smooth on U. Then we get

$$f_{j,k}(0) = \frac{1}{2} \cdot \frac{\partial^2 f}{\partial x_j \partial x_k}(0).$$

Now we assume that f is written as

$$f(y) = \sum_{l<r} y_l^2 + \sum_{j,k\geq r} y_j y_k F_{j,k}(y), \tag{4.15}$$

where $r \in \{0, 1, \ldots, n-1\}$ and every $F_{j,k}$ is smooth function. It is clear that the equation (4.15) holds for $r = 0$. From now, let us show that (4.15) holds for $r+1$ under the assumption that (4.15) holds for r.

We may suppose $F_{r,r}(0) \neq 0$ (if necessary, we had better change variables). Then $F_{r,r}$ has no zeros in a sufficiently small neighborhood of 0. Furthermore, since $f(y)$ must not be negative, we may suppose $F_{r,r}(0) > 0$. Define new variables $\tilde{y}_1, \ldots, \tilde{y}_n$ as

$$\tilde{y}_k := \begin{cases} y_k, & (k \neq r) \\ \sqrt{F_{r,r}(y)}\left(y_r + \dfrac{1}{F_{r,r}(y)} \displaystyle\sum_{j>r} y_j F_{j,r}(y)\right). & (k = r) \end{cases}$$

Substituting there new variables, we have the equation (4.15) for $r+1$. □

Remark 4.2.8

From the proofs of Proposition 4.2.5 and Lemma 4.2.7, we can find a_0 in (4.12). The Jacobian matrix of Ψ at $x = 0$ is written as

$$J\Psi(0) = \begin{pmatrix} \dfrac{\partial x_1}{\partial y_1}(0) & \cdots & \dfrac{\partial x_1}{\partial y_n}(0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_n}{\partial y_1}(0) & \cdots & \dfrac{\partial x_n}{\partial y_n}(0) \end{pmatrix}.$$

By a simple calculation, we get

$$\begin{pmatrix} 2 & & O \\ & \ddots & \\ O & & 2 \end{pmatrix} = {}^t J\Psi(0) \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1}(0) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}(0) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_n}(0) \end{pmatrix} J\Psi(0),$$

and thus $|J\Psi(0)|^2 = 2^n / |\mathrm{Hess}\, f(0)|$. By (4.11), (4.13) and (4.14), we obtain

$$a_0 = \frac{(2\pi)^{n/2} \cdot \varphi(0)}{\sqrt{|\mathrm{Hess}\, f(0)|}}.$$

In particular, if $f(x) = x_1^2 + \cdots + x_n^2$, then $a_0 = \sqrt{\pi^n} \cdot \varphi(0)$.

It is well-known that, if the matrix

$$M = \begin{pmatrix} \mu_{1,1} & \cdots & \mu_{1,n} \\ \vdots & \ddots & \vdots \\ \mu_{n,1} & \cdots & \mu_{n,n} \end{pmatrix}$$

is positive-definite, then every diagonal element $\mu_{j,j}$ $(j = 1, \ldots, n)$ is positive.

Since $|\mathrm{Hess}\, f(0)| \neq 0$ and f does not have a saddle point at $x = 0$, the Hessian matrix $\mathrm{Hess}\, f(0)$ is positive-definite. Consequently, we obtain that

$$\frac{\partial^2 f}{\partial x_j^2}(0) > 0$$

for each $j = 1, \ldots, n$. The inequality implies that f is convenient. Furthermore, the principle face of the Newton polyhedron $\Gamma_+(f)$ is $\tau_* = \{\alpha = (\alpha_1, \ldots, \alpha_n);\ \alpha_1 + \cdots + \alpha_n = 2\}$ and the principle part of f is

$$f_*(x) = \sum_{\alpha_1 + \cdots + \alpha_n = 2} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha.$$

Remark 4.2.9

In this case, the Newton polyhedron of f is expressed as $\Gamma_+(f) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n;\ \alpha_1 + \cdots + \alpha_n \geq 2\}$ and its principle face is $\tau_* = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n;\ \alpha_1 + \cdots + \alpha_n = 2\}$. Then one can immediately get $d(f) = 2/n$ and $m(f) = 1$. On the other hand, the expansion (4.12) imply that $\beta(f) = n/2$ and $\eta(f) = 0$. Therefore, the equations $\beta(f) = -1/d(f)$ and $\eta(f) = m(f)$ hold.

Chapter 5 Main theorem of this paper

§5.1 Known results on Laplace integrals

Next, we consider the case that f is degenerate, that is, the matrix (3.1) is not regular. One variable case is already shown in Proposition 4.1.5 under some assumptions.

Now we assume thet U is a sufficiently small neighborhood U of the origin and introduce the condition (C) as below:
(C) f belongs to the Kamimoto-Nose class $\hat{\mathcal{E}}(U)$ and

$$\nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \dots, \frac{\partial f_\gamma}{\partial x_n}\right) \neq (0, \dots, 0)$$

on $U \cap (\mathbb{R} \setminus \{0\})^n$ for every compact face γ of $\Gamma_+(f)$.

The condition (C) is sometimes called *nondegenerate over* $\mathbb{R}$ *with respect to the Newton polyhedron* $\Gamma_+(f)$. Let us see the example below.

Example 5.1.1

Let $n = 2$ and $g(x_1, x_2) := (x_1 - x_2)^2(x_1 + x_2)^2$. This function is real-analytic, and thus, $g \in \hat{\mathcal{E}}(U)$. Of course, g is always non-negative on $\mathbb{R}^2$. Consider the Newton polyhedron of g. An edge $\gamma = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2_+; \alpha_1 + \alpha_2 = 4\}$ is compact and the γ-part is $g_\gamma(x_1, x_2) = f(x) = (x_1 - x_2)^2(x_1 + x_2)^2$, however, the gradient of g_γ vanishes on $\{(u, u) \in \mathbb{R}^2; u \in \mathbb{R}\}$ and $\{(v, -v) \in \mathbb{R}^2; v \in \mathbb{R}\}$. Therefore, $g(x_1, x_2) := (x_1 - x_2)^2(x_1 + x_2)^2$ is degenerate over $\mathbb{R}$ with respect to $\Gamma_+(g)$.

Putting $y_1 := x_1 - x_2$ and $y_2 := x_1 + x_2$ to the definition of f above, we can make a new function $f(y_1, y_2) = y_1^2 y_2^2$. The only compact face of f is a vertex $\gamma = \{(2, 2)\}$ and the γ-part of f is $f_\gamma(y_1, y_2) = f(y_1, y_2) = y_1^2 y_2^2$. Partial derivatives of f are

$$\frac{\partial f}{\partial y_1}(y) = 2y_1 y_2^2, \quad \frac{\partial f}{\partial y_2}(y) = 2y_1^2 y_2.$$

Thus, the set $\{(y_1, y_2) \in \mathbb{R}^2; \nabla f(y) = (0, 0)\}$ does not intersect to $(\mathbb{R} \setminus \{0\})^2$, that is, this function satisfies the condition (C). The argument above implies that the condition (C) (in fact, the Kamimoto-Nose class $\hat{\mathcal{E}}(U)$) is not closed in the sense of changing of coordinates.

Let $f_1(x_1, x_2) := x_1^2 + x_1^4 + x_1^2 \cdot \exp(-1/x_2^2)$, $f_2(x_1, x_2) := -x_1^2$, $f_3(x_1, x_2) := (x_1 - x_2)^2 + \exp(-1/x_2^2)$ and consider $f_1(x_1, x_2) + f_2(x_1, x_2)$. Then one can find that $\hat{\mathcal{E}}(U)$ is not closed in the sense of summation.

Theorem 5.1.2 (Arnold, Gusein-Zade, Varchenko, [1])

Let U be a sufficiently small open neighborhood of the origin in $\mathbb{R}^n$. Suppose that f is real-analytic on U, the Newton polyhedron of f is not an empty set, the support of φ is contained in U, f has only one critical zero point at 0 in Supp(φ) and f is non-negative on Supp(φ) and satisfies the condition (C). Then the asymptotic expansion (4.1) holds as $t \to \infty$, where each $C_{\alpha,k}(\varphi)$ is a constant which depends on φ and $\{\alpha\}$ belongs to finitely many negative arithmetic progressions. Furthermore, $\beta(f)$ equals to $-1/d(f)$ and $\eta(f)$ equals to $m(f)$.

Consider the case that:
(M) f has the global minimum at the origin and $f(0) \neq 0$.
Under such an assumption, we can get the asymptotic expansion of $L(t;\varphi)$ as following:

$$L(t;\varphi) \sim e^{-tf(0)} \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha} (\log t)^{k-1}. \tag{5.1}$$

It is clear that the right-hand side of (5.1) does not converge when $f(0) < 0$ and is rapidly decreasing when $f(0) > 0$ as $t \to \infty$.

§5.2 Main Results on Laplace integrals

Theorem 5.2.1 (Main Theorem)

Suppose that $f \in \hat{\mathcal{E}}(U)$ and $\varphi \in C^{\infty}(U)$ satisfy the conditions (A), (B) and (C), that is,

- *Supp(φ) is contained in a sufficiently small open neighborhood of the origin U,*
- *$f(x) \geq 0$ and the origin is the only critical zero point in Supp(φ),*
- *$\nabla f_{\gamma}(x) \neq 0$ for every compact face $\gamma \subset \Gamma_{+}(f)$ and any $x \in U \cap (\mathbb{R} \setminus \{0\})^n$.*

Then one can get the asymptotic expansion:

$$\int_{\mathbb{R}^n} e^{-tf(x)} \varphi(x) dx \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi)\, t^{\alpha} (\log t)^{k-1}, \tag{5.2}$$

as $t \to \infty$, where each $C_{\alpha,k}(\varphi)$ is a constant and $\{\alpha\}$ belongs to finitely many negative arithmetic progressions. Moreover, the following equations hold:

$$\beta(f) = -\frac{1}{d(f)}, \tag{5.3}$$

$$\eta(f) = m(f), \tag{5.4}$$

where $\beta(f)$ and $\eta(f)$ are indices defined in Chapter 4 and $d(f)$ and $m(f)$ are constants defined in §1.2.

We prove this theorem in §9.3.

Remark 5.2.2

There exists a positive constant C such that

$$L(t;\varphi) \leq Ct^{-1/d(f)}(\log t)^{m(f)-1} \tag{5.5}$$

for sufficiently large $t > 1$.

This fact follows from (5.2), (5.3) and (5.4).

Compare this property to Theorem 3.2.1 or 3.2.2. In cases of oscillatory integrals, f can be negative near the origin, and thus, the equations (5.3) and (5.4) don't always hold. On the other hand, in cases of Laplace integrals, the condition (ii) in Theorem 3.2.1 always holds under the condition (B). When we consider the case (M), we should put $F(x) = f(x) - f(0)$. Then F has the global minimum value 0 at the origin.

Remark 5.2.3

The limitation

$$\lim_{t\to\infty} t^{1/d(f)}(\log t)^{-m(f)+1}L(t;\varphi) \tag{5.6}$$

exists. In fact, (5.6) equals to $C_{\beta(f),\eta(f)}(\varphi)$.

Chapter 6 Fans and toric varieties

§6.1 Fans associated with polyhedra

In this section, we always assume that P is a nonempty n-dimensional convex polyhedron satisfying $P+\mathbb{R}^n_+ \subset P \subset \mathbb{R}^n_+$. Now we introduce the following definitions and lemmas.

Definition 6.1.1

A k-dimensional cone σ is a *rational polyhedral cone* if there are linearly independent vectors $a^1(\sigma),\dots,a^k(\sigma)$ suct that $\sigma=\{\upsilon_1 a^1(\sigma)+\dots+\upsilon_k a^k(\sigma);\ \upsilon_j\ge 0 \text{ for } j=1,\dots,k\}$. The set of such vectors $\{a^1(\sigma),\dots,a^k(\sigma)\}$ is called a *skeleton* of σ. If a cone σ satisfies $\sigma\cap(-\sigma)=\{0\}$, σ is called a *strongly convex cone*.

Definition 6.1.2

Σ is a *fan* if Σ consists of finitely many cones and satisfies the following (i)-(iii):
(i) Every $\sigma\in\Sigma$ is a strongly convex rational polyhedral cone.
(ii) If $\sigma\in\Sigma$ and τ is a face of σ, then τ is also an element of Σ.
(iii) If both σ and τ are elements of Σ, then $\sigma\cap\tau$ is a face of each cone σ, τ.

For an arbitrary fan Σ, the *support* of Σ is defined by

$$|\Sigma| := \bigcup_{\sigma\in\Sigma}\sigma.$$

For $k=0,\dots,n$, we denote by $\Sigma^{(k)}$ the set of k-dimensional cones in Σ.

Definition 6.1.3

We define $\mathfrak{F}(P)$ as the set of nonempty faces of the polyhedron P,

$$\gamma(I,\sigma) := \bigcap_{j\in I} H(a^j(\sigma), l(a^j(\sigma)))\cap P,$$

where $I\in\mathfrak{P}(\{1,\dots,n\})$, $\sigma\in\Sigma^{(n)}$ and $\{a^1(\sigma),\dots,a^n(\sigma)\}$ is a skeleton of σ, and

$$I(\gamma,\sigma) := \{j\in\{1,\dots,n\};\ \gamma\subset H(a^j(\sigma), l(a^j(\sigma)))\},$$

where $\gamma\in\mathfrak{F}(P)$, $\sigma\in\Sigma^{(n)}$. If $I=\varnothing$, we define $\gamma(\varnothing,\sigma):=P$.

Here, the notation $\mathfrak{P}(\{1,\dots,n\})$ is same as in Definition 2.2.1.

Example 6.1.4

Let $n=2$, $P:=\{(\alpha_1,\alpha_2)\in\mathbb{R}^2_+;\ \alpha_1\ge 2,\ \alpha_2\ge 2 \text{ and } \alpha_1+\alpha_2\ge 6\}$, $\sigma_1:=\{(\alpha_1,\alpha_2)\in\mathbb{R}^2_+;\ \alpha_2\ge\alpha_1\}$ and $\sigma_2:=\{(\alpha_1,\alpha_2)\in\mathbb{R}^2_+;\ \alpha_1\ge\alpha_2\}$. Then a skeletons of σ_1 and σ_2 are

$$a^1(\sigma_1)=(0,1),\quad a^2(\sigma_1)=(1,1),\quad a^1(\sigma_2)=(1,0),\quad a^2(\sigma_2)=(1,1).$$

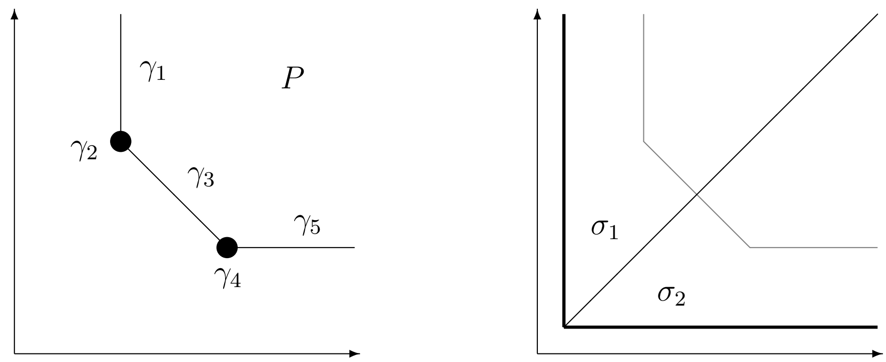

Figure 2: The polyhedron P and its faces [left], cones σ_1 and σ_2 [right]

Write faces γ_j $(j = 1, \ldots, 5)$ as

$$\begin{aligned}
\gamma_1 &:= \{(2, \alpha_2);\ \alpha_2 \geq 4\}, \\
\gamma_2 &:= \{(2, 4)\}, \\
\gamma_3 &:= \{(\alpha_1, \alpha_2);\ \alpha_1 + \alpha_2 = 6 \text{ and } 2 \leq \alpha_1 \leq 4\}, \\
\gamma_4 &:= \{(4, 2)\}, \\
\gamma_5 &:= \{(\alpha_1, 2);\ \alpha_1 \geq 4\}.
\end{aligned}$$

(See also Figure 2.) By the definitions above, one can find the following facts.
(i) $I(\gamma_j, \sigma_k)$ and $\gamma(I, \sigma_k)$ are as below.

$I(\gamma_j, \sigma_k)$	γ_1	γ_2	γ_3	γ_4	γ_5
σ_1	$\varnothing$	$\{2\}$	$\{2\}$	$\{1,2\}$	$\{1\}$
σ_2	$\{1\}$	$\{1,2\}$	$\{2\}$	$\{2\}$	$\varnothing$

$\gamma(I, \sigma_k)$	$\{1\}$	$\{2\}$	$\{1,2\}$
σ_1	γ_5	γ_3	γ_4
σ_2	γ_1	γ_3	γ_2

(ii) The tables in (i) yield that faces $\gamma(I(\gamma_j, \sigma_k), \sigma_k)$ are as below.

$\gamma(I(\gamma_j, \sigma_k), \sigma_k)$	γ_1	γ_2	γ_3	γ_4	γ_5
σ_1	P	γ_3	γ_3	γ_4	γ_5
σ_2	γ_1	γ_2	γ_3	γ_3	P

(iii) $\Sigma^{(n)}(\gamma_1) = \Sigma^{(n)}(\gamma_2) = \{\sigma_2\}$, $\Sigma^{(n)}(\gamma_3) = \{\sigma_1, \sigma_2\}$ and $\Sigma^{(n)}(\gamma_4) = \Sigma^{(n)}(\gamma_5) = \{\sigma_1\}$.

Lemma 6.1.5

Let $\sigma \in \Sigma^{(n)}$, $\gamma \in \mathfrak{F}(P)$ and $I \in \mathfrak{P}(\{1, \ldots, n\})$. Then
(i) $\gamma \subset \gamma(I(\gamma, \sigma), \sigma)$ *and* $dim(\gamma) \leq n - \#I(\gamma, \sigma)$.
(ii) $\gamma = \gamma(I, \sigma) \ \Rightarrow\ I \subset I(\gamma, \sigma) \ \Rightarrow\ dim(\gamma) \leq n - \#I$.

Proof.

(i): For each $j \in I(\gamma, \sigma)$, the face γ is contained in $H(a^j(\sigma), l(a^j(\sigma))) \cap P$. Hence, γ is contained in $\gamma(I(\gamma,\sigma),\sigma)$. Since vectors $a^1(\sigma),\dots,a^n(\sigma)$ are linearly independent, we get $dim(\gamma(I,\sigma)) = n - \#I$, in particular, $\dim(\gamma) \le dim(\gamma(I(\gamma,\sigma),\sigma)) = n - \#I(\gamma,\sigma)$.

(ii): If $\gamma = \gamma(I,\sigma)$, then every $j \in I$ satisfies $\gamma \subset H(a^j(\sigma), l(a^j(\sigma))) \cap P$. This implies $I \subset I(\gamma,\sigma)$. Since $\gamma \subset \gamma(I(\gamma,\sigma),\sigma)$, the same discussion as (i) yields $dim(\gamma) \le dim(\gamma(I(\gamma,\sigma),\sigma)) \le n - \#I(\gamma,\sigma) \le n - \#I$ for any $I \subset I(\gamma,\sigma)$. $\square$

Definition 6.1.6

We denote by $(\mathbb{R}^n)^*$ the dual space of $\mathbb{R}^n$ with respect to the standard inner product and we can equate $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ in the natural way. For $a \in (\mathbb{R}^n)^*$, we define $l(a)$ as

$$l(a) := \min\{\langle a,\alpha\rangle;\ \alpha \in P\} \tag{6.1}$$

and

$$\gamma(a) := \{\alpha \in P;\ \langle a,\alpha\rangle = l(a)\}.$$

Then one can immediately show that $\gamma(a) = H(a, l(a)) \cap P$. Now we consider the equivalence relation $\sim$ in $(\mathbb{R}^n)^*$ defined as

$$a \sim b \ \Leftrightarrow\ \gamma(a) = \gamma(b).$$

Moreover, for every face γ, there is an equivalence class γ^* defined by

$$\gamma^* := \{a \in (\mathbb{R}^n)^*;\ \gamma(a) = \gamma \text{ and } a_j \ge 0 \text{ for } j = 1,\dots,n\},$$

and, in particular, $P^* := \{0\}$. From the definition above, one can get

$$\gamma^* = \{a \in (\mathbb{R}^n)^*;\ \gamma = H(a, l(a)) \cap P \text{ and } a_j \ge 0 \text{ for } j = 1,\dots,n\}.$$

Define the closure $\overline{\gamma^*}$ of γ^* as

$$\overline{\gamma^*} := \{a \in (\mathbb{R}^n)^*;\ \gamma \subset H(a, l(a)) \cap P \text{ and } a_j \ge 0 \text{ for } j = 1,\dots,n\}.$$

Lemma 6.1.7

Let $\sigma \in \Sigma^{(n)}$, $\gamma \in \mathfrak{F}(P)$, $I \in \mathfrak{P}(\{1,\dots,n\})$ and γ^* be as above. Define $\Sigma^{(n)}(\gamma) := \{\sigma \in \Sigma^{(n)};\ dim(\gamma) = n - \#I(\gamma,\sigma)\}$. Then
(i) $\#I(\gamma,\sigma) = dim(\gamma^* \cap \sigma)$.
(ii) $\Sigma^{(n)}(\gamma) = \{\sigma \in \Sigma^{(n)};\ dim(\gamma^* \cap \sigma) = dim(\gamma^*)\} \neq \varnothing$.
(iii) *For any $\sigma \in \Sigma^{(n)}(\gamma)$, one has $\gamma = \gamma(I(\gamma,\sigma),\sigma)$.*

Proof.

(i): For any $j \in I(\gamma,\sigma)$, the face γ is contained in the hyperplane $H(a^j(\sigma), l(a^j(\sigma)))$, and thus $a^j(\sigma) \in \overline{\gamma^*}$. Since $\{a^j(\sigma)\}$ is a skeleton of the cone σ, we have $a^j(\sigma) \in \overline{\gamma^*} \cap \sigma$. The converse also holds, that is, $j \in I(\gamma, \sigma)$ is equivalent to $a^j(\sigma) \in \overline{\gamma^*} \cap \sigma$.

(ii): From (i), it is obvious that $dim(\gamma)$ equals to $n - dim(\gamma^* \cap \sigma)$ for $\sigma \in \Sigma^{(n)}(\gamma)$. Since $dim(\gamma^*) = n - dim(\gamma)$, one can get $dim(\gamma^* \cap \sigma) = dim(\gamma^*)$. The condition that the support of the fan Σ is $\mathbb{R}^n_+$ implies that there exists a cone $\sigma \in \Sigma^{(n)}$ such that $dim(\gamma^* \cap \sigma) = dim(\gamma^*)$, that is, the set $\Sigma^{(n)}(\gamma)$ is not empty.

(iii): Lemma 6.1.5 (i) and the previous facts yield that $dim(\gamma) \leq n - \#I(\gamma, \sigma) \leq dim(\gamma(I(\gamma, \sigma), \sigma)) \leq n - dim(\gamma^*) = dim(\gamma)$ for any $\sigma \in \Sigma^{(n)}(\gamma)$. Hence $dim(\gamma)$ is equal to $dim(\gamma(I(\gamma, \sigma), \sigma))$, and moreover, $\gamma = \gamma(I(\gamma, \sigma), \sigma)$ holds, because γ is contained in $\gamma(I(\gamma, \sigma), \sigma)$. □

Remark 6.1.8

It is clear that $\Sigma^{(n)}(P) = \Sigma^{(n)}$.

Definition 6.1.9

Let Σ_0 be the set of all $\overline{\gamma^*}$. We call Σ_0 the *fan associated with* P. Moreover, Σ is called a *simplicial subdivision* of Σ_0 if the following conditions (i)-(iii) hold:

(i) $|\Sigma| = |\Sigma_0|$.

(ii) Every cone σ of Σ lies in some cone of Σ_0.

(iii) A skeleton of an arbitrary cone σ of Σ can be completed to a base of the lattice dual to $\mathbb{Z}^n$.

Remark 6.1.10

An arbitrary fan Σ_0 associated with P satisfies $|\Sigma_0| = \mathbb{R}^n_+$ and has a simplicial subdivision Σ.

From now, Σ_0 is the fan associated with P and Σ is a simplicial subdivision of Σ_0 unless otherwise noted. For a n-dimensional vector $a^j(\sigma)$, we usually write $a^j(\sigma) = (a^j_1(\sigma), \ldots, a^j_n(\sigma))$.

§6.2 Real toric varieties

In this section, the polyhedron P is always the Newton polyhedron $\Gamma_+(f)$ of f.

Definition 6.2.1

Let $\sigma \in \Sigma$ be a n-dimensional cone. We associate a copy of $\mathbb{R}^n$ which is denoted by $\mathbb{R}^n(\sigma)$. Define the map $\pi(\sigma) : \mathbb{R}^n(\sigma) \to \mathbb{R}^n$ by $(x_1, \ldots x_n) = \pi(\sigma)(y_1, \ldots, y_n)$ with

$$x_k := \prod_{j=1}^{n} y_j^{a^j_k(\sigma)} = y_1^{a^1_k(\sigma)} \cdots y_n^{a^n_k(\sigma)},$$

for $k = 1, \ldots, n$, where $\{a^1(\sigma), \ldots, a^n(\sigma)\}$ is a skeleton of the cone σ.

Definition 6.2.2

Let Y_Σ be the union of $\mathbb{R}^n(\sigma)$ for σ which are glued along the image of $\pi(\sigma)$. We call the manifold Y_Σ the *(real) toric variety* associated with the fan Σ.

Remark 6.2.3

The following facts are known.

(i) Y_Σ is an n-dimensional real algebraic manifold.

(ii) The map $\pi : Y_\Sigma \to \mathbb{R}^n$ defined on each $\mathbb{R}^n(\sigma)$ as $\pi(\sigma) : \mathbb{R}^n(\sigma) \to \mathbb{R}^n$ is proper.

(iii) The set of points in $\mathbb{R}^n(\sigma)$ in which $\pi(\sigma)$ is not an isomorphism is a union of coordinate planes.

(iv) The Jacobian of the mapping $\pi(\sigma)$ equals to

$$J_{\pi(\sigma)} = \vartheta \cdot \prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle - 1},$$

where ϑ is equal to 1 or -1.

<u>Proof</u>.

Here, we prove only (iv). Proofs of the others are fererred in [6], [5].

From the definition of the Jacobian and the determinant, we have

$$\begin{aligned}
|J_{\pi(\sigma)}(y)| &= \begin{vmatrix} a_1^1(\sigma) y_1^{a_1^1(\sigma)-1} y_2^{a_1^2(\sigma)} \cdots y_n^{a_1^n(\sigma)} & \ldots & a_1^n(\sigma) y_1^{a_1^1(\sigma)} y_2^{a_1^2(\sigma)} \cdots y_n^{a_1^n(\sigma)-1} \\ a_2^1(\sigma) y_1^{a_2^1(\sigma)-1} y_2^{a_2^2(\sigma)} \cdots y_n^{a_2^n(\sigma)} & \ldots & a_1^n(\sigma) y_1^{a_2^1(\sigma)} y_2^{a_2^2(\sigma)} \cdots y_n^{a_2^n(\sigma)-1} \\ \vdots & \ddots & \vdots \\ a_n^1(\sigma) y_1^{a_n^1(\sigma)-1} y_2^{a_n^2(\sigma)} \cdots y_n^{a_n^n(\sigma)} & \ldots & a_n^n(\sigma) y_1^{a_n^1(\sigma)} y_2^{a_n^2(\sigma)} \cdots y_n^{a_n^n(\sigma)-1} \end{vmatrix} \\
&= \left(\prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle} \right) \cdot \begin{vmatrix} a_1^1(\sigma)/y_1 & \ldots & a_1^n(\sigma)/y_n \\ \vdots & \ddots & \vdots \\ a_n^1(\sigma)/y_1 & \ldots & a_n^n(\sigma)/y_n \end{vmatrix} \\
&= \left(\prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle - 1} \right) \cdot \begin{vmatrix} a_1^1(\sigma) & \ldots & a_1^n(\sigma) \\ \vdots & \ddots & \vdots \\ a_n^1(\sigma) & \ldots & a_n^n(\sigma) \end{vmatrix}. \qquad (6.2)
\end{aligned}$$

The determinant of the last matrix in (6.2) equals to 1 or -1, because of the construction of the simplicial subdivision (the condition (iii) in Definition 6.1.9). $\square$

Chapter 7 Local zeta functions and transforms

§7.1 Laplace transform and Mellin transform

First, we introduce the Laplace transform and the Mellin transform. Some properties of these transforms have already been known, thus, in this paper, we omit proofs of them. Details are in [15], [18], [4] and so on.

Definition 7.1.1

Let a, b be real constants with $a < b$. Assume that g is continuous and locally integrable on $\mathbb{R}$ and satisfies the following conditions (i) and (ii).
(i) $g(y) = \mathsf{O}\left(e^{(a+\varepsilon)y}\right)$ as $y \to +\infty$.
(ii) $g(y) = \mathsf{O}\left(e^{(b-\varepsilon)y}\right)$ as $y \to -\infty$.
Then the *Laplace transform* of g is defined by

$$(\mathcal{L}g)(s) = G(s) := \int_{-\infty}^{\infty} e^{-sy} g(y) dy, \tag{7.1}$$

where $s \in \mathbb{C}$ with $a < \Re(s) < b$.

This integral is sometimes called the two-sided Laplace transform or the bilateral Laplace transform. Usually the integral defined by

$$\tilde{G}(s) := \int_{0}^{\infty} e^{-sy} g(y) dy,$$

is called the Laplace transform of g. However, we use the definition (7.1) in this paper.

The integral in (7.1) converges absolutely on the set $\{s \in \mathbb{C};\, a < \Re(s) < b\}$. Indeed, $\mathcal{L}g$ is holomorphic in the strip.

The inverse transform of the (two-sided) Laplace transform is expressed

$$(\mathcal{L}^{-1}G)(x) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{sx} G(s) ds,$$

where $\lambda = \Re(s)$ and the contour path is contained in the region of convergence of G that is, $a < \lambda < b$. The expression above is independent of λ.

On the other hand, the Mellin transform is defined as the integral below.

Definition 7.1.2

Let a, b be real constants with $a < b$. Assume that f is continuous and locally integrable on $\mathbb{R}_+$ and satisfies the following conditions (i) and (ii).
(i) $f(x) = \mathsf{O}\left(x^{-(a+\varepsilon)}\right)$ as $x \to +0$.

(ii) $f(x) = \mathsf{O}\left(x^{-(b-\varepsilon)}\right)$ as $x \to -\infty$.
Then the *Mellin transform* of f is defined by

$$(\mathcal{M}f)(s) = F(s) := \int_0^\infty x^{s-1} f(x) dx,$$

where $s \in \mathbb{C}$ with $a < \Re(s) < b$.

Now, substituting $y = -\log x$ and $f(x) = g(-\log x)$, one can immediately get

$$\begin{aligned} (\mathcal{L}g)(s) &= \int_{-\infty}^{\infty} e^{-sy} g(y) dy \\ &= -\int_\infty^0 e^{-s(-\log x)} g(-\log x) \frac{dx}{x} \\ &= \int_0^\infty x^{s-1} f(x) dx \ = \ (\mathcal{M}f)(s). \end{aligned}$$

This equation shows that there is a strong relationship between the Laplace transform and the Mellin transform and the conditions (i), (ii) in Definition 7.1.2 stand to reason.

The inverse transform of the Mellin transform is

$$(\mathcal{M}^{-1}F)(x) := \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} x^{-s} F(s) ds.$$

where $a < \lambda < b$. This is sometimes expressed as

$$(\mathcal{M}^{-1}F)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-(\lambda + i\nu)} F(\lambda + i\nu) d\nu.$$

The integral is also independent of the constant λ, while it satisfies $a < \lambda < b$. Here, the function F is required that F is analytic in the strip $\{s \in \mathbb{C};\ a < \Re(s) < b\}$.

By the definition of the gamma function, it is clear that $\Gamma(s)$ is the Mellin transform of $f(x) = \exp(-x)$ for $\Re(s) > 0$. It is known that, if $f(x) = \cos(ax)$, then $(\mathcal{M}f)(s) = a^{-s}\Gamma(s)\cos(\pi s/2)$ for $a > 0$ and $0 < \Re(s) < 1$. See Appendix 1 and [17], [14], [18] for more information.

§7.2 Relationship between $Z(s;\varphi)$ and $L(t;\varphi)$

Let U be a sufficiently small open neighborhood of the origin. For $s \in \mathbb{C}$, define

$$Z(s;\varphi) := \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx,$$

which is called the *local zeta function*, and

$$Z_\pm(s;\varphi) = \int_{\mathbb{R}^n} \left(f(x)_\pm\right)^s \varphi(x) dx,$$

where $f(x)_+ = \max\{f(x), 0\}$ and $f(x)_- = \max\{-f(x), 0\}$. Since $|f(x)|$ is equal to $f(x)_+ + f(x)_-$ and $f(x)_+ \cdot f(x)_- \equiv 0$, we obtain $Z(s;\varphi) = Z_+(s;\varphi) + Z_-(s;\varphi)$.

Define $W_u := \{x \in \mathbb{R}^n;\ f(x) = u\}$ for $u \in \mathbb{R}$ and

$$\mathrm{K}(f, \varphi, u) := \int_{W_u} \varphi(x)\omega,$$

where ω is a differential form which satisfies $du \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$. This function $\mathrm{K}(f, \varphi, u)$, which is often denoted by $\mathrm{K}(u)$, is called the *Gelfand-Leray function*. If $f(x) \geq 0$, then K is identically zero on $(-\infty, 0)$.

Changing the integral variables, we have

$$\begin{aligned}
\int_0^\infty u^s \mathrm{K}(u) du &= \int_0^\infty u^s \left(\int_{W_u} \varphi(x)\omega \right) du \\
&= \int_0^\infty \int_{W_u} f(x)^s \varphi(x)\, du \wedge \omega \\
&= \int_{\mathbb{R}^n} (f(x)_+)^s \varphi(x) dx \;=\; Z_+(s;\varphi).
\end{aligned}$$

This equation means that the local zeta function is expressed as the Mellin transform of the Gelfand-Leray function. Since $\mathrm{K} \equiv 0$ on $(-\infty, 0)$,

$$Z_-(s;\varphi) = \int_0^\infty 0\, du = 0.$$

Indeed, since f_- is identically zero under the condition (B), we have $Z_-(s;\varphi) \equiv 0$.

On the other hand, similarly calculating, we get

$$\begin{aligned}
\int_{-\infty}^\infty e^{-tu} \mathrm{K}(u) du &= \int_0^\infty e^{-tu} \left(\int_{W_u} \varphi(x)\omega \right) du \\
&= \int_0^\infty \int_{W_u} e^{-tf(x)} \varphi(x)\, df \wedge \omega \\
&= \int_{\mathbb{R}^n} e^{-tf(x)} \varphi(x) dx \;=\; L(t;\varphi).
\end{aligned}$$

This means that the Laplace integral is the Laplace transform of the Gelfand-Leray function.

We will analyze the local zeta function more precisely in Chapter 8.

Chapter 8 Analysis of the local zeta function

§8.1 The function f_σ

Proposition 8.1.1

For an arbitrary $\sigma \in \Sigma^{(n)}$, there exists a function f_σ which is smooth on $\pi(\sigma)^{-1}(U)$ such that $f_\sigma(0) \neq 0$ and

$$f(\pi(\sigma)(y)) = \left(\prod_{j=1}^{n} y_j^{l(a^j(\sigma))} \right) f_\sigma(y), \tag{8.1}$$

for $y \in \pi(\sigma)^{-1}(U)$.

Proof.

Putting $x = \pi(\sigma)(y)$ and $P = \Gamma_+(f)$ into the equation (2.5), one can easily get

$$f(\pi(\sigma)(y)) = \sum_{p \in S} \left(\prod_{j=1}^{n} y_j^{\langle a^j(\sigma), p \rangle} \right) \psi_p(\pi(\sigma)(y)),$$

where $S \in \mathcal{S}[\Gamma_+(f)]$ with $\mathcal{V}(\Gamma_+(f)) \subset \mathcal{S}[\Gamma_+(f)]$ and $\psi_p \in C^\infty(U)$. Define

$$f_{p,\sigma}(y) := \left(\prod_{j=1}^{n} y_j^{\langle a^j(\sigma), p \rangle - l(a^j(\sigma))} \right) \psi_p(\pi(\sigma)(y)) \text{ and } f_\sigma(y) := \sum_{p \in S} f_{p,\sigma}(y). \tag{8.2}$$

Then, we obtain the equation (8.1). We remark that the definition of $l(a)$ and the condition that p belongs to $\Gamma_+(f)$, imply $\langle a^j(\sigma), p \rangle - l(a^j(\sigma))$ is a non-negative integer for every j.

Next, let us show $f_\sigma(0) \neq 0$. If there exists a number j such that $\langle a^j(\sigma), p \rangle - l(a^j(\sigma)) > 0$, then $f_{p,\sigma}(0)$ equals to 0. Now, $\gamma(\{1, \ldots, n\}, \sigma)$ is a vertex of $\Gamma_+(f)$, which is denoted by $p(\sigma)$. Since S contains the vertex $p(\sigma)$, $f_\sigma(0)$ equals to $f_{p(\sigma),\sigma}(0) = \psi_{p(\sigma)}(0) \neq 0$. □

Define $T_I^*(\mathbb{R}^n) := \{y \in \mathbb{R}^n;\ y_j = 0 \text{ if and only if } j \in I\}$ for $I \in \mathfrak{P}(\{1, \ldots, n\})$.

Theorem 8.1.2

Let f be a function satisfying conditions (B) (in P.29) and (C) and assume $\pi(\sigma)(T_I^*(\mathbb{R}^n)) = 0$. Then $\nabla f_\sigma \neq (0, \ldots, 0)$ on the set $\{y \in T_I^*(\mathbb{R}^n);\ f_\sigma(y) = 0\}$.

From this theorem, it is clear that f is negative at some zero points in U. However, f must be non-negative on the support of φ. This is a contradiction. Thus we see that f_σ does not vanish on $U \setminus \{0\}$. Furthermore, by the equation (8.1), it can be seen that every $l(a^j(\sigma))$ is even integer and $f \circ \pi(\sigma)$ does not vanish on $\pi(\sigma)^{-1}(U) \cap (\mathbb{R} \setminus \{0\})^n$.

Proof of Theorem 8.1.2.

First, we show that $\gamma = \gamma(I, \sigma)$ is compact. Let $y \in T_I^*(\mathbb{R}^n)$ and $J := \{1, \ldots, n\} \setminus I$. Then, for every $k \in \{1, \ldots, n\}$, there exists $j \in I$ such that $a_k^j(\sigma) > 0$. On the other hand, every valid pair (a, l) for $\Gamma_+(f)$ defining γ must satisfy $a_k = 0$ for any $k \in V(\gamma)$, which is defined in Definition 2.2.4. Thus one can see that $W(\gamma) = \{1, \ldots, n\}$ and the face γ is compact.

By the equation (2.8), we get

$$f_\gamma(x) = \sum_{p \in \gamma \cap S} x^p \psi_p(0),$$

for any compact face γ, where S is a finite subset of $\mathbb{Z}_+^n$. Since $\langle a^j(\sigma), p\rangle = l(a^j(\sigma))$ for $j \in I$ and $p \in \gamma$, one can see that f_γ has the *quasihomogeneous property*:

$$f_\gamma(t^{a_1^j(\sigma)} x_1, \ldots, t^{a_n^j(\sigma)} x_n) = t^{l(a^j(\sigma))} f_\gamma(x). \tag{8.3}$$

Taking the derivative of the left-hand side of (8.3) with respect to t, we obtain

$$\frac{\partial}{\partial t} f_\gamma(t^{a_1^j(\sigma)} x_1, \ldots, t^{a_n^j(\sigma)} x_n) = \sum_{k=1}^n a_k^j(\sigma) t^{a_k^j(\sigma)-1} x_k \frac{\partial f_\gamma}{\partial x_k}(t^{a_1^j(\sigma)} x_1, \ldots, t^{a_n^j(\sigma)} x_n). \tag{8.4}$$

The right-hand side of (8.4) tends to

$$\sum_{k=1}^n a_k^j(\sigma) x_k \frac{\partial f_\gamma}{\partial x_k}(x).$$

as $t \to 1$. Even if $a_k^j(\sigma) = 0$ for some k, similar arguments are available. On the other hand, taking the derivative of the right-hand side of (8.3) with respect to t, we have

$$\frac{\partial}{\partial t} t^{l(a^j(\sigma))} f_\gamma(x) = l(a^j(\sigma)) t^{l(a^j(\sigma))-1} f_\gamma(x)$$

and this tends to $l(a^j(\sigma)) f_\gamma(x)$ as $t \to 1$, for $j \in I$. Accordingly, we get the *Euler identity*:

$$\sum_{k=1}^n a_k^j(\sigma) x_k \frac{\partial f_\gamma}{\partial x_k}(x) = l(a^j(\sigma)) f_\gamma(x). \tag{8.5}$$

Next, we show the following lemma.

Lemma 8.1.3

For a face γ of $\Gamma_+(f)$, a cone $\sigma \in \Sigma^{(n)}$ and $y \in \pi(\sigma)^{-1}(U)$, the following equation holds:

$$f_\gamma(\pi(\sigma)(y)) = \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))}\right) f_\sigma(T_I(y)). \tag{8.6}$$

Proof.

From the definitions of $\pi(\sigma)$ and $W(\gamma)$, one can easily get $\pi(\sigma)\circ T_I = T_{W(\gamma)}\circ\pi(\sigma)$. This fact and the equations (2.8) and (8.2) yield

$$\begin{aligned} f_\gamma(\pi(\sigma)(y)) &= \sum_{p\in\gamma\cap S}\left(\prod_{j=1}^n y_j^{\langle a^j(\sigma),p\rangle}\right)\psi_p((T_{W(\gamma)}\circ\pi(\sigma))(y)) \\ &= \left(\prod_{j\in I} y_j^{l(a^j(\sigma))}\right)\sum_{p\in\gamma\cap S}\left(\prod_{j\in J} y_j^{\langle a^j(\sigma),p\rangle}\right)\psi_p((T_{W(\gamma)}\circ\pi(\sigma))(y)) \\ &= \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))}\right)\sum_{p\in\gamma\cap S}\left(\prod_{j\in J} y_j^{\langle a^j(\sigma),p\rangle - l(a^j(\sigma))}\right)\psi_p((\pi(\sigma)\circ T_I)(y)) \\ &= \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))}\right) f_\sigma(T_I(y)). \end{aligned}$$

The last equation is from the condition that, for any $j\in I$, $(a^j(\sigma), l(a^j(\sigma)))$ is valid pair for the Newton polyhedron $\Gamma_+(f)$ defining γ. $\square$

Let us return the proof of the theorem. Taking the partial derivatives of (8.6) with respect to y_j $(j\in J)$ and putting $x=\pi(\sigma)(y)$, we have

$$\frac{\partial f_\gamma}{\partial y_j}(\pi(\sigma)(y)) = \sum_{k=1}^n\left(\prod_{\ell=1}^n y_\ell^{a_k^\ell(\sigma)}\right)\cdot\frac{a_k^j(\sigma)}{y_j}\cdot\frac{\partial f_\gamma}{\partial x_k}(x)$$

and

$$\begin{aligned} &\frac{\partial}{\partial y_j}\left(\left(\prod_{\ell=1}^n y_\ell^{l(a^\ell(\sigma))}\right) f_\sigma(T_I(y))\right) \\ = \;& l(a^j(\sigma))\left(\prod_{\ell=1}^n y_\ell^{l(a^\ell(\sigma))-\delta_{j,\ell}}\right) f_\sigma(T_I(y)) + \left(\prod_{\ell=1}^n y_\ell^{l(a^\ell(\sigma))}\right)\frac{\partial f_\sigma}{\partial y_j} f_\sigma(T_I(y)). \end{aligned}$$

Here, $\delta_{j,\ell}$ is Kronecker's delta. These equations follow that

$$\sum_{k=1}^n a_k^j(\sigma) x_k \frac{\partial f_\gamma}{\partial x_k} = \left(\prod_{\ell=1}^n y_\ell^{l(a^\ell(\sigma))}\right)\left(l(a^j(\sigma))(f_\sigma\circ T_I)(y) + y_j\frac{\partial}{\partial y_j}(f_\sigma\circ T_I)(y)\right) \quad (8.7)$$

Now we suppose that there exists a point $b\in T_I^*(\mathbb{R}^n)$ such that

$$f_\sigma(b)=0 \text{ and } \frac{\partial f_\sigma}{\partial y_j}(b)=0$$

for a certain $j\in J$ and define $U_I(b) := \{x\in U;\ x=\pi(\sigma)(T_I^r(b)) \text{ for } r\in\mathbb{R}\setminus\{0\}\}$. Since the k-th coordinate of $x=\pi(\sigma)(T_I^r(b))$ is expressed by

$$x_k = \left(\prod_{j\in I} r^{a_k^j(\sigma)}\right)\cdot\left(\prod_{j\in J} b_j^{a_k^j(\sigma)}\right)$$

for any $k \in \{1, \ldots, n\}$. Since b_j is not equal to 0 for $j \in J$, x_k does not equal to 0 for every k, and thus, the set $U_I(b)$ is a subset of $(\mathbb{R} \setminus \{0\})^n$. Putting $y = T_I^r(b)$ ($r \in \mathbb{R} \setminus \{0\}$ is arbitrary), one can get

$$f_\sigma(T_I(y)) = f_\sigma(T_I(b)) = f_\sigma(b) = 0,$$

because the point b belongs to $T_I^*(\mathbb{R}^n)$. Since $\pi(\sigma)(y)$ is an element of $U_I(b)$ and (8.6) holds, $f_\gamma \equiv 0$ on the set $U_I(b)$. Hence, the equations (8.5) and (8.7) yield that, for any $j \in \{1, \ldots, n\}$ and $x \in U_I(b)$,

$$\sum_{k=1}^{n} a_k^j(\sigma) x_k \frac{\partial f_\gamma}{\partial x_k}(x).$$

Define the matrix A_σ by

$$A_\sigma := \begin{pmatrix} a_1^1(\sigma) & \cdots & a_n^1(\sigma) \\ \vdots & \ddots & \vdots \\ a_1^n(\sigma) & \cdots & a_n^n(\sigma) \end{pmatrix}.$$

Since the vectors $a^1(\sigma), \ldots, a^n(\sigma)$ are linealy independent, the matrix A_σ is invertible. Multiplying A_σ^{-1} on the left to

$$A_\sigma \begin{pmatrix} x_1 \dfrac{\partial f_\gamma}{\partial x_1}(x) \\ \vdots \\ x_n \dfrac{\partial f_\gamma}{\partial x_n}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

we have

$$x_j \frac{\partial f_\gamma}{\partial x_j}(x) = 0,$$

for any $j \in \{1, \ldots, n\}$, and thus,

$$\frac{\partial f_\gamma}{\partial x_j}(x) = 0$$

for $x \in U_I(b)$ and $j \in \{1, \ldots, n\}$. This is a contradiction to the condition (C). □

§8.2 Poles of $Z(s; \varphi)$

Lemma 8.2.1

Let $б(y_1, \ldots, y_n; \mu)$ *be a smooth function of* y *with a complex parameter* μ *and* *Supp*$(б)$ *is compact in* $\mathbb{R}^n$. *Then the function* Λ *defined by*

$$\Lambda(\tau_1, \ldots, \tau_n; \mu) := \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^{n} y_j^{\tau_j} \right) б(y_1, \ldots, y_n; \mu) dy_1 \cdots dy_n$$

can be analytically continued at all the complex values of $\tau_1, \ldots, \tau_n$ and μ as a meromorphic function. Furthermore, all poles of the integral $\Lambda(\tau_1, \ldots, \tau_n; \mu)$ are simple and lie on $\tau_j = -1, -2, \ldots$ for $j = 1, \ldots, n$.

Proof.

Integrating by parts, one can easily obtain. □

Remind that the local zeta function is defined by

$$Z(s;\varphi) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx$$

and it is easy to see that $Z(s;\varphi)$ is decomposed into

$$Z(s;\varphi) = \sum_{\theta \in \{-1,1\}^n} \int_{\mathbb{R}^n_+} f(\theta x)^s \varphi(\theta x) dx,$$

where $\theta x = (\theta_1 x_1, \ldots, \theta_n x_n)$. Now, to know properties of $Z(s;\varphi)$, we had better analyze $\tilde{Z}(s;\varphi)$ defined by

$$\tilde{Z}(s;\varphi) := \int_{\mathbb{R}^n_+} f(x)^s \varphi(x) dx.$$

Generally, the functions $Z_\pm(s;\varphi)$ are defined by

$$Z_\pm(s;\varphi) = \int_{\mathbb{R}^n_+} (f(x)_\pm)^s \varphi(x) dx,$$

respectively. However, condition (B) implies that $Z_-(s;\varphi)$ is constantly equal to 0.

Theorem 8.2.2

Let f and φ be smooth functions satisfying conditions (A), (B) and (C) in P.29 and P.36. If Supp(φ) is contained in a sufficiently small neighborhood of the origin, then the function $Z(s;\varphi)$ can be analytically continued to the complex plane as a meromorphic function, which is also denoted by the same symbol. Moreover, the poles of $Z(s;\varphi)$ is contained the following set:

$$\left\{ -\frac{\langle a \rangle + \nu}{l(a)} ; \nu \in \mathbb{Z}_+, a \in \Sigma^{(1)} \text{ with } l(a) > 0 \right\}, \tag{8.8}$$

where $l(a)$ is as in (6.1) with $P = \Gamma_+(f)$.

Proof.

Let $\tilde{Y}_\Sigma := Y_\Sigma \cap \pi^{-1}(\mathbb{R}^n_+)$ and $s \in \mathbb{C}$ satisfying $\Re(s) > 0$. Now we consider $\{\chi_\sigma\}$ such that

(i) For every $\sigma \in \Sigma^{(n)}$, $Supp(\chi_\sigma)$ is contained in $\mathbb{R}^n(\sigma)$ and $\chi_\sigma \equiv 1$ in some neighborhood of the origin,
(ii) $\sum_{\sigma\in\Sigma^{(n)}} \chi_\sigma \equiv 1$ on $Supp(\chi \circ \pi)$.
It can be easily seen that there exists $\{\chi_\sigma\}$ satisfying (i) and (ii). Then the local zeta function $\tilde{Z}(s;\varphi)$ is expressed as

$$\begin{aligned}
\tilde{Z}(s;\varphi) &= \int_{\mathbb{R}^n_+} f(x)^s \varphi(x)dx \\
&= \int_{\tilde{Y}_\Sigma} f(\pi(\sigma)(y))^s \varphi(\pi(\sigma)(y))|J_\pi(y)|dy \\
&= \sum_{\sigma\in\Sigma^{(n)}} \int_{\mathbb{R}^n_+} f(\pi(\sigma)(y))^s \varphi(\pi(\sigma)(y))\chi_\sigma(y)\,|J_{\pi(\sigma)}(y)|dy \\
&= \sum_{\sigma\in\Sigma^{(n)}} \int_{\mathbb{R}^n_+} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y)\right)^s \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle -1}\right| \varphi_\sigma(y)dy,
\end{aligned}$$

where dy is a volume element in Y_Σ and $\varphi_\sigma(y) := \varphi(\pi(\sigma)(y))\chi_\sigma(y)$. Next, we consider $\{\tilde{\chi}_\kappa : \mathbb{R}^n \to \mathbb{R}_+\}$ such that
(i) For each k, $Supp(\tilde{\chi}_\kappa)$ is sufficiently small,
(ii) $\sum_\kappa \tilde{\chi}_\kappa \equiv 1$ on $Supp(\varphi_\sigma)$.
Define $\psi_\kappa(y) := \varphi_\sigma(y)\tilde{\chi}_\kappa(y)$ and

$$\begin{aligned}
\zeta_{\sigma,k}(s) &:= \int_{\mathbb{R}^n_+} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y)\right)^s \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle -1}\right| \psi_\kappa(y)dy \\
&= \int_{\mathbb{R}^n_+} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))s+\langle a^j(\sigma)\rangle -1}\right) f_\sigma(y)^s \psi_\kappa(y)dy. \qquad (8.9)
\end{aligned}$$

Then we get

$$\tilde{Z}(s;\varphi) = \sum_{\sigma\in\Sigma^{(n)}} \sum_\kappa \zeta_{\sigma,\kappa}(s).$$

Lemma 8.2.1 yields that every $\zeta_{\sigma,\kappa}(s)$ can be analytically continued to the complex plane as a meromorphic function and the poles are contained in the set

$$\left\{-\frac{\langle a^j(\sigma)\rangle + \nu}{l(a^j(\sigma))};\, \nu \in \mathbb{Z}_+, j \in \{1,\ldots,n\} \text{ with } l(a^j(\sigma)) \neq 0\right\},$$

and thus, $Z(s;\varphi)$ can be also analytically continued. □

Now define the value $\tilde{\beta}(f)$ by

$$\tilde{\beta}(f) := \max\left\{-\frac{\langle a\rangle}{l(a)};\, a \in \Sigma^{(1)} \text{ with } l(a) > 0\right\}.$$

and consider the lemma below.

Lemma 8.2.3

For $a \in \Sigma^{(1)}$ with $l(a) > 0$, we define the point $q(a)$ by $q(a) := H(a, l(a)) \cap \{(t, \ldots, t); t > 0\}$. Then one has

$$q(a) = \left(\frac{l(a)}{\langle a \rangle}, \ldots, \frac{l(a)}{\langle a \rangle} \right). \tag{8.10}$$

In addition, if $l(a) > 0$, then the following conditions are equivalent:
(i) $\tilde{\beta}(f) = -\langle a \rangle / l(a)$.
(ii) $q_* = q(a)$.
(iii) $q_* \in H(a, l(a))$.

Proof.

The equation (8.10) and the equivalence of (ii) and (iii) are trivial. Hence, we should prove (i) $\Leftrightarrow$ (ii).

Remind $q_* = (d(f), \ldots, d(f))$. By the definition of the function $l(\cdot)$, we obtain that $l(a) \leq \langle a, (d(f), \ldots, d(f)) \rangle = \langle a \rangle \cdot d(f)$. Therefore, $-\langle a \rangle / l(a)$ is at most $-1/d(f)$.

If $q_* = q(a)$ for $a \in \Sigma^{(1)}$ with $l(a) > 0$, then $-\langle a \rangle / l(a) = -1/d(f)$, that is, such a vector $a \in \Sigma^{(1)}$ satisfies the condition (i).

Next, we assume that $\tilde{a} \in \Sigma^{(1)}$ satisfies $q_* \neq q(\tilde{a})$ and show that the condition (i) is not satisfied for such a vector $\tilde{a}$. From the argument above, one can find $-\langle \tilde{a} \rangle / l(\tilde{a}) < -1/d(f)$. On the other hand, there exists a valid pair (a_0, l_0) defining the principle face τ_*, that is, $H^+(a_0, l_0) \supset \Gamma_+(f)$ and $\tau_* = H(a_0, l_0) \cap \Gamma_+(f)$. Then we get $l(a_0) = l_0$ and $\langle a_0, (d(f), \ldots, d(f)) \rangle = l_0$. They imply that $-\langle a_0 \rangle / l(a_0) = -1/d(f) > -\langle \tilde{a} \rangle / l(\tilde{a})$. Now we get a conclusion. $\square$

Lemma 8.2.4

The value $m(f)$ is equal to $\max\{\#A(\sigma); \sigma \in \Sigma^{(n)}\}$, where

$$A(\sigma) := \left\{ j \in \{1, \ldots, n\};\, l(a^j(\sigma)) \text{ and } \tilde{\beta}(f) = -\frac{\langle a^j(\sigma) \rangle}{l(a^j(\sigma))} \right\}.$$

Proof.

From the proof of the lemma above, one can easily get $A(\sigma) = \{j \in \{1, \ldots, n\}; q_* \in H(a^j(\sigma), l(a^j(\sigma)))\} = \{j \in \{1, \ldots, n\}; \tau_* \subset H(a^j(\sigma), l(a^j(\sigma)))\}$. By the definition of $I(\cdot, \cdot)$, we have $A(\sigma) = I(\tau_*, \sigma)$. Lemma 6.1.5 (i) leads $n - m(f) = dim(\tau_*) \leq n - \#I(\tau_*, \sigma) = n - \#A(\sigma)$, that is, $\#A(\sigma) \leq m(f)$ for an arbitrary $\sigma \in \Sigma^{(n)}$. By Lemma 6.1.7 (ii), there exists a cone $\sigma_0 \in \Sigma^{(n)}$ such that $dim(\tau_*) = n - \#A(\sigma_0)$, that is, $\#A(\sigma_0) = m(f)$. $\square$

Proposition 8.2.5

Under the same assumption as Theorem 8.2.2, the largest element of (8.8) is $-1/d(f)$*. If the local zeta function* $Z(s;\varphi)$ *has a pole at* $s=-1/d(f)$*, then the order of the pole is at most* $m(f)$*.*

Proof.

Applying Lemma 8.2.1 to (8.9), one can obtain that, if $\zeta_{\sigma,\kappa}(s)$ has a pole at $s=\tilde{\beta}(f)$, then the upper bound of the order is $\#A(\sigma)$. Consequently, if $Z(s;\varphi)$ has a pole at $s=\tilde{\beta}(f)$, then the upper bound of the order is $\max\{\#A(\sigma)\}=m(f)$. $\square$

§8.3 The first coefficient of $Z(s;\varphi)$

In this section, we consider the first coefficient of the local zeta function. Put

$$\begin{aligned} \text{Б}(f,\varphi) = \text{Б} &:= \lim_{s\to -1/d(f)} (s+1/d(f))^{m(f)} Z(s;\varphi), \\ \tilde{\text{Б}} &:= \lim_{s\to -1/d(f)} (s+1/d(f))^{m(f)} \tilde{Z}(s;\varphi), \\ Z^{(\sigma)}(s) &:= \sum_{\kappa} \zeta_{\sigma,\kappa}(s), \end{aligned}$$

and $\Sigma_*^{(n)} := \{\sigma\in\Sigma^{(n)};\, A(\sigma)=m(f)\}$. Sometimes Б is denoted by $\text{Б}(f,\varphi)$ and it is obvious that

$$\text{Б} = \sum_{\theta\in\{-1,1\}^n} \text{Б}(f_\theta,\varphi_\theta),$$

where $f_\theta(x)=f(\theta x)$ and $\varphi_\theta(x)=\varphi(\theta x)$ for $x\in\mathbb{R}_+^n$. This implies that we had better consider the coefficient $\tilde{\text{Б}}$ to know about Б.

Now, this set $\Sigma_*^{(n)}$ is not empty. By the definitions of τ_*, $m(f)$ and $A(\sigma)$, If $\sigma\in\Sigma_*^{(n)}$, then one can immediately get

$$\tau_* = \bigcap_{j\in A(\sigma)} H(a^j(\sigma), l(a^j(\sigma))) \cap \Gamma_+(f).$$

In other words, $\gamma=\gamma(I,\sigma)$ holds, where $\gamma=\tau_*$ and $I=A(\sigma)$. Notice that the order of the pole of $Z^{(\sigma)}(s)$ at $s=-1/d(f)$ is less than $m(f)$ if $\sigma\in\Sigma^{(n)}\setminus\Sigma_*^{(n)}$. Since

$$\tilde{Z}(s;\varphi) = \sum_{\sigma\in\Sigma^{(n)}} Z^{(\sigma)}(s),$$

the first coefficient $\tilde{\text{Б}}$ is expressed as

$$\tilde{\text{Б}} = \sum_{\sigma\in\Sigma_*^{(n)}} \lim_{s\to -1/d(f)} (s+1/d(f)) Z^{(\sigma)}(s) =: \sum_{\sigma\in\Sigma_*^{(n)}} \text{Б}_\sigma.$$

Lemma 8.3.1

If f satisfies conditions (B) and (C), then there exists a cone $\sigma \in \Sigma_*^{(n)}$ such that $f_\sigma \circ T_{A(\sigma)}$ does not vanish on $\mathbb{R}_+^n \cap \pi(\sigma)^{-1}(U)$, where U is a sufficiently small open neighborhood of the origin in $\mathbb{R}^n$.

Proof.

Theorem 8.1.2 shows that f_σ has no zero point near $y = 0$. □

Lemma 8.3.2

Let Φ be a smooth function defined on $\mathbb{R}$ and λ be a natural number. Then

$$\lim_{w \to -\lambda} (w + \lambda) \int_0^\infty y^w \Phi(y) dy = \frac{1}{(\lambda - 1)!} \Phi^{(\lambda-1)}(0).$$

Here, w is a complex number near $-\lambda$. In particular, when $\lambda = 1$,

$$\lim_{w \to -1} (w + 1) \int_0^\infty y^w \Phi(y) dy = \Phi(0).$$

Proof.

Trivial. □

Theorem 8.3.3

The first coefficient $\tilde{Б}$ is expressed by

$$\tilde{Б} = \left(\prod_{j \in A(\sigma)} \frac{1}{l(a^j(\sigma))} \right) \int_{\mathbb{R}_+^{n-m(f)}} \frac{(\varphi \circ \pi(\sigma))(T_{A(\sigma)})}{f_\sigma(T_{A(\sigma)}(y))^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{\Xi_j(\sigma)} dy_j, \tag{8.11}$$

where $\Xi_j(\sigma) = -l(a^j(\sigma))/d(f) + \langle a^j(\sigma) \rangle - 1$ and

$$\prod_{j \notin A(\sigma)} y_j^{\Xi_j(\sigma)} dy_j = \left(\prod_{j \notin A(\sigma)} y_j^{\Xi_j(\sigma)} \right) \cdot \left(\prod_{j \notin A(\sigma)} dy_j \right).$$

In particular, when $m(f) = n$, the first coefficient is specified as

$$\tilde{Б} = \left(\prod_{j=1}^{n} \frac{1}{l(a^j(\sigma))} \right) \cdot \frac{\varphi(0)}{f_\sigma(0)^{1/d(f)}}. \tag{8.12}$$

Proof.

First, we assume $m(f) < n$. Using (8.9) and the lemma above, we will show

$$\tilde{Б}_\sigma = \sum_\kappa G^{(\kappa)}(\sigma), \tag{8.13}$$

where

$$G^{(\kappa)}(\sigma)=\left(\prod_{j\in A(\sigma)}\frac{1}{l(a^j(\sigma))}\right)\int_{\mathbb{R}_+^{n-m(f)}}\frac{\psi_\kappa(T_{A(\sigma)}(y))}{f_\sigma(T_{A(\sigma)}(y))^{1/d(f)}}\prod_{j\notin A(\sigma)}y_j^{\Xi_j(\sigma)}dy_j.$$

Then (8.13) and the equation $\sum\psi_\kappa=\phi_\sigma$ yield the expression (8.11).

Put $w_j=l(a^j(\sigma))s+\langle a^j(\sigma)\rangle-1$ for $j\in A(\sigma)$. If s tends to $-1/d(f)$, then w_j goes to -1. Define operators $X_1,\ldots,X_n$ as following:
(i) If $j\in A(\sigma)$ and $\Phi_{j-1}\in C^\infty$, then

$$\Phi_j(\tilde{y})=X_j(\Phi_{j-1}):=\lim_{s\to-1/d(f)}\left(s+\frac{1}{d(f)}\right)\int_0^\infty\Phi_{j-1}(y)dy_j,$$

where $\tilde{y}=(y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_n)$.
(ii) If $j\notin A(\sigma)$, then $\Phi_j=X_j(\Phi_{j-1}):=\Phi_{j-1}$.
Notice that, for $j\in A(\sigma)$,

$$\begin{aligned}X_j(\Phi_{j-1})(\tilde{y}) &= \lim_{sto-1/d(f)}\left(s+\frac{1}{d(f)}\right)\int_0^\infty y_j^{l(a^j(\sigma))s+\langle a^j(\sigma)\rangle-1}\tilde{\Phi}_{j-1}(y)dy_j\\ &= \frac{1}{l(a^j(\sigma))}\lim_{w_j\to-1}(w_j+1)y_j^{w_j}\tilde{\Phi}_{j-1}(y)dy_j\\ &= \frac{\tilde{\Phi}_{j-1}(T_{\{j\}}(y))}{l(a^j(\sigma))},\end{aligned}$$

where

$$y_j^{l(a^j(\sigma))s+\langle a^j(\sigma)\rangle-1}\tilde{\Phi}_{j-1}(y)=\Phi_{j-1}(y).$$

Putting

$$\Phi_0(y):=\left(\prod_{j=1}^n y_j^{l(a^j(\sigma))s+\langle a^j(\sigma)\rangle-1}\right)f_\sigma(y)^{-1/d(f)}\psi_\kappa(y),$$

we have

$$\begin{aligned}\Phi_n(y) &= \big((X_n\circ\cdots\circ X_1)(\Phi_0)\big)(y)\\ &= \left(\prod_{j\in A(\sigma)}\frac{1}{l(a^j(\sigma))}\right)\frac{\psi_\kappa(T_{A(\sigma)}(y))}{f_\sigma(T_{A(\sigma)}(y))^{1/d(f)}}\prod_{j\notin A(\sigma)}y_j^{\Xi_j(\sigma)}.\end{aligned}$$

Indeed, (8.2) implies that f_σ does not have any effect on the limitation as $s\to-1/d(f)$ and terms which satisfy $\langle a^j(\sigma),p\rangle>l(a^j(\sigma))$ also have no effect.

When $m(f)=n$, similar calculation gives the conclusion. $\square$

Remark 8.3.4

The equations (8.11) and (8.12) are rewritten as

$$\tilde{Б} = \left(\prod_{j \in A(\sigma)} \frac{1}{l(a^j(\sigma))} \right) \int_{\mathbb{R}_+^{n-m(f)}} \frac{(\varphi \circ \pi(\sigma))(T_{A(\sigma)}(y))}{(f_{\tau_*} \circ \pi(\sigma))(T^1_{A(\sigma)}(y))^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{\langle a^j(\sigma) \rangle - 1} dy_j \quad (8.11)'$$

and

$$\tilde{Б} = \left(\prod_{j=1}^{n} \frac{1}{l(a^j(\sigma))} \right) \frac{\varphi(0)}{f_{\tau_*}(1, \ldots, 1)^{1/d(f)}} = \left(\prod_{j=1}^{n} \frac{1}{l(a^j(\sigma))} \right) \frac{(d(f)!)^{n/d(f)} \varphi(0)}{(\partial^{q_*} f)(0)^{1/d(f)}}, \quad (8.12)'$$

respectively. Notice that the principle face τ_* is a vertex $q_* = (d(f), \ldots, d(f))$, and thus, $d(f)$ is an integer, when $m(f) = n$.

From the equations (8.11) and (8.12), it is clear that $\Re(Б) > 0$ if $\Re(\varphi(0)) > 0$.

In particular, consider the case $m(f) = n$ and $\varphi(0) \neq 0$. Then we can assume that the value $\varphi(0)$ equals to 1 without loss of generality. Indeed, we should regard the new smooth function $\varphi(x)/\varphi(0)$ as new φ. From (8.12) or (8.12)', one can see that:

(i) $\Re(\tilde{Б}) > 0$ if and only if $\Re(\varphi(0)) > 0$,
(ii) $\Re(\tilde{Б}) < 0$ if and only if $\Re(\varphi(0)) < 0$,
(iii) $\Im(\tilde{Б}) > 0$ if and only if $\Im(\varphi(0)) > 0$,
(iv) $\Im(\tilde{Б}) < 0$ if and only if $\Im(\varphi(0)) < 0$.

These properties imply that the first coefficient is not equal to 0 unless $\varphi(0)$ does not equal to 0.

Remark 8.3.5

If τ_* is compact, then $\pi(\sigma) \circ T_{A(\sigma)}(\mathbb{R}^n) = 0$ and (8.11)' is rewritten as

$$\tilde{Б} = \left(\prod_{j \in A(\sigma)} \frac{1}{l(a^j(\sigma))} \right) \int_{\mathbb{R}_+^{n-m(f)}} \frac{\varphi(0)}{(f_{\tau_*} \circ \pi(\sigma))(T^1_{A(\sigma)}(y))^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{\langle a^j(\sigma) \rangle - 1} dy_j. \quad (8.11)''$$

Remark 8.3.6

Let $n = 1$ and $P = [k, \infty)$ ($k \in \mathbb{Z}_+$). Then the principle face is the point $\{k\}$ and we have

$$Б(f_{+1}, \varphi_{+1}) = \frac{1}{k} \cdot \frac{(k!)^{1/k} \varphi(0)}{(f^{(k)}(0))^{1/k}}$$

from (8.12)', where

$$f_{+1}(x) = \begin{cases} f(x) & (x \geq 0), \\ 0 & (x < 0), \end{cases} \qquad \varphi_{+1}(x) = \begin{cases} \varphi(x) & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

Similarly we get

$$Б(f_{-1},\varphi_{-1}) = \frac{1}{k}\cdot\frac{(k!)^{1/k}\varphi(0)}{(f^{(k)}(0))^{1/k}},$$

where

$$f_{-1}(x) = \begin{cases} 0 & (x \geq 0), \\ f(x) & (x < 0), \end{cases} \qquad \varphi_{-1}(x) = \begin{cases} 0 & (x \geq 0), \\ \varphi(x) & (x < 0). \end{cases}$$

Therefore,

$$Б = Б(f_{+1},\varphi_{+1}) + Б(f_{-1},\varphi_{-1}) = \frac{2}{k}\cdot\frac{(k!)^{1/k}\varphi(0)}{(f^{(k)}(0))^{1/k}}.$$

This is equal to (4.10).

Chapter 9 Proof of the main theorem

§9.1 Asymptotic expansion of the Gelfand-Leray function

Before the proof of the main theorem, we show the asymptotic expansion of the Gelfand-Leray function K as below and some important lemmas.

Theorem 9.1.1

Let $0 < s_1 < s_2 < \ldots$. Suppose that $Z(s;\varphi)$ has poles at $s = -s_j$, the order of the pole at $s = -s_j$ is k_j for every $j = 1, 2, \ldots$ and $b_{j,k}$ is the coefficient of $(s+s_j)^{-k}$ of $Z(s;\varphi)$. Then one has

$$\mathrm{K}(u) \sim \sum_{j=1}^{\infty}\sum_{k=1}^{k_j} a_{j,k} u^{s_j-1}(\log u)^{k-1}, \tag{9.1}$$

as $u \to +0$, and

$$a_{j,k} = \frac{(-1)^{k-1}}{(k-1)!}\cdot b_{j,k}.$$

Outline of the proof.

Details are written in [7], [12].

First, we should choose $\lambda > 0$ large enough. Notice that the integral

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} u^{-s-1}Z(s;\varphi)ds$$

converges in the sense of distributions. Applying Cauchy integral formula, for $\tilde{\lambda} \in \mathbb{R}$ with $\tilde{\lambda} < \lambda$ and $\tilde{\lambda} \neq s_j$ for any j, one can get

$$\begin{aligned} \mathrm{K}(u) &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} u^{-s-1}Z(s;\varphi)ds \\ &= \frac{1}{2\pi i}\int_{\tilde{\lambda}-i\infty}^{\tilde{\lambda}+i\infty} u^{-s-1}Z(s;\varphi)ds + \sum_{\tilde{\lambda}<-s_j<\lambda} \frac{1}{2\pi i}\int_{\Upsilon_j} u^{-s-1}Z(s;\varphi)ds, \end{aligned} \tag{9.2}$$

where Υ_j is the boundary of a sufficirntly small open neighborhood of $s = -s_j$. To simplify, we suppose each neighborhood of $s = -s_j$ is simply connected. Now, the first term in the right-hand size of (9.2) can be eatimated as

$$\left|\frac{1}{2\pi i}\int_{\tilde{\lambda}-i\infty}^{\tilde{\lambda}+i\infty} u^{-s-1}Z(s;\varphi)ds\right| \leq R\cdot u^{-\lambda-1},$$

where R is a sufficiently large (positive) constant. On the other hand, by Cauchy's residue theorem, the last term in (9.2) gives

$$\sum_{\tilde{\lambda}<-s_j<\lambda} \frac{1}{2\pi i} \int_{\Upsilon_j} u^{-s-1} Z(s;\varphi) ds = \sum_{\tilde{\lambda}<-s_j} u^{s_j-1} \sum_{k=1}^{k_j} \frac{(-1)^{k-1} b_{j,k}}{(k-1)!} \cdot (\log u)^{k-1}.$$

Consequently, the asymptotic expansion (9.1) holds. □

In particular, a_{1,k_1} equals to $(-1)^{k_1-1}\text{Б}(f,\varphi)/(k_1-1)!$. Therefore, the coefficient of the leading term in (9.1) is not 0 if and only if the coefficient of $(s+s_1)^{-k_1}$ in the Laurent series of $Z(s;\varphi)$ is not 0.

§9.2 Lemmas for the main theorem

Lemma 9.2.1

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfies $\chi(u) \equiv 1$ on $[-\delta,\delta]$ for a sufficiently small $\delta > 0$ and $0 \le \chi(u) \le 1$ for all $u \in \mathbb{R}$. Then

$$\mathcal{I}_m(t;p) := \int_0^\infty e^{-tu} u^p (\log u)^m \chi(u) du - \left(\frac{d}{dp}\right)^m \left(\frac{\Gamma(p+1)}{t^{p+1}}\right)$$

is rapidly decreasing as $t \to \infty$, if $p > -1$ and $m \in \mathbb{Z}_+$.

Proof.

Changing the integral variables, we get

$$\int_0^\infty e^{-tu} u^p du = \frac{\Gamma(p+1)}{t^{p+1}}. \tag{9.3}$$

By exchanging the integral and derivatives, $\mathcal{I}_m(t;p)$ is calculated as

$$\begin{aligned} \mathcal{I}_m(t;p) &= \int_0^\infty e^{-tu} u^p (\log u)^m (\chi(u)-1) du \\ &= \int_\delta^\infty e^{-tu} u^p (\log u)^m (\chi(u)-1) du \\ &= \int_0^\infty e^{-t(\delta+u)} (\delta+u)^p (\log(\delta+u))^m (\chi(\delta+u)-1) du \\ &= e^{-t\delta} \int_0^\infty e^{-tu} (\delta+u)^p (\log(\delta+u))^m (\chi(\delta+u)-1) du. \end{aligned} \tag{9.4}$$

For sufficiently large $u > 0$, we see that $|(\delta+u)^p(\chi(\delta+u)(\log(\delta+u))^m - 1)|$ is less than $\exp(tu/2)$, thus the absolute value of the integral in (9.4) is less than

$$R \int_0^\infty e^{-tu/2} du = \frac{2}{t},$$

where R is a large constant which is independent of t. The estimate shows that $\mathcal{I}_m(t;p)$ is rapidly decreasing as $t \to \infty$. $\square$

Lemma 9.2.2

Let m be a non-negative integer and $p > -1$. Then one has

$$\int_0^\infty e^{-tu} u^p (\log u)^m du = \left(\frac{d}{dp}\right)^m \left(\frac{\Gamma(p+1)}{t^{p+1}}\right) = \frac{1}{t^{p+1}} \sum_{j=0}^{m} c_j^{[m]}(p) \cdot (\log t)^j, \tag{9.5}$$

as $t \to \infty$, where

$$c_j^{[m]}(p) = (-1)^j \binom{m}{j} \int_0^\infty e^{-u} u^p (\log u)^{m-j} du. \tag{9.6}$$

In particular, $c_m^{[m]}(p) = (-1)^m \Gamma(p+1)$.

Proof.

Let us prove this lemma by induction on m.

From the definition of $c_j^{[m]}$, it is trivial that (9.5) holds for $m = 0$. When $m = 1$, we get

$$\frac{d}{dp}\left(\frac{\Gamma(p+1)}{t^{p+1}}\right) = -\frac{\log t}{t^{p+1}} \int_0^\infty e^{-u} u^p du + \frac{1}{t^{p+1}} \int_0^\infty e^{-u} u^p \log u \, du.$$

Next, let us assume that the lemma holds for $m = k$ ($k \geq 1$) and show for $m = k+1$. Similarly as above, we have

$$\begin{aligned}
& \frac{d}{dp}\left(\frac{1}{t^{p+1}} \sum_{j=0}^{k} c_j^{[k]}(p) \cdot (\log t)^j\right) \\
= \; & -\frac{\log t}{t^{p+1}} \sum_{j=0}^{k} c_j^{[k]}(p) \cdot (\log t)^j + \frac{1}{t^{p+1}} \sum_{j=0}^{k} \frac{d}{dp} c_j^{[k]}(p) \cdot (\log t)^j \\
= \; & \frac{1}{t^{p+1}} \sum_{j=0}^{k} \left((-1)^{j+1} \binom{k}{j} \int_0^\infty e^{-u} u^p (\log u)^{k-j} du\right) (\log t)^{j+1} \\
& + \frac{1}{t^{p+1}} \sum_{j=0}^{k} \left((-1)^j \binom{k}{j} \int_0^\infty e^{-u} u^p (\log u)^{k-j+1} du\right) (\log t)^j \\
= \; & \frac{1}{t^{p+1}} \left((-1)^{k+1} \int_0^\infty e^{-u} u^p du\right) (\log t)^{k+1} + \frac{1}{t^{p+1}} \left(\int_0^\infty e^{-u} u^p (\log u)^{k+1} du\right) \\
& + \frac{1}{t^{p+1}} \sum_{j=1}^{k} \left((-1)^j \left(\binom{k}{j} + \binom{k}{j-1}\right) \int_0^\infty e^{-u} u^p (\log u)^{k-j+1} du\right) (\log t)^j
\end{aligned}$$

$$= \frac{1}{t^{p+1}}\left(c_{k+1}^{[k+1]}(p)\cdot(\log t)^{k+1} + c_0^{[k+1]}(p) + \sum_{j=1}^{k} c_j^{[k+1]}(p)\cdot(\log t)^j\right).$$

At the end, we should show the integral in (9.6) converges. Since $\exp(-u) \le 1$ on the closed interval $[0,1]$, we get

$$\begin{aligned}
\left|\int_0^1 e^{-u}u^p(\log u)^m du\right| &\le (-1)^m \int_0^1 u^p(\log u)^m du \\
&= (-1)^m\left(\left[\frac{u^{p+1}}{p+1}(\log u)^m\right]_{u=0}^1 - \frac{m}{p+1}\int_0^1 u^p(\log u)^{m-1}du\right) \\
&= \cdots = \frac{m!}{(p+1)^m}\int_0^1 u^p du = \frac{m!}{(p+1)^{m+1}}.
\end{aligned}$$

Since $\log u$ is non-negative and less than u for $u \ge 1$, we obtain

$$\begin{aligned}
\left|\int_1^\infty e^{-u}u^p(\log u)^m du\right| &\le \int_1^\infty e^{-u}u^{p+m}du \\
&\le \int_0^\infty e^{-u}u^{p+m}ds = \Gamma(p+m+1).
\end{aligned}$$

Now we get the conclusion. □

§9.3 Proof of Theorem 5.2.1

Let N be a sufficiently large natural number and $\varepsilon > 0$ be a sufficiently small number such that $s_{N+1} - s_N < \varepsilon$. Rewrite (9.1) to

$$\mathrm{K}(u) = \sum_{j=1}^{N}\sum_{k=1}^{k_j} a_{j,k}u^{s_j-1}(\log u)^{k-1} + \mathsf{o}\left(u^{s_N-1}\right),$$

as $u \to +0$. Lemmas 9.2.1 and 9.2.2 show that, for $j = 1, 2, \ldots$,

$$\begin{aligned}
\int_0^\infty e^{-tu}u^{s_j-1}(\log u)^{k-1}\chi(u)du &= \int_0^\infty e^{-tu}u^{s_j-1}(\log u)^{k-1}du + \mathcal{I}_{k-1}(t; s_j-1) \\
&= \sum_{\ell=0}^{k-1}\frac{1}{t^{s_j}}\cdot c_\ell^{[k-1]}(s_j-1)\cdot(\log t)^\ell + \mathsf{o}\left(t^{-(s_N+\varepsilon)}\right).
\end{aligned}$$

If the support of φ is sufficiently small, then the maximum of f is also small and $\mathrm{K}(u)$ is constantly equal to 0 for large $u > 0$. Finnaly we obtain

$$\int_0^\infty e^{-tu}\mathrm{K}(u)\chi(u)du$$

$$
\begin{aligned}
&= \int_0^\infty e^{-tu}\left(\sum_{j=1}^{N}\sum_{k=1}^{k_j} a_{j,k} u^{s_j-1}(\log u)^{k-1} + \mathsf{o}\left(u^{s_N-1}\right)\right) du \\
&= \sum_{j=1}^{N}\sum_{k=1}^{k_j}\left(\sum_{\ell=0}^{k-1} \frac{a_{j,k}}{t^{s_j}} c_\ell^{[k-1]}(s_j-1)\cdot(\log t)^\ell + \mathsf{o}\left(t^{-(s_N+\varepsilon)}\right)\right) + \mathsf{o}\left(t^{-(s_N+\varepsilon)}\right) \\
&= \sum_{j=1}^{N}\sum_{\ell=1}^{k_j}\sum_{k=\ell}^{k_j} a_{j,k} c_{\ell-1}^{[k]}(s_j-1)\cdot t^{-s_j}(\log t)^{\ell-1} + \mathsf{o}\left(t^{-(s_N+\varepsilon)}\right), \qquad (9.7)
\end{aligned}
$$

as $t \to \infty$, and thus,

$$L(t;\varphi) = \mathsf{O}\left(t^{-s_1}(\log t)^{k_1-1}\right).$$

Moreover, the coefficient of $t^{-s_1}(\log t)^{k_1-1}$ in (9.7) is specified as

$$
\begin{aligned}
a_{1,k_1} c_{k_1-1}^{[k_1-1]}(s_1-1) &= \frac{\Gamma(s_1)}{(k_1-1)!}\cdot b_{1,k_1} \\
&= \frac{\Gamma(s_1)}{(k_1-1)!}\cdot \text{Б}(f,\varphi).
\end{aligned}
$$

To know Б(f,φ), look back §8.3. □

Appendix

Here, we introduce several propositions which have been already known or can be easily obtained.

Appendix 1

About the Mellin transformation, the following facts are known. See also [17], [14], [18] and so on. $\longrightarrow$ *cf. §7.1.*

(i) $(\mathcal{M}f^{(m)})(s) = \Gamma(m+1-s)\cdot(\mathcal{M}f)(s-m)/\Gamma(1-s)$. *In particular,* $(\mathcal{M}f')(s) = -(s-1)\cdot(\mathcal{M}f)(s-1)$.

(ii) *Let* $g(s) = (\mathcal{M}f)(s)$. *Then*

(a) *If* $\tilde{f}(x) = f(ax)$ *for* $a > 0$, *then* $(\mathcal{M}\tilde{f})(s) = a^{-s}g(s)$.

(b) *If* $\tilde{f}(x) = f(x^a)$ *for* $a > 0$, *then* $(\mathcal{M}\tilde{f})(s) = g(s/a)/a$.

(c) *If* $\tilde{f}(x) = x^a f(x)$, *then* $(\mathcal{M}\tilde{f})(s) = g(s+a)$.

(d) *If* $\tilde{f}(x) = f(1/x)$, *then* $(\mathcal{M}\tilde{f})(s) = g(-s)$.

(e) *If* $\tilde{f}(x) = (\log x)^m f(x)$, *then* $(\mathcal{M}\tilde{f})(s) = g^{(m)}(s)$.

(iii) *Define functions* f_j $(j = 1, \ldots, 8)$ *by*

$$f_1(x) := \begin{cases} 1 & (0 \le x \le 1) \\ 0 & (x > 1), \end{cases} \qquad f_2(x) := \begin{cases} 0 & (0 \le x \le 1) \\ 1 & (x > 1), \end{cases}$$

$$f_3(x) := \begin{cases} x^a & (0 \le x \le 1) \\ 0 & (x > 1), \end{cases} \qquad f_4(x) := \begin{cases} 0 & (0 \le x \le 1) \\ x^a & (x > 1), \end{cases}$$

$$f_5(x) := \begin{cases} \log x & (0 \le x \le 1) \\ 0 & (x > 1), \end{cases} \qquad f_6(x) := \begin{cases} 0 & (0 \le x \le 1) \\ \log x & (x > 1), \end{cases}$$

Then each $(\mathcal{M}f_j)(s)$ *is calculated as below.*

$$(\mathcal{M}f_1)(s) = \frac{1}{s} \quad (\Re(s) > 0), \qquad (\mathcal{M}f_2)(s) = -\frac{1}{s} \quad (\Re(s) < 0),$$

$$(\mathcal{M}f_3)(s) = \frac{1}{s+a} \quad (\Re(s) > -\Re(a)), \quad (\mathcal{M}f_4)(s) = -\frac{1}{s+a} \quad (\Re(s) < -\Re(a)),$$

$$(\mathcal{M}f_5)(s) = -\frac{1}{s^2} \quad (\Re(s) > 0), \qquad (\mathcal{M}f_6)(s) = \frac{1}{s^2} \quad (\Re(s) < 0).$$

Appendix 2

Calculating with Mathematica, one can get the approximate values of

$$\int_0^\infty e^{-u}u^p(\log u)^m du.$$

are as below. $\longrightarrow$ *cf. P.61, (9.6).*

p	$m=0$	1	2	3	4	5
-0.9	9.51351	-99.1665	1998.67	-59996	2.4×10^6	-1.2×10^8
-0.8	4.59084	-24.2812	249.013	-3747.14	74990.7	-1.9×10^6
-0.7	2.99157	-10.478	73.3325	-738.647	9870.36	-164584
-0.6	2.21816	-5.68156	30.6906	-232.793	2339.55	-29281.1
-0.5	1.77245	-3.48023	15.5802	-94.7686	765.092	-7669.52
-0.4	1.48919	-2.29428	8.94962	-45.3095	306.604	-2564.85
-0.3	1.29806	-1.58366	5.61085	-24.1747	141.363	-1014.92
-0.2	1.16423	-1.12349	3.7613	-13.9542	72.2398	-454.082
-0.1	1.06863	-0.806737	2.66351	-8.53425	39.9613	-223.032
0	1	-0.577216	1.97811	-5.44487	23.5615	-117.839
0.1	0.951351	-0.40314	1.5344	-3.57675	14.658	-65.9802
0.2	0.918169	-0.265387	1.24037	-2.3883	9.56259	-38.6929
0.3	0.897471	-0.151844	1.04365	-1.59662	6.52221	-23.5257
0.4	0.887264	-0.0544643	0.913105	-1.04536	4.64961	-14.684
0.5	0.886227	0.0323384	0.829627	-0.643769	3.47149	-9.30277
0.6	0.893515	0.112625	0.781218	-0.33682	2.72416	-5.88978
0.7	0.908639	0.189495	0.760281	-0.089717	2.2554	-3.63176
0.8	0.931384	0.265436	0.762053	0.120498	1.9749	-2.0669
0.9	0.961766	0.342566	0.783685	0.309698	1.82811	-0.922708
1	1	0.422784	0.823681	0.489462	1.78198	-0.0320378

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Symbols

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