REPRESENTATIONS OF CLANS AND THE BASIC RELATIVE INVARIANTS

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REPRESENTATIONS OF CLANS AND THE BASIC RELATIVE INVARIANTS

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Abstract. The basic relative invariants on clans are generalizations of the principal minors of real symmetric matrices obtained by focusing on the relative invariance. In this paper, we present an explicit expression of the basic relative invariants on the clans extended by representations of a given clan. We also get an explicit expression of the corresponding parameters of one-dimensional representations in a matrix form which we call the multiplier matrix.

Introduction

Non-associative algebras called clans introduced by Vinberg [10] are significant algebraic objects in studying homogeneous convex domains. Among them, those which have the unit element correspond to homogeneous open convex cones containing no entire line (homogeneous cones for short in what follows). Moreover, we know by Ishi [4] that any homogeneous cone is described as a positivity set of the irreducible polynomials called the basic relative invariants. On the other hand, homogeneous cones provide many examples of non-reductive prehomogeneous vector spaces (see Kimura [7] for definition), and the basic relative invariants are keys to analysis on those spaces. These facts form the backgrounds of our study.

In the previous paper [9], we construct a clan from a symmetric cone Ω paired with a representation φ of the associated Euclidean Jordan algebra, and calculate the basic relative invariants of the resulting clan. The crucial facts there are that φ is automatically a representation of the associated clan in the sense of Ishi [5] and that the quadratic map associated with φ is Ω-positive. In this paper, we generalize the results of [9] by starting with the homogeneous cone Ω corresponding to an arbitrary clan $(V, \triangle)$ with unit element. However, the quadratic map associated with a representation of V is no longer Ω-positive in general, but Ω^*-positive, where Ω^* is the dual cone of Ω. This forces us to switch to a representation (φ, E) of the dual clan $(V, \triangledown)$ corresponding to Ω^*. As in the previous paper [9], we construct a clan V_E^0 from V and φ, and obtain an explicit expression of the basic relative invariants of V_E^0. We note that in [9] we are led to divide the cases in order to have an explicit expression of the basic relative invariants of V_E^0 according to the non-regularity states of the Jordan algebra representation in the sense of Clerc [1]. This phenomenon is captured in this paper by introducing the notation of $\boldsymbol{\varepsilon}$-representations which are related to the range of the quadratic map associated with φ.

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We now describe the body of this paper. Let V be a clan with unit element e_0 associated with a homogeneous cone Ω, and an inner product $\langle\cdot|\cdot\rangle$ of V given by an admissible linear form. We know that the split solvable Lie group H generated by exponentials of the left multiplication operators of V acts simply transitively on Ω. We fix a complete system $c_1,\dots,c_r$ of orthogonal primitive idempotents and denote the corresponding normal decomposition by $\bigoplus_{j\le k} V_{kj}$. The dual clan product $\triangledown$ is defined through

$$\big\langle x \triangledown y \,|\, z\big\rangle = \langle y\,|\,x \vartriangle z\rangle \quad (x,y,z\in V).$$

Let E be a Euclidean vector space with inner product $\langle\cdot|\cdot\rangle_E$ and φ a linear map from V to the vector space $\mathcal{L}(E)$ of linear operators on E. We call φ a selfadjoint representation of the clan $(V,\triangledown)$ if $\varphi(x)$ is a selfadjoint operator for every $x\in V$ and if the following condition is satisfied:

$$\varphi(x\triangledown y) = \overline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x) \quad (x,y\in V),$$

where $\overline{\varphi}(x)$ (resp. $\underline{\varphi}(x)$) is the upper (resp. lower) triangular part of $\varphi(x)$ (see (2.1)). Denoting by Q the bilinear map $E\times E\to V$ associated with φ defined through

$$\langle Q(\xi,\eta)\,|\,x\rangle = \langle \varphi(x)\xi\,|\,\eta\rangle_E \quad (\xi,\eta\in E,\ x\in V),$$

we introduce a product $\vartriangle$ in the space $V_E := E\oplus V$ by

$$(\xi+x)\vartriangle(\eta+y) := \underline{\varphi}(x)\eta + (Q(\xi,\eta) + x\vartriangle y) \quad (\xi,\eta\in E,\ x,y\in V).$$

Then $(V_E,\vartriangle)$ is indeed a clan (Theorem 3.1). If $\dim E>0$, we make an adjunction of a unit element e to V_E and obtain a clan $V_E^0 := \mathbb{R}e\oplus V_E$. Putting $u := e-e_0$, we also have $V_E^0 = \mathbb{R}u\oplus V_E$ and denote a general element v of V_E^0 by $v=\lambda u+\xi+x$ without any comments. In Proposition 4.1, we calculate $\operatorname{Det} R_v^0$ for the right multiplication operators R_v^0 of V_E^0 to obtain

$$\operatorname{Det} R^0_{\lambda u+\xi+x} = \lambda^{1+\dim E-\dim V}\operatorname{Det} R_{\lambda x-\frac12 Q[\xi]},$$

where R is the right multiplication operator of V and $Q[\xi] := Q(\xi,\xi)$. Let $\Delta_j(x)$ $(j=1,\dots,r)$ be the basic relative invariants of V. Then we see in Proposition 4.2 and Theorem 4.3 that the basic relative invariants $P_j(v)$ $(j=0,1,\dots,r)$ of V_E^0 are described as

$$(0.1)\qquad \begin{cases} P_0(\lambda u+\xi+x) = \lambda, \\ P_j(\lambda u+\xi+x) = \lambda^{-\alpha_j}\Delta_j(\lambda x - \frac12 Q[\xi]) \quad (j=1,\dots,r), \end{cases}$$

where α_j are non-negative integers. The determination of α_j will be done in Section 5. As an application of Theorem 4.3, we are able to give an expression of the positive integers $n_1,\dots,n_r$ appearing in the formula

$$\operatorname{Det} R_x = \Delta_1(x)^{n_1}\cdots\Delta_r(x)^{n_r} \quad (x\in V).$$

The row vector $\underline{n}=(n_1,\dots,n_r)$ is called the basic index of V in this paper. Using the normal decomposition that we are fixing, we put $m_j := \sum_{k\ge j}\dim V_{kj}$ and $\underline{m} = (m_1,\dots,m_r)$. Then the basic index $\underline{n}$ is described in Theorem 4.4 as

$$\underline{n} = \underline{m}\sigma^{-1},$$

where $\sigma=\sigma_V$ is the multiplier matrix of V (see (1.4) for definition).

In order to describe the non-negative integers α_j that appeared in (0.1), we define an $\boldsymbol{\varepsilon}$-representation ($\boldsymbol{\varepsilon} \in \{0,1\}^r$) to be a representation of $(V, \triangledown)$ such that the range $Q[E]$ of the associated quadratic map $Q[\xi]$ is equal to the closure $\overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$ of the H-orbit $\mathcal{O}_{\boldsymbol{\varepsilon}}$ through $\varepsilon_1 c_1 + \cdots + \varepsilon_r c_r$. Using the results due to Graczyk and Ishi [3], we see in Proposition 2.4 that for every representation φ, there exists a unique $\boldsymbol{\varepsilon}(\varphi) \in \{0,1\}^r$ such that φ is an $\boldsymbol{\varepsilon}(\varphi)$-representation. Then we see in Theorem 5.1 that $\boldsymbol{\alpha} = {}^t(\alpha_1, \ldots, \alpha_r)$ is given by

$$\boldsymbol{\alpha} = \sigma_V(\mathbf{1} - \boldsymbol{\varepsilon}),$$

where $\mathbf{1} = {}^t(1, \ldots, 1)$. Moreover the multiplier matrix σ^0 of V_E^0 is written as

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ \sigma_V \boldsymbol{\varepsilon} & \sigma_V \end{pmatrix}.$$

Our final objective is to determine the multiplier matrix σ_V of V, and this completes Theorems 4.3, 4.4 and 5.1. Let $V^{[k]}$ and $E^{[k]}$ $(k = 1, \ldots, r-1)$ be the subspaces of V defined respectively by

$$V^{[k]} = \bigoplus_{k<l\le m\le r} V_{ml}, \quad E^{[k]} = \bigoplus_{m>k} V_{mk}.$$

Then $V^{[k]}$ is a subclan and we have $E^{[k]} \triangledown V^{[k]} \subset E^{[k]}$. The latter property enables us to define a representation $\mathcal{R}^{[k]}$ of $V^{[k]}$ on $E^{[k]}$ by $\mathcal{R}^{[k]}(x)\xi := \xi \triangledown x$ (Proposition 2.3). For each k, there exists $\boldsymbol{\varepsilon}^{[k]} = \boldsymbol{\varepsilon}(\mathcal{R}^{[k]}) \in \{0,1\}^{r-k}$ such that $(\mathcal{R}^{[k]}, E^{[k]})$ is an $\boldsymbol{\varepsilon}^{[k]}$-representation. Putting

$$\mathcal{E}_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boldsymbol{\varepsilon}^{[k]} & I_{r-k} \end{pmatrix} \quad (k = 1, \ldots, r-1),$$

we see in Theorem 5.3 that the multiplier matrix σ_V of V is given by

$$\sigma_V = \mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1.$$

We organize this paper as follows. Section 1 collects definitions and facts about clans and homogeneous cones. In Section 2, we define a representation φ of the dual clan and study its basic properties. In particular, we attach an $\boldsymbol{\varepsilon} \in \{0,1\}^r$ to φ. Section 3 is devoted to describing the clans V_E and V_E^0. At the end of this section, we describe an inductive structure of V, which is used in later sections during the induction arguments. In Section 4, we express the basic relative invariants $P_j(v)$ $(j = 0, 1, \ldots, r)$ of V_E^0 with non-negative integers α_j as in (0.1). In the last section, Section 5, we write down α_j in terms of the multiplier matrix σ_V of V and the $\boldsymbol{\varepsilon} \in \{0,1\}^r$ attached to φ. Finally, we obtain an explicit expression of σ_V.

1. Preliminaries

Let V be a finite-dimensional real vector space with a bilinear product $\triangle$. We do not assume the existence of unit element for the moment. For $x \in V$, we denote by

L_x the left multiplication operator $L_x y = x \triangle y$ $(y \in V)$. The pair $(V, \triangle)$ (or simply V) is called a *clan* if the following three conditions are satisfied:

(C1) $(V, \triangle)$ is left-symmetric: $L_x L_y - L_y L_x = L_{x \triangle y - y \triangle x}$ for all $x, y \in V$,
(C2) there exists $s \in V^*$ such that $s(x \triangle y)$ defines an inner product in V,
(C3) for each $x \in V$, the operator L_x has only real eigenvalues.

Linear forms s with the property (C2) are said to be *admissible*.

Let V be a clan. By Vinberg [10, p. 369], V has a principal idempotent c by which V can be decomposed as

$$V = V_{(1)} \oplus V_{(1/2)},$$

where

$$V_{(1)} := \{x \in V;\ L_c x = x\}, \quad V_{(1/2)} := \left\{x \in V;\ L_c x = \tfrac{1}{2}x\right\}.$$

Denoting by R_x the right multiplication operator $R_x y = y \triangle x$ $(x, y \in V)$, we also have

$$V_{(1)} = \{x \in V;\ R_c x = x\}, \quad V_{(1/2)} = \{x \in V;\ R_c x = 0\}.$$

We note here that if V has a unit element e, then $c = e$ and evidently we have $V_{(1)} = V$ and $V_{(1/2)} = \{0\}$. The following multiplication rules hold:

$$V_{(1)} \triangle V_{(1)} \subset V_{(1)}, \quad V_{(1)} \triangle V_{(1/2)} \subset V_{(1/2)},$$
$$V_{(1/2)} \triangle V_{(1)} = \{0\}, \quad V_{(1/2)} \triangle V_{(1/2)} \subset V_{(1)}.$$

Clearly $V_{(1)}$ itself is a clan with unit element c. Let r be the rank of the clan $V_{(1)}$ and let $c_1, \dots, c_r$ be a complete system of orthogonal primitive idempotents in $V_{(1)}$, so that we have $c_1 + \cdots + c_r = c$. Then, after relabeling $c_1, \dots, c_r$ if necessary, we have the following decomposition of $V_{(1)}$:

$$V_{(1)} = \bigoplus_{1 \le j \le k \le r} V_{kj}, \qquad V_{jj} = \mathbb{R}c_j \quad (j = 1, \dots, r),$$
$$V_{kj} := \left\{x \in V_{(1)};\ L_{c_i} x = \tfrac{1}{2}(\delta_{ij} + \delta_{ik})x,\ R_{c_i} x = \delta_{ij} x\ (i = 1, \dots, r)\right\}\ (j < k).$$

The multiplication rules are

$$V_{ji} \triangle V_{lk} = \{0\} \quad (\text{if } i \ne k, l), \quad V_{kj} \triangle V_{ji} \subset V_{ki},$$
$$V_{ji} \triangle V_{ki} \subset V_{jk} \text{ or } V_{kj} \quad (\text{according to } j \ge k \text{ or } j \le k). \tag{1.1}$$

We assume from now on that V has a unit element e_0. By (C1) and (C3), the space $\mathfrak{h} := \{L_x;\ x \in V\}$ of left multiplication operators forms a split solvable Lie algebra. We note here that $\mathfrak{h}$ is linearly isomorphic to V. Let $H := \exp \mathfrak{h}$ be the connected and simply connected Lie group corresponding to $\mathfrak{h}$. We denote by Ω the H-orbit in V through e_0. We know that Ω is a proper open convex cone in V, and H acts on Ω simply transitively.

We fix a complete system $c_1, \dots, c_r$ of orthogonal primitive idempotents and denote the corresponding normal decomposition of V by

$$V = \bigoplus_{1 \le j \le k \le r} V_{kj}. \tag{1.2}$$

By introducing the lexicographic order in (1.2), we see that every L_x $(x \in V)$ is simultaneously represented by a lower triangular matrix. Then for each $h \in H$, there exist unique $h_{jj} > 0$ $(j = 1, \dots, r)$ and $v_{kj} \in V_{kj}$ $(1 \le j < k \le r)$ such that by setting $T_{jj} := (2 \log h_{jj}) L_{c_j}$ and $L_j := \sum_{k>j} L_{v_{kj}}$, we have

$$h = (\exp T_{11})(\exp L_1)(\exp T_{22}) \cdots (\exp L_{r-1})(\exp T_{rr}). \tag{1.3}$$

A function f on Ω is said to be relatively invariant under the action of H if there exists a one-dimensional representation χ of H with which we have $f(hx) = \chi(h) f(x)$ for all $h \in H$ and $x \in \Omega$. To each such χ, there corresponds an r-tuple $\underline{\tau} := (\tau_1, \dots, \tau_r) \in \mathbb{R}^r$ so that

$$\chi(h) = (h_{11})^{2\tau_1} \cdots (h_{rr})^{2\tau_r} \quad (\text{for } h \text{ as in (1.3)}).$$

We call $\underline{\tau}$ the multiplier of a relative invariant function f and write $\chi = \chi_{\underline{\tau}}$.

Theorem 1.1 (Ishi [4]). *There exist irreducible relatively H-invariant polynomial functions $\Delta_1, \dots, \Delta_r$ by which any relatively H-invariant polynomial function p on V is written as*

$$p(x) = (\mathrm{const}) \cdot \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad \big((n_1, \dots, n_r) \in \mathbb{Z}^r_{\ge 0}\big).$$

Moreover Ω is described as

$$\Omega = \{x \in V;\ \Delta_1(x) > 0, \dots, \Delta_r(x) > 0\}.$$

The polynomials $\Delta_1(x), \dots, \Delta_r(x)$ are called the *basic relative invariants* of the cone Ω. They are also called the basic relative invariants of the clan V. We assume that the numbering of the basic relative invariants is given by the procedure of Ishi [4] according to $c_1, \dots, c_r$. For $j = 1, \dots, r$, let $\underline{\sigma}_j = (\sigma_{j1}, \dots, \sigma_{jr})$ be the multiplier of the relative invariant $\Delta_j(x)$, and we place them in an $r \times r$ matrix as

$$\sigma_V := \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk}). \tag{1.4}$$

In this paper, we call σ_V the *multiplier matrix* of the clan V. We note that by the procedure of Ishi [4], σ_V is a lower triangular matrix with all $\sigma_{jk} \in \mathbb{Z}_{\ge 0}$ and $\sigma_{jj} = 1$ $(j = 1, \dots, r)$. In particular, σ_V is invertible. We put $d_j := \deg \Delta_j$ for $j = 1, \dots, r$. Then by definition, we have

$$d_j = \sigma_{j1} + \cdots + \sigma_{jj} \quad (j = 1, \dots, r). \tag{1.5}$$

For every $\boldsymbol{\varepsilon} := {}^t(\varepsilon_1, \dots, \varepsilon_r) \in \{0,1\}^r$, we put $c_{\boldsymbol{\varepsilon}} := \varepsilon_1 c_1 + \cdots + \varepsilon_r c_r$. Then $c_{\boldsymbol{\varepsilon}} \in \overline{\Omega}$ and we denote by $\mathcal{O}_{\boldsymbol{\varepsilon}}$ the H-orbit in V through $c_{\boldsymbol{\varepsilon}}$. Note that $\mathcal{O}_{\mathbf{1}} = \Omega$, where $\mathbf{1} = {}^t(1, \dots, 1)$. Then by Ishi [3, Theorem 3.5], the H-orbit decomposition of $\overline{\Omega}$ is described as

$$\overline{\Omega} = \bigsqcup_{\boldsymbol{\varepsilon} \in \{0,1\}^r} \mathcal{O}_{\boldsymbol{\varepsilon}}.$$

Now we assume that the inner product $\langle \cdot | \cdot \rangle$ of V is given by an admissible linear form s_0. Let us define a bilinear product $\triangledown$ in V through

$$\big\langle x \triangledown y \,|\, z \big\rangle = \langle y \,|\, x \vartriangle z \rangle \quad (x, y, z \in V). \tag{1.6}$$

Then it turns out that the product $\triangledown$ defines a clan structure in V. The clan $(V, \triangledown)$ is called the *dual clan* of $(V, \triangle)$. The linear form s_0 is also an admissible linear form for $(V, \triangledown)$. In fact, we have $s_0(x \triangledown y) = \left\langle x \triangledown y \,|\, e_0 \right\rangle = \langle y \,|\, x\rangle$. Moreover it is easy to see from (1.6) that e_0 is also a unit element of $(V, \triangledown)$. The cone corresponding to $(V, \triangledown)$ is the dual cone Ω^* of Ω with respect to the inner product $\langle \cdot|\cdot\rangle$, where

$$\Omega^* := \left\{x \in V;\ \langle x \,|\, y\rangle > 0 \text{ for all } y \in \overline{\Omega}\backslash\{0\}\right\}.$$

Let $L^{\triangledown}_x \colon V \ni y \mapsto x \triangledown y$ be the left multiplication operator by $x \in V$ of $(V, \triangledown)$.

Proposition 1.2. *The following relationships hold between* $\triangle$ *and* $\triangledown$.

(1) *For* $x, y \in V$, *we have* $x \triangle y + x \triangledown y = y \triangle x + y \triangledown x$.
(2) *For* $i = 1, \dots, r$, *one has* $L^{\triangledown}_{c_i} = L_{c_i}$.

Proof. (1) For any $z \in V$, we have by (C1)

$$\begin{aligned}\langle x \triangle y - y \triangle x \,|\, z\rangle =& s_0\big((x \triangle y - y \triangle x) \triangle z\big) = s_0\big(x \triangle (y \triangle z) - y \triangle (x \triangle z)\big)\\ =& \langle x \,|\, y \triangle z\rangle - \langle y \,|\, x \triangle z\rangle = \left\langle y \triangledown x - x \triangledown y \,|\, z\right\rangle.\end{aligned}$$

Hence, the assertion is proved.
(2) Suppose $x_{kj} \in V_{kj}$ $(j \leq k)$. Then for any $y \in V$, we have

$$\left\langle c_i \triangledown x_{kj} \,|\, y\right\rangle = \langle x_{kj} \,|\, c_i \triangle y\rangle = \langle x_{kj} \,|\, c_i \triangle y_{kj}\rangle = \frac{1}{2}(\delta_{ij} + \delta_{ik}) \langle x_{kj} \,|\, y\rangle,$$

where y_{kj} is the V_{kj}-component of y. Thus we get $L^{\triangledown}_{c_i} x_{kj} = \frac{1}{2}(\delta_{ij} + \delta_{ik}) x_{kj} = L_{c_i} x_{kj}$ for any $x_{kj} \in V_{kj}$. This shows $L^{\triangledown}_{c_i} = L_{c_i}$. □

Proposition 1.2 (2) shows that $c_1, \dots, c_r$ form also a complete system of orthogonal primitive idempotents of the dual clan $(V, \triangledown)$. We denote by $R^{\triangledown}_x$ the right multiplication operator of $(V, \triangledown)$ by $x \in V$. By (1) and (2) of Proposition 1.2, we get $R^{\triangledown}_{c_i} x_{kj} = \delta_{ik} x_{kj}$ for any $x_{kj} \in V_{kj}$ $(j \leq k)$ and $i = 1, \dots, r$. Thus we have

$$V_{kj} = \left\{x \in V;\ L^{\triangledown}_{c_i} x = \tfrac{1}{2}(\delta_{ij} + \delta_{ik}) x,\ R^{\triangledown}_{c_i} x = \delta_{ik} x \ (i = 1, \dots, r)\right\}.$$

This implies that the decomposition (1.2) also serves as a normal decomposition of $(V, \triangledown)$ relative to $c_1, \dots, c_r$ with the multiplication rules

$$\begin{gathered} V_{ji} \triangledown V_{lk} = \{0\} \quad (\text{if } j \neq k, l), \quad V_{ji} \triangledown V_{kj} \subset V_{ki}, \\ V_{ki} \triangledown V_{kj} \subset V_{ji} \text{ or } V_{ij} \quad (\text{according to } i \leq j \text{ or } i \geq j). \end{gathered} \tag{1.7}$$

2. Representations of clans

We keep to the clan notation used in Section 1. Let E be a real Euclidean vector space with inner product $\langle \cdot|\cdot\rangle_E$. We denote by $\mathcal{L}(E)$ the vector space of linear operators on E. For a linear map $\varphi \colon V \to \mathcal{L}(E)$, let $\underline{\varphi}$ and $\overline{\varphi}$ be the "lower triangular part" and the "upper triangular part" of φ respectively associated with

$c_1, \ldots, c_r$ given by

$$
\begin{aligned}
\underline{\varphi}(x) &:= \frac{1}{2}\sum_{j=1}^{r} x_j\varphi(c_j) + \sum_{j<k}\varphi(c_k)\varphi(x_{kj})\varphi(c_j), \\
\overline{\varphi}(x) &:= \frac{1}{2}\sum_{j=1}^{r} x_j\varphi(c_j) + \sum_{j<k}\varphi(c_j)\varphi(x_{kj})\varphi(c_k),
\end{aligned}
\tag{2.1}
$$

where we write $x = \sum x_j c_j + \sum_{j<k} x_{kj}$ according to (1.2). A linear map $\varphi\colon V \to \mathcal{L}(E)$ is called a selfadjoint representation of the clan $(V, \triangledown)$ if $\varphi(x)$ is a selfadjoint operator for every $x \in V$ and if the following condition is satisfied:

$$\varphi(x \triangledown y) = \overline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x) \quad (x, y \in V). \tag{2.2}$$

We always require that $\varphi(e_0)$ is the identity operator. In this paper, we only consider selfadjoint representations of $(V, \triangledown)$. Thus we often drop the adjective selfadjoint for simplicity.

Let (φ, E) be a representation of $(V, \triangledown)$.

Proposition 2.1. *The following statements hold.*

1. $\varphi(c_j)$ $(j = 1, \ldots, r)$ *are mutually orthogonal projection operators.*
2. *Each* $\underline{\varphi}(x)$ *can be simultaneously expressed as a lower triangular matrix by an appropriate choice of orthonormal basis of* E.
3. *For all* $x \in V$, *one has* $\underline{\varphi}(x)^* = \overline{\varphi}(x)$ *and* $\underline{\varphi}(x) + \overline{\varphi}(x) = \varphi(x)$.

Proof. (1) Recalling $c_j \triangledown c_k = \delta_{jk}c_j$, we have by (2.2)

$$\delta_{jk}\varphi(c_j) = \varphi(c_j \triangledown c_k) = \tfrac{1}{2}\big(\varphi(c_j)\varphi(c_k) + \varphi(c_k)\varphi(c_j)\big). \tag{2.3}$$

Letting $k = j$, we obtain $\varphi(c_j) = \varphi(c_j)^2$, so that $\varphi(c_j)$ is a projection operator. Next we assume that $j \neq k$. Then (2.3) implies $\varphi(c_j)\varphi(c_k) = -\varphi(c_k)\varphi(c_j)$. Multiplying both sides from the left by $\varphi(c_j)$, we get $\varphi(c_j)\varphi(c_k) = \varphi(c_k)\varphi(c_j)$. From this, we conclude $\varphi(c_j)\varphi(c_k) = 0$.
(2) Let us put $E_j := \varphi(c_j)E$ $(j = 1, \ldots, r)$. Since $\varphi(c_j)$'s are mutually orthogonal, we have an orthogonal direct sum $E = \bigoplus_j E_j$. Form an orthonormal basis of E by first taking the one from E_1, then from E_2, …, and finally from E_r. By this choice of orthonormal basis, every operator $\underline{\varphi}(x)$ for $x \in V_{kj}$ $(j < k)$ is represented by a strictly lower triangular matrix. Since each $\underline{\varphi}(c_j)$ is clearly diagonal, we conclude that $\underline{\varphi}(x)$ is lower triangular.
(3) Since $\varphi(x)$ is selfadjoint for any $x \in V$, the first assertion follows from (2.1). The second assertion is proved by putting $y = e_0$ in (2.2). □

Associated with φ, we define a symmetric bilinear map $Q\colon E \times E \to V$ through

$$\langle \varphi(x)\xi \,|\, \eta \rangle_E = \langle Q(\xi, \eta) \,|\, x \rangle \quad (\xi, \eta \in E,\ x \in V). \tag{2.4}$$

For simplicity, we put $Q[\xi] := Q(\xi, \xi)$ and $Q[E] := \{Q[\xi];\ \xi \in E\}$.

Proposition 2.2. *One has the following properties:*

1. $x \vartriangle Q(\xi, \eta) = Q(\underline{\varphi}(x)\xi, \eta) + Q(\xi, \underline{\varphi}(x)\eta)$ $(x \in V,\ \xi, \eta \in E)$,

(2) *Q is Ω-positive, that is, $Q[\xi]\in\overline{\Omega}\backslash\{0\}$ for all $\xi\in E\backslash\{0\}$.*

Proof. (1) For any $y\in V$, we have by (2.2) and Proposition 2.1 (3)

$$\begin{aligned}\langle x\bigtriangleup Q(\xi,\eta)\,|\,y\rangle &= \big\langle Q(\xi,\eta)\,|\,x\bigtriangledown y\big\rangle = \langle \varphi(x\bigtriangledown y)\xi\,|\,\eta\rangle_E\\ &= \langle \overline{\varphi}(x)\varphi(y)\xi+\varphi(y)\underline{\varphi}(x)\xi\,|\,\eta\rangle_E\\ &= \langle \varphi(y)\xi\,|\,\underline{\varphi}(x)\eta\rangle_E+\langle \varphi(y)\underline{\varphi}(x)\xi\,|\,\eta\rangle_E\\ &= \big\langle Q(\xi,\underline{\varphi}(x)\eta)+Q(\underline{\varphi}(x)\xi,\eta)\,|\,y\big\rangle.\end{aligned}$$

Hence we obtain the assertion.
(2) The formula in (1) gives rise to

$$(\exp L_x)Q(\xi,\eta)=Q\big((\exp\underline{\varphi}(x))\xi,(\exp\underline{\varphi}(x))\eta\big)\quad (x\in V,\ \xi,\eta\in E). \tag{2.5}$$

For each $x\in\Omega^*$, we take $x_0\in V$ such that $(\exp L^{\bigtriangledown}_{x_0})e_0=x$. Since $L^{\bigtriangledown}_x=(L_x)^*$ by (1.6), we have

$$\langle Q[\xi]\,|\,x\rangle=\langle(\exp L_{x_0})Q[\xi]\,|\,e_0\rangle=\big\langle Q[(\exp\underline{\varphi}(x_0))\xi]\,|\,e_0\big\rangle=\|(\exp\underline{\varphi}(x_0))\xi\|_E^2\geq 0.$$

Hence $Q[\xi]\in\overline{\Omega}$. Moreover, since $\exp\underline{\varphi}(x_0)$ is invertible, we see that $Q[\xi]=0$ if and only if $\xi=0$. Hence the proposition is proved. ☐

Let $\boldsymbol{\varepsilon}\in\{0,1\}^r$. A representation (φ,E) of $(V,\bigtriangledown)$ is called an *$\boldsymbol{\varepsilon}$-representation* if the associated symmetric bilinear map Q satisfies $Q[E]=\overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$. Any $\boldsymbol{\varepsilon}$-representation arises from the right multiplication operators as we now show. For $k=1,2,\dots,r-1$, let $V^{[k]}$ and $E^{[k]}$ be the subspaces of V defined respectively by

$$V^{[k]}:=\bigoplus_{k<l\leq m\leq r}V_{ml},\quad E^{[k]}:=\bigoplus_{m>k}V_{mk}. \tag{2.6}$$

We note that the multiplication rules (1.7) yield that for any $k=1,\dots,r-1$, $V^{[k]}$ is a subclan of $(V,\bigtriangledown)$ and $E^{[k]}\bigtriangledown V^{[k]}\subset E^{[k]}$. The latter property allows us to define $\mathcal{R}^{[k]}\colon V^{[k]}\to\mathcal{L}(E^{[k]})$ by

$$\mathcal{R}^{[k]}(x)\xi=\xi\bigtriangledown x\quad (x\in V^{[k]},\ \xi\in E^{[k]}\text{ and }k=1,2,\dots,r-1).$$

Proposition 2.3. *For each k, the pair $(\mathcal{R}^{[k]},E^{[k]})$ is a selfadjoint representation of $(V^{[k]},\bigtriangledown)$.*

Proof. We only prove the proposition for $k=1$, and the proof for general k is similar. For simplicity, we write

$$V'=V^{[1]},\quad (\varphi',E')=(\mathcal{R}^{[1]},E^{[1]}). \tag{2.7}$$

Let $\xi\in E'$ and $x\in V'$. By (1.1), we have $\xi\bigtriangleup x=0$, so that Proposition 1.2 (1) yields

$$\varphi'(x)\xi=\xi\bigtriangledown x=x\bigtriangleup\xi+x\bigtriangledown\xi=(L_x+L^{\bigtriangledown}_x)\xi. \tag{2.8}$$

Since $L^{\bigtriangledown}_x=(L_x)^*$, we see that $\varphi'(x)$ is selfadjoint. Furthermore for all $x,y\in V'$ and $\xi\in E'$, we have by (C1) and (2.8)

$$\begin{aligned}\varphi'(x\bigtriangledown y)\xi &= x\bigtriangledown(\xi\bigtriangledown y)+(\xi\bigtriangledown x-x\bigtriangledown\xi)\bigtriangledown y\\ &= x\bigtriangledown(\xi\bigtriangledown y)+(x\bigtriangleup\xi)\bigtriangledown y.\end{aligned} \tag{2.9}$$

Taking the lower and the upper triangular part of $\varphi'(x)$ in (2.8), we see that $\underline{\varphi}'(x) = L_x$ and $\overline{\varphi}'(x) = L_x^{\triangledown}$. Then the last term of (2.9) is equal to $\big(\overline{\varphi}'(x)\varphi'(y)+\varphi'(y)\underline{\varphi}'(x)\big)\xi$. Since $\varphi'(e_0)$ is obviously the identity operator, the pair (φ', E') is now a selfadjoint representation of the clan $(V', \triangledown)$. □

For $k = 1, 2, \dots, r-1$, let $\pi_k\colon V \to V^{[k]}$ be the orthogonal projection. By the multiplication rules in (1.7), we see that $\pi_k(x \triangledown y) = \pi_k(x) \triangledown \pi_k(y)$. Thus $\mathcal{R}_k := \mathcal{R}^{[k]} \circ \pi_k$ defines a selfadjoint representation of the clan $(V, \triangledown)$. Take any $\boldsymbol{\varepsilon} = {}^t(\varepsilon_1, \dots, \varepsilon_r) \in \{0,1\}^r$ with $\boldsymbol{\varepsilon} \neq \mathbf{0}$. We put $E_{\boldsymbol{\varepsilon}} := \bigoplus_{\varepsilon_j=1} E^{[j]}$ and define a linear map $\mathcal{R}_{\boldsymbol{\varepsilon}}\colon V \to \mathcal{L}(E_{\boldsymbol{\varepsilon}})$ by

$$\mathcal{R}_{\boldsymbol{\varepsilon}}(x)\left(\sum\nolimits_{\varepsilon_j=1} \xi_j\right) := \sum\nolimits_{\varepsilon_j=1} \mathcal{R}_j(x)\xi_j \quad (x \in V,\ \xi_j \in E^{[j]}).$$

Clearly, $(\mathcal{R}_{\boldsymbol{\varepsilon}}, E_{\boldsymbol{\varepsilon}})$ is a selfadjoint representation of $(V, \triangledown)$. If $\boldsymbol{\varepsilon} = \mathbf{0}$, then $\mathcal{R}_{\mathbf{0}}$ is defined to be the zero-dimensional zero-representation. Let $Q_{\boldsymbol{\varepsilon}}$ be the symmetric bilinear map associated with $\mathcal{R}_{\boldsymbol{\varepsilon}}$. Then, by Graczyk and Ishi [2, Proposition 3.5], we have $Q_{\boldsymbol{\varepsilon}}[E_{\boldsymbol{\varepsilon}}] = \overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$ and hence $\mathcal{R}_{\boldsymbol{\varepsilon}}$ is an $\boldsymbol{\varepsilon}$-representation.

Now let (φ, E) be any selfadjoint representation of $(V, \triangledown)$ and Q the corresponding bilinear map. The Riesz measure μ_Q associated with the quadratic map $Q[\xi]$ is, by definition, the image of the Lebesgue measure $d\xi$ on E by $Q[\xi]$ (cf. Graczyk and Ishi [2]). In other words, we have

$$\int_V f(x)\mu_Q(dx) = \int_E f(Q[\xi])d\xi$$

for any measurable function f on V. On the other hand by (2.4) and (2.5), we obtain

$$\operatorname{Det}\varphi(h^*x) = \operatorname{Det}\big(h^*\varphi(x)h\big) = (\operatorname{Det} h)^2 \operatorname{Det}\varphi(x)$$

for any $x \in V$ and $h \in H$. Thus $\operatorname{Det}\varphi(x)$ is a relatively H-invariant polynomial and its multiplier $\boldsymbol{l} = {}^t(l_1, \dots, l_r) \in \mathbb{Z}_{\geq 0}^r$ satisfies

$$\operatorname{Det}\varphi(\lambda_1 c_1 + \cdots + \lambda_r c_r) = (\lambda_1)^{l_1} \cdots (\lambda_r)^{l_r} \quad (\lambda_1, \dots, \lambda_r \in \mathbb{R})$$

with $l_j = \dim \varphi(c_j)E$ $(j = 1, \dots, r)$. Let $\mathscr{R}_{\boldsymbol{s}}$ $(\boldsymbol{s} \in \mathbb{R}^r)$ be the Gindikin–Riesz distribution defined in Ishi [3] (in that paper, it is simply called the Riesz distribution). Then, by [2, (3.29)], we have

$$\mu_Q = \pi^{\dim E/2}\mathscr{R}_{\boldsymbol{l}/2}. \tag{2.10}$$

Since Q is Ω-positive, the measure μ_Q is a positive measure, and hence $\mathscr{R}_{\boldsymbol{l}/2}$ is also a positive measure. Let Ξ be the Gindikin–Wallach set (cf. Ishi [3]), which is the set of $\boldsymbol{s}$ for $\mathscr{R}_{\boldsymbol{s}}$ to be a positive measure. By (2.10) and [3, Theorem 6.2], we obtain $\boldsymbol{l}/2 \in \Xi$. Putting $d_{kj} := \dim V_{kj}$ for $1 \leq j < k \leq r$, we define $\boldsymbol{l}^{(i)} \in \mathbb{R}^r$ $(i = 1, \dots, r)$ inductively by $\boldsymbol{l}^{(1)} := \boldsymbol{l}$ and, for $2 \leq i \leq r$,

$$\boldsymbol{l}^{(i)} := \begin{cases} \boldsymbol{l}^{(i-1)} - {}^t(0, \dots, 0, d_{i+1,i}, \dots, d_{ri}) & \text{if } l_{i-1}^{(i-1)} > 0, \\ \boldsymbol{l}^{(i-1)} & \text{if } l_{i-1}^{(i-1)} \leq 0. \end{cases}$$

Further we define $\boldsymbol{\varepsilon}(\varphi) = {}^t(\varepsilon_1, \dots, \varepsilon_r) \in \{0,1\}^r$ by

$$\varepsilon_i = \begin{cases} 1 & \text{if } l_i^{(i)} > 0, \\ 0 & \text{if } l_i^{(i)} \leq 0 \end{cases} \qquad (i = 1, \dots, r).$$

Then by [2, p. 183], $\mathscr{R}_{\boldsymbol{l}/2}$ is a measure on $\mathcal{O}_{\boldsymbol{\varepsilon}(\varphi)}$. Using Proposition 3.10 and Theorem 3.13 of [2], we see that the support of $\mathscr{R}_{\boldsymbol{l}/2}$ is equal to $\overline{\mathcal{O}_{\boldsymbol{\varepsilon}(\varphi)}}$, the closure of $\mathcal{O}_{\boldsymbol{\varepsilon}(\varphi)}$. These observations together with (2.10) give the following proposition.

Proposition 2.4. *φ is an $\boldsymbol{\varepsilon}(\varphi)$-representation.*

3. Clans defined by representations of a clan

In this section, we define a clan starting from a representation of a clan. Let $(V, \triangle)$ be a clan of rank r with unit element e_0 and (φ, E) a representation of the dual clan $(V, \triangledown)$ on a real Euclidean vector space E with inner product $\langle \cdot | \cdot \rangle_E$. First we assume that $\dim E > 0$. Let us keep to the notation used in the previous sections. Put $V_E := E \oplus V$ and we define a product $\triangle$ on V_E by

$$(\xi + x) \triangle (\eta + y) = \underline{\varphi}(x)\eta + (Q(\xi, \eta) + x \triangle y) \quad (\xi, \eta \in E,\ x, y \in V).$$

Theorem 3.1. *The algebra $(V_E, \triangle)$ is a clan with an admissible linear form s' given by $s'(\xi + x) = s_0(x)$ $(\xi \in E,\ x \in V)$.*

Once we have the following Lemma 3.2, we can prove Theorem 3.1 in exactly the same way as [9, Theorem 3.2]. Thus we omit the proof.

Lemma 3.2. *Put $[x \triangle y] := x \triangle y - y \triangle x$. Then one has $\underline{\varphi}([x \triangle y]) = [\underline{\varphi}(x), \underline{\varphi}(y)]$ for $x, y \in V$.*

Proof. Let us also put $[y \triangledown x] := y \triangledown x - x \triangledown y$. By Proposition 1.2 (1), we have $[x \triangle y] = [y \triangledown x]$. Then by (2.2) and Proposition 2.1 (3), we obtain

$$\begin{aligned} \varphi([x \triangle y]) = \varphi([y \triangledown x]) &= \overline{\varphi}(y)\varphi(x) + \varphi(x)\underline{\varphi}(y) - \overline{\varphi}(x)\varphi(y) - \varphi(y)\underline{\varphi}(x) \\ &= [\underline{\varphi}(x), \underline{\varphi}(y)] + ([\underline{\varphi}(x), \underline{\varphi}(y)])^*. \end{aligned}$$

Taking the lower triangular part, we get the lemma. □

Now we consider the algebra $V_E^0 := \mathbb{R}e \oplus V_E$ obtained by the adjunction of a unit element e to the clan V_E. Since $\dim E > 0$, we have $u := e - e_0 \neq 0$ and hence $V_E^0 = \mathbb{R}u \oplus V_E$. By this decomposition, the clan product of V_E^0 is written as

$$\begin{aligned} &(\lambda u + \xi + x) \triangle (\mu u + \eta + y) \\ &\qquad = (\lambda\mu)u + (\mu\xi + \tfrac{1}{2}\lambda\eta + \underline{\varphi}(x)\eta) + (Q(\xi, \eta) + x \triangle y), \end{aligned} \tag{3.1}$$

where $\lambda, \mu \in \mathbb{R}$, $\xi, \eta \in E$ and $x, y \in V$. As an admissible linear form of V_E^0, we take s^0 given by $s^0(\lambda u + \xi + x) := \lambda + s_0(x)$. We call $(V_E^0, \triangle)$ the clan obtained by the clan V and the representation (φ, E).

Next we assume that $\dim E = 0$. In this case, we take a one-dimensional extension $V^0 = \mathbb{R}u \oplus V$ of V and define a product $\triangle$ by

$$(\lambda u + x) \triangle (\mu u + y) = (\lambda\mu)u + x \triangle y \quad (\lambda, \mu \in \mathbb{R},\ x, y \in V).$$

Clearly the algebra $(V^0, \triangle)$ is a clan. This clan coincides with the clan $(V_E^0, \triangle)$ in (3.1) with E formally equal to $\{0\}$. Therefore we also use the notation V_E^0 in this particular case.

Before closing this section, we write down here some formulas needed in the later sections during the induction arguments. Let the subclan V' and the representation (φ', E') be as in (2.7). The associated symmetric bilinear map $Q' \colon E' \times E' \to V'$ is given by

$$Q'(\xi', \eta') = \xi' \triangle \eta' \quad (\xi', \eta' \in E'). \tag{3.2}$$

In fact, we have for any $x' \in V'$

$$\langle Q'(\xi', \eta') \,|\, x' \rangle = \langle \varphi'(x')\xi' \,|\, \eta' \rangle = \big\langle \xi' \triangledown x' \,|\, \eta' \big\rangle = \langle x' \,|\, \xi' \triangle \eta' \rangle \,.$$

Let $\mathfrak{h}' := \{L_{x'};\ x' \in V'\}$ be the Lie algebra of left multiplication operators of V' and $H' := \exp \mathfrak{h}'$ the corresponding Lie group. Any $h \in H$ is written as in (1.3). Putting

$$\xi' = v_{21} + \cdots + v_{r1} \in E', \quad h' = \exp T_{22} \exp L_2 \cdots \exp T_{rr} \in H',$$

we have $h = (\exp T_{11})(\exp L_{\xi'})h'$.

Lemma 3.3. *Let $y = y_{11}c_1 + \eta' + y' \in \mathbb{R}c_1 \oplus E' \oplus V'$. Then*

$$hy = y_{11}(h_{11})^2 c_1 + h_{11}(y_{11}\xi' + h'\eta') + \left(\frac{1}{2} y_{11} Q'[\xi'] + Q'(\xi', h'\eta') + h'y'\right).$$

Proof. We first note that the multiplication rules (1.1) tell us that

$$h'c_1 = c_1, \quad h'\eta' \in E', \quad h'y' \in V'.$$

Next we have again by (1.1)

$$R_{c_1}|_{E'} = \mathrm{id}_{E'}, \quad E' \triangle E' \subset V', \quad E' \triangle V' = \{0\},$$

so that recalling (3.2)

$$\begin{aligned}
&(\exp L_{\xi'})c_1 = c_1 + \xi' \triangle c_1 + \frac{1}{2}\xi' \triangle (\xi' \triangle c_1) = c_1 + \xi' + \frac{1}{2} Q'[\xi'], \\
&(\exp L_{\xi'})\eta' = \eta' + Q'(\xi', \eta'), \quad (\exp L_{\xi'})y' = y'.
\end{aligned}$$

Finally,

$$(\exp T_{11})c_1 = (h_{11})^2 c_1, \quad (\exp T_{11})\eta' = h_{11}\eta', \quad (\exp T_{11})y' = y'.$$

These observations yield that

$$\begin{aligned}
hc_1 &= \exp T_{11}(c_1 + \xi' + \tfrac{1}{2}Q'[\xi']) = (h_{11})^2 c_1 + h_{11}\xi' + \tfrac{1}{2}Q'[\xi'], \\
h\eta' &= \exp T_{11}(h'\eta' + Q'(\xi', h'\eta')) = h_{11}h'\eta' + Q'(\xi', h'\eta'), \\
hy' &= h'y'.
\end{aligned}$$

Hence we obtain the formula in the lemma. □

Let $(\widetilde{V}, \triangle)$ be the clan obtained by V' and φ'. Then the linear map $\iota \colon \widetilde{V} \to V$ defined by

$$\iota \colon \widetilde{V} \ni \lambda u + \xi' + x' \longmapsto \lambda c_1 + \xi' + x' \in V \tag{3.3}$$

is a clan isomorphism and hence $(\widetilde{V},\triangle)$ is isomorphic to $(V,\triangle)$. Thus this $(\widetilde{V},\triangle)$ can be identified with $(V,\triangle)$.

4. Basic relative invariants

We keep to the notation of the previous sections. In this section, we study the basic relative invariants of the clan V_E^0. To do so, we consider the right multiplication operators of the clan $(V_E^0,\triangle)$. The reason for this is that by Ishi and Nomura [6], the basic relative invariants are the irreducible factors of the determinant of the right multiplication operators.

Let $(V_E^0,\triangle)$ be the clan obtained by the clan V of rank r and the representation (φ,E) of $(V,\bigtriangledown)$ introduced in Section 3. Let R_v^0 be the right multiplication operators by $v\in V_E^0$ of the clan $(V_E^0,\triangle)$. By (3.1), we have

$$R^0_{\lambda u+\xi+x}=\begin{pmatrix}\lambda & 0 & 0\\ \frac{1}{2}\xi & \lambda\,\mathrm{id}_E & R^0_\xi\\ 0 & R^0_\xi & R_x\end{pmatrix}\quad(\lambda u+\xi+x\in V_E^0),$$

where R_x is the right multiplication operator of V, and we note that $R^0_\xi(V)\subset E$ and $R^0_\xi(E)\subset V$. As in [9, Proposition 4.1], we have the following proposition.

Proposition 4.1. *For $\lambda u+\xi+x\in V_E^0$, one has*

$$\operatorname{Det}R^0_{\lambda u+\xi+x}=\lambda^{1+\dim E-\dim V}\operatorname{Det}\bigl(R_{\lambda x-\frac{1}{2}Q[\xi]}\bigr).$$

Let $\Delta_1(x),\dots,\Delta_r(x)$ be the basic relative invariants of V. Then we have

$$\operatorname{Det}R_x=\Delta_1(x)^{n_1}\cdots\Delta_r(x)^{n_r}\quad(x\in V),\tag{4.1}$$

with positive integers $n_1,\dots,n_r$. Proposition 4.1 together with (4.1) tells us that the basic relative invariants of $(V_E^0,\triangle)$ are exhausted by the polynomial λ and the irreducible factors of $\Delta_j(\lambda x-\frac{1}{2}Q[\xi])$ $(j=1,\dots,r)$.

Proposition 4.2. *For each $j=1,\dots,r$, the only possible factor of the polynomial $\Delta_j(\lambda x-\frac{1}{2}Q[\xi])$ is λ^{α_j} for some non-negative integer α_j.*

Proof. First, we note that any basic relative invariant is a homogeneous polynomial. Let d_j be the homogeneous degree of Δ_j and we put

$$\widetilde{P}_j(\lambda u+\xi+x):=\Delta_j(\lambda x-\tfrac{1}{2}Q[\xi]).$$

Since $\lambda u+\xi+x\mapsto\lambda x-\frac{1}{2}Q[\xi]$ is a quadratic map, the polynomial $\widetilde{P}_j(\lambda u+\xi+x)$ is a homogeneous polynomial of degree $2d_j$. In particular, the degree of each monomial of $\widetilde{P}_j(\lambda u+\xi+x)$ with respect to λ is equal to that of x. Moreover since $\Delta_j(\lambda x)=\lambda^{d_j}\Delta_j(x)$, the degree of the polynomial $\widetilde{P}_j(\lambda u+\xi+x)$ with respect to λ is d_j and the coefficient of λ^{d_j} is the irreducible polynomial $\Delta_j(x)$. Hence $\widetilde{P}_j(\lambda u+\xi+x)$ is written as

$$\widetilde{P}_j(\lambda u+\xi+x)=\lambda^{d_j}\Delta_j(x)+\lambda^{d_j-1}p_j^{(1)}(x,\xi)+\cdots+p_j^{(d_j)}(x,\xi),$$

where $p_j^{(k)}(x,\xi)$ are polynomials of x and ξ of degree d_j+k. In particular, the degree of the polynomial $p_j^{(k)}(x,\xi)$ with respect to x is strictly lower than d_j. Since $\Delta_j(x)$ is irreducible, $\widetilde{P}_j(\lambda u+\xi+x)$ is factorized as

$$\widetilde{P}_j(\lambda u+\xi+x)=\lambda^{\alpha_j}\big(\lambda^{d_j-\alpha_j}\Delta_j(x)+\lambda^{d_j-\alpha_j-1}p_j^{(1)}(x,\xi)+\cdots+p_j^{(d_j-\alpha_j)}(x,\xi)\big), \tag{4.2}$$

where α_j is the maximal integer such that $p_j^{(d_j-\alpha_j)}\neq 0$. Thus $\lambda^{-\alpha_j}\widetilde{P}_j(\lambda u+\xi+x)$ is irreducible. The proof is now completed. □

Propositions 4.1 and 4.2 immediately give the following theorem.

Theorem 4.3. *The basic relative invariants $P_j(\lambda u+\xi+x)$ $(j=0,1,\dots,r)$ of V_E^0 are given by*

$$\begin{cases} P_0(\lambda u+\xi+x)=\lambda, \\ P_j(\lambda u+\xi+x)=\lambda^{-\alpha_j}\Delta_j(\lambda x-\tfrac{1}{2}Q[\xi]) \quad (j\geq 1). \end{cases}$$

We will determine the non-negative integers α_j in Section 5.

Let us return to (4.1). Theorem 4.3 enables us to give an answer to the question of expressing $n_1,\dots,n_r$ in terms of the data of the clan V. Considering the degree of (4.1), we have

$$\dim V=n_1d_1+\cdots+n_rd_r. \tag{4.3}$$

We set $\underline{n}:=(n_1,\dots,n_r)$ in the form of row vector and call $\underline{n}$ the *basic index* of V. Let

$$m_k:=\sum_{l\geq k}\dim V_{lk}\quad (k=1,\dots,r), \tag{4.4}$$

and we put them also in the form of row vector as $\underline{m}:=(m_1,\dots,m_r)$. We note that $m_k=1+\dim E^{[k]}$ for any k. In what follows, we write the bold symbol $\boldsymbol{x}$ for the column vector ${}^t(x_1,\dots,x_r)$. Now we have the following theorem.

Theorem 4.4. *Let $\sigma=\sigma_V$ be the multiplier matrix of V. Then one has*

$$\underline{n}=\underline{m}\sigma^{-1}.$$

Proof. We shall prove the theorem by induction on r. Let V' be the subclan and (φ',E') the representation in (2.7). Then we have the decomposition $V=\mathbb{R}c_1\oplus E'\oplus V'$. Let Q' be the symmetric bilinear map associated with φ', and Δ'_j $(j=2,\dots,r)$ the basic relative invariants of $(V',\triangle)$. Applying Theorem 4.3 to the clan V obtained by V' and (φ',E') via (3.3), we have for some $\alpha'_j\in\mathbb{Z}_{\geq 0}$

$$\Delta_1(x)=x_{11},\quad \Delta_j(x)=(x_{11})^{-\alpha'_j}\Delta'_j(x_{11}x'-\tfrac{1}{2}Q'[\xi'])\quad (j=2,\dots,r), \tag{4.5}$$

where $x=x_{11}c_1+\xi'+x'$ ($x_{11}\in\mathbb{R}$, $\xi'\in E'$, $x'\in V'$). We denote by σ' the multiplier matrix of V' and put $d'_j=\deg\Delta'_j$ $(j=2,\dots,r)$. Comparing the degree in (4.5), we see that σ is described as

$$\sigma=\begin{pmatrix} 1 & 0 \\ \boldsymbol{d}'-\boldsymbol{\alpha}' & \sigma' \end{pmatrix}. \tag{4.6}$$

Let R' be the right multiplication operators of V' and $\underline{n}'$ the basic index of V'. By (4.3), we have $\dim V' = n_2'd_2' + \cdots + n_r'd_r' = \underline{n}'\boldsymbol{d}'$. Since $V' = \bigoplus_{2\le j\le k\le r} V_{kj}$ is the normal decomposition of V', we have $m_k' := \sum_{l\ge k} \dim V_{lk} = m_k$ for $k = 2, 3, \ldots, r$. Now Proposition 4.1 applied to the situation (3.3) together with (4.5) gives

$$\begin{aligned}\operatorname{Det} R_{x_{11}c_1+\xi'+x'} &= (x_{11})^{1+\dim E'-\dim V'} \operatorname{Det} R'_{x_{11}x'-\frac{1}{2}Q'[\xi']}\\ &= \Delta_1(x)^{1+\dim E'+\underline{n}'(\boldsymbol{\alpha}'-\boldsymbol{d}')}\Delta_2(x)^{n_2'}\cdots\Delta_r(x)^{n_r'}.\end{aligned}$$

This tells us that

$$\underline{n} = (1+\dim E' + \underline{n}'(\boldsymbol{\alpha}'-\boldsymbol{d}'),\, \underline{n}') = (m_1 + \underline{n}'(\boldsymbol{\alpha}'-\boldsymbol{d}'),\, \underline{n}'). \tag{4.7}$$

Then by the induction hypothesis $\underline{n}'\sigma' = \underline{m}'$, we obtain by (4.6) and (4.7)

$$\underline{n}\sigma = (m_1 + \underline{n}'(\boldsymbol{\alpha}'-\boldsymbol{d}'),\, \underline{n}')\begin{pmatrix}1 & 0\\ \boldsymbol{d}'-\boldsymbol{\alpha}' & \sigma'\end{pmatrix} = (m_1,\, \underline{m}') = \underline{m}.$$

The proof is now completed. □

5. MULTIPLIER MATRIX

In this section, we calculate the non-negative integers α_j that appeared in Proposition 4.2. We keep to the notation of the previous sections. Let us put $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\varphi)$. Since $Q[E] = \overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$, we consider the polynomials $\Delta_j(\lambda x - x_{\boldsymbol{\varepsilon}})$ ($\lambda \in \mathbb{R}$, $x \in V$, $x_{\boldsymbol{\varepsilon}} \in \overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$). If $x \in \Omega$, then putting $x = he_0$ with $h \in H$, we have by the relative invariance

$$\Delta_j(\lambda x - x_{\boldsymbol{\varepsilon}}) = \Delta_j(x)\Delta_j(\lambda e_0 - h^{-1}x_{\boldsymbol{\varepsilon}}). \tag{5.1}$$

Let us put

$$\mathcal{P}_j^{\boldsymbol{\varepsilon}}(\lambda, x_{\boldsymbol{\varepsilon}}) := \Delta_j(\lambda e_0 - x_{\boldsymbol{\varepsilon}}) = \lambda^{d_j} + \lambda^{d_j-1}q_j^{(1)}(x_{\boldsymbol{\varepsilon}}) + \cdots + q_j^{(d_j)}(x_{\boldsymbol{\varepsilon}}),$$

where $q_j^{(k)}$ ($k = 1, \ldots, d_j$) are polynomial functions on $\overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$ of degree k. By the coefficient comparison of (5.1) with (4.2) relative to λ, the polynomials $q_j^{(k)}(x_{\boldsymbol{\varepsilon}})$ are the zero-polynomials on $\overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$ for $k = d_j - \alpha_j + 1, \ldots, d_j$, and the polynomial $q_j^{(d_j-\alpha_j)}(x_{\boldsymbol{\varepsilon}})$ is non-zero. In particular, $\lambda^{-\alpha_j}\mathcal{P}_j^{\boldsymbol{\varepsilon}}(\lambda, x_{\boldsymbol{\varepsilon}})$ is an irreducible polynomial. By (5.1), we see that $\lambda^{-\alpha_j}\Delta_j(\lambda x - x_{\boldsymbol{\varepsilon}})$ is also irreducible.

Theorem 5.1. *Let $\sigma = \sigma_V$ be the multiplier matrix of V. Then with $\boldsymbol{\varepsilon}$ as above, one has $\boldsymbol{\alpha} = \sigma_V(\mathbf{1}-\boldsymbol{\varepsilon})$, that is,*

$$\alpha_j = \sum_{k=1}^{r} \sigma_{jk}(1-\varepsilon_k) \quad (j = 1, \ldots, r). \tag{5.2}$$

Moreover, if σ^0 is the multiplier matrix of the clan $(V_E^0, \triangle)$, then

$$\sigma^0 = \begin{pmatrix}1 & 0\\ \sigma\boldsymbol{\varepsilon} & \sigma\end{pmatrix}. \tag{5.3}$$

Proof. We shall prove the theorem by induction on rank r. Let V' and (φ', E') be as in (2.7). We put $e_1 = e_0 - c_1$, which is the unit element of V'. By (4.6), the multiplier matrix $\sigma' := \sigma_{V'}$ of V' is equal to the $(r-1) \times (r-1)$ matrix $\sigma' = (\sigma_{jk})_{2 \le j,k \le r}$. We consider the Lie algebra $\mathfrak{h}' = \{L_{x'};\ x' \in V'\}$ of left multiplication operators of V' and the corresponding Lie group $H' = \exp \mathfrak{h}'$. Let $\Omega' := H'e_1$, the homogeneous cone associated with $(V', \triangle)$. For each $\boldsymbol{\delta} = {}^t(\delta_2, \dots, \delta_r) \in \{0,1\}^{r-1}$, we put $c'_{\boldsymbol{\delta}} := \delta_2 c_2 + \cdots + \delta_r c_r$. Then $c'_{\boldsymbol{\delta}} \in \overline{\Omega'}$ and let $\mathcal{O}'_{\boldsymbol{\delta}} := H'c'_{\boldsymbol{\delta}} \subset \overline{\Omega'}$. Moreover, let $\widetilde{\alpha}_j \colon \{0,1\}^{r-1} \to \mathbb{Z}_{\ge 0}$ and polynomials $\widetilde{\mathcal{P}}^{\boldsymbol{\delta}}_j(\lambda, y_{\boldsymbol{\delta}})$ $(\lambda \in \mathbb{R},\ y_{\boldsymbol{\delta}} \in \overline{\mathcal{O}'_{\boldsymbol{\delta}}})$ be

$$\widetilde{\alpha}_j(\boldsymbol{\delta}) := \sum_{k=2}^{r} \sigma_{jk}(1 - \delta_k), \quad \widetilde{\mathcal{P}}^{\boldsymbol{\delta}}_j(\lambda, y_{\boldsymbol{\delta}}) := \Delta'_j(\lambda e_1 - y_{\boldsymbol{\delta}}) \quad (j = 2, \dots, r). \tag{5.4}$$

We note that $\widetilde{\boldsymbol{\alpha}}(\boldsymbol{\delta}) = \sigma'(\mathbf{1} - \boldsymbol{\delta})$. By the induction hypothesis, there exist irreducible polynomials $\widetilde{\mathcal{F}}^{\boldsymbol{\delta}}_j(\lambda, y_{\boldsymbol{\delta}})$ $(j = 2, \dots, r)$ such that

$$\widetilde{\mathcal{P}}^{\boldsymbol{\delta}}_j(\lambda, y_{\boldsymbol{\delta}}) = \lambda^{\widetilde{\alpha}_j(\boldsymbol{\delta})} \widetilde{\mathcal{F}}^{\boldsymbol{\delta}}_j(\lambda, y_{\boldsymbol{\delta}}) \quad (\lambda \in \mathbb{R},\ y_{\boldsymbol{\delta}} \in \overline{\mathcal{O}'_{\boldsymbol{\delta}}}). \tag{5.5}$$

For $j = 1, \dots, r$, let us consider the polynomial functions $\mathcal{P}^{\boldsymbol{\varepsilon}}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ $(\lambda \in \mathbb{R},\ x_{\boldsymbol{\varepsilon}} \in \overline{\mathcal{O}_{\boldsymbol{\varepsilon}}})$ defined by

$$\mathcal{P}^{\boldsymbol{\varepsilon}}_j(\lambda, x_{\boldsymbol{\varepsilon}}) = \Delta_j(\lambda e_0 - x_{\boldsymbol{\varepsilon}}). \tag{5.6}$$

In order to know what power of λ is factored out from $\mathcal{P}^{\boldsymbol{\varepsilon}}_j(\lambda, x_{\boldsymbol{\varepsilon}})$, it is clearly sufficient by continuity that we argue by restricting the variable $x_{\boldsymbol{\varepsilon}}$ to $\mathcal{O}_{\boldsymbol{\varepsilon}}$. Thus we assume $x_{\boldsymbol{\varepsilon}} \in \mathcal{O}_{\boldsymbol{\varepsilon}}$ and take $h \in H$ such that $x_{\boldsymbol{\varepsilon}} = hc_{\boldsymbol{\varepsilon}}$. Let Q' be the symmetric bilinear map associated with φ' and we put $\boldsymbol{\varepsilon}' := \boldsymbol{\varepsilon}(\varphi') = {}^t(\varepsilon'_2, \dots, \varepsilon'_r) \in \{0,1\}^{r-1}$. By the induction hypothesis and (4.5), we have

$$\Delta_j(x) = (x_{11})^{-\widetilde{\alpha}_j(\boldsymbol{\varepsilon}')} \Delta'_j(x_{11}x' - \tfrac{1}{2}Q'[\xi']) \quad (j = 2, \dots, r), \tag{5.7}$$

where $x = x_{11}c_1 + \xi' + x' \in V$. Let us put $\widetilde{\boldsymbol{\varepsilon}} = {}^t(\varepsilon_2, \dots, \varepsilon_r) \in \{0,1\}^{r-1}$. Applying Lemma 3.3 to $y = c_{\boldsymbol{\varepsilon}}$, we obtain

$$x_{\boldsymbol{\varepsilon}} = hc_{\boldsymbol{\varepsilon}} = \varepsilon_1(h_{11})^2 c_1 + \varepsilon_1 h_{11}\xi' + \left(h'c'_{\widetilde{\boldsymbol{\varepsilon}}} + \frac{\varepsilon_1}{2}Q'[\xi']\right).$$

Putting $y_{\tilde{\boldsymbol{\varepsilon}}} = h'c'_{\widetilde{\boldsymbol{\varepsilon}}}$, we have

$$\lambda e_0 - x_{\boldsymbol{\varepsilon}} = \big(\lambda - \varepsilon_1(h_{11})^2\big)c_1 - \varepsilon_1 h_{11}\xi' + \left(\lambda e_1 - y_{\tilde{\boldsymbol{\varepsilon}}} - \frac{\varepsilon_1}{2}Q'[\xi']\right). \tag{5.8}$$

(i) The case $j = 1$. In this case, we have $\mathcal{P}^{\boldsymbol{\varepsilon}}_1(\lambda, x_{\boldsymbol{\varepsilon}}) = \lambda - \varepsilon_1(h_{11})^2$. If $\varepsilon_1 = 0$ then $\mathcal{P}^{\boldsymbol{\varepsilon}}_1(\lambda, x_{\boldsymbol{\varepsilon}}) = \lambda$, and if $\varepsilon_1 = 1$ then $\mathcal{P}^{\boldsymbol{\varepsilon}}_1(\lambda, x_{\boldsymbol{\varepsilon}})$ does not have the factor λ. Hence in both cases we have $\alpha_1 = 1 - \varepsilon_1$. Since $\sigma_{1k} = \delta_{1k}$ $(k = 1, \dots, r)$, we obtain (5.2) for α_1.

(ii) The case $j = 2, \dots, r$. (a) We first assume that $\varepsilon_1 = 0$. In this case, (5.8) reduces to $\lambda e_0 - x_{\boldsymbol{\varepsilon}} = \lambda c_1 + (\lambda e_1 - y_{\tilde{\boldsymbol{\varepsilon}}})$. Using (5.7) and (5.5), we obtain

$$\begin{aligned}\Delta_j\big(\lambda c_1 + (\lambda e_1 - y_{\tilde{\boldsymbol{\varepsilon}}})\big) &= \lambda^{-\widetilde{\alpha}_j(\boldsymbol{\varepsilon}')} \Delta'_j\big(\lambda(\lambda e_1 - y_{\tilde{\boldsymbol{\varepsilon}}})\big) = \lambda^{-\widetilde{\alpha}_j(\boldsymbol{\varepsilon}') + d'_j} \widetilde{\mathcal{P}}^{\tilde{\boldsymbol{\varepsilon}}}_j(\lambda, y_{\tilde{\boldsymbol{\varepsilon}}}) \\ &= \lambda^{-\widetilde{\alpha}_j(\boldsymbol{\varepsilon}') + d'_j + \widetilde{\alpha}_j(\tilde{\boldsymbol{\varepsilon}})} \widetilde{\mathcal{F}}^{\tilde{\boldsymbol{\varepsilon}}}_j(\lambda, y_{\tilde{\boldsymbol{\varepsilon}}}).\end{aligned}$$

Here the induction hypothesis for (5.3) says $\sigma_{j1} = \sum_{k=2}^r \sigma_{jk}\varepsilon'_k$. By using (1.5) for d'_j, we can rewrite σ_{j1} as $\sigma_{j1} = -\widetilde{\alpha}_j(\boldsymbol{\varepsilon}') + d'_j$. Thus we get

$$\alpha_j = \sigma_{j1} + \widetilde{\alpha}_j(\widetilde{\boldsymbol{\varepsilon}}) = \sum_{k=1}^r \sigma_{jk}(1-\varepsilon_k) \quad (j=2,\dots,r).$$

(b) Next let us consider the case $\varepsilon_1 = 1$. We assume that λ is in a small neighborhood U_1 of 0 and h_{11} in a small neighborhood U_2 of 1, so that putting $a_\lambda := -(\lambda-(h_{11})^2)$, we have $a_\lambda > 0$. Then by (5.7) and (5.8)

$$\begin{aligned}\Delta_j(\lambda e_0 - x_{\boldsymbol{\varepsilon}}) &= (-a_\lambda)^{-\widetilde{\alpha}_j(\varepsilon')}\Delta'_j\big((-a_\lambda)(\lambda e_1 - y_{\tilde{\varepsilon}} - \tfrac{1}{2}Q'[\xi']) - \tfrac{1}{2}Q'[h_{11}\xi']\big)\\ &= (-a_\lambda)^{-\widetilde{\alpha}_j(\varepsilon')}\Delta'_j\big(-\lambda(a_\lambda e_1 + \tfrac{1}{2}Q'[\xi']) + a_\lambda y_{\tilde{\varepsilon}}\big).\end{aligned}$$

Since $a_\lambda > 0$ and $Q'[\xi'] \in \overline{\mathcal{O}'_{\boldsymbol{\varepsilon}'}} \subset \overline{\Omega'}$, we have $a_\lambda e_1 + \frac{1}{2}Q'[\xi'] \in \Omega'$ for any $\lambda \in U_1$ and $h_{11} \in U_2$. Thus for each such λ and h_{11}, there exists a unique $g_\lambda \in H'$ so that $g_\lambda e_1 = a_\lambda e_1 + \frac{1}{2}Q'[\xi']$. The one-dimensional representation associated with Δ'_j being $\chi_{\underline{\sigma}'_j}$, we have $\chi_{\underline{\sigma}'_j}(g_\lambda) = \Delta'_j(a_\lambda e_1 + \frac{1}{2}Q'[\xi'])$. Using the relative H-invariance of Δ'_j and (5.5), we obtain

$$\begin{aligned}\Delta_j(\lambda e_0 - x_{\boldsymbol{\varepsilon}}) &= (-a_\lambda)^{-\widetilde{\alpha}_j(\varepsilon')}\Delta'_j(-\lambda g_\lambda e_1 + a_\lambda y_{\tilde{\varepsilon}})\\ &= (-a_\lambda)^{-\widetilde{\alpha}_j(\varepsilon')}\chi_{\underline{\sigma}'_j}(g_\lambda)\Delta'_j(-\lambda e_1 + a_\lambda g_\lambda^{-1} y_{\tilde{\varepsilon}})\\ &= (-1)^{2d'_j}(-a_\lambda)^{-\widetilde{\alpha}_j(\varepsilon')}\Delta'_j\big((-a_\lambda)e_1 - \tfrac{1}{2}Q'[\xi']\big)\Delta'_j(\lambda e_1 - a_\lambda g_\lambda^{-1} y_{\tilde{\varepsilon}})\\ &= \lambda^{\widetilde{\alpha}_j(\widetilde{\varepsilon})}\widetilde{\mathcal{F}}_j^{\boldsymbol{\varepsilon}'}(-a_\lambda, \tfrac{1}{2}Q'[\xi'])\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(\lambda, a_\lambda g_\lambda^{-1} y_{\tilde{\varepsilon}}).\end{aligned}$$

To continue, we introduce a rational function $\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ defined by

$$\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}}) := \lambda^{-\widetilde{\alpha}_j(\widetilde{\varepsilon})}\mathcal{P}_j^{\boldsymbol{\varepsilon}}(\lambda, x_{\boldsymbol{\varepsilon}}) = \widetilde{\mathcal{F}}_j^{\boldsymbol{\varepsilon}'}(-a_\lambda, \tfrac{1}{2}Q'[\xi'])\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(\lambda, a_\lambda g_\lambda^{-1} y_{\tilde{\varepsilon}}).$$

We shall show that $\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ is actually an irreducible polynomial. Since $\mathcal{P}_j^{\boldsymbol{\varepsilon}}(\lambda, x_{\boldsymbol{\varepsilon}})$ is a polynomial, it is sufficient to prove the existence of a non-zero limit of $\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ as $\lambda \to 0$. Since both of $\widetilde{\mathcal{F}}_j^{\boldsymbol{\varepsilon}'}$ and $\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}$ are polynomial functions, and since the map $g\colon U_1 \ni \lambda \mapsto g_\lambda^{-1} \in H'$ is continuous as well as $\lambda \mapsto g_\lambda$, we obtain

$$\lim_{\lambda \to 0}\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}}) = \widetilde{\mathcal{F}}_j^{\boldsymbol{\varepsilon}'}(-a_0, \tfrac{1}{2}Q'[\xi'])\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(0, a_0 g_0^{-1} y_{\tilde{\varepsilon}}).$$

In order to see that this limit is non-zero, we put $h_{11} = 1$ and $\xi' = 0$. Then we have $a_0 = 1$ and $g_0 e_1 = e_1$, that is, g_0 is the unit element of H'. By (5.5) and (5.4), we have

$$\widetilde{\mathcal{F}}_j^{\boldsymbol{\varepsilon}'}(-1, 0) = (-1)^{-\widetilde{\alpha}_j(\boldsymbol{\varepsilon}')}\widetilde{\mathcal{P}}_j^{\boldsymbol{\varepsilon}'}(-1, 0) = (-1)^{d'_j - \widetilde{\alpha}_j(\boldsymbol{\varepsilon}')}.$$

On the other hand, since $\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(\lambda, y_{\tilde{\varepsilon}})$ does not have the factor of λ, we can take $z_{\widetilde{\varepsilon}} \in \mathcal{O}'_{\widetilde{\boldsymbol{\varepsilon}}}$ such that $\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(0, z_{\widetilde{\varepsilon}}) \neq 0$. Thus we obtain

$$\mathcal{F}_j(0, c_1 + z_{\widetilde{\varepsilon}}) = \widetilde{\mathcal{F}}_j^{\boldsymbol{\varepsilon}'}(-1, 0)\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(0, z_{\widetilde{\varepsilon}}) = (-1)^{d'_j - \widetilde{\alpha}_j(\boldsymbol{\varepsilon}')}\widetilde{\mathcal{F}}_j^{\widetilde{\boldsymbol{\varepsilon}}}(0, z_{\widetilde{\varepsilon}}) \neq 0.$$

Hence $\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ does not have the factor of λ. Since U_1 and U_2 are open sets and since we now know that $\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ is a polynomial, the function $\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ is extended to $\mathbb{R}\times\mathcal{O}_{\boldsymbol{\varepsilon}}$ and does not have the factor of λ. Therefore $\mathcal{P}_j^{\boldsymbol{\varepsilon}}(\lambda, x_{\boldsymbol{\varepsilon}}) = \lambda^{\widetilde{\alpha}_j(\widetilde{\boldsymbol{\varepsilon}})}\mathcal{F}_j(\lambda, x_{\boldsymbol{\varepsilon}})$ is an irreducible factorization. This shows

$$\alpha_j = \widetilde{\alpha}_j(\widetilde{\boldsymbol{\varepsilon}}) = \sum_{k=1}^{r}\sigma_{jk}(1-\varepsilon_k) \quad (j=2,\dots,r).$$

It remains to prove (5.3). Since $\boldsymbol{\alpha} = \sigma(\mathbf{1}-\boldsymbol{\varepsilon})$ and $\boldsymbol{d} = \sigma\mathbf{1}$ by (1.5), we obtain by (4.6)

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ \boldsymbol{d}-\boldsymbol{\alpha} & \sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sigma\boldsymbol{\varepsilon} & \sigma \end{pmatrix}.$$

This completes the proof. □

Remark 5.2. If we put $x_{\boldsymbol{\varepsilon}} := \sum_{k=1}^{r}\varepsilon_k\lambda_k c_k$ $(\lambda_k > 0)$ in (5.6), then we have

$$\mathcal{P}_j^{\boldsymbol{\varepsilon}}(\lambda, x_{\boldsymbol{\varepsilon}}) = \prod_{k=1}^{r}(\lambda-\varepsilon_k\lambda_k)^{\sigma_{jk}} = \lambda^{\sum_{\varepsilon_k=0}\sigma_{jk}}\prod_{\varepsilon_k=1}(\lambda-\lambda_k)^{\sigma_{jk}}.$$

But this only implies that $\alpha_j \le \sum_{\varepsilon_k=0}\sigma_{jk} = \sum_{k=1}^{r}\sigma_{jk}(1-\varepsilon_k)$, since, in general, the restriction to a lower dimensional set of an irreducible polynomial need not be irreducible.

Let $\underline{n}$ be the basic index of V. By Proposition 4.1 and Theorem 4.3, the determinant of the right multiplication operators R_v^0 of the clan V_E^0 is described as

$$\operatorname{Det} R_v^0 = P_0(v)^{1+\dim E-\dim V+\underline{n}\boldsymbol{\alpha}}P_1(v)^{n_1}\cdots P_r(v)^{n_r} \quad (v\in V_E^0).$$

Let us verify here that $1+\dim E-\dim V+\underline{n}\boldsymbol{\alpha}\ge 1$. Indeed we first note that Ishi [3, Lemma 3.3 (ii)] tells us that $\dim\overline{\mathcal{O}_{\boldsymbol{\varepsilon}}} = \underline{m}\boldsymbol{\varepsilon}$. Then the fact that $Q[E] = \overline{\mathcal{O}_{\boldsymbol{\varepsilon}}}$ together with Theorem 4.4 implies

$$\dim E \ge \dim\overline{\mathcal{O}_{\boldsymbol{\varepsilon}}} = \underline{m}\boldsymbol{\varepsilon} = \underline{n}\sigma\boldsymbol{\varepsilon}.$$

Recalling $\dim V = \underline{n}\boldsymbol{d} = \underline{n}(\sigma\mathbf{1})$ and Theorem 5.1, we obtain

$$\dim E - \dim V + \underline{n}\boldsymbol{\alpha} \ge \underline{n}\sigma\boldsymbol{\varepsilon} - \underline{n}\sigma\mathbf{1} + \underline{n}\sigma(\mathbf{1}-\boldsymbol{\varepsilon}) = 0.$$

Finally we determine the multiplier matrix $\sigma = \sigma_V$ of a given clan V. This completes Theorems 4.3, 4.4 and 5.1. Let $V^{[k]}$ be the subclan defined in (2.6) and $(\mathcal{R}^{[k]}, E^{[k]})$ the representation of $(V^{[k]}, \triangledown)$ appearing in Proposition 2.3. For $k=1,2,\dots,r-1$, we put $\boldsymbol{\varepsilon}^{[k]} := \boldsymbol{\varepsilon}(\mathcal{R}^{[k]}) \in \{0,1\}^{r-k}$ and consider the $r\times r$ matrix $\mathcal{E}_k$ defined by

$$\mathcal{E}_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boldsymbol{\varepsilon}^{[k]} & I_{r-k} \end{pmatrix} \quad (k=1,\dots,r-1).$$

Theorem 5.3. *The multiplier matrix $\sigma = \sigma_V$ of the clan V is given by*

$$\sigma = \mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1.$$

Proof. We shall prove the theorem by induction on rank r. Let V' and (φ', E') be as in (2.7). Then we have $V = \mathbb{R}c_1 \oplus E' \oplus V'$. By the induction hypothesis, the multiplier matrix σ' of V' is described as

$$\sigma' = \mathcal{E}'_{r-1}\mathcal{E}'_{r-2}\cdots\mathcal{E}'_2, \quad \mathcal{E}'_k = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boldsymbol{\varepsilon}^{[k]} & I_{r-k} \end{pmatrix} \quad (k = 2, 3, \dots, r-1). \tag{5.9}$$

Let us put $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\varphi')$. Applying Theorem 5.1 with V regarded as in (3.3), we have by (5.9)

$$\sigma = \begin{pmatrix} 1 & 0 \\ \sigma'\boldsymbol{\varepsilon} & \sigma' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \boldsymbol{\varepsilon} & I_{r-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}'_{r-1} \end{pmatrix}\cdots\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}'_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \boldsymbol{\varepsilon} & I_{r-1} \end{pmatrix}.$$

By noting $\mathcal{E}_k = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}'_k \end{pmatrix}$ $(k = 2, 3, \dots, r-1)$, the proof is completed. □

Remark 5.4. Let Ω be a symmetric cone with rank $r \geq 3$. The corresponding clan is $V = \mathrm{Herm}(r, \mathbb{K})$ $(r \geq 3,\ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$. Since the basic relative invariants $\Delta_k(x)$ $(k = 1, \dots, r)$ are the left upper corner principal minors $\det^{[k]}(x)$, the multiplier matrix $\sigma = (\sigma_{jk})$ of V is given by $\sigma = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}$. For a positive integer p, we put $E_p := \mathrm{Mat}(r \times p, \mathbb{K})$ and define a linear map $\varphi_p \colon V \to \mathcal{L}(E_p)$ by

$$\varphi_p(x)\xi := x\xi \quad (x \in V,\ \xi \in E_p),$$

where the multiplication on the right hand side is the ordinary matrix multiplication. Then (φ_p, E_p) is a representation of $(V, \triangledown)$. If we put

$$\boldsymbol{\varepsilon}(p) := \begin{cases} {}^t(\underbrace{1, \dots, 1}_{p}, 0, \dots, 0) & (1 \leq p < r), \\ {}^t(1, \dots, 1) & (r \leq p), \end{cases}$$

then φ_p is an $\boldsymbol{\varepsilon}(p)$-representation. By Theorem 5.1, we have

$$\boldsymbol{\alpha} = \sigma(\mathbf{1} - \boldsymbol{\varepsilon}(p)) = \begin{cases} {}^t(\underbrace{0, \dots, 0}_{p}, 1, 2, \dots, r-p) & (1 \leq p < r), \\ {}^t(0, \dots, 0) & (r \leq p). \end{cases}$$

This $\boldsymbol{\alpha}$ is first obtained in [9]. We note that the representation is regular if and only if $\boldsymbol{\varepsilon}(p) = \mathbf{1}$.

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