Unitary transformations and multivariate special orthogonal polynomials

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Unitary transformations and multivariate special orthogonal polynomials

GENKI SHIBUKAWA

Dedicated to my family and ancestors.

Contents

Chapter 1

Introduction

Investigations into special orthogonal polynomial systems by using unitary transformations have a long history. We actually historically know that the composition of Laplace and Cayley transforms provides correspondence between Laguerre and power polynomials. In addition, Shen [She] established a connection between Laguerre polynomials and circular Jacobi polynomials using a Fourier transform, and Koornwinder [Ko] found a link between Laguerre and Meixner-Pollaczek polynomials by using a Mellin transform.

Let us describe the picture in the one variable case more precisely. We put $\alpha > 1$, $(\alpha)_m := \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+m-1)$, $\binom{m}{k} = (-1)^k \frac{(-m)_k}{k!}$, $\mathcal{D} := \{w \in \mathbb{C} \mid |w| < 1\}$, $T := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, $H := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, $\partial H = \mathbb{R}$, $\Sigma := \{\sigma \in \mathbb{C} \mid \sigma^{-1} = \overline{\sigma}\}$, m is the Lebesgue measure on $\mathbb{C}$. Further, we introduce the following function spaces and their complete orthogonal bases.

(1) $f_m^{(\alpha)}$; power polynomials

$$\begin{aligned}
\mathcal{H}^2_\alpha(\mathcal{D}) &:= \{f : \mathcal{D} \longrightarrow \mathbb{C} \mid f \text{ is analytic in } \mathcal{D} \text{ and } \|f\|^2_{\alpha,\mathcal{D}} < \infty\},\\
\|f\|^2_{\alpha,\mathcal{D}} &:= \frac{\alpha-1}{\pi}\int_{\mathcal{D}} |f(w)|^2(1-|w|^2)^{\alpha-2}\, m(dw),\\
f_m^{(\alpha)}(w) &:= \frac{(\alpha)_m}{m!} w^m.
\end{aligned}$$

(2) $F_m^{(\alpha)}$; Cayley transform of the power polynomials

$$\begin{aligned}
\mathcal{H}^2_\alpha(T) &:= \{F : T \longrightarrow \mathbb{C} \mid F \text{ is analytic in } T \text{ and } \|F\|^2_{\alpha,T} < \infty\},\\
\|F\|^2_{\alpha,T} &:= \frac{\alpha-1}{4\pi}\int_{T} |F(z)|^2 x^{\alpha-2}\, m(dz),\\
F_m^{(\alpha)}(z) &:= \frac{(\alpha)_m}{m!}\left(\frac{1+z}{2}\right)^{-\alpha}\left(\frac{z-1}{z+1}\right)^m.
\end{aligned}$$

(3) $\psi_m^{(\alpha)}$; exponential multiplied by Laguerre polynomials

$$
\begin{aligned}
L^2_\alpha(\mathbb{R}_{>0}) &:= \{\psi : \mathbb{R}_{>0} \longrightarrow \mathbb{C} \mid \|\psi\|^2_{\alpha,\mathbb{R}_{>0}} < \infty\},\\
\|\psi\|^2_{\alpha,\mathbb{R}_{>0}} &:= \frac{2^\alpha}{\Gamma(\alpha)} \int_0^\infty |\psi(u)|^2 u^{\alpha-1}\, du,\\
\psi_m^{(\alpha)}(u) &:= e^{-u} L_m^{(\alpha-1)}(2u) = \frac{(\alpha)_m}{m!} e^{-u} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{1}{(\alpha)_k} (2u)^k.
\end{aligned}
$$

(4) $q_m^{(\alpha)}(s)$; Meixner-Pollaczek polynomials

$$
\begin{aligned}
L^2_\alpha(\mathbb{R}) &:= \{q : \mathbb{R} \longrightarrow \mathbb{C} \mid \|q\|^2_{\alpha,\mathbb{R}} < \infty\},\\
\|q\|^2_{\alpha,\mathbb{R}} &:= \frac{1}{2\pi} \frac{2^\alpha}{\Gamma(\alpha)} \int_{-\infty}^\infty |q(s)|^2 \left|\Gamma\left(is + \frac{\alpha}{2}\right)\right|^2 ds,\\
q_m^{(\alpha)}(s) &:= i^{-m} P_m^{(\frac{\alpha}{2})}\left(s; \frac{\pi}{2}\right) = \frac{(\alpha)_m}{m!} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\left(\frac{\alpha}{2} + is\right)_k}{(\alpha)_k} 2^k.
\end{aligned}
$$

(5) $\Psi_m^{(\alpha)}(t)$; Modified Fourier transform of the Laguerre polynomials

$$
\begin{aligned}
H^2_\alpha(\partial H) := \Big\{\Psi : \mathbb{R} \longrightarrow \mathbb{C} \mid \|\Psi\|^2_{\alpha,\partial H} < \infty \text{ and } \Psi \text{ is continued analytically to } H\\
\text{as a holomorphic function which satisfies with}\\
\sup_{0<y<\infty} \frac{1}{2\pi} \int_0^\infty |\Psi(x+iy)|^2\, dx < \infty\Big\},\\
\|\Psi\|^2_{\alpha,\partial H} := \frac{\Gamma\left(\frac{\alpha+1}{2}\right)^2}{2\pi} \frac{2^\alpha}{\Gamma(\alpha)} \int_{-\infty}^\infty |\Psi(t)|^2\, dt,\\
\Psi_m^{(\alpha)}(t) := (1-it)^{-\frac{\alpha+1}{2}} \frac{(\alpha)_m}{m!} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\left(\frac{\alpha+1}{2}\right)_k}{(\alpha)_k} \left(\frac{2}{1-it}\right)^k.
\end{aligned}
$$

(6) $\phi_m^{(\alpha)}(\sigma)$; circular Jacobi polynomials

$$
\begin{aligned}
H^2_\alpha(\Sigma) := \{\phi : \Sigma \longrightarrow \mathbb{C} \mid \phi \text{ is continued analytically to } \mathcal{D} \text{ as a holomorphic function}\\
\text{and } \|\phi\|^2_{\alpha,\Sigma} < \infty\},\\
\|\phi\|^2_{\alpha,\Sigma} := \frac{\Gamma\left(\frac{\alpha+1}{2}\right)^2}{2\pi i} \frac{1}{\Gamma(\alpha)} \int_\Sigma |\phi(\sigma)|^2 (1-\sigma)^{\frac{\alpha-1}{2}} (1-\overline{\sigma})^{\frac{\alpha-1}{2}} \frac{m(d\sigma)}{\sigma},\\
\phi_m^{(\alpha)}(\sigma) := \frac{(\alpha)_m}{m!} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\left(\frac{\alpha+1}{2}\right)_k}{(\alpha)_k} (1-\sigma)^k.
\end{aligned}
$$

We remark that

$$
\|f_m^{(\alpha)}\|^2_{\alpha,\mathcal{D}} = \|F_m^{(\alpha)}\|^2_{\alpha,T} = \|\psi_m^{(\alpha)}\|^2_{\alpha,\mathbb{R}_{>0}} = \|q_m^{(\alpha)}\|^2_{\alpha,\mathbb{R}} = \|\Psi_m^{(\alpha)}\|^2_{\alpha,\partial H} = \|\phi_m^{(\alpha)}\|^2_{\alpha,\Sigma} = \frac{(\alpha)_m}{m!}.
$$

Furthermore, the following unitary isomorphisms are known.

<u>Modified Cayley transform 1</u>

$$C_\alpha^{-1} : \mathcal{H}_\alpha^2(T) \xrightarrow{\simeq} \mathcal{H}_\alpha^2(\mathcal{D}),\ \ (C_\alpha^{-1}F)(w) := (1-w)^{-\alpha} F\left(\frac{1+w}{1-w}\right).$$

<u>Modified Cayley transform 2</u>

$$\mathcal{C}_\alpha^{-1} : H_\alpha^2(\partial H) \xrightarrow{\simeq} H_\alpha^2(\Sigma),\ \ (\mathcal{C}_\alpha^{-1}\Psi)(\sigma) := \left(\frac{1-\sigma}{2}\right)^{-\frac{\alpha+1}{2}} \Psi\left(i\frac{1+\sigma}{1-\sigma}\right).$$

<u>Modified Laplace transform</u>

$$\mathcal{L}_\alpha : L_\alpha^2(\mathbb{R}_{>0}) \xrightarrow{\simeq} \mathcal{H}_\alpha^2(T),\ \ (\mathcal{L}_\alpha\psi)(z) := \frac{2^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-zu} u^{\alpha-1}\psi(u)\, du.$$

<u>Modified Mellin transform</u>

$$\mathcal{M}_\alpha : L_\alpha^2(\mathbb{R}_{>0}) \xrightarrow{\simeq} L_\alpha^2(\mathbb{R}),\ \ (\mathcal{M}_\alpha\psi)(s) := \frac{1}{\Gamma\left(is+\frac{\alpha}{2}\right)} \int_0^\infty u^{is+\frac{\alpha}{2}-1}\psi(u)\, du.$$

<u>Modified (inverse) Fourier transform</u>

$$\mathcal{F}_\alpha^{-1} : L_\alpha^2(\mathbb{R}_{>0}) \xrightarrow{\simeq} H_\alpha^2(\partial H),\ \ (\mathcal{F}_\alpha^{-1}\psi)(t) := \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^\infty e^{itu} u^{\frac{\alpha-1}{2}}\psi(u)\, du.$$

To summarize, we obtain the following picture given by the unitary transformations.

$$\begin{array}{ccccccc}
 & & \mathbf{(6)} & & \mathbf{(5)} & & \\
 & & \phi_m^{(\alpha)} & \longleftarrow & \Psi_m^{(\alpha)} & & \\
 & & \cap & & \cap & & \\
 & & H_\alpha^2(\Sigma) & \xleftarrow[\mathcal{C}_\alpha^{-1}]{\simeq} & H_\alpha^2(\partial H) & & \\
 & & & & \wr\uparrow \mathcal{F}_\alpha^{-1} & & \\
\mathcal{H}_\alpha^2(\mathcal{D}) & \xleftarrow[C_\alpha^{-1}]{\simeq} & \mathcal{H}_\alpha^2(T) & \xleftarrow[\mathcal{L}_\alpha]{\simeq} & L_\alpha^2(\mathbb{R}_{>0}) & \xrightarrow[\mathcal{M}_\alpha]{\simeq} & L_\alpha^2(\mathbb{R}). \\
\cup & & \cup & & \cup & & \cup \\
f_m^{(\alpha)} & \longleftarrow & F_m^{(\alpha)} & \longleftarrow & \psi_m^{(\alpha)} & \longmapsto & q_m^{(\alpha)} \\
\mathbf{(1)} & & \mathbf{(2)} & & \mathbf{(3)} & & \mathbf{(4)}
\end{array}$$

Moreover, some parts of the picture have been generalized to the multivariate case. That is

$$\mathcal{H}_\alpha^2(\mathcal{D})^K \xleftarrow[C_\alpha^{-1}]{\simeq} \mathcal{H}_\alpha^2(T_\Omega)^K \xleftarrow[\mathcal{L}_\alpha]{\simeq} L_\alpha^2(\Omega)^K \xrightarrow[\mathcal{M}_\alpha]{\simeq} L_\alpha^2(\mathbb{R}^r)^{\mathfrak{S}_r}. \tag{1.0.1}$$

Here, as we will introduce the notations in Chapter 2, Ω is the symmetric cone and T_Ω is the associated tube domain with Ω. From this picture, similar to the one variable case, we

obtain the well-known correspondence between Laguerre and spherical polynomials. A link between Laguerre and Meixner-Pollaczek polynomials from a spherical Fourier transform has recently been established that was also established by Davidson, Olafsson and Zhang [DOZ], and Faraut and Wakayama [FW1]. This setting is not only beneficial for introducing the above orthogonal systems, but also studying their fundamental properties (orthogonality, generating functions, difference or differential equations and recurrence formulas).

This thesis has two purposes. The first is to study a multivariate analogue of the results obtained by Shen [She]. Namely, we consider a modified Fourier transform of $L^2_\alpha(\Omega)^K$ and multivariate Laguerre polynomials. Using this unitary isomorphism and the modified Cayley transform, we introduce some new multivariate special orthogonal polynomials, which are a multivariate analogue of circular Jacobi polynomials. These polynomials, which we call multivariate circular Jacobi (MCJ) polynomials, are generalizations of the spherical (zonal) polynomials that are different from the Jack or Macdonald polynomials, which are well known as an extension of spherical polynomials. We also remark that the weight function of their orthogonality relation coincides with the circular Jacobi ensemble defined by Bourgade et al. [BNR]. Furthermore, we provide a generating function for the MCJ polynomials and a differential equation that is satisfied by the modified Cayley transform of the MCJ polynomials.

The second purpose is to introduce some multivariate discrete orthogonal polynomials within the setting (1.0.1) that are multivariate analogues of Meixner, Charlier and Krawtchouk polynomials, and to establish their main properties, that is, duality, degenerate limits, generating functions, orthogonality relations, difference equations and recurrence formulas. A particularly important and interesting result is that "the generating function of the generating function" for the Meixner polynomials coincides with the generating function of the Laguerre polynomials. We derive the above properties for the multivariate Meixner, Charlier and Krawtchouk polynomials from some properties of the multivariate Laguerre polynomials and the unitary picture (1.0.1) by using this key lemma. This scheme has previously not been known even for the one variable case. It is also interesting to note that there is correspondence between Laguerre and Meixner polynomials. The former has orthogonality defined by the integral on the symmetric cone and the latter is defined by the summation on partitions.

Let us now describe the content of the following chapters. The basic definitions and fundamental properties of Jordan algebras and symmetric cones, and lemmas for analysis on symmetric cones and tube domains have been presented in the first section of Chapter 2, so that they can be referred to later. The next section presents a compilation of basic facts for the multivariate Laguerre polynomials and their unitary picture. In particular, we construct the unitary isomorphism between $L^2_\alpha(\Omega)^K$ and $L^2_{\alpha,\theta}(\mathbb{R}^r)^{\mathfrak{S}_r}$. Since we do not need to use the Gutzmer formula, our construction is much simpler than [FW1] and is regarded as a multiple analogue of Koornwinder's construction [Ko]. Based on these preparations, in Chapter 3, we

complete the picture (1.0.1) as follows

$$\begin{tikzcd}
 & H^2_{\alpha,\nu}(\Sigma)^K & H^2_{\alpha,\nu}(V)^K \arrow[l, "\simeq", "\mathcal{C}^{-1}_{\alpha,\nu}"'] & & \\
 & & & & \\
\mathcal{H}^2_\alpha(\mathcal{D})^K \arrow[r, "\simeq", "C^{-1}_\alpha"'] & \mathcal{H}^2_\alpha(T_\Omega)^K \arrow[r, "\simeq", "\mathcal{L}_\alpha"'] & L^2_\alpha(\Omega)^K \arrow[uu, "\wr", "\mathcal{F}^{-1}_{\alpha,\nu}"'] \arrow[r, "\simeq", "\mathcal{M}_{\alpha,\theta}"'] & L^2_{\alpha,\theta}(\mathbb{R}^r)^{\mathfrak{S}_r}.
\end{tikzcd}$$

In addition, using the above picture, we obtain the MCJ polynomials and their fundamental properties which are one of the main results in this thesis. The other main results on some multivariate discrete orthogonal polynomials are discussed in Chapter 4.

Finally, in the appendix, we extend the results for the operator ordering problem that is to related the Meixner-Pollaczek polynomials in [Ko], [HZ].

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Chapter 2

Preliminaries

Throughout the paper, we denote the ring of rational integers by $\mathbb{Z}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$, the partition set of length r by $\mathscr{P}$

$$\mathscr{P} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geq 0} \mid m_1 \geq \cdots \geq m_r\}. \tag{2.0.1}$$

For any vector $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we put

$$\operatorname{Re} \mathbf{s} := (\operatorname{Re} s_1, \ldots, \operatorname{Re} s_r), \tag{2.0.2}$$
$$|\mathbf{s}| := s_1 + \cdots + s_r, \tag{2.0.3}$$
$$\|\mathbf{s}\| := (|s_1|, \ldots, |s_r|). \tag{2.0.4}$$

Moreover, for $\mathbf{m} \in \mathscr{P}$

$$\mathbf{m}! := m_1! \cdots m_r!$$

and we set $\delta := (r-1, r-2, \ldots, 1, 0)$. Refer to Faraut and Koranyi [FK] for the details in this chapter.

2.1 Analysis on symmetric cones

Let Ω be an irreducible symmetric cone in V which is a finite dimensional simple Euclidean Jordan algebra of dimension n as a real vector space and rank r. The classification of irreducible symmetric cones is well-known. Namely, there are four families of classical irreducible symmetric cones $\Pi_r(\mathbb{R}), \Pi_r(\mathbb{C}), \Pi_r(\mathbb{H})$, the cones of all $r \times r$ positive definite matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the Lorentz cones Λ_r and an exceptional cone $\Pi_3(\mathbb{O})$ (see [FK] p. 97). Also, let $V^{\mathbb{C}}$ be its complexification. For $w, z \in V^{\mathbb{C}}$, we define

$$\begin{aligned} L(w)z &:= wz, \\ w\Box z &:= L(wz) + [L(w), L(z)], \\ P(w,z) &:= L(w)L(z) + L(z)L(w) - L(wz), \\ P(w) &:= P(w,w) = 2L(w)^2 - L(w^2). \end{aligned}$$

We denote the Jordan trace and determinant of the complex Jordan algebra $V^{\mathbb{C}}$ by $\operatorname{tr} x$ and by $\Delta(x)$ respectively.

Fix a Jordan frame $\{c_1, \ldots, c_r\}$ that is a complete system of orthogonal primitive idempotents in V and define the following subspaces:

$$V_j := \{x \in V \mid L(c_j)x = x\},$$
$$V_{jk} := \left\{ x \in V \,\middle|\, L(c_j)x = \frac{1}{2}x \text{ and } L(c_k)x = \frac{1}{2}x \right\}.$$

Then, $V_j = \mathbb{R}e_j$ for $j = 1, \ldots, r$ are 1-dimensional subalgebras of V, while the subspaces V_{jk} for $j, k = 1, \ldots, r$ with $j < k$ all have a common dimension $d = \dim_{\mathbb{R}} V_{jk}$. Then, V has the Peirce decomposition

$$V = \left(\bigoplus_{j=1}^{r} V_j \right) \oplus \left(\bigoplus_{j<k} V_{jk} \right),$$

which is the orthogonal direct sum. It follows that $n = r + \frac{d}{2}r(r-1)$. Let $G(\Omega)$ denote the automorphism group of Ω and let G be the identity component in $G(\Omega)$. Then, G acts transitively on Ω and $\Omega \cong G/K$ where $K \in G$ is the isotropy subgroup of the unit element, $e \in V$. K is also the identity component in $\operatorname{Aut}(V)$.

For any $x \in V$, there exists $k \in K$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that

$$x = k \sum_{j=1}^{r} \lambda_j c_j, \quad (\lambda_1 \geq \cdots \geq \lambda_r).$$

From this polar decomposition, we obtain the following integral formula (see [FK] Theorem VI. 2.3).

Lemma 2.1.1. *Let f be an integrable function on V. We have*

$$\int_V f(x)\, dx = \widetilde{c_0} \int_{K \times \mathbb{R}^r} f(k\lambda) \prod_{1 \leq p < q \leq r} |\lambda_p - \lambda_q|^d \, dk d\lambda_1 \cdots d\lambda_r. \tag{2.1.1}$$

Here, dx is the Euclidean measure associated with the Euclidean structure on V given by $(u|v) = \operatorname{tr}(uv)$, dk is the normalized Haar measure on the compact group K, $\lambda = \sum_{j=1}^{r} \lambda_j c_j$ and $\widetilde{c_0}$ is defined by

$$\widetilde{c_0} := (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d}{2}j+1\right)} = \frac{(2\pi)^{\frac{n-r}{2}}}{r!} \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)}. \tag{2.1.2}$$

In particular, for $f \in L^1(V)^K$

$$\int_V f(x)\, dx = \widetilde{c_0} \int_{\mathbb{R}^r} f(\lambda_1, \ldots, \lambda_r) \prod_{1 \leq p < q \leq r} |\lambda_p - \lambda_q|^d \, d\lambda_1 \cdots d\lambda_r. \tag{2.1.3}$$

As in the case of V, we also have the following spectral decomposition for $V^{\mathbb{C}}$. Every z in $V^{\mathbb{C}}$ can be written

$$z = u \sum_{j=1}^{r} \lambda_j c_j,$$

with u in U which is the identity component of $Str(V^{\mathbb{C}}) \cap U(V^{\mathbb{C}})$, $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Moreover, we define the spectral norm of $z \in V^{\mathbb{C}}$ by $|z| = \lambda_1$ and introduce open unit ball $\mathcal{D} \in V^{\mathbb{C}}$ as follows.

$$\mathcal{D} = \{z \in V^{\mathbb{C}} \mid |z| < 1\}.$$

We define Σ as the set of invertible elements in $V^{\mathbb{C}}$ such that $z^{-1} = \overline{z}$, which coincides with the Shilov boundary of $\mathcal{D}$. For Σ, the following result is well known (see [FK] Proposition X.2.3).

Lemma 2.1.2. *For $z \in V^{\mathbb{C}}$, the following properties are equivalent:*
(i) $z \in \Sigma$,
(ii) $z = e^{i\theta} = \sum_{j=1}^{r} e^{i\theta_j} c_j$ *with* $\theta = \sum_{j=1}^{r} \theta_j c_j \in V$,
(iii) $z \in \overline{c^{-1}(V)}$,
where $c^{-1}(t) := (t - ie)(t + ie)^{-1} = e - 2i(t + ie)^{-1}$ *is called the inverse Cayley transform.*

We will later need the following integral formula on Σ to describe the MCJ polynomials.

Lemma 2.1.3. *Let μ denote the measure associated with the Riemannian structure on Σ induced by the Euclidean structure of $V^{\mathbb{C}}$.*
(1) *If ϕ is an integrable function on Σ, then*

$$\int_{\Sigma} \phi(\sigma)\, d\mu(\sigma) = 2^n \int_{V} \phi(c^{-1}(t)) |\Delta(e - it)^{-\frac{n}{r}}|^2\, dt. \tag{2.1.4}$$

(2) *If Ψ is an integrable function on V, then*

$$\int_{V} \Psi(t)\, dt = 2^n \int_{\Sigma} \Psi(c(\sigma)) |\Delta(e - \sigma)^{-\frac{n}{r}}|^2\, d\mu(\sigma). \tag{2.1.5}$$

Here, c is a Cayley transform defined by $c(\sigma) := i(e + \sigma)(e - \sigma)^{-1} = -ie + 2i(e - \sigma)^{-1}$.
(3) *If Ψ is an integrable function on V and a K-invariant, then*

$$\int_{\Sigma} \phi(\sigma)\, d\mu(\sigma) = \widetilde{c_0} \int_{\mathcal{S}^r} \phi(e^{i\theta}) \prod_{1 \leq p < q \leq r} |e^{i\theta_p} - e^{i\theta_q}|^d\, d\theta_1 \cdots d\theta_r. \tag{2.1.6}$$

Here, $\mathcal{S}^r$ is the direct product of r copies of S^1.

Proof. (1) is Proposition X.2.4 of [FK] itself and (2) also immediately follows from some proposition. Hence, we only prove (3).

Let $\phi \in L^1(\Sigma)^K$. Since for any $k \in K$

$$c^{-1}(kt) = (k(t - ie))(k(t + ie))^{-1} = k((t + ie)(t - ie)^{-1}) = kc^{-1}(t),$$

from Lemma 2.1.1, we have

$$\begin{aligned}\int_\Sigma \phi(\sigma)\,d\mu(\sigma) &= 2^n \int_V \phi(c^{-1}(t))\Delta(e+t^2)^{-\frac{n}{r}}\,dt \\ &= 2^n \widetilde{c_0} \int_{K\times\mathbb{R}^r} \phi(c^{-1}(k\lambda))\Delta(e+(kt)^2)^{-\frac{n}{r}} \prod_{1\le p<q\le r} |\lambda_p-\lambda_q|^d\,dk d\lambda_1\cdots d\lambda_r \\ &= 2^n \widetilde{c_0} \int_{\mathbb{R}^r} \phi(c^{-1}(\lambda))\Delta(e+\lambda^2)^{-\frac{n}{r}} \prod_{1\le p<q\le r} |\lambda_p-\lambda_q|^d\,d\lambda_1\cdots d\lambda_r.\end{aligned}$$

If we put $\lambda_j = -\cot\left(\frac{\theta_j}{2}\right)$, then

$$\lambda = -\sum_{j=1}^r \cot\left(\frac{\theta_j}{2}\right) c_j = i\sum_{j=1}^r \frac{1+e^{i\theta_j}}{1-e^{i\theta_j}} c_j = i\left(\sum_{j=1}^r (1+e^{i\theta_j})c_j\right)\left(\sum_{l=1}^r (1-e^{i\theta_l})c_l\right)^{-1} = c(e^{i\theta}).$$

Therefore,

$$\begin{aligned}\int_\Sigma \phi(\sigma)\,d\mu(\sigma) &= 2^{n-r}\widetilde{c_0}\int_{\mathcal{S}^r} \phi(e^{i\theta}) \prod_{j=1}^r \sin\left(\frac{\theta_j}{2}\right)^{2\left(\frac{n}{r}-1\right)} \prod_{1\le p<q\le r} \left|\frac{\sin\left(\frac{1}{2}(\theta_p-\theta_q)\right)}{\sin\left(\frac{\theta_p}{2}\right)\sin\left(\frac{\theta_q}{2}\right)}\right|^d d\theta_1\cdots d\theta_r \\ &= \widetilde{c_0}\int_{\mathcal{S}^r} \phi(e^{i\theta}) \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d\,d\theta_1\cdots d\theta_r.\end{aligned}$$

□

For $j=1,\dots,r$, let $e_j := c_1+\cdots+c_j$, and set

$$V^{(j)} := \{x\in V \mid L(e_j)x = x\}.$$

Denote the orthogonal projection of V onto the subalgebra $V^{(j)}$ by P_j, and define

$$\Delta_j(x) := \delta_j(P_j x)$$

for $x\in V$, where δ_j denotes the Koecher norm function for $V^{(j)}$. In particular, $\delta_r = \Delta$. Then, Δ_j is a polynomial on V that is homogeneous of degree j. Let $\mathbf{s} := (s_1,\dots,s_r)\in\mathbb{C}^r$ and define the function $\Delta_{\mathbf{s}}$ on V by

$$\Delta_{\mathbf{s}}(x) := \Delta(x)^{s_r}\prod_{j=1}^{r-1}\Delta_j(x)^{s_j-s_{j+1}}. \tag{2.1.7}$$

That is the generalized power function on V. Furthermore, for $\mathbf{m}\in\mathscr{P}$, $\Delta_{\mathbf{m}}$ becomes a polynomial function on V, which is homogeneous of degree $|\mathbf{m}|$.

The gamma function Γ_Ω for the symmetric cone Ω is defined, for $\mathbf{s} \in \mathbb{C}^r$, with $\operatorname{Re} s_j > \frac{d}{2}(j-1)\,(j=1,\dots,r)$ by

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\mathrm{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}}\, dx. \tag{2.1.8}$$

Its evaluation gives

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - \frac{d}{2}(j-1)\right). \tag{2.1.9}$$

Hence, Γ_Ω extends analytically as a meromorphic function on $\mathbb{C}^r$.

Lemma 2.1.4. *For any $y \in \Omega, \beta \in \mathbb{C}, \operatorname{Re}\beta > 0$ and $\operatorname{Re} s_j > \frac{d}{2}(j-1)$:*

$$\int_\Omega e^{-(\beta y|x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}}\, dx = \Gamma_\Omega(\mathbf{s}) \Delta_{\mathbf{s}}((\beta y)^{-1}). \tag{2.1.10}$$

For $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in \mathscr{P}$, we define the generalized shifted factorial by

$$(\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})}. \tag{2.1.11}$$

It follows from (2.1.9) that

$$(\mathbf{s})_{\mathbf{m}} = \prod_{j=1}^{r} \left(s_j - \frac{d}{2}(j-1)\right)_{m_j}. \tag{2.1.12}$$

Lemma 2.1.5. *If $\mathbf{s} \in \mathbb{C}^r, \mathbf{m}, \mathbf{k} \in \mathscr{P}$ and $\mathbf{m} \supset \mathbf{k}$, then*

$$\left|\frac{(\mathbf{s})_{\mathbf{m}}}{(\mathbf{s})_{\mathbf{k}}}\right| \le \frac{(\|\mathbf{s}\| + d(r-1))_{\mathbf{m}}}{(\|\mathbf{s}\| + d(r-1))_{\mathbf{k}}}. \tag{2.1.13}$$

Proof. We remark that for any $s \in \mathbb{C}, N \in \mathbb{Z}_{\ge 0}$ and $j = 1, \dots, r$, the following is satisfied.

$$\left|s + N - \frac{d}{2}(j-1)\right| \le |s| + N + d(r-1) - \frac{d}{2}(j-1) = |s| + N + \frac{d}{2}(2r-j-1).$$

Hence,

$$\begin{aligned}
\left|\frac{(\mathbf{s})_{\mathbf{m}}}{(\mathbf{s})_{\mathbf{k}}}\right| &= \prod_{j=1}^{r} \left|\left(s_j + k_j - \frac{d}{2}(j-1)\right)_{m_j - k_j}\right| \\
&\le \prod_{j=1}^{r} \left(|s_j| + k_j + d(r-1) - \frac{d}{2}(j-1)\right)_{m_j - k_j} \\
&= \frac{(\|\mathbf{s}\| + d(r-1))_{\mathbf{m}}}{(\|\mathbf{s}\| + d(r-1))_{\mathbf{k}}}.
\end{aligned}$$

□

Corollary 2.1.6. *If* $\mathbf{s} \in \mathbb{C}^r, \mathbf{m} \in \mathscr{P}$, *then*

$$|(\mathbf{s})_{\mathbf{m}}| \leq (\|\mathbf{s}\| + d(r-1))_{\mathbf{m}} \leq \prod_{j=1}^{r} (|s_j| + d(r-1))_{m_j}. \tag{2.1.14}$$

The space, $\mathcal{P}(V)$, of the polynomial ring on V has the following decomposition.

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m} \in \mathscr{P}} \mathcal{P}_{\mathbf{m}},$$

where each $\mathcal{P}_{\mathbf{m}}$ are mutually inequivalent, and finite dimensional irreducible G-modules. Further, their dimensions are denoted by $d_{\mathbf{m}}$. For $d_{\mathbf{m}}$, the following formula is known (see, [Up] Lemma 2.6 or [FK] p. 315).

Lemma 2.1.7. *For any* $\mathbf{m} \in \mathscr{P}$,

$$d_{\mathbf{m}} = \frac{c(-\rho)}{c(\rho - \mathbf{m})c(\mathbf{m} - \rho)} \tag{2.1.15}$$

$$= \prod_{1 \leq p < q \leq r} \frac{m_p - m_q + \frac{d}{2}(q-p)}{\frac{d}{2}(q-p)} \frac{B\left(m_p - m_q, \frac{d}{2}(q-p-1)+1\right)}{B\left(m_p - m_q, \frac{d}{2}(q-p+1)\right)} \tag{2.1.16}$$

$$= \frac{\Gamma\left(\frac{d}{2}\right)^r}{\Gamma\left(\frac{d}{2}r\right)} \prod_{j=1}^{r-1} \frac{1}{\Gamma\left(\frac{d}{2}j\right)^2} \prod_{1 \leq p < q \leq r} (m_p - m_q + \frac{d}{2}(q-p)) \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p+1)\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p-1)+1\right)}. \tag{2.1.17}$$

Here, $\rho = (\rho_1, \ldots, \rho_r)$, $\rho_j := \frac{d}{4}(2j - r - 1)$, *and* c *is the Harish-Chandra function:*

$$c(\mathbf{s}) = \prod_{1 \leq p < q \leq r} \frac{B\left(s_q - s_p, \frac{d}{2}\right)}{B\left(\frac{d}{2}(q-p), \frac{d}{2}\right)}.$$

In particular, for $d = 2$

$$d_{\mathbf{m}} = \prod_{1 \leq p < q \leq r} \left(\frac{m_p - m_q + q - p}{q - p}\right)^2 = s_{\mathbf{m}}(1, \ldots, 1)^2. \tag{2.1.18}$$

Here, $s_{\mathbf{m}}$ *is the Schur polynomial corresponding to* $\mathbf{m} \in \mathscr{P}$ *defined by*

$$s_{\mathbf{m}}(\lambda_1, \ldots, \lambda_r) := \frac{\det\left(\lambda_j^{m_k + r - k}\right)}{\det\left(\lambda_j^{r-k}\right)}.$$

The following lemma is necessary to evaluate the Fourier transform of the multivariate Laguerre polynomial.

Lemma 2.1.8 ([FK] Theorem XI. 2.3)**.** *For $p \in \mathcal{P}_{\mathbf{m}}$, $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$, and $y \in \Omega + iV$,*

$$\int_{\Omega} e^{-(y|x)} p(x)\Delta(x)^{\alpha - \frac{n}{r}}\, dx = \Gamma_{\Omega}(\mathbf{m}+\alpha)\Delta(y)^{-\alpha} p(y^{-1}). \tag{2.1.19}$$

Here, α is regarded as $(\alpha, \dots, \alpha) \in \mathbb{C}^r$.

For each $\mathbf{m} \in \mathscr{P}$, the spherical polynomial of weight $|\mathbf{m}|$ on Ω is defined by

$$\Phi_{\mathbf{m}}^{(d)}(x) := \int_K \Delta_{\mathbf{m}}(kx)\, dk. \tag{2.1.20}$$

We often omit multiplicity d of $\Phi_{\mathbf{m}}^{(d)}(x)$. The algebra of all K-invariant polynomials on V, denoted by $\mathcal{P}(V)^K$, decomposes as

$$\mathcal{P}(V)^K = \bigoplus_{\mathbf{m} \in \mathscr{P}} \mathbb{C}\Phi_{\mathbf{m}}.$$

By analytic continuation to the complexification $V^{\mathbb{C}}$ of V, we can extend $\operatorname{tr}, \Delta$ and $\Phi_{\mathbf{m}}$ to polynomial functions on $V^{\mathbb{C}}$.

Remark 2.1.9. (1) Since $\Phi_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}^K$, for $x = k\sum_{j=1}^r \lambda_j c_j$, $\Phi_{\mathbf{m}}(x)$ can be expressed by

$$\Phi_{\mathbf{m}}(\lambda_1, \dots, \lambda_r) := \Phi_{\mathbf{m}}\left(\sum_{j=1}^r \lambda_j c_j\right) (= \Phi_{\mathbf{m}}(x)).$$

$\Phi_{\mathbf{m}}(x)$ also has the following expression (see [F]).

$$\Phi_{\mathbf{k}}^{(d)}(\lambda_1, \dots, \lambda_r) = \frac{P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1, \dots, \lambda_r)}{P_{\mathbf{k}}^{(\frac{2}{d})}(1, \dots, 1)}. \tag{2.1.21}$$

Here, $P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1, \dots, \lambda_r)$ is an r-variable Jack polynomial (see [M], Chapter. VI.10). In particular, since $P_{\mathbf{k}}^{(1)}(\lambda_1, \dots, \lambda_r) = s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r)$, $\Phi_{\mathbf{m}}^{(2)}$ becomes the Schur polynomial.

$$\Phi_{\mathbf{m}}^{(2)}(\lambda_1, \dots, \lambda_r) = \frac{s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r)}{s_{\mathbf{m}}(1, \dots, 1)} = \frac{\delta!}{\prod_{p<q}(m_p - m_q + q - p)} s_{\mathbf{m}}(\lambda_1, \dots, \lambda_r). \tag{2.1.22}$$

(2) When $r = 2$, $\Phi_{\mathbf{m}}^{(d)}$ has the following hypergeometric expression (see [Sa]).

$$\begin{aligned}\Phi_{m_1,m_2}^{(d)}(\lambda_1, \lambda_2) &= \lambda_1^{m_1}\lambda_2^{m_2}\, {}_2F_1\left(\begin{matrix} -(m_1 - m_2), \frac{d}{2} \\ d \end{matrix}; \frac{\lambda_1 - \lambda_2}{\lambda_1}\right) \\ &= \lambda_1^{m_1}\lambda_2^{m_2} \frac{\left(\frac{d}{2}\right)_{m_1 - m_2}}{(d)_{m_1 - m_2}}\, {}_2F_1\left(\begin{matrix} -(m_1 - m_2), \frac{d}{2} \\ -(m_1 - m_2) - \frac{d}{2} + 1 \end{matrix}; \frac{\lambda_2}{\lambda_1}\right).\end{aligned}$$

We remark that the function $\Phi_{\mathbf{m}}(e+x)$ is a K-invariant polynomial of degree $|\mathbf{m}|$ and define the generalized binomial coefficients $\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}$ by using the following expansion.

$$\Phi_{\mathbf{m}}^{(d)}(e+x) = \sum_{|\mathbf{k}|\leq|\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \Phi_{\mathbf{k}}^{(d)}(x). \tag{2.1.23}$$

For $\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}$, we also often omit $\frac{d}{2}$. The fact that if $\mathbf{k} \not\subset \mathbf{m}$, then $\binom{\mathbf{m}}{\mathbf{k}} = 0$, is well known. Hence, we have

$$\Phi_{\mathbf{m}}(e+x) = \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(x). \tag{2.1.24}$$

Lemma 2.1.10. *For $z = u\sum_{j=1}^{r}\lambda_j c_j$ with $u \in U$, $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and $\mathbf{m} \in \mathscr{P}$, we have*

$$|\Phi_{\mathbf{m}}(z)| \leq \lambda_1^{m_1}\cdots\lambda_r^{m_r} \leq \lambda_1^{|\mathbf{m}|} = \Phi_{\mathbf{m}}(\lambda_1). \tag{2.1.25}$$

Lemma 2.1.11. *For any $\alpha \in \mathbb{C}, z \in \overline{\mathcal{D}}, w \in \mathcal{D}$, we have*

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z)\Phi_{\mathbf{m}}(w) = \Delta(w)^{-\alpha}\int_K \Delta(kw^{-1}-z)^{-\alpha}\,dk. \tag{2.1.26}$$

The spherical function, $\varphi_{\mathbf{s}}$, on Ω for $\mathbf{s} \in \mathbb{C}^r$ is defined by

$$\varphi_{\mathbf{s}}(x) := \int_K \Delta_{\mathbf{s}+\rho}(kx)\,dk. \tag{2.1.27}$$

We remark that for $x \in \Omega$

$$\varphi_{\mathbf{s}}(x^{-1}) = \varphi_{-\mathbf{s}}(x) \tag{2.1.28}$$

and for $x \in \Omega, \mathbf{m} \in \mathscr{P}$

$$\Phi_{\mathbf{m}}(x) = \varphi_{\mathbf{m}-\rho}(x). \tag{2.1.29}$$

Let $\mathbb{D}(\Omega)$ be the algebra of G-invariant differential operators on Ω, $\mathcal{P}(V)^K$ be the space of K-invariant polynomials on V, and $\mathcal{P}(V\times V)^G$ be the space of polynomials on $V\times V$, which are invariant in the sense that

$$p(gx,\xi) = p(x, g^*\xi), \quad (g \in G).$$

Here, we write g^* for the adjoint of an element g (i.e., $(gx|y) = (x|g^*y)$ for all $x, y \in V$). The spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$. Thus, we denote its eigenvalues by $\gamma(D)(\mathbf{s})$, that is, $D\varphi_{\mathbf{s}} = \gamma(D)(\mathbf{s})\varphi_{\mathbf{s}}$.

The symbol σ_D of a partial differential operator D which acts on the variable $x \in V$ is defined by

$$De^{(x|\xi)} = \sigma_D(x,\xi)e^{(x|\xi)} \quad (x,\xi \in V).$$

Differential operator D on Ω is invariant under G if and only if its symbol σ_D belongs to $\mathcal{P}(V\times V)^G$. In addition, the map $D \mapsto \sigma_D$ establishes a linear isomorphism from $\mathbb{D}(\Omega)$ onto

$\mathcal{P}(V \times V)^G$. Moreover, the map $D \mapsto \sigma_D(e,u)$ is a vector space isomorphism from $\mathbb{D}(\Omega)$ onto $\mathcal{P}(V)^K$. In particular, for $\mathbf{k} \in \mathscr{P}, \mathbf{s} \in \mathbb{C}^r$, we put

$$\gamma_{\mathbf{k}}(\mathbf{s}) := \gamma(\Phi_{\mathbf{k}}(\partial_x))(\mathbf{s}) = \Phi_{\mathbf{k}}(\partial_x)\varphi_{\mathbf{s}}(x)|_{x=e}. \tag{2.1.30}$$

Here, $\Phi_{\mathbf{k}}(\partial_x)$ is a unique G-invariant differential operator, which is satisfied with

$$\sigma_{\Phi_{\mathbf{k}}(\partial_x)}(e,\xi) = \Phi_{\mathbf{k}}(\xi) \in \mathcal{P}(V)^K, \quad \text{i.e., } \Phi_{\mathbf{k}}(\partial_x)e^{(x|\xi)}|_{x=e} = \Phi_{\mathbf{k}}(\xi)e^{\operatorname{tr}\xi}.$$

We remark that $\Phi_k(\partial_x) = \partial_x^k$ and $\gamma_k(s) = s(s-1)\cdots(s-k+1)$ in the $r=1$ case, and for any $\alpha \in \mathbb{C}$, $\mathbf{k} \in \mathscr{P}$, we have

$$\gamma_{\mathbf{k}}(\alpha - \rho) = (-1)^{|\mathbf{k}|}(-\alpha)_{\mathbf{k}}. \tag{2.1.31}$$

The function γ_D is an r variable symmetric polynomial and map $D \mapsto \gamma_D$ is an algebra isomorphism from $\mathbb{D}(\Omega)$ onto algebra $\mathcal{P}(\mathbb{R}^r)^{\mathfrak{S}_r}$, which is a special case of the Harish-Chandra isomorphism.

Lemma 2.1.12. *If $\beta \in \mathbb{C}, \operatorname{Re}\beta > 0, \operatorname{Re} s_j > \frac{d}{4}(r-1)$, then for all $\mathbf{k} \in \mathscr{P}$, we have*

$$\int_\Omega e^{-\beta \operatorname{tr} u}\Phi_{\mathbf{k}}(u)\varphi_{\mathbf{s}}(u)\Delta(u)^{-\frac{n}{r}}\,du = (-1)^{|\mathbf{k}|}\beta^{-|\mathbf{s}+\mathbf{k}|}\Gamma_\Omega(\mathbf{s}+\rho)\gamma_{\mathbf{k}}(-\mathbf{s}). \tag{2.1.32}$$

Here, we choose the branch of $\beta^{-|\mathbf{s}|}$, which takes the value 1 at $\beta = 1$.

Proof. We remark that the left hand side of (2.1.32) is converges absolutely under the assumptions. Hence,

$$\begin{aligned}
\int_\Omega e^{-\beta \operatorname{tr} u}\Phi_{\mathbf{k}}(u)\varphi_{\mathbf{s}}(u)\Delta(u)^{-\frac{n}{r}}\,du &= (-\beta)^{-|\mathbf{k}|}\int_\Omega \Phi_{\mathbf{k}}(\partial_x)e^{-(\beta x|u)}|_{x=e}\varphi_{\mathbf{s}}(u)\Delta(u)^{-\frac{n}{r}}\,du \\
&= (-\beta)^{-|\mathbf{k}|}\Phi_{\mathbf{k}}(\partial_x)\int_\Omega\int_K e^{-(\beta x|u)}\Delta_{\mathbf{s}+\rho}(ku)\Delta(u)^{-\frac{n}{r}}\,dkdu\Big|_{x=e} \\
&= (-\beta)^{-|\mathbf{k}|}\Phi_{\mathbf{k}}(\partial_x)\int_K\int_\Omega e^{-(\beta kx|u)}\Delta_{\mathbf{s}+\rho}(u)\Delta(u)^{-\frac{n}{r}}\,dudk\Big|_{x=e}.
\end{aligned}$$

By Lemma 2.1.4,

$$\int_\Omega e^{-(\beta kx|u)}\Delta_{\mathbf{s}+\rho}(u)\Delta(u)^{-\frac{n}{r}}\,du = \Gamma_\Omega(\mathbf{s}+\rho)\Delta_{\mathbf{s}+\rho}((\beta kx)^{-1}) = \beta^{-|\mathbf{s}|}\Gamma_\Omega(\mathbf{s}+\rho)\Delta_{\mathbf{s}+\rho}((kx)^{-1}).$$

Therefore,

$$\begin{aligned}
\int_\Omega e^{-\beta \operatorname{tr} u}\Phi_{\mathbf{k}}(u)\varphi_{\mathbf{s}}(u)\Delta(u)^{-\frac{n}{r}}\,du &= (-1)^{|\mathbf{k}|}\beta^{-|\mathbf{s}+\mathbf{k}|}\Gamma_\Omega(\mathbf{s}+\rho)\Phi_{\mathbf{k}}(\partial_x)\int_K \Delta_{\mathbf{s}+\rho}(kx^{-1})\,dk\Big|_{x=e} \\
&= (-1)^{|\mathbf{k}|}\beta^{-|\mathbf{s}+\mathbf{k}|}\Gamma_\Omega(\mathbf{s}+\rho)\Phi_{\mathbf{k}}(\partial_x)\varphi_{-\mathbf{s}}(x)|_{x=e}.
\end{aligned}$$

□

If a K-invariant function ψ is analytic in the neighborhood of e, it admits a spherical Taylor expansion near e:

$$\psi(e+x) = \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \{\Phi_{\mathbf{k}}(\partial_x)\psi(x)|_{x=e}\}\Phi_{\mathbf{k}}(x).$$

By the definition of $\gamma_{\mathbf{k}}$, we have

$$\varphi_{\mathbf{s}}(e+x) = \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{s})\Phi_{\mathbf{k}}(x).$$

Since $\Phi_{\mathbf{m}} = \varphi_{\mathbf{m}-\rho}$,

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho).$$

For a complex number α, we define the following differential operator on Ω:

$$D_\alpha = \Delta(x)^{1+\alpha}\Delta(\partial_x)\Delta(x)^{-\alpha}.$$

For this operator, we have

$$\gamma(D_\alpha)(\mathbf{s}) = \prod_{j=1}^{r}\left(s_j - \alpha + \frac{d}{4}(r-1)\right). \tag{2.1.33}$$

The operators $D_{j\frac{d}{2}}$, $j = 0,\dots,r-1$ generate algebra $\mathbb{D}(\Omega)$.

Lemma 2.1.13. *For all $\mathbf{k}\in\mathscr{P}$, there exist some constant $C>0$ and integer N such that for any $\mathbf{s}\in\mathbb{C}^r$*

$$|\gamma_{\mathbf{k}}(\mathbf{s})| \le C\prod_{l=1}^{r}\left(|s_l| + \frac{d}{4}(r-1)\right)^N. \tag{2.1.34}$$

$$|\gamma_{\mathbf{k}}(\mathbf{s}-\rho)| \le C\prod_{l=1}^{r}\left(|s_l| + \frac{d}{2}(r-1)\right)^N. \tag{2.1.35}$$

Proof. Since algebra $\mathbb{D}(\Omega)$ is generated by $D_{j\frac{d}{2}}$, $j=0,\dots,r-1$, for $\Phi_{\mathbf{k}}(\partial_x)\in\mathbb{D}(\Omega)$,

$$\Phi_{\mathbf{k}}(\partial_x) = \sum_{l_0,\dots,l_{r-1};\text{finite}} a_{l_0,\dots,l_{r-1}} D_{0\frac{d}{2}}^{l_0}\cdots D_{(r-1)\frac{d}{2}}^{l_{r-1}}.$$

Here, we remark that for $j=0,\dots,r-1$

$$|\gamma(D_{\frac{d}{2}(j-1)})(\mathbf{s})| = \left|\prod_{l=1}^{r}\left(s_l + \frac{d}{4}(r-1) - \frac{d}{2}(j-1)\right)\right| \le \prod_{l=1}^{r}\left(|s_l| + \frac{d}{4}(r-1)\right).$$

Therefore,

$$|\gamma_{\mathbf{k}}(\mathbf{s})| \le \sum_{l_0,\dots,l_{r-1};\text{finite}} |a_{l_0,\dots,l_{r-1}}|\gamma(D_{0\frac{d}{2}})(\mathbf{s})^{l_0}\cdots\gamma(D_{(r-1)\frac{d}{2}})(\mathbf{s})^{l_{r-1}} \le C\prod_{l=1}^{r}\left(|s_l| + \frac{d}{4}(r-1)\right)^N.$$

We immediately derive (2.1.35) from (2.1.34). □

Lemma 2.1.14. *For all $\mathbf{s} \in \mathbb{C}^r$, there exist some $l \in \mathbb{Z}_{\geq 0}$ and some constant $C > 0$ such that for all $l \subset \mathbf{k} \in \mathscr{P}$*

$$|\gamma_{\mathbf{k}}(-\mathbf{s})| \leq C \frac{\Gamma_\Omega(\mathrm{Re}\,(\mathbf{s}) + \rho + \mathbf{k})}{|\Gamma_\Omega(\mathbf{s})|} \leq C \left| \frac{\Gamma_\Omega(\mathrm{Re}\,(\mathbf{s}) + \rho)}{\Gamma_\Omega(\mathbf{s})} \right| (\|\mathrm{Re}\,(\mathbf{s})\| + d(r-1))_{\mathbf{k}}. \tag{2.1.36}$$

Proof. For fixed $\mathbf{s} \in \mathbb{C}^r$, we assume the integral

$$\int_\Omega |e^{-\operatorname{tr} u} \Phi_l(u) \Delta_{\mathbf{s}+\rho}(u) \Delta(u)^{-\frac{n}{r}}|\, du$$

converges absolutely. Hence, for all $l \subset \mathbf{k} \in \mathscr{P}$,

$$\int_\Omega |e^{-\operatorname{tr} u} \Phi_{\mathbf{k}}(u) \Delta_{\mathbf{s}+\rho}(u) \Delta(u)^{-\frac{n}{r}}|\, du < \infty.$$

By Lemma 2.1.12,

$$\gamma_{\mathbf{k}}(-\mathbf{s}) = \frac{(-1)^{|\mathbf{k}|}}{\Gamma_\Omega(\mathbf{s})} \int_\Omega e^{-\operatorname{tr} u} \Phi_{\mathbf{k}}(u) \Delta_{\mathbf{s}+\rho}(u) \Delta(u)^{-\frac{n}{r}}\, du.$$

Hence,

$$\begin{aligned} |\gamma_{\mathbf{k}}(-\mathbf{s})| &\leq \frac{1}{|\Gamma_\Omega(\mathbf{s})|} \int_\Omega e^{-\operatorname{tr} u} \Phi_{\mathbf{k}}(u) |\Delta_{\mathbf{s}+\rho}(u)| \Delta(u)^{-\frac{n}{r}}\, du \\ &\leq \frac{1}{|\Gamma_\Omega(\mathbf{s})|} \int_K \int_\Omega e^{-\operatorname{tr} u} \Delta_{\mathbf{k}}(u) \Delta_{\mathrm{Re}\,(\mathbf{s})+\rho}(ku) \Delta(u)^{-\frac{n}{r}}\, du dk. \end{aligned}$$

Since K is a compact group and $\int_K dk = 1$, there exists some $\widetilde{k} \in K$ such that

$$\int_K \int_\Omega e^{-\operatorname{tr} u} \Delta_{\mathbf{k}}(u) \Delta_{\mathrm{Re}\,(\mathbf{s})+\rho}(ku) \Delta(u)^{-\frac{n}{r}}\, du dk \leq \int_\Omega e^{-\operatorname{tr} u} \Delta_{\mathbf{k}}(u) \Delta_{\mathrm{Re}\,(\mathbf{s})+\rho}(\widetilde{k}u) \Delta(u)^{-\frac{n}{r}}\, du.$$

Moreover, since $\Delta_{\mathbf{k}}$ is a homogeneous degree $|\mathbf{k}|$ polynomial and $\widetilde{k} \in K$ is a linear transformation on V, there exists some constant $C > 0$ such that

$$\int_\Omega e^{-\operatorname{tr} u} \Delta_{\mathbf{k}}(\widetilde{k}u) \Delta_{\mathrm{Re}\,(\mathbf{s})+\rho}(u) \Delta(u)^{-\frac{n}{r}}\, du \leq C \int_\Omega e^{-\operatorname{tr} u} \Delta_{\mathrm{Re}\,(\mathbf{s})+\rho+\mathbf{k}}(u) \Delta(u)^{-\frac{n}{r}}\, du.$$

Therefore, by using the definition of Γ_Ω, we have

$$\begin{aligned} |\gamma_{\mathbf{k}}(-\mathbf{s})| &\leq \frac{C}{|\Gamma_\Omega(\mathbf{s})|} \int_\Omega e^{-\operatorname{tr} u} \Delta_{\mathrm{Re}\,(\mathbf{s})+\rho+\mathbf{k}}(u) \Delta(u)^{-\frac{n}{r}}\, du \\ &= C \frac{\Gamma_\Omega(\mathrm{Re}\,(\mathbf{s}) + \rho + \mathbf{k})}{|\Gamma_\Omega(\mathbf{s})|} \\ &\leq C \left| \frac{\Gamma_\Omega(\mathrm{Re}\,(\mathbf{s}) + \rho)}{\Gamma_\Omega(\mathbf{s})} \right| |(\mathrm{Re}\,(\mathbf{s}) + \rho)_{\mathbf{k}}| \\ &\leq C \left| \frac{\Gamma_\Omega(\mathrm{Re}\,(\mathbf{s}) + \rho)}{\Gamma_\Omega(\mathbf{s})} \right| (\|\mathrm{Re}\,(\mathbf{s})\| + d(r-1))_{\mathbf{k}}. \end{aligned}$$

The final inequality follows from

$$
\begin{aligned}
|(\mathrm{Re}\,(\mathbf{s})+\rho)_{\mathbf{k}}| &= \prod_{j=1}^{r}\left|\left(\mathrm{Re}\,(\mathbf{s})+\rho_j-\frac{d}{2}(j-1)\right)_{k_j}\right| \\
&\le \prod_{j=1}^{r}\left(|\mathrm{Re}\,(\mathbf{s})|+\frac{d}{2}(r-1)\right)_{k_j} \le (\|\mathrm{Re}\,(\mathbf{s})\|+d(r-1))_{\mathbf{k}}.
\end{aligned}
$$

□

Lemma 2.1.15. *For all* $\mathbf{m},\mathbf{k}\in\mathscr{P}$*, we have*

$$\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\ge 0. \tag{2.1.37}$$

Proof. Since $\gamma_{\mathbf{k}}(\mathbf{m}-\rho)=\frac{1}{d_{\mathbf{k}}}\left(\frac{n}{r}\right)_{\mathbf{k}}\binom{\mathbf{m}}{\mathbf{k}}$ and $d_{\mathbf{k}},\left(\frac{n}{r}\right)_{\mathbf{k}}>0$, it suffices to show $\binom{\mathbf{m}}{\mathbf{k}}\ge 0$ for all $\mathbf{m},\mathbf{k}\in\mathscr{P}$. From [OO], generalized binomial coefficients are written as

$$\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}=\frac{P^*_{\mathbf{k}}\left(\mathbf{m};\frac{d}{2}\right)}{H_{\left(\frac{d}{2}\right)}(\mathbf{k})},$$

where $P^*_{\mathbf{k}}\left(\mathbf{m};\frac{d}{2}\right)$ is the shifted Jack polynomial and $H_{\left(\frac{d}{2}\right)}(\mathbf{k})>0$ is a deformation of the hook length. Moreover, by using (5.2) in [OO]

$$P^*_{\mathbf{k}}\left(\mathbf{m};\frac{d}{2}\right)=\frac{\frac{d}{2}\text{-}\dim\mathbf{m}/\mathbf{k}}{\frac{d}{2}\text{-}\dim\mathbf{m}}|\mathbf{m}|(|\mathbf{m}|-1)\cdots(|\mathbf{m}|-|\mathbf{k}|+1).$$

Further, the positivity of the generalized dimensions of the skew Young diagram, $\frac{d}{2}\text{-}\dim\mathbf{m}/\mathbf{k}$, follows from (5.1) of [OO] and Chapter VI. 6 of [M]. Therefore, we obtain the positivity of the shifted Jack polynomial and the conclusion. □

Theorem 2.1.16. (1) *For* $w\in\mathcal{D},\mathbf{k}\in\mathscr{P},\alpha\in\mathbb{C}$*, we have*

$$(\alpha)_{\mathbf{k}}\Delta(e-w)^{-\alpha}\Phi_{\mathbf{k}}(w(e-w)^{-1})=\sum_{\mathbf{x}\in\mathscr{P}}d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w). \tag{2.1.38}$$

Here, we choose the branch of $\Delta(e-w)^{-\alpha}$ *which takes the value* 1 *at* $w=0$*.*
(2) *For* $w\in V^{\mathbb{C}},\mathbf{k}\in\mathscr{P}$*, a* K*-invariant analytic function* $e^{\mathrm{tr}\,w}\Phi_{\mathbf{k}}(w)$ *has the following expansion.*

$$e^{\mathrm{tr}\,w}\Phi_{\mathbf{k}}(w)=\sum_{\mathbf{x}\in\mathscr{P}}d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w). \tag{2.1.39}$$

Proof. (1) We take $w=u\sum_{j=1}^{r}\lambda_j c_j\in\mathcal{D}$ with $u\in U$ and $1>\lambda_1\ge\ldots\ge\lambda_r\ge 0$. By Lemmas 2.1.10 and 2.1.13, there exist some $C>0$ and $N\in\mathbb{Z}_{\ge 0}$ such that

$$
\begin{aligned}
\sum_{\mathbf{x}\in\mathscr{P}}\left|d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w)\right| &\le \sum_{\mathbf{x}\in\mathscr{P}}d_{\mathbf{x}}\frac{|(\alpha)_{\mathbf{x}}|}{\left(\frac{n}{r}\right)_{\mathbf{x}}}|\gamma_{\mathbf{k}}(\mathbf{x}-\rho)||\Phi_{\mathbf{x}}(w)| \\
&\le C\prod_{l=1}^{r}\sum_{x_l\ge 0}\frac{(|\alpha|+d(r-1))_{x_l}}{x_l!}\left(x_l+\frac{d}{2}(r-1)\right)^N\lambda_l^{x_l}<\infty.
\end{aligned}
$$

Therefore, the right hand side of (2.1.38) converges absolutely. By analytic continuation, it is sufficient to show the assertion when $\operatorname{Re}\alpha > \frac{d}{2}(r-1)$ and $w \in \Omega \cap (e-\Omega) \subset \mathcal{D}$.

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(z)\Phi_{\mathbf{x}}(w)\Big|_{z=e} &= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{k}}(\partial_z)\Phi_{\mathbf{x}}(z)|_{z=e}\Phi_{\mathbf{x}}(w)\\ &= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(z)\Phi_{\mathbf{x}}(w)\Big|_{z=e} &= \Phi_{\mathbf{k}}(\partial_z)\Delta(w)^{-\alpha}\int_K \Delta(kw^{-1}-z)^{-\alpha}\,dk\Big|_{z=e}\\ &= \Delta(w)^{-\alpha}\int_K \Phi_{\mathbf{k}}(\partial_z)\Delta(kw^{-1}-z)^{-\alpha}\big|_{z=e}\,dk.\end{aligned}$$

Here, from $kw^{-1}-z \in T_\Omega$ for all $k \in K$ and Lemma 2.1.8,

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\Delta(kw^{-1}-z)^{-\alpha}\big|_{z=e} &= \Phi_{\mathbf{k}}(\partial_z)\frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-(x|kw^{-1}-z)}\Delta(x)^{\alpha}\Delta(x)^{-\frac{n}{r}}\,dx\Big|_{z=e}\\ &= \frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega \Phi_{\mathbf{k}}(\partial_z)e^{(x|z)}|_{z=e}e^{-(x|kw^{-1})}\Delta(x)^{\alpha}\Delta(x)^{-\frac{n}{r}}\,dx\\ &= \frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega \Phi_{\mathbf{k}}(x)e^{-(kx|(w^{-1}-e))}\Delta(x)^{\alpha}\Delta(x)^{-\frac{n}{r}}\,dx\\ &= (\alpha)_{\mathbf{k}}\Delta(w^{-1}-e)^{-\alpha}\Phi_{\mathbf{k}}((w^{-1}-e)^{-1}).\end{aligned}$$

Therefore,

$$\begin{aligned}\Phi_{\mathbf{k}}(\partial_z)\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(z)\Phi_{\mathbf{x}}(w)\Big|_{z=e} &= \Delta(w)^{-\alpha}\int_K (\alpha)_{\mathbf{k}}\Delta(w^{-1}-e)^{-\alpha}\Phi_{\mathbf{k}}((w^{-1}-e)^{-1})\,dk\\ &= (\alpha)_{\mathbf{k}}\Delta(e-w)^{-\alpha}\Phi_{\mathbf{k}}(w(e-w)^{-1}).\end{aligned}$$

(2) Since the right hand side of (2.1.39) converges absolutely due to a similar argument of (1), we have

$$\begin{aligned}e^{\operatorname{tr} w}\Phi_{\mathbf{k}}(w) &= \lim_{\alpha\to\infty}(\alpha)_{\mathbf{k}}\Delta\left(e-\frac{w}{\alpha}\right)^{-\alpha}\Phi_{\mathbf{k}}\left(\frac{w}{\alpha}\left(e-\frac{w}{\alpha}\right)^{-1}\right)\\ &= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\lim_{\alpha\to\infty}(\alpha)_{\mathbf{x}}\Phi_{\mathbf{x}}\left(\frac{w}{\alpha}\right)\\ &= \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w).\end{aligned}$$

□

Next, we preview the gradient for a $\mathbb{C}$-valued and V-valued function f on simple Euclidean Jordan algebra V. In this parts, we refer to [Di]. For differentiable function $f : V \to \mathbb{R}$ and $x, u \in V$, we define the gradient, $\nabla f(x) \in V$, of f by

$$(\nabla f(x)|u) = D_u f(x) = \frac{d}{dt} f(x + tu)\Big|_{t=0}.$$

For a $\mathbb{C}$-valued function $f = f_1 + if_2$, we define $\nabla f = \nabla f_1 + i\nabla f_2$. For $z = x + iy \in V^{\mathbb{C}}$, we define $D_z = D_x + iD_y$. Moreover, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V and $x = \sum_{j=1}^n x_j e_j \in V^{\mathbb{C}}$, then

$$\nabla f(x) = \sum_{j=1}^n \frac{\partial f(x)}{\partial x_j} e_j.$$

We remark that this expression is independent of the choice of an orthonormal basis of V.

For a V-valued function $f : V \to V$ expressed by $f(x) = \sum_{j=1}^r f_j(x) e_j$, we define ∇f by

$$\nabla f(x) = \sum_{j,l=1}^n \frac{\partial f_j(x)}{\partial x_l} e_j e_l.$$

That is also well defined. Let us present some derivation formulas.

Lemma 2.1.17. (1) *The product rule of differentiation: For V-valued function f, h, we have*

$$\mathrm{tr}\ (\nabla(f(x)h(x))) = \mathrm{tr}\ (\nabla f(x))h(x) + f(x)\,\mathrm{tr}\ (\nabla h(x)). \tag{2.1.40}$$

For $\mathbb{C}$-valued functions f, h,

$$\nabla(f(x)h(x)) = (\nabla f(x))h(x) + f(x)(\nabla h(x)). \tag{2.1.41}$$

(2)

$$\nabla x = \frac{n}{r} e. \tag{2.1.42}$$

(3) *For any invertible element $x \in V^{\mathbb{C}}$,*

$$\mathrm{tr}\ (x\nabla)x^{-1} := \mathrm{tr}\ (x(\nabla x^{-1})) = -\frac{n}{r}\,\mathrm{tr}\ x^{-1}. \tag{2.1.43}$$

(4) *For $\beta \in \mathbb{C}$ and an invertible element $x \in V^{\mathbb{C}}$,*

$$\nabla(\Delta(x)^\beta) = \beta\Delta(x)^\beta x^{-1}. \tag{2.1.44}$$

(1), (2), and (4) are well known (see [FK], [Di], and [FW1]). (3) follows from (1), (2), and $\nabla(xx^{-1}) = \nabla(e) = 0$.

The following recurrence formulas for the spherical functions, some of which involve the gradient, are also well known (see [Di] and [FW1]).

Lemma 2.1.18. *Let* $\mathbf{s} \in \mathbb{C}^r$ *and* $x \in V^{\mathbb{C}}$. *Put*

$$a_j(\mathbf{s}) := \frac{c(\mathbf{s})}{c(\mathbf{s}+\epsilon_j)} = \prod_{k \neq j} \frac{s_j - s_k + \frac{d}{2}}{s_j - s_k}. \tag{2.1.45}$$

Then,

$$(\mathrm{tr}\ x)\varphi_{\mathbf{s}}(x) = \sum_{j=1}^{r} a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(x), \tag{2.1.46}$$

$$(\mathrm{tr}\ \nabla)\varphi_{\mathbf{s}}(x) = \sum_{j=1}^{r} \left(s_j + \frac{d}{4}(r-1)\right) a_j(-\mathbf{s})\varphi_{\mathbf{s}-\epsilon_j}(x), \tag{2.1.47}$$

$$(\mathrm{tr}\ (x^2\nabla))\varphi_{\mathbf{s}}(x) = \sum_{j=1}^{r} \left(s_j - \frac{d}{4}(r-1)\right) a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(x). \tag{2.1.48}$$

Finally, we provide a Plancherel theorem, which is needed to investigate the MCJ polynomials.

Lemma 2.1.19. *Put*

$$\begin{aligned} L^2(\Omega) &:= \{\psi : \Omega \longrightarrow \mathbb{C} \mid \|\psi\|_\Omega^2 < \infty\}, \\ H^2(V) &:= \{\Psi : V \longrightarrow \mathbb{C} \mid \Psi \textit{ is continued analytically to } H_\Omega \textit{ as a holomorphic function} \\ &\qquad \textit{and } \|\Psi\|_V^2 < \infty\}. \end{aligned}$$

Here,

$$\|\psi\|_\Omega^2 := \int_\Omega |\psi(u)|^2\, du, \quad \|\Psi\|_V^2 := \frac{1}{(2\pi)^n}\int_V |\Psi(t)|^2\, dt.$$

The (inverse) Fourier transform of an integrable function, ψ, *on* Ω *is defined as*

$$(F^{-1}\psi)(t) := \int_\Omega e^{i(t|u)}\psi(u)\, du. \tag{2.1.49}$$

We have

$$F^{-1} : L^2(\Omega) \xrightarrow{\simeq} H^2(V) \ \ (\text{unitary}). \tag{2.1.50}$$

In particular,

$$F^{-1} : L^2(\Omega)^K \xrightarrow{\simeq} H^2(V)^K \ \ (\text{unitary}). \tag{2.1.51}$$

Proof. From Theorem IX.4.1 in [FK], we have

$$\widetilde{F}^{-1} : L^2(\Omega) \xrightarrow{\simeq} H^2(H_\Omega) \ \ (\text{unitary}),$$

where

$$
\begin{aligned}
H^2(H_\Omega) &:= \{\widetilde{\Psi} : H_\Omega := V + i\Omega \longrightarrow \mathbb{C} \mid \widetilde{\Psi} \text{ is analytic in } H_\Omega \text{ and } \|\widetilde{\Psi}\|^2_{H_\Omega} < \infty\},\\
\|\widetilde{\Psi}\|^2_{H_\Omega} &:= \sup_{y\in\Omega} \frac{1}{(2\pi)^n} \int_V |\widetilde{\Psi}(x+iy)|^2\, dx,\\
\widetilde{F}^{-1}(\psi)(z) &:= \int_\Omega e^{i(z|u)}\psi(u)\, du.
\end{aligned}
$$

Moreover, from Corollary IX.4.2 in [FK], for function $\widetilde{\Psi} \in H^2(H_\Omega)$, $y \in \Omega$, we write $\widetilde{\Psi}_y(x) := \widetilde{\Psi}(x+iy)$; then,

$$
\lim_{y\to 0, y\in\Omega} \widetilde{\Psi}_y = \widetilde{\Psi}_0, \quad \widetilde{\Psi}_0(t) := \int_\Omega e^{i(t|u)}\psi(u)\, du = F^{-1}(\psi)(t),
$$

exists in $L^2(V)$ and the map $\widetilde{\Psi} \mapsto \widetilde{\Psi}_0$ is an isometric embedding of $H^2(H_\Omega)$ into $L^2(V)$. Hence, the map $F^{-1} : \psi \mapsto \widetilde{\Psi}_0$ is unitary. The surjectivity of this map follows from the above facts and the definition of $H^2(V)$.

Furthermore, since the inverse Fourier transform F^{-1} and the action of K are commutative, the above unitary isomorphism also holds for the K-invariant spaces. □

2.2 Multivariate Laguerre polynomials and their unitary picture

In this section, we promote a unitary picture associated with the multivariate Laguerre polynomials and provide some fundamental lemmas based on [FK], [FW1].

First, we recall some function spaces and their complete orthogonal basis as in the case of one variable. Let $\alpha > 2\frac{n}{r} - 1$, $\mathbf{m} \in \mathscr{P}$, $T_\Omega := \Omega + iV$, and $\mathfrak{S}_r$ be the symmetric group of order r.

(1) $f^{(\alpha)}_{\mathbf{m}}$; spherical polynomials

$$
\begin{aligned}
\mathcal{H}^2_\alpha(\mathcal{D})^K &:= \{f : \mathcal{D} \longrightarrow \mathbb{C} \mid f \text{ is } K\text{-invariant and analytic in } \mathcal{D}, \text{ and } \|f\|^2_{\alpha,\mathcal{D}} < \infty\},\\
\|f\|^2_{\alpha,\mathcal{D}} &:= \frac{1}{\pi^n} \frac{\Gamma_\Omega(\alpha)}{\Gamma_\Omega\left(\alpha - \frac{n}{r}\right)} \int_{\mathcal{D}} |f(w)|^2 h(w)^{\alpha - \frac{2n}{r}}\, m(dw),\\
h(w) &:= \mathrm{Det}(I_{V^{\mathbb{C}}} - 2w\Box\overline{w} + P(w)P(\overline{w}))^{\frac{r}{2n}},\\
f^{(\alpha)}_{\mathbf{m}}(w) &:= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(u).
\end{aligned}
$$

Here, Det stands for the usual determinant of a complex linear operator on $V^{\mathbb{C}}$.

(2) $F_{\mathbf{m}}^{(\alpha)}$; Cayley transform of the spherical polynomials

$$\begin{aligned}
\mathcal{H}_{\alpha}^{2}(T_{\Omega})^{K} &:= \{F : T_{\Omega} \longrightarrow \mathbb{C} \mid F \text{ is } K\text{-invariant and analytic in } T_{\Omega}, \text{ and } \|F\|_{\alpha,T_{\Omega}}^{2} < \infty\}, \\
\|F\|_{\alpha,T_{\Omega}}^{2} &:= \frac{1}{(4\pi)^{n}} \frac{\Gamma_{\Omega}(\alpha)}{\Gamma_{\Omega}\left(\alpha - \frac{n}{r}\right)} \int_{T_{\Omega}} |F(z)|^{2} \Delta(x)^{\alpha - \frac{2n}{r}}\, m(dz), \\
F_{\mathbf{m}}^{(\alpha)}(z) &:= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Delta\left(\frac{e+z}{2}\right)^{-\alpha} \Phi_{\mathbf{m}}((z-e)(z+e)^{-1}).
\end{aligned}$$

(3) $\psi_{\mathbf{m}}^{(\alpha)}$; Multivariate Laguerre polynomials (to multiply exponential)

$$\begin{aligned}
L_{\alpha}^{2}(\Omega)^{K} &:= \{\psi : \Omega \longrightarrow \mathbb{C} \mid \psi \text{ is } K\text{-invariant and } \|\psi\|_{\alpha,\Omega}^{2} < \infty\}, \\
\|\psi\|_{\alpha,\Omega}^{2} &:= \frac{2^{r\alpha}}{\Gamma_{\Omega}(\alpha)} \int_{\Omega} |\psi(u)|^{2} \Delta(u)^{\alpha - \frac{n}{r}}\, du, \\
\psi_{\mathbf{m}}^{(\alpha)}(u) &:= e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\alpha - \frac{n}{r})}(2u).
\end{aligned}$$

Here, $L_{\mathbf{m}}^{\left(\alpha - \frac{n}{r}\right)}(u)$ is the multivariate Laguerre polynomial defined by

$$\begin{aligned}
L_{\mathbf{m}}^{\left(\alpha - \frac{n}{r}\right)}(u) &:= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{(-1)^{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}(u) \\
&= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{\mathbf{k}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m} - \rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}} (\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}(u).
\end{aligned}$$

(4) $q_{m}^{(\alpha)}(s)$; Multivariate Meixner-Pollaczek polynomials

$$\begin{aligned}
L_{\alpha}^{2}(\mathbb{R}^{r})^{\mathfrak{S}_{r}} &:= \{q : \mathbb{R}^{r} \longrightarrow \mathbb{C} \mid q \text{ is } \mathfrak{S}_{r}\text{-invariant and } \|q\|_{\alpha,\mathbb{R}}^{2} < \infty\}, \\
\|q\|_{\alpha,\mathbb{R}^{r}}^{2} &:= \frac{1}{(2\pi)^{r}} \frac{2^{r\alpha}}{\Gamma_{\Omega}(\alpha)} \int_{\mathbb{R}^{r}} |q(\mathbf{s})|^{2} \left|\Gamma_{\Omega}\left(i\mathbf{s} + \frac{\alpha}{2} + \rho\right)\right|^{2} \frac{m(d\mathbf{s})}{|c(i\mathbf{s})|^{2}}, \\
q_{\mathbf{m}}^{(\alpha)}(s) &:= i^{-|\mathbf{m}|} P_{\mathbf{m}}^{(\frac{\alpha}{2})}\left(\mathbf{s}; \frac{\pi}{2}\right).
\end{aligned}$$

Here, $P_{\mathbf{m}}^{(\alpha)}(\mathbf{s};\theta)$ is the multivariate Meixner-Pollaczek polynomial defined by

$$\begin{aligned}
P_{\mathbf{m}}^{(\alpha)}(\mathbf{s};\theta) &:= e^{i|\mathbf{m}|\theta} d_{\mathbf{m}} \frac{(2\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(-i\mathbf{s} - \alpha)}{(2\alpha)_{\mathbf{k}}} (1 - e^{-2i\theta})^{|\mathbf{k}|} \\
&= e^{i|\mathbf{m}|\theta} d_{\mathbf{m}} \frac{(2\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m} - \rho)\gamma_{\mathbf{k}}(-i\mathbf{s} - \alpha)}{\left(\frac{n}{r}\right)_{\mathbf{k}} (2\alpha)_{\mathbf{k}}} (1 - e^{-2i\theta})^{|\mathbf{k}|}.
\end{aligned}$$

We remark that

$$\|f_{\mathbf{m}}^{(\alpha)}\|_{\alpha,\mathcal{D}}^{2} = \|F_{\mathbf{m}}^{(\alpha)}\|_{\alpha,T_{\Omega}}^{2} = \|\psi_{\mathbf{m}}^{(\alpha)}\|_{\alpha,\Omega}^{2} = \|q_{\mathbf{m}}^{(\alpha)}\|_{\alpha,\mathbb{R}^{r}}^{2} = d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}$$

and the orthogonality relations of $\psi_{\mathbf{m}}^{(\alpha)}$ and $q_{\mathbf{m}}^{(\alpha)}$ also hold for $\alpha > \frac{n}{r} - 1$.

Next, similar to the one variable case, we will consider some unitary isomorphisms.

<u>Modified Cayley transform 1</u>

$$C_\alpha^{-1} : \mathcal{H}_\alpha^2(T_\Omega)^K \xrightarrow{\simeq} \mathcal{H}_\alpha^2(\mathcal{D})^K, \ \ (C_\alpha^{-1}F)(w) := \Delta(e-w)^{-\alpha} F\left((e+w)(e-w)^{-1}\right).$$

<u>Modified Laplace transform</u>

$$\mathcal{L}_\alpha : L_\alpha^2(\Omega)^K \xrightarrow{\simeq} \mathcal{H}_\alpha^2(T_\Omega)^K, \ \ (\mathcal{L}_\alpha\psi)(z) := \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)} \int_\Omega e^{-(z|u)} \Delta(u)^{\alpha - \frac{n}{r}} \psi(u)\, du.$$

<u>Modified Mellin transform</u>

$$\mathcal{M}_\alpha : L_\alpha^2(\Omega)^K \xrightarrow{\simeq} L_\alpha^2(\mathbb{R}^r)^{\mathfrak{S}_r}, \ \ (\mathcal{M}_\alpha\psi)(\mathbf{s}) := \frac{1}{\Gamma_\Omega\left(i\mathbf{s} + \rho + \frac{\alpha}{2}\right)} \int_\Omega \varphi_{i\mathbf{s}}(u) \Delta(u)^{\frac{\alpha}{2} - \frac{n}{r}} \psi(u)\, du.$$

To summarize the above, we obtain the following picture.

$$\begin{array}{ccccccc}
\mathcal{H}_\alpha^2(\mathcal{D})^K & \underset{C_\alpha^{-1}}{\xleftarrow{\simeq}} & \mathcal{H}_\alpha^2(T_\Omega)^K & \underset{\mathcal{L}_\alpha}{\xleftarrow{\simeq}} & L_\alpha^2(\Omega)^K & \underset{\mathcal{M}_\alpha}{\xrightarrow{\simeq}} & L_\alpha^2(\mathbb{R}^r)^{\mathfrak{S}_r}. \\
\cup & & \cup & & \cup & & \cup \\
f_{\mathbf{m}}^{(\alpha)} & \longleftarrow & F_{\mathbf{m}}^{(\alpha)} & \longleftarrow & \psi_{\mathbf{m}}^{(\alpha)} & \longmapsto & q_{\mathbf{m}}^{(\alpha)} \\
(\mathbf{1}) & & (\mathbf{2}) & & (\mathbf{3}) & & (\mathbf{4})
\end{array}$$

Furthermore, the link between $\psi_{\mathbf{m}}^{(\alpha)}$ and $q_{\mathbf{m}}^{(\alpha)}$ obtained by the modified Mellin transform is extended to the correspondence between $\psi_{\mathbf{m}}^{(\alpha)}$ and $q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s}) := e^{-i|\mathbf{m}|\theta} P_{\mathbf{m}}^{(\frac{\alpha}{2})}(\mathbf{s};\theta)$.

Theorem 2.2.1. *We put $\alpha > \frac{n}{r} - 1, 0 < \theta < 2\pi$ and*

$$\begin{aligned}
L_{\alpha,\theta}^2(\mathbb{R}^r)^{\mathfrak{S}_r} &:= \{q : \mathbb{R}^r \longrightarrow \mathbb{C} \mid q \textit{ is } \mathfrak{S}_r\textit{-invariant and } \|q\|_{\alpha,\theta,\mathbb{R}^r}^2 < \infty\}, \\
\|q\|_{\alpha,\theta,\mathbb{R}^r}^2 &:= \frac{1}{(2\pi)^r} \frac{(2\sin\theta)^{r\alpha}}{\Gamma_\Omega(\alpha)} \int_{\mathbb{R}^r} |q(\mathbf{s})|^2 e^{(2\theta-\pi)|\mathbf{s}|} \left|\Gamma_\Omega\left(i\mathbf{s} + \frac{\alpha}{2} + \rho\right)\right|^2 \frac{m(d\mathbf{s})}{|c(i\mathbf{s})|^2}, \\
\mathcal{M}_{\alpha,\theta}(\psi)(\mathbf{s}) &:= (1 - i\cot\theta)^{i|\mathbf{s}| + \frac{\alpha}{2}r} \mathcal{M}_\alpha(e^{i\cot\theta\,\mathrm{tr}\,u}\psi)(\mathbf{s}) \\
&= \frac{(1 - i\cot\theta)^{i|\mathbf{s}| + \frac{\alpha}{2}r}}{\Gamma_\Omega\left(i\mathbf{s} + \frac{\alpha}{2} + \rho\right)} \int_\Omega e^{i\cot\theta\,\mathrm{tr}\,u} \varphi_{i\mathbf{s}}(u) \Delta(u)^{\frac{\alpha}{2} - \frac{n}{r}} \psi(u)\, du.
\end{aligned}$$

Here, we choose the branch of $(1 - i\cot\theta)^{i|\mathbf{s}| + \frac{\alpha}{2}r}$, which takes the value 1 at $\theta = \frac{\pi}{2}$. Then, we have

$$\begin{array}{cccc}
\mathcal{M}_{\alpha,\theta} : & L_\alpha^2(\Omega)^K & \xrightarrow{\simeq} & L_{\alpha,\theta}^2(\mathbb{R}^r)^{\mathfrak{S}_r} \quad \text{(unitary)}. \\
& \cup & & \cup \\
& \psi_{\mathbf{m}}^{(\alpha)} & \longmapsto & q_{\mathbf{m}}^{(\alpha,\theta)}
\end{array}$$

Proof. Let us evaluate $(\mathcal{M}_\alpha \circ e^{i\cot\theta \operatorname{tr} u})(\psi_{\mathbf{m}}^{(\alpha)})(\mathbf{s})$. By using this assumption, we can apply Lemma 2.1.12 to this one. Thus, we have

$$\begin{aligned}\mathcal{M}_\alpha(e^{i\cot\theta \operatorname{tr} u}e^{-\operatorname{tr} u}\Phi_{\mathbf{k}})(\mathbf{s}) &= \frac{1}{\Gamma_\Omega\left(i\mathbf{s}+\frac{\alpha}{2}+\rho\right)}\int_\Omega e^{-(1-i\cot\theta)\operatorname{tr} u}\Phi_{\mathbf{k}}(u)\varphi_{i\mathbf{s}+\frac{\alpha}{2}}(u)\Delta(u)^{-\frac{n}{r}}\,du \\ &= (-1)^{|\mathbf{k}|}(1-i\cot\theta)^{-i|\mathbf{s}|-\frac{\alpha}{2}r-|\mathbf{k}|}\gamma_{\mathbf{k}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}(\mathcal{M}_\alpha \circ e^{i\cot\theta \operatorname{tr} u})(\psi_{\mathbf{m}}^{(\alpha)})(\mathbf{s}) &= d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k}}\frac{(-2)^{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\mathcal{M}_\alpha(e^{i\cot\theta \operatorname{tr} u}e^{-\operatorname{tr} u}\Phi_{\mathbf{k}})(\mathbf{s}) \\ &= (1-i\cot\theta)^{-i|\mathbf{s}|-\frac{\alpha}{2}r}q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s}).\end{aligned}$$

Since $\psi \mapsto e^{i\cot\theta \operatorname{tr} u}\psi$ is the unitary transform on $L^2_\alpha(\Omega)$, $\mathcal{M}_\alpha \circ e^{i\cot\theta \operatorname{tr} u}$ is also the unitary isomorphism from $L^2_\alpha(\Omega)$ onto $L^2_\alpha(\mathbb{R}^r)^{\mathfrak{S}_r}$. Therefore,

$$\|\psi_{\mathbf{m}}^{(\alpha)}\|^2_{\alpha,\Omega} = \|(\mathcal{M}_\alpha \circ e^{i\cot\theta \operatorname{tr} u})\psi_{\mathbf{m}}^{(\alpha)}\|^2_{\alpha,\mathbb{R}^r} = \|(1-i\cot\theta)^{-i|\mathbf{s}|-\frac{\alpha}{2}r}q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})\|^2_{\alpha,\mathbb{R}^r}. \tag{2.2.1}$$

Finally, if we adjust the weight function and norm, we obtain the conclusion. □

We obtain the generating function for $q_{\mathbf{m}}^{(\alpha,\theta)}$ as an application of this theorem.

Lemma 2.2.2. (1) *For any $\alpha \in \mathbb{C}, u \in \Omega$ and $z \in \mathcal{D}$, we have*

$$\sum_{\mathbf{m}\in\mathscr{P}} L_{\mathbf{m}}^{\left(\alpha-\frac{n}{r}\right)}(u)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\int_K e^{-(ku|z(e-z)^{-1})}\,dk. \tag{2.2.2}$$

Here, we define the branch by $\Delta(e)^{-\alpha} = 1$.
(2) *Let $z = u'\sum_{j=1}^r a_jc_j \in \mathcal{D}$ with $u' \in U$, $1 > a_1 \geq \ldots \geq a_r \geq 0$, $\alpha \in \mathbb{C}, 0 < \theta < 2\pi$ and $\mathbf{s} \in \mathbb{R}^r$. If $a_1 < \frac{1}{3}$, then*

$$\sum_{\mathbf{m}\in\mathscr{P}} q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\varphi_{-i\mathbf{s}-\frac{\alpha}{2}}((e-e^{-2i\theta}z)(e-z)^{-1}). \tag{2.2.3}$$

Proof. (1) By referring to [FK], (2.2.2) holds for $\alpha > \frac{n}{r} - 1 = d(r-1)$. Moreover, the right hand side of (2.2.2) is well defined for any $\alpha \in \mathbb{C}$. Hence, by analytic continuation, it is sufficient to show the absolute convergence of the left hand side under the assumption. By

Lemmas 2.1.5, 2.1.10 and 2.1.11,

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} |L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)\Phi_{\mathbf{m}}(z)| &\le \sum_{\mathbf{m}\in\mathscr{P}}\sum_{\mathbf{k}\subset\mathbf{m}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \binom{\mathbf{m}}{\mathbf{k}} \frac{(-1)^{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}(u) \right| \Phi_{\mathbf{m}}(a_1) \\
&\le \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \frac{1}{(|\alpha|+d(r-1))_{\mathbf{k}}} \Phi_{\mathbf{k}}(u) \\
&\quad \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \\
&= (1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\frac{a_1}{1-a_1}u\right) \\
&= (1-a_1)^{-r|\alpha|-dr(r-1)} e^{\frac{a_1}{1-a_1}\operatorname{tr} u} < \infty. \qquad (2.2.4)
\end{aligned}$$

(2) From the proof of (1), for $\alpha > \frac{n}{r} - 1$, we have

$$\begin{aligned}
\mathcal{M}_{\alpha,\theta}\left(\sum_{\mathbf{m}\in\mathscr{P}} |\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)|\right)(\mathbf{s}) &= \mathcal{M}_{\alpha,\theta}\left(\sum_{\mathbf{m}\in\mathscr{P}} |e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(2u)\Phi_{\mathbf{m}}(z)|\right)(\mathbf{s}) \\
&\le (1-a_1)^{-r|\alpha|-dr(r-1)} \mathcal{M}_{\alpha,\theta}(e^{-\frac{1-3a_1}{1-a_1}\operatorname{tr} u})(\mathbf{s}).
\end{aligned}$$

By $1-3a_1 > 0$, we can apply Lemma 2.1.12 to $\mathcal{M}_{\alpha,\theta}(e^{-\frac{1-3a_1}{1-a_1}\operatorname{tr} u})(\mathbf{s})$. Thus,

$$\mathcal{M}_{\alpha,\theta}\left(\sum_{\mathbf{m}\in\mathscr{P}} |\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)|\right)(\mathbf{s}) \le (1-a_1)^{-r|\alpha|-dr(r-1)} \left(1 - \frac{a_1}{1-a_1}(1-e^{-2i\theta})\right)^{-i|\mathbf{s}|-\frac{\alpha}{2}r}.$$

Hence, the exchange of integration and summation is justified and we obtain

$$\mathcal{M}_{\alpha,\theta}\left(\sum_{\mathbf{m}\in\mathscr{P}} \psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\right)(\mathbf{s}) = \sum_{\mathbf{m}\in\mathscr{P}} \mathcal{M}_{\alpha,\theta}(\psi_{\mathbf{m}}^{(\alpha)})(\mathbf{s})\Phi_{\mathbf{m}}(z) = \sum_{\mathbf{m}\in\mathscr{P}} q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})\Phi_{\mathbf{m}}(z).$$

On the other hand,

$$\mathcal{M}_{\alpha,\theta}\left(\sum_{\mathbf{m}\in\mathscr{P}} \psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\right)(\mathbf{s}) = \Delta(e-z)^{-\alpha}\mathcal{M}_{\alpha,\theta}\left(\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dk\right)(\mathbf{s}).$$

We remark that if $z \in \mathcal{D}$, then $(e+z)(e-z)^{-1} \in T_\Omega$, $\mathcal{M}_{\alpha,\theta}\left(\int_K |e^{-(ku|(e+z)(e-z)^{-1})}\,dk|\right)(\mathbf{s})$ is convergent under these conditions. Therefore, by Lemma 2.1.4,

$$\mathcal{M}_{\alpha,\theta}\left(\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dk\right)(\mathbf{s}) = \varphi_{-i\mathbf{s}-\frac{\alpha}{2}}((e-e^{-2i\theta}z)(e-z)^{-1}).$$

Hence, for $\alpha > \frac{n}{r} - 1$, it can be seen that this assertion holds.

Finally, from a similar argument to that in (1), it is sufficient to prove the absolute convergence of the generating function of $q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})$ for all $\alpha \in \mathbb{C}$.

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} |q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})\Phi_{\mathbf{m}}(z)| &\leq \sum_{\mathbf{m}\in\mathscr{P}} \sum_{\mathbf{k}\subset\mathbf{m}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)}{(\alpha)_{\mathbf{k}}} (1-e^{-2i\theta})^{\mathbf{k}} \right| \Phi_{\mathbf{m}}(a_1) \\
&\leq \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \frac{\left|\gamma_{\mathbf{k}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\right|}{(|\alpha|+d(r-1))_{\mathbf{k}}} (2\sin\theta)^{\mathbf{k}} \\
&\quad \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \\
&\leq \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{\left|\gamma_{\mathbf{k}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\right|}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\frac{2a_1}{1-a_1}\sin\theta\right).
\end{aligned}$$

Moreover, by Lemma 2.1.14, there exists some constants, $C_1, C_2 > 0$ and $l \in \mathbb{Z}_{\geq 0}$, such that

$$\sum_{\mathbf{m}\in\mathscr{P}} |q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})\Phi_{\mathbf{m}}(z)| \leq C_1 + C_2 \frac{\Gamma_\Omega\left(\frac{\alpha}{2}+\rho\right)}{|\Gamma_\Omega(i\mathbf{s})|} \sum_{l\subset\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{\left(\frac{\alpha}{2}+\rho\right)_{\mathbf{k}}}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\frac{2a_1}{1-a_1}\sin\theta\right) < \infty.$$

□

Let us consider the operators $D_\alpha^{(j)}$ for $j = 1, 2, 3$. The operator $D_\alpha^{(1)}$ is a first order differential operator on the domain $\mathcal{D}$:

$$D_\alpha^{(1)} := 2\,\mathrm{tr}\,(w\nabla_w). \tag{2.2.5}$$

Since this is the Euler operator,

$$D_\alpha^{(1)} f_{\mathbf{m}}^{(\alpha)}(w) = 2|\mathbf{m}| f_{\mathbf{m}}^{(\alpha)}(w).$$

The operators $D_\alpha^{(2)}$ and $D_\alpha^{(3)}$ are respectively defined by $C_\alpha^{-1} D_\alpha^{(2)} = D_\alpha^{(1)} C_\alpha^{-1}$ and $\mathcal{L}_\alpha D_\alpha^{(3)} = D_\alpha^{(2)} \mathcal{L}_\alpha$. Hence, $D_\alpha^{(2)} F_{\mathbf{m}}^{(\alpha)}(w) = 2|\mathbf{m}| F_{\mathbf{m}}^{(\alpha)}(w)$ and

$$D_\alpha^{(3)} \psi_{\mathbf{m}}^{(\alpha)}(u) = 2|\mathbf{m}| \psi_{\mathbf{m}}^{(\alpha)}(u). \tag{2.2.6}$$

Moreover, they have the following expressions.

$$D_\alpha^{(2)} = \mathrm{tr}\,((z^2-e)\nabla_z + \alpha(z-\alpha e)), \tag{2.2.7}$$

$$D_\alpha^{(3)} = \mathrm{tr}\,(-u\nabla_u^2 - \alpha\nabla_u + u - \alpha e). \tag{2.2.8}$$

Lemma 2.2.3. (1)

$$\begin{aligned}
D_\alpha^{(3)} \varphi_{\mathbf{s}}(u) &= \sum_{j=1}^{r} a_j(\mathbf{s}) \varphi_{\mathbf{s}+\epsilon_j}(u) - r\alpha\varphi_{\mathbf{s}}(u) \\
&\quad - \sum_{j=1}^{r} \left(s_j + \frac{d}{4}(r-1)\right)\left(s_j + \alpha - \frac{d}{4}(r-1) - 1\right) a_j(-\mathbf{s}) \varphi_{\mathbf{s}-\epsilon_j}(u).
\end{aligned} \tag{2.2.9}$$

(2)

$$D_\alpha^{(3)}\Phi_{\mathbf{x}}(u) = \sum_{j=1}^{r} \tilde{a_j}(\mathbf{x})\Phi_{\mathbf{x}+\epsilon_j}(u) - r\alpha\Phi_{\mathbf{x}}(u)$$
$$- \sum_{j=1}^{r}\left(x_j + \frac{d}{2}(r-j)\right)\left(x_j + \alpha - 1 - \frac{d}{2}(j-1)\right)\tilde{a_j}(-\mathbf{x})\Phi_{\mathbf{x}-\epsilon_j}(u). \quad (2.2.10)$$

Here,

$$\tilde{a_j}(\mathbf{x}) := a_j(\mathbf{x}-\rho) = \prod_{k\neq j}\frac{x_j - x_k - \frac{d}{2}(j-k-1)}{x_j - x_k - \frac{d}{2}(j-k)}. \quad (2.2.11)$$

(3) *For any* $C \in \mathbb{C}$,

$$e^{C\,\mathrm{tr}\,u}D_\alpha^{(3)}e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u) = (1-C^2)\sum_{j=1}^{r}\tilde{a_j}(\mathbf{x})\Phi_{\mathbf{x}+\epsilon_j}(u)$$
$$+ \sum_{j=1}^{r}(C(2x_j+\alpha)-\alpha)\Phi_{\mathbf{x}}(u)$$
$$- \sum_{j=1}^{r}\left(x_j + \frac{d}{2}(r-j)\right)\left(x_j + \alpha - 1 - \frac{d}{2}(j-1)\right)\tilde{a_j}(-\mathbf{x})\Phi_{\mathbf{x}-\epsilon_j}(u). \quad (2.2.12)$$

(2.2.9) is a corollary of Lemma 3.18 in [FW1]. However, since Faraut and Wakayama's lemma is incorrect in terms of the sign, we re-prove it.

Proof. (1) The modified Laplace transform of $\varphi_{\mathbf{s}}$ is given by

$$(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z) = \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-(z|u)}\varphi_{\mathbf{s}}(u)\Delta(u)^{\alpha-\frac{n}{r}}\,du = 2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha}(z).$$

Thus, from the definition of $D_\alpha^{(3)}$ and Lemma 2.1.18,

$$\mathcal{L}_\alpha(D_\alpha^{(3)}\varphi_{\mathbf{s}})(z) = D_\alpha^{(2)}(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z)$$
$$= 2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}D_\alpha^{(2)}\varphi_{-\mathbf{s}-\alpha}(z)$$
$$= \sum_{j=1}^{r}\left(s_j + \alpha - \frac{d}{4}(r-1)\right)a_j(\mathbf{s}+\alpha)2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha-\epsilon_j}(z)$$
$$- r\alpha 2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha}(z)$$
$$- \sum_{j=1}^{r}\left(s_j + \frac{d}{4}(r-1)\right)a_j(-\mathbf{s}-\alpha)2^{r\alpha}\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\alpha)}\varphi_{-\mathbf{s}-\alpha+\epsilon_j}(z).$$

Since

$$\varphi_{-\mathbf{s}-\alpha\pm\epsilon_j}(z) = \frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\mathbf{s}+\alpha+\rho\pm\epsilon_j)}\mathcal{L}_\alpha(\varphi_{\mathbf{s}\pm\epsilon_j})(z),$$

we have

$$\begin{aligned}
D_\alpha^{(2)}(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z) &= \sum_{j=1}^r \left(s_j+\alpha-\frac{d}{4}(r-1)\right) a_j(\mathbf{s})\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\mathbf{s}+\alpha+\rho+\epsilon_j)}\mathcal{L}_\alpha(\varphi_{\mathbf{s}+\epsilon_j})(z) \\
&\quad - r\alpha(\mathcal{L}_\alpha\varphi_{\mathbf{s}})(z) \\
&\quad - \sum_{j=1}^r \left(s_j+\frac{d}{4}(r-1)\right) a_j(-\mathbf{s})\frac{\Gamma_\Omega(\mathbf{s}+\alpha+\rho)}{\Gamma_\Omega(\mathbf{s}+\alpha+\rho-\epsilon_j)}\mathcal{L}_\alpha(\varphi_{\mathbf{s}-\epsilon_j})(z) \\
&= \mathcal{L}_\alpha\Bigg(\sum_{j=1}^r a_j(\mathbf{s})\varphi_{\mathbf{s}+\epsilon_j}(u) - r\alpha\varphi_{\mathbf{s}}(u) \\
&\quad - \sum_{j=1}^r \left(s_j+\frac{d}{4}(r-1)\right)\left(s_j+\alpha-\frac{d}{4}(r-1)-1\right) a_j(-\mathbf{s})\varphi_{\mathbf{s}-\epsilon_j}(u)\Bigg)(z).
\end{aligned}$$

(2) Put $\mathbf{s} = \mathbf{m} - \rho$ in (2.2.9).

(3) By

$$e^{C\,\mathrm{tr}\,u}\nabla_u e^{-C\,\mathrm{tr}\,u} = -Ce. \quad e^{C\,\mathrm{tr}\,u}\,\mathrm{tr}\,(u\nabla_u^2)e^{-C\,\mathrm{tr}\,u} = C^2\,\mathrm{tr}\,u$$

and the product rule of differentiation, we remark that

$$\begin{aligned}
e^{C\,\mathrm{tr}\,u}\,\mathrm{tr}\,(u\nabla_u^2)e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u) &= \mathrm{tr}\,(u\nabla_u^2)\Phi_{\mathbf{x}}(u) \\
&\quad + 2\,\mathrm{tr}\,(ue^{C\,\mathrm{tr}\,u}\nabla_u(e^{-C\,\mathrm{tr}\,u})\nabla_u(\Phi_{\mathbf{x}}(u))) \\
&\quad + e^{C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u)\,\mathrm{tr}\,u\nabla_u^2 e^{-C\,\mathrm{tr}\,u} \\
&= \left\{\mathrm{tr}\,(u\nabla_u^2 + C^2\,\mathrm{tr}\,u) - 2C|\mathbf{x}|\right\}\Phi_{\mathbf{x}}(u)
\end{aligned}$$

and

$$\begin{aligned}
e^{C\,\mathrm{tr}\,u}\,\mathrm{tr}\,(\nabla_u)e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u) &= \Phi_{\mathbf{x}}(u)\,\mathrm{tr}\,(e^{C\,\mathrm{tr}\,u}\nabla_u e^{-C\,\mathrm{tr}\,u}) + \mathrm{tr}\,(\nabla_u)\Phi_{\mathbf{x}}(u) \\
&= -Cr\Phi_{\mathbf{x}}(u) + \mathrm{tr}\,(\nabla_u)\Phi_{\mathbf{x}}(u).
\end{aligned}$$

Hence,

$$e^{C\,\mathrm{tr}\,u}D_\alpha^{(3)}e^{-C\,\mathrm{tr}\,u}\Phi_{\mathbf{x}}(u) = D_\alpha^{(3)}\Phi_{\mathbf{x}}(u) - C^2\,\mathrm{tr}\,u\Phi_{\mathbf{x}}(u) + C(2|\mathbf{x}| + r\alpha)\Phi_{\mathbf{x}}(u).$$

Therefore, from (2.2.10) and (2.1.46), we have the conclusion. □

Chapter 3

Multivariate circular Jacobi polynomials

In the first section of this chapter, we complete the picture (2.2.1) below

$$
\begin{array}{ccccccc}
 & & \mathbf{(6)} & & \mathbf{(5)} & & \\
 & & \phi_{\mathbf{m}}^{(\alpha,\nu)} & \longleftarrow & \Psi_{\mathbf{m}}^{(\alpha,\nu)} & & \\
 & & \cap & & \cap & & \\
 & & H^2_{\alpha,\nu}(\Sigma)^K & \xleftarrow[\mathcal{C}^{-1}_{\alpha,\nu}]{\simeq} & H^2_{\alpha,\nu}(V)^K & & \\
 & & & & \wr\uparrow \mathcal{F}^{-1}_{\alpha,\nu} & & \\
\mathcal{H}^2_\alpha(\mathcal{D})^K & \xleftarrow[C^{-1}_\alpha]{\simeq} & \mathcal{H}^2_\alpha(T_\Omega)^K & \xleftarrow[\mathcal{L}_\alpha]{\simeq} & L^2_\alpha(\Omega)^K & \xrightarrow[\mathcal{M}_{\alpha,\theta}]{\simeq} & L^2_{\alpha,\theta}(\mathbb{R}^r)^{\mathfrak{S}_r} \\
\cup & & \cup & & \cup & & \cup \\
f_{\mathbf{m}}^{(\alpha)} & \longleftarrow & F_{\mathbf{m}}^{(\alpha)} & \longleftarrow & \psi_{\mathbf{m}}^{(\alpha)} & \longmapsto & q_{\mathbf{m}}^{(\alpha,\theta)} \\
\mathbf{(1)} & & \mathbf{(2)} & & \mathbf{(3)} & & \mathbf{(4)}
\end{array}
\tag{3.0.1}
$$

and introduce a new multivariate orthogonal polynomial, $\phi_{\mathbf{m}}^{(d)}(\sigma;\alpha,\nu) = \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)$, which is a 2-parameter deformation of the spherical polynomial. This is also regarded as a multivariate analogue of the circular Jacobi polynomial. Hence, we call this polynomial the *multivariate circular Jacobi (MCJ) polynomial* that degenerates to a 1-parameter deformation of the usual circular Jacobi polynomial, $\phi_m^{(\alpha)}(e^{i\theta})$, in the one variable case. Further, the weight function of its orthogonality relation coincides with the circular Jacobi ensemble defined by Bourgade et al. [BNR].

We derive the generating function of $\phi_{\mathbf{m}}^{(\alpha,\nu)}$ in section 3.2 and the pseudo-differential equation for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$ in the section 3.3. Moreover, we study the one variable case in more detail in section 3.4. Finally, we describe future work for $\phi_{\mathbf{m}}^{(\alpha,\nu)}$.

Unless otherwise specified, we have assumed $\alpha > \frac{n}{r} - 1 = \frac{d}{2}(r-1)$ and $\nu \in \mathbb{R}$ in this chapter.

3.1 Definitions and orthogonality

Following Chapter 2, we introduce some function space, functions that become complete orthogonal bases and unitary transformations required to provide the MCJ polynomials.
(5) $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$; Modified Fourier transform of $\psi_{\mathbf{m}}^{(\alpha)}$

$$
\begin{aligned}
H^2_{\alpha,\nu}(V)^K &:= \{\Psi : V \longrightarrow \mathbb{C} \mid \Psi \in H^2(V) \text{ is } K\text{-invariant and } \|\Psi\|^2_{\alpha,\nu,V} < \infty\},\\
\|\Psi\|^2_{\alpha,\nu,V} &:= \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2 \|\Psi\|^2_V\\
&= \frac{\widetilde{c_0}}{(2\pi)^n}\frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2 \int_{\mathbb{R}^r} |\Psi(\lambda)|^2 \prod_{1\le p<q\le r} |\lambda_p-\lambda_q|^d \, d\lambda_1\cdots d\lambda_r,\\
\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t) &:= \Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\widetilde{\Psi_{\mathbf{m}}^{(\alpha,\nu)}}(t),\\
\widetilde{\Psi_{\mathbf{m}}^{(\alpha,\nu)}}(t) &:= d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(2(e-it)^{-1}).
\end{aligned}
$$

Here, $\lambda = \sum_{j=1}^r \lambda_j c_j$ and we choose the branch of $\Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}$ which takes the value 1 at $t=0$.
(6) $\phi_{\mathbf{m}}^{(\alpha,\nu)}$; MCJ polynomials

$$
\begin{aligned}
H^2_{\alpha,\nu}(\Sigma)^K &:= \{\phi : \Sigma \longrightarrow \mathbb{C} \mid \phi \text{ is } K\text{-invariant and continued analytically to } \mathcal{D}\\
&\qquad\qquad \text{as a holomorphic function which satisfies with } \|\phi\|^2_{\alpha,\nu,\Sigma} < \infty\},\\
\|\phi\|^2_{\alpha,\nu,\Sigma} &:= \frac{1}{(2\pi)^n}\frac{1}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2 \int_\Sigma |\phi(\sigma)|^2 |\Delta(e-\sigma)^{\frac{1}{2}(\alpha-\frac{n}{r})+i\nu}|^2 \, d\mu(\sigma)\\
&= \frac{\widetilde{c_0}}{(2\pi)^n}\frac{1}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2\\
&\quad\cdot \int_{\mathcal{S}^r} |\phi(e^{i\theta})|^2 \prod_{j=1}^r |(1-e^{i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2 \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d \, d\theta_1\cdots d\theta_r.
\end{aligned}
$$

Here, we define the multivariate circular Jacobi polynomial by

$$
\phi_{\mathbf{m}}^{(d)}(\sigma;\alpha,\nu) = \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma) := d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(e-\sigma). \quad (3.1.1)
$$

The main purpose of this section is to show that these polynomials form the complete orthogonal basis of $H^2_{\alpha,\nu}(\Sigma)^K$ and to explicitly write their orthogonal relations. To achieve that purpose, we introduce a modified Fourier transform $\mathcal{F}_\alpha^{-1}$ for a function ψ on Ω and the

second inverse modified Cayley transform $\mathcal{C}_{\alpha,\nu}^{-1}$ as follows.

$$(\mathcal{F}_\alpha^{-1}\psi)(t) := \frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)}(F^{-1}(\Delta(u)^{\frac{1}{2}(\alpha-\frac{n}{r})}\psi))(t) \tag{3.1.2}$$

$$= \frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)}\int_\Omega e^{i(t|u)}\psi(u)\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)}\,du, \tag{3.1.3}$$

$$(\mathcal{C}_{\alpha,\nu}^{-1}\Psi)(\sigma) := \Delta(e-ic(\sigma))^{\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu}\Psi(c(\sigma)) = \Delta\left(\frac{e-\sigma}{2}\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\Psi(c(\sigma)). \tag{3.1.4}$$

These give the following unitary isomorphisms.

Theorem 3.1.1. (1)

$$\begin{array}{rccc} \mathcal{F}_{\alpha,\nu}^{-1} := \mathcal{F}_{\alpha+2i\nu}^{-1}: & L_\alpha^2(\Omega)^K & \xrightarrow{\simeq} & H_{\alpha,\nu}^2(V)^K \quad \text{(unitary)}. \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & \psi_{\mathbf{m}}^{(\alpha)} & \longmapsto & \Psi_{\mathbf{m}}^{(\alpha,\nu)} \end{array}$$

In particular, $\{\Psi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$ form the complete orthogonal basis of $H_{\alpha,\nu}^2(V)^K$ and for all $\mathbf{m},\mathbf{n}\in\mathscr{P}$,

$$\frac{1}{(2\pi)^n}\int_V \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\overline{\Psi_{\mathbf{n}}^{(\alpha,\nu)}(t)}\,dt = d_{\mathbf{m}}\frac{\Gamma_\Omega(\alpha+\mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{1}{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2}\delta_{\mathbf{mn}}. \tag{3.1.5}$$

(2)

$$\begin{array}{rccc} \mathcal{C}_{\alpha,\nu}^{-1}: & H_{\alpha,\nu}^2(V)^K & \xrightarrow{\simeq} & H_{\alpha,\nu}^2(\Sigma)^K \quad \text{(unitary)}. \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & \Psi_{\mathbf{m}}^{(\alpha,\nu)} & \longmapsto & \phi_{\mathbf{m}}^{(\alpha,\nu)} \end{array}$$

Furthermore, the MCJ polynomials form the complete orthogonal basis of $H_{\alpha,\nu}^2(\Sigma)^K$ and for all $\mathbf{m},\mathbf{n}\in\mathscr{P}$,

$$\begin{aligned} &\frac{1}{(2\pi)^n}\int_\Sigma \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)\overline{\phi_{\mathbf{n}}^{(\alpha,\nu)}(\sigma)}|\Delta(e-\sigma)^{\frac{1}{2}(\alpha-\frac{n}{r})+i\nu}|^2\,d\mu(\sigma) \\ &= \frac{\widetilde{c_0}}{(2\pi)^n}\int_{\mathcal{S}^r}\phi_{\mathbf{m}}^{(\alpha,\nu)}(e^{i\theta})\overline{\phi_{\mathbf{n}}^{(\alpha,\nu)}(e^{i\theta})}\prod_{j=1}^r|(1-e^{i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2\prod_{1\le k<l\le r}|e^{i\theta_k}-e^{i\theta_l}|^d\,d\theta_1\cdots d\theta_r \\ &= d_{\mathbf{m}}\frac{\Gamma_\Omega(\alpha+\mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{1}{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2}\delta_{\mathbf{mn}}. \end{aligned} \tag{3.1.6}$$

Proof. (1) Observing that

$$\begin{array}{ccc} L_\alpha^2(\Omega)^K & \xrightarrow{\simeq} & L^2(\Omega)^K \quad \text{(unitary)}, \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \psi & \longmapsto & \left(\frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\right)^{\frac{1}{2}}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi \end{array}$$

$$\begin{array}{ccc} H^2_{\alpha,\nu}(V)^K & \xrightarrow{\simeq} & H^2(V)^K \qquad \text{(unitary).} \\ \cup & & \cup \\ \Psi & \longmapsto & \frac{2^{\frac{r\alpha}{2}}}{\Gamma_\Omega(\alpha)^{\frac{1}{2}}}\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\Psi \end{array}$$

In addition to using Lemma 2.1.19, we immediately obtain the following unitary isomorphism $\mathcal{F}^{-1}_{\alpha,\nu}$.

$$\begin{array}{ccccccc} L^2_\alpha(\Omega)^K & \xrightarrow{\simeq} & L^2(\Omega)^K & \xrightarrow{\simeq} & H^2(V)^K & \xrightarrow{\simeq} & H^2_{\alpha,\nu}(V)^K. \\ \cup & & \cup & & \cup & & \cup \\ \psi & \longmapsto & \left(\frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\right)^{\frac{1}{2}}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi & \longmapsto & F^{-1}\left(\frac{2^{\frac{r\alpha}{2}}}{\Gamma_\Omega(\alpha)^{\frac{1}{2}}}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi\right) & \longmapsto & \mathcal{F}^{-1}_{\alpha,\nu}(\psi) \end{array}$$

Next, we evaluate the modified Fourier transform of $\psi^{(\alpha)}_{\mathbf{m}}$ which forms the complete orthogonal basis for $L^2_\alpha(\Omega)^K$. From Lemma 2.1.8, we obtain

$$\mathcal{F}^{-1}_{\alpha,\nu}(e^{-\operatorname{tr} u}\Phi_{\mathbf{k}})(t) = \left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}\Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\Phi_{\mathbf{k}}((e-it)^{-1}).$$

Hence,

$$\begin{aligned} \mathcal{F}^{-1}_{\alpha,\nu}(\psi^{(\alpha)}_{\mathbf{m}})(t) &= \frac{1}{\Gamma\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)}F^{-1}(\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi^{(\alpha)}_{\mathbf{m}})(t) \\ &= d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-2)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\frac{1}{(\alpha)_{\mathbf{k}}} \\ &\quad\cdot\frac{1}{\Gamma\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)}\int_\Omega e^{i(t|u)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}e^{-\operatorname{tr} u}\Phi_{\mathbf{k}}(u)\,du \\ &= \Psi^{(\alpha,\nu)}_{\mathbf{m}}(t). \end{aligned}$$

Finally, we have

$$\begin{aligned} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\delta_{\mathbf{mn}} &= (\psi^{(\alpha)}_{\mathbf{m}},\psi^{(\alpha)}_{\mathbf{n}})_{\alpha,\Omega} \\ &= (\mathcal{F}^{-1}_{\alpha,\nu}(\psi^{(\alpha)}_{\mathbf{m}}),\mathcal{F}^{-1}_{\alpha,\nu}(\psi^{(\alpha)}_{\mathbf{n}}))_{\alpha,\nu,V} \\ &= \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\frac{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)+i\nu\right|^2}{(2\pi)^n}\int_V \Psi^{(\alpha,\nu)}_{\mathbf{m}}(t)\overline{\Psi^{(\alpha,\nu)}_{\mathbf{n}}(t)}\,dt. \end{aligned}$$

(2) We remark that the inverse Cayley transform c^{-1} is a holomorphic bijection of H_Ω onto $\mathcal{D}$ and the inverse map of $\mathcal{C}^{-1}_{\alpha,\nu}$ is given by

$$(\mathcal{C}_{\alpha,\nu}\phi)(t) := \Delta\left(\frac{e-c^{-1}(t)}{2}\right)^{\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu}\phi(c^{-1}(t)) = \Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\phi(c^{-1}(t)). \quad (3.1.7)$$

Hence, since $\{\Psi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$ form the complete orthogonal basis of $H^2_{\alpha,\nu}(V)^K$, it is sufficient to show the statement for $\{\Psi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$ and $\{\phi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$.

By the definition, we have

$$(\mathcal{C}_{\alpha,\nu}^{-1}\Psi_{\mathbf{m}}^{(\alpha,\nu)})(\sigma) = \Delta\left(\frac{e-\sigma}{2}\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\Delta(e-ic(\sigma))^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\widetilde{\Psi_{\mathbf{m}}^{(\alpha,\nu)}}(c(\sigma)) = \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma),$$

and from (2.1.5) of Lemma 2.1.3, we have

$$\begin{aligned}\int_V |\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)|^2\,dt &= 2^n\int_\Sigma |\Delta(e-\sigma)^{-\frac{n}{r}}|^2|\Psi_{\mathbf{m}}^{(\alpha,\nu)}(c(\sigma))|^2\,d\mu(\sigma)\\ &= 2^n\int_\Sigma |\Delta(e-\sigma)^{-\frac{n}{r}}|^2\left|\Delta\left(\frac{e-\sigma}{2}\right)^{\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu}\right|^2|(\mathcal{C}_{\alpha,\nu}^{-1}\Psi_{\mathbf{m}}^{(\alpha,\nu)})(\sigma)|^2\,d\mu(\sigma)\\ &= 2^{-r\alpha}\int_\Sigma |\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)|^2|\Delta(e-\sigma)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2\,d\mu(\sigma).\end{aligned}$$

Therefore, we obtain

$$(\phi_{\mathbf{m}}^{(\alpha,\nu)},\phi_{\mathbf{n}}^{(\alpha,\nu)})_{\alpha,\nu,\Sigma} = (\mathcal{C}_{\alpha,\nu}^{-1}(\Psi_{\mathbf{m}}^{(\alpha,\nu)}),\mathcal{C}_{\alpha,\nu}^{-1}(\Psi_{\mathbf{n}}^{(\alpha,\nu)}))_{\alpha,\nu,\Sigma} = (\Psi_{\mathbf{m}}^{(\alpha,\nu)},\Psi_{\mathbf{n}}^{(\alpha,\nu)})_{\alpha,\nu,V} = d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\delta_{\mathbf{mn}}.$$

(3.1.6) follows from (2.1.6) of Lemma 2.1.3 immediately. □

Remark 3.1.2. (1) The weight function of the left hand side for (3.1.6)

$$\prod_{j=1}^r(1-e^{i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}(1-e^{-i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu}\prod_{1\le p<q\le r}|e^{i\theta_p}-e^{i\theta_q}|^d$$

coincides with the circular Jacobi ensemble defined by [BNR].
(2) When $\alpha=\frac{n}{r},\nu=0$, (3.1.1) and (3.1.6) degenerate to

$$\phi_{\mathbf{m}}^{(\frac{n}{r},0)}(e^{i\theta}) = d_{\mathbf{m}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\Phi_{\mathbf{k}}(e-e^{i\theta}) = d_{\mathbf{m}}\Phi_{\mathbf{m}}(e^{i\theta}), \tag{3.1.8}$$

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_\Sigma \phi_{\mathbf{m}}^{(\frac{n}{r},0)}(\sigma)\overline{\phi_{\mathbf{n}}^{(\frac{n}{r},0)}(\sigma)}\,d\mu(\sigma)\\ &= \frac{\widetilde{c_0}}{(2\pi)^n}\int_{\mathcal{S}^r}\phi_{\mathbf{m}}^{(\frac{n}{r},0)}(e^{i\theta})\overline{\phi_{\mathbf{n}}^{(\frac{n}{r},0)}(e^{i\theta})}\prod_{1\le p<q\le r}|e^{i\theta_p}-e^{i\theta_q}|^d\,d\theta_1\cdots d\theta_r = d_{\mathbf{m}}\frac{1}{\Gamma_\Omega\left(\frac{n}{r}\right)}\delta_{\mathbf{mn}}.\end{aligned} \tag{3.1.9}$$

Therefore, $\phi_{\mathbf{m}}^{(\alpha,\nu)}(e^{i\theta})$ is regarded as a 2-parameter deformation of the spherical polynomial.

As a generalization of spherical polynomial $\Phi_{\mathbf{m}}^{(d)}$, the Jack polynomial $P_{\mathbf{m}}^{(\frac{2}{d})}$ which is a generalization for multiplicity d is well known (see [M], Chapter VI). This multivariate special orthogonal polynomial system is derived as the simultaneous eigenfunctions of some

commuting differential operators. On the other hand, using the unitary picture, we obtain another extension $\phi_m^{(\alpha,\nu)}$, which is different from the Jack polynomial, that is, instead of the multiplicity d, we consider deformations for real 2-parameters α and ν.
(3) From (2.1.31), for any $\theta \in \mathbb{R}$ and $\mathbf{m} \in \mathscr{P}$, we have

$$\phi_{\mathbf{m}}^{(\alpha,\nu)}(e^{i\theta}e) = q_{\mathbf{m}}^{\left(\alpha,-\frac{\theta}{2}\right)}\left(\nu - i\left(\frac{n}{2r}+\rho\right)\right). \tag{3.1.10}$$

3.2 Generating function

By using these unitary isomorphisms, we present the generating functions of MCJ polynomials from (2.2.2).

Theorem 3.2.1. *We assume $z = u\sum_{j=1}^r a_j c_j \in \mathcal{D}$ with $u \in U$, $1 > a_1 \geq \ldots \geq a_r \geq 0$ and $a_1 < \frac{1}{3}$.*
(1) *For all $t \in V$,*

$$\sum_{\mathbf{m}\in\mathscr{P}} \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\int_K \Delta((e+z)(e-z)^{-1} - ikt)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk. \tag{3.2.1}$$

(2) *For any $\sigma \in \Sigma$,*

$$\sum_{\mathbf{m}\in\mathscr{P}} \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\int_K \Delta((e-z)^{-1} - (z(e-z)^{-1})k\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk. \tag{3.2.2}$$

Proof. (1) From a similar argument to that in (2) of Lemma 2.2.2,

$$\begin{aligned}\mathcal{F}_{\alpha,\nu}^{-1}\left(\sum_{\mathbf{m}\in\mathscr{P}} |\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)|\right)(t) &\leq (1-a_1)^{-r|\alpha|-dr(r-1)}\mathcal{F}_{\alpha,\nu}^{-1}(e^{-\frac{1-3a_1}{1-a_1}\operatorname{tr} u})(t)\\ &= (1-a_1)^{-r|\alpha|-dr(r-1)}\Delta\left(\frac{1-3a_1}{1-a_1}e - it\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu} < \infty.\end{aligned}$$

Hence, the exchange of integration and summation is justified and we obtain

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\Phi_{\mathbf{m}}(z) &= \sum_{\mathbf{m}\in\mathscr{P}} \mathcal{F}_{\alpha,\nu}^{-1}(\psi_{\mathbf{m}}^{(\alpha)})(t)\Phi_{\mathbf{m}}(z)\\ &= \mathcal{F}_{\alpha,\nu}^{-1}\left(\sum_{\mathbf{m}\in\mathscr{P}} \psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\right)(t)\\ &= \Delta(e-z)^{-\alpha}\mathcal{F}_{\alpha,\nu}^{-1}\left(\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dk\right)(t).\end{aligned}$$

Moreover, by Lemma 2.1.8,

$$\begin{aligned}
\mathcal{F}_{\alpha,\nu}^{-1}\left(\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dk\right)(t) &= \frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)} \\
&\quad\cdot\int_\Omega e^{i(t|u)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dkdu \\
&= \frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)} \\
&\quad\cdot\int_K\int_\Omega e^{-(u|k(e+z)(e-z)^{-1}-it)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\,dudk \\
&= \int_K \Delta(k(e+z)(e-z)^{-1}-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk \\
&= \int_K \Delta((e+z)(e-z)^{-1}-ikt)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk.
\end{aligned}$$

(2) Applying the modified Cayley transform $\mathcal{C}_{\alpha,\nu}^{-1}$ to (3.2.1), we obtain

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}}\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)\Phi_{\mathbf{m}}(z) &= \Delta(e-z)^{-\alpha}\mathcal{C}_{\alpha,\nu}^{-1}\left(\int_K \Delta((e+z)(e-z)^{-1}-ikt)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk\right)(\sigma) \\
&= \Delta(e-z)^{-\alpha}\Delta\left(\frac{e-\sigma}{2}\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu} \\
&\quad\cdot\int_K \Delta((e+z)(e-z)^{-1}-ikc(\sigma))^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk.
\end{aligned} \tag{3.2.3}$$

Since

$$(e+z)(e-z)^{-1}-ikc(\sigma) = 2z(e-z)^{-1}+2k(e-\sigma)^{-1},$$

we have

$$\begin{aligned}
&\Delta(e-\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\int_K \Delta(z(e-z)^{-1}+k(e-\sigma)^{-1})^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk \\
&= \int_K \Delta(e+(k(z(e-z)^{-1}))(e-\sigma))^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk \\
&= \int_K \Delta(e+(z(e-z)^{-1})(e-k\sigma))^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk \\
&= \int_K \Delta((e-z)^{-1}-(z(e-z)^{-1})k\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk.
\end{aligned}$$

□

3.3 Differential equation for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$

Considering some pseudo-differential operator that is defined by

$$\mathrm{tr}\ (\nabla_t^{-1})e^{(t|u)} = \mathrm{tr}\ (u^{-1})e^{(t|u)} \quad (\text{any } t \in V, u \in \Omega), \tag{3.3.1}$$

we obtain the explicit (pseudo-) differential equations for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$ as follows.

Theorem 3.3.1. *The operator $D_{\alpha,\nu}^{(5)}$ on V is defined by the relation $D_{\alpha,\nu}^{(5)}\mathcal{F}_{\alpha,\nu}^{-1} = \mathcal{F}_{\alpha,\nu}^{-1}D_{\alpha}^{(3)}$. Then, we obtain*

$$D_{\alpha,\nu}^{(5)} = \mathrm{tr}\left(-i(e+t^2)\nabla_t + \left(2\nu - i\frac{n}{r}\right)t - \alpha e + i\left(\frac{1}{4}\left(\alpha - \frac{n}{r}\right)^2 + \nu^2\right)\nabla_t^{-1}\right), \tag{3.3.2}$$

and

$$D_{\alpha,\nu}^{(5)}\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t) = 2|\mathbf{m}|\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t). \tag{3.3.3}$$

Proof. From Theorem 3.1.1, to prove (3.3.2), it suffices to show the relation for complete orthogonal basis $\psi_{\mathbf{m}}^{(\alpha)}$ for $L_{\alpha}^2(\Omega)^K$

$$(\mathcal{F}_{\alpha,\nu}^{-1}D_{\alpha}^{(3)}\psi_{\mathbf{m}}^{(\alpha)})(t) = \widetilde{D_{\alpha,\nu}^{(5)}}((\mathcal{F}_{\alpha,\nu}^{-1}\psi_{\mathbf{m}}^{(\alpha)})(t)),$$

where $D_{\alpha,\nu}^{(5)}$ is the operator on the right hand side of (3.3.2).

By the very definition of the modified Fourier transform $\mathcal{F}_{\alpha,\nu}^{-1}$ and the inner product of $L_{\alpha}^2(\Omega)$, we can write

$$(\mathcal{F}_{\alpha,\nu}^{-1}\psi)(t) = (e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|\overline{\psi})_{L_{\alpha}^2(\Omega)}, \tag{3.3.4}$$

and from Lemma 3.13 in [FW1], $D_{\alpha}^{(3)} = \overline{D_{\alpha}^{(3)}}$ is a self-adjoint operator with respect to the measure $\Delta(u)^{\alpha-\frac{n}{r}}\,du$. Hence,

$$\begin{aligned}(\mathcal{F}_{\alpha,\nu}^{-1}D_{\alpha}^{(3)}\psi_{\mathbf{m}}^{(\alpha)})(t) &= (e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|\overline{D_{\alpha}^{(3)}\psi_{\mathbf{m}}^{(\alpha)}})_{L_{\alpha}^2(\Omega)}\\ &= (D_{\alpha}^{(3)}(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})|\overline{\psi_{\mathbf{m}}^{(\alpha)}})_{L_{\alpha}^2(\Omega)}.\end{aligned}$$

Furthermore, based on Lemma 2.1.17, let us perform

$$\begin{aligned}\mathrm{tr}\,(u\nabla_u^2)(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}) &= \mathrm{tr}\,(u(\nabla_u^2 e^{i(t|u)}))\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\\ &\quad + 2\,\mathrm{tr}\,(u(\nabla_u e^{i(t|u)})(\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}))\\ &\quad + e^{i(t|u)}\,\mathrm{tr}\,(u\nabla_u^2\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\ &= e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\left\{\mathrm{tr}\,(-ut^2) - i\left(\left(\alpha-\frac{n}{r}\right)-2i\nu\right)\mathrm{tr}\,(t)\right.\\ &\quad \left. + \left(\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu\right)\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu\right)\mathrm{tr}\,(u^{-1})\right\},\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(\alpha\nabla_u)(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}) &= \alpha\operatorname{tr}(\nabla_u e^{i(t|u)})\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu} \\ &\quad + \alpha e^{i(t|u)}\operatorname{tr}(\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}) \\ &= \operatorname{tr}\left(i\alpha t - \alpha\left(\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu\right)u^{-1}\right)e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}.\end{aligned}$$

Here, we remark that for any $p\in\mathbb{Z}_{\geq 0}$ $\operatorname{tr}((\nabla_u^p e^{i(t|u)})) = \operatorname{tr}((it)^p)e^{i(t|u)}$ and

$$\begin{aligned}\operatorname{tr}(u\nabla_u^2\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}) &= \left(-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu\right) \\ &\quad \left\{\operatorname{tr}(u(\nabla_u u^{-1}))\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu} + \operatorname{tr}(\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\right\} \\ &= \left(\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu\right)\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu\right)\operatorname{tr}(u^{-1})\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\end{aligned}$$

and

$$\operatorname{tr}((\nabla_u e^{i(t|u)})(u\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})) = \left(-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu\right)\operatorname{tr}(it)e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}.$$

Hence,

$$\begin{aligned}D_\alpha^{(3)}(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}) &= \operatorname{tr}\,(-u\nabla_u^2-\alpha\nabla_u+u-\alpha e)(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}) \\ &= \operatorname{tr}\left((e+t^2)u+\left(2\nu-i\frac{n}{r}\right)t-\alpha e+\left(\frac{1}{4}\left(\alpha-\frac{n}{r}\right)^2+\nu^2\right)u^{-1}\right) \\ &\quad \cdot e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu} \\ &= \widetilde{D_{\alpha,\nu}^{(5)}}e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}.\end{aligned}$$

Therefore,

$$\begin{aligned}(\mathcal{F}_{\alpha,\nu}^{-1}D_\alpha^{(3)}\psi_{\mathbf{m}}^{(\alpha)})(t) &= (\widetilde{D_{\alpha,\nu}^{(5)}}e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\overline{|\psi_{\mathbf{m}}^{(\alpha)}})_{L_\alpha^2(\Omega)} \\ &= \widetilde{D_{\alpha,\nu}^{(5)}}(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\overline{|\psi_{\mathbf{m}}^{(\alpha)}})_{L_\alpha^2(\Omega)} \\ &= \widetilde{D_{\alpha,\nu}^{(5)}}((\mathcal{F}_{\alpha,\nu}^{-1}\psi)(t)).\end{aligned}$$

The second equality is justified by $e^{i(t|u)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi_{\mathbf{m}}^{(\alpha)}\in L^1(\Omega)$. Finally, since $\mathcal{F}_{\alpha,\nu}^{-1}$ is an isomorphism from the space $L_\alpha^2(\Omega)^K$ onto $H_{\alpha,\nu}^2(V)^K$, we obtain $D_{\alpha,\nu}^{(5)} = \widetilde{D_{\alpha,\nu}^{(5)}}$.

On the other hand, for (3.3.3), from the definition of $D_{\alpha,\nu}^{(5)}$ and (2.2.6), we have

$$D_{\alpha,\nu}^{(5)}\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t) = D_{\alpha,\nu}^{(5)}\mathcal{F}_{\alpha,\nu}^{-1}(\psi_{\mathbf{m}}^{(\alpha)})(t) = \mathcal{F}_{\alpha,\nu}^{-1}(D_\alpha^{(3)}\psi_{\mathbf{m}}^{(\alpha)})(t) = 2|\mathbf{m}|\mathcal{F}_{\alpha,\nu}^{-1}(\psi_{\mathbf{m}}^{(\alpha)})(t) = 2|\mathbf{m}|\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t).$$

□

We can also define $D^{(6)}_{\alpha,\nu}$ by the relation $D^{(6)}_{\alpha,\nu}\mathcal{C}^{-1}_{\alpha,\nu} = \mathcal{C}^{-1}_{\alpha,\nu}D^{(5)}_{\alpha,\nu}$. However, since there are difficulties in deriving the modified Cayley transform of ∇_t^{-1}, we have not been able to obtain the explicit expression for $D^{(6)}_{\alpha,\nu}$ like that in the above theorem. On the other hand, when $\alpha = \frac{n}{r}, \nu = 0$, $\Psi^{(\frac{n}{r},0)}_{\mathbf{m}}(t)$ becomes

$$\begin{aligned}\Psi^{(\frac{n}{r},0)}_{\mathbf{m}}(t) &= \Delta(e-it)^{-\frac{n}{r}} d_{\mathbf{m}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\Phi_{\mathbf{k}}(2(e-it)^{-1}) \\ &= \Delta(e-it)^{-\frac{n}{r}} d_{\mathbf{m}}\Phi_{\mathbf{m}}(c^{-1}(t)). \qquad (3.3.5)\\ & \qquad (3.3.6)\end{aligned}$$

Further, the term of the pseudo-differential operator vanishes for $D^{(5)}_{\frac{n}{r},0}$. Hence,

$$D^{(5)}_{\frac{n}{r},0} = -i\,\mathrm{tr}\,((e+t^2)\partial_t) - \frac{n}{r}\,\mathrm{tr}\,(e+it). \qquad (3.3.7)$$

Therefore, we obtain the following explicit expression for $D^{(6)}_{\frac{n}{r},0}$ from the modified Cayley transform of $D^{(5)}_{\frac{n}{r},0}$.

$$D^{(6)}_{\frac{n}{r},0} = 2\,\mathrm{tr}\,(\sigma\nabla_\sigma). \qquad (3.3.8)$$

3.4 One variable case

In this section, we have assumed that $r = 1$.

First, we remark that (3.1.1) becomes

$$\begin{aligned}\phi^{(\alpha,\nu)}_m(\sigma) &:= \frac{(\alpha)_m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{\left(\frac{1}{2}(\alpha+1)+i\nu\right)_k}{(\alpha)_k}(1-\sigma)^k \qquad (3.4.1)\\ &= \frac{(\alpha)_m}{m!}{}_2F_1\left(\begin{matrix}-m, \frac{1}{2}(\alpha+1)+i\nu\\ \alpha\end{matrix};1-\sigma\right) \qquad (3.4.2)\\ &= \frac{\left(\frac{\alpha-1}{2}-i\nu\right)}{m!}{}_2F_1\left(\begin{matrix}-m, \frac{1}{2}(\alpha+1)+i\nu\\ -m-\frac{\alpha-3}{2}+i\nu\end{matrix};\sigma\right). \qquad (3.4.3)\end{aligned}$$

and for $\alpha > 0, \nu \in \mathbb{R}$, (3.1.6) degenerates to

$$\frac{1}{2\pi i}\int_\Sigma \phi^{(\alpha,\nu)}_m(\sigma)\overline{\phi^{(\alpha,\nu)}_n(\sigma)}|(1-\sigma)^{\frac{\alpha-1}{2}+i\nu}|^2\,\frac{m(d\sigma)}{\sigma} = \frac{\Gamma(\alpha+m)}{m!}\frac{1}{\left|\Gamma(\frac{\alpha+1}{2}+i\nu)\right|^2}\delta_{mn}. \qquad (3.4.4)$$

That is a 1-parameter deformation of the usual circular Jacobi polynomial that coincides with $\phi^{(\alpha,0)}_m(\sigma)$(See [As], [Is]). In particular, $\phi^{(1,0)}_m(\sigma) = \sigma^m$ and

$$\frac{1}{2\pi i}\int_\Sigma \phi^{(1,0)}_m(\sigma)\overline{\phi^{(1,0)}_n(\sigma)}\,\frac{m(d\sigma)}{\sigma} = \frac{1}{2\pi}\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \delta_{mn}.$$

We also remark that the rank 1 case of (3.1.10) is

$$\phi_m^{(\alpha,\nu)}(e^{i\theta}) = q_m^{\left(\alpha,-\frac{\theta}{2}\right)}\left(\nu+\frac{1}{2i}\right) = e^{m\frac{i\theta}{2}}P_m^{\left(\frac{\alpha}{2}\right)}\left(\nu+\frac{1}{2i};-\frac{\theta}{2}\right), \tag{3.4.5}$$

That means if θ is regarded as a parameter and ν is regarded as a variable for the circular Jacobi polynomial, then we can consider the circular Jacobi polynomial to be the Meixner-Pollaczek polynomial. Moreover, the generating function of $\phi_m^{(\alpha,\nu)}(\sigma)$ is given by

$$\sum_{m\geq 0}\phi_m^{(\alpha,\nu)}(\sigma)z^m = (1-z)^{-\frac{1}{2}(\alpha-1)+i\nu}(1-\sigma z)^{-\frac{1}{2}(\alpha+1)-i\nu}. \tag{3.4.6}$$

Although, we have not been given an explicit expression for the differential relation of $\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)$ in the multivariate case, we obtain the following explicit result in the one variable case from the differential equation of ${}_2F_1$.

Proposition 3.4.1. *If*

$$D_{\alpha,\nu} := \sigma(1-\sigma)\partial_\sigma^2 + \left\{\left(-m+\frac{3}{2}+i\nu\right)(1-\sigma) - \frac{\alpha}{2}(1+\sigma)\right\}\partial_\sigma + m\left(\frac{1}{2}(\alpha+1)+i\nu\right), \tag{3.4.7}$$

then

$$D_{\alpha,\nu}\phi_m^{(\alpha,\nu)}(\sigma) = 0. \tag{3.4.8}$$

3.5 Concluding remarks

We have investigated the fundamental properties of MCJ polynomials, that is, orthogonality and the generating function etc. However, as we have not succeeded in obtaining a differential equation for $\phi_{\mathbf{m}}^{(\alpha,\nu)}$ similar to Proposition 3.4.1, we can not derive a modified Cayley transform of tr ∇_u^{-1}.

It is also important to consider the generalization of MCJ polynomials for multiplicity d. Actually, this generalization has been obtained by Baker and Forrester [BF] for multivariate Laguerre polynomials which are a modified Fourier transform of the Cayley transform of MCJ polynomials. In addition, we can consider MCJ polynomials and their orthogonality without using the analysis on the symmetric cones as follows.

Let $n := r + \frac{d}{2}r(r-1)$,

$$d_{\mathbf{m}} := \frac{\Gamma\left(\frac{d}{2}\right)^r}{\Gamma\left(\frac{d}{2}r\right)}\prod_{j=1}^{r-1}\frac{1}{\Gamma\left(\frac{d}{2}j\right)^2}\prod_{1\leq p<q\leq r}\left(m_p-m_q+\frac{d}{2}(q-p)\right)\frac{\Gamma\left(m_p-m_q+\frac{d}{2}(q-p+1)\right)}{\Gamma\left(m_p-m_q+\frac{d}{2}(q-p-1)+1\right)}, \tag{3.5.1}$$

$$\Gamma_\Omega(\mathbf{s}) := (2\pi)^{\frac{n-r}{2}}\prod_{j=1}^{r}\Gamma\left(s_j-\frac{d}{2}(j-1)\right),$$

$$(\mathbf{s})_{\mathbf{k}} := \prod_{j=1}^{r}\left(s_j-\frac{d}{2}(j-1)\right)_{k_j}.$$

Further, $P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1,\dots,\lambda_r)$ is an r-variable Jack polynomial and

$$\Phi_{\mathbf{k}}^{(d)}(\lambda_1,\dots,\lambda_r) := \frac{P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1,\dots,\lambda_r)}{P_{\mathbf{k}}^{(\frac{2}{d})}(1,\dots,1)}. \tag{3.5.2}$$

Furthermore, we introduce the generalized (Jack) binomial coefficients based on [OO] by

$$\Phi_{\mathbf{m}}^{(d)}(1+\lambda_1,\dots,1+\lambda_r) = \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \Phi_{\mathbf{k}}^{(d)}(\lambda_1,\dots,\lambda_r).$$

Definition 3.5.1. We define the generalized MCJ polynomial as follows.

$$\begin{aligned}\phi_{\mathbf{m}}^{(d)}(e^{i\theta};\alpha,\nu) = \phi_{\mathbf{m}}^{(d)}(e^{i\theta_1},\dots,e^{i\theta_r};\alpha,\nu) := d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \\ \cdot \Phi_{\mathbf{k}}^{(d)}(1-e^{i\theta_1},\dots,1-e^{i\theta_r}).\end{aligned} \tag{3.5.3}$$

Therefore, we present the following conjecture.

Conjecture 3.5.2. *If $\alpha > \frac{n}{r}-1$, $\nu\in\mathbb{R}$, $d>0$, then*

$$\begin{aligned}&\frac{\widetilde{c_0}}{(2\pi)^n}\int_{\mathcal{S}^r} \phi_{\mathbf{m}}^{(d)}(e^{i\theta};\alpha,\nu)\overline{\phi_{\mathbf{n}}^{(d)}(e^{i\theta};\alpha,\nu)} \prod_{j=1}^{r} |(1-e^{i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2 \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d \, d\theta_1\cdots d\theta_r \\ &= d_{\mathbf{m}} \frac{\Gamma_\Omega(\alpha+\mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2} \delta_{\mathbf{mn}}.\end{aligned} \tag{3.5.4}$$

For some special case, we prove the conjecture.

Proposition 3.5.3. (1) *If $d=1,2,4$ or $r=2, d\in\mathbb{Z}_{>0}$ or $r=3, d=8$, then this conjecture is true.*
(2) *The case of $\alpha=\frac{n}{r}$ and $\nu=0$ is also true.*

Proof. (1) It follows immediately from Theorem 3.1.1 and the classification of irreducible symmetric cones.
(2) We remark that when $\alpha=\frac{n}{r}, \nu=0$,

$$\begin{aligned}\phi_{\mathbf{m}}^{(d)}\left(e^{i\theta};\frac{n}{r},0\right) &= d_{\mathbf{m}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \frac{P_{\mathbf{k}}^{(\frac{2}{d})}(1-e^{i\theta_1},\dots,1-e^{i\theta_r})}{P_{\mathbf{k}}^{(\frac{2}{d})}(1,\dots,1)} \\ &= d_{\mathbf{m}} \frac{P_{\mathbf{m}}^{(\frac{2}{d})}(e^{i\theta_1},\dots,e^{i\theta_r})}{P_{\mathbf{m}}^{(\frac{2}{d})}(1,\dots,1)}.\end{aligned} \tag{3.5.5}$$

For the Jack polynomial, the following formulas are known (see (6.4) in [OO] and (10.38) in [M] respectively).

$$P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1,\dots,1) = \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)} \prod_{1\le p<q\le r} \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p+1)\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p)\right)}, \tag{3.5.6}$$

$$\begin{aligned}\|P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}\|_{r,\frac{2}{d}}^2 &:= \frac{1}{(2\pi)^r}\frac{1}{r!}\int_{\mathcal{S}^r} |P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(e^{i\theta_1},\dots,e^{i\theta_r})|^2 \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d \, d\theta_1\cdots d\theta_r \\ &= \prod_{1\le p<q\le r} \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p+1)\right)\Gamma\left(m_p - m_q + \frac{d}{2}(q-p-1)+1\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p)\right)\Gamma\left(m_p - m_q + \frac{d}{2}(q-p)+1)\right)}.\end{aligned} \tag{3.5.7}$$

Hence, from (3.5.1), (3.5.6) and (3.5.7), we have

$$d_{\mathbf{m}}\|P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}\|_{r,\frac{2}{d}}^2 = \frac{\Gamma\left(\frac{d}{2}r\right)}{\Gamma\left(\frac{d}{2}\right)^r} P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1,\dots,1)^2. \tag{3.5.8}$$

Therefore, by (3.5.5) and the orthogonality of the Jack polynomial, we obtain

$$\begin{aligned}&\frac{\widetilde{c_0}}{(2\pi)^n}\int_{\mathcal{S}^r} \phi_{\mathbf{m}}^{(d)}\left(e^{i\theta};\frac{n}{r},0\right)\overline{\phi_{\mathbf{n}}^{(d)}\left(e^{i\theta};\frac{n}{r},0\right)} \prod_{1\le p<q\le r}|e^{i\theta_p}-e^{i\theta_q}|^d\,d\theta_1\cdots d\theta_r \\ &= \frac{1}{(2\pi)^n}\frac{(2\pi)^{\frac{n-r}{2}}}{r!}\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)}\right\}\frac{d_{\mathbf{m}}^2}{P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1,\dots,1)^2}(2\pi)^r r!\|P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}\|_{r,\frac{2}{d}}^2\delta_{\mathbf{mn}} \\ &= \frac{d_{\mathbf{m}}}{(2\pi)^{\frac{n-r}{2}}}\prod_{j=1}^{r-1}\frac{1}{\Gamma\left(\frac{d}{2}j\right)}\delta_{\mathbf{mn}} = d_{\mathbf{m}}\frac{1}{\Gamma_\Omega\left(\frac{n}{r}\right)}\delta_{\mathbf{mn}}.\end{aligned}$$

□

Although a proof of a general case would be desirable, our method in this thesis cannot be applied to a general case. It may be necessary to consider a method of quantum integrable systems, that is, to construct some commuting families of differential or pseudo-differential operators whose simultaneous eigenfunctions become MCJ polynomials. We are not aware of any studies of constructions of commuting families of pseudo-differential operators thus far. Hence, we think our conjecture is likely to be an important target of investigations into quantum integrable systems. Since multivariate Laguerre polynomials have been studied by using degenerate double affine Hecke algebra [Ka], we are also interested in similar algebraic treatment of MCJ polynomials which are the composition of modified Cayley and Fourier transforms of multivariate Laguerre polynomials.

Finally, we would like to raise the issue of applications for MCJ polynomials. In particular, since the weight function of the orthogonality relation for MCJ polynomials coincides with a circular Jacobi ensemble, we expect an application to the random matrix model whose density function is a circular Jacobi ensemble. However, further details on this are goals of future work.

Chapter 4

Multivariate Meixner, Charlier and Krawtchouk polynomials

The standard Meixner, Charlier and Krawtchouk polynomials of single discrete variable are defined by

$$
\begin{aligned}
M_m(x;\alpha,c) &:= {}_2F_1\left(\begin{matrix}-m,-x\\ \alpha\end{matrix};1-\frac{1}{c}\right)=\sum_{k=0}^{m}\frac{k!}{(\alpha)_k}\binom{m}{k}\binom{x}{k}\left(1-\frac{1}{c}\right)^k,\\
C_m(x;a) &:= {}_2F_0\left(\begin{matrix}-m,-x\\ -\end{matrix};-\frac{1}{a}\right)=\sum_{k=0}^{m}\binom{m}{k}\binom{x}{k}\left(-\frac{1}{a}\right)^k,\\
K_m(x;p,N) &:= {}_2F_1\left(\begin{matrix}-m,-x\\ -N\end{matrix};\frac{1}{p}\right)=\sum_{k=0}^{m}\frac{k!}{(-N)_k}\binom{m}{k}\binom{x}{k}\left(\frac{1}{p}\right)^k,
\end{aligned}
$$

respectively. These polynomials have been generalized to the multivariate case [DG], [Gr1], [Gr2], and [Il]. Although these multivariate discrete orthogonal polynomials are types written in Aomoto-Gelfand hypergeometric series, we introduce other types of multivariate Meixner, Charlier and Krawtchouk polynomials in this chapter, which are defined by generalized binomial coefficients. Moreover, we provide their fundamental properties, that is, duality, degenerate limits, generating functions, orthogonality relations, difference equations and recurrence formulas. The most basic result in these properties is Theorem 4.2.2, which states that the generating function of the generating functions for the multivariate Meixner polynomials

$$
\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\left\{\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z)\right\}\Phi_{\mathbf{x}}(w)
$$

coincides with the generating function for the multivariate Laguerre polynomials

$$
\sum_{\mathbf{m}\in\mathscr{P}} e^{\operatorname{tr} w}L_{\mathbf{m}}^{\left(\alpha-\frac{n}{r}\right)}\left(\left(\frac{1}{c}-1\right)w\right)\Phi_{\mathbf{m}}(z).
$$

Even though this result has not been known even for one variable, many properties for our multivariate discrete special orthogonal polynomials follow from this and the unitary picture (3.0.1).

In this chapter, we assume that $\mathbf{m}, \mathbf{n}, \mathbf{x}, \mathbf{y} \in \mathscr{P}, \alpha \in \mathbb{C}, 0 < c, p < 1, a > 0, N \in \mathbb{Z}_{\geq 0}$ unless otherwise specified.

4.1 Definitions

Definition 4.1.1. We define the multivariate Meixner, Charlier and Krawtchouk polynomials as follows.

$$M_{\mathbf{m}}(\mathbf{x};\alpha,c) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|} \tag{4.1.1}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{(\alpha)_{\mathbf{k}}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|} \tag{4.1.2}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}} (\alpha)_{\mathbf{k}}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|}, \tag{4.1.3}$$

$$C_{\mathbf{m}}(\mathbf{x};a) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \left(\frac{n}{r}\right)_{\mathbf{k}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(-\frac{1}{a}\right)^{|\mathbf{k}|} \tag{4.1.4}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho) \left(-\frac{1}{a}\right)^{|\mathbf{k}|} \tag{4.1.5}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \left(-\frac{1}{a}\right)^{|\mathbf{k}|}, \tag{4.1.6}$$

$$K_{\mathbf{m}}(\mathbf{x};p,N) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(-N)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{x}}{\mathbf{k}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} \quad (\mathbf{m} \subset N = (N,\ldots,N)) \tag{4.1.7}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{(-N)_{\mathbf{k}}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} \tag{4.1.8}$$

$$= \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(\mathbf{x}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}} (-N)_{\mathbf{k}}} \left(\frac{1}{p}\right)^{|\mathbf{k}|}. \tag{4.1.9}$$

When $r = 1$, these polynomials become the usual Meixner, Charlier and Krawtchouk polynomials. By the definition, we immediately obtain a duality property for these polynomials.

Proposition 4.1.2. (1) *For all* $\mathbf{m}, \mathbf{x} \in \mathscr{P}$, *we have*

$$M_{\mathbf{m}}(\mathbf{x};\alpha,c) = M_{\mathbf{x}}(\mathbf{m};\alpha,c). \tag{4.1.10}$$

(2) *For all* $\mathbf{m}, \mathbf{x} \in \mathscr{P}$, *we have*

$$C_{\mathbf{m}}(\mathbf{x}; a) = C_{\mathbf{x}}(\mathbf{m}; a). \tag{4.1.11}$$

(3) *For all* $N \supset \mathbf{m}, \mathbf{x} \in \mathscr{P}$, *we have*

$$K_{\mathbf{m}}(\mathbf{x}; p, N) = K_{\mathbf{x}}(\mathbf{m}; p, N). \tag{4.1.12}$$

We also obtain the following relations by the definitions.

Proposition 4.1.3. (1)

$$M_{\mathbf{m}}\left(\mathbf{x}; -N, \frac{p}{p-1}\right) = K_{\mathbf{m}}(\mathbf{x}; p, N). \tag{4.1.13}$$

(2)

$$\lim_{\alpha\to\infty} M_{\mathbf{m}}\left(\mathbf{x}; \alpha, \frac{a}{a+\alpha}\right) = C_{\mathbf{m}}(\mathbf{x}; a). \tag{4.1.14}$$

(3)

$$\lim_{N\to\infty} K_{\mathbf{m}}\left(\mathbf{x}; \frac{a}{N}, N\right) = C_{\mathbf{m}}(\mathbf{x}; a). \tag{4.1.15}$$

Actually, (1) follows from the definitions. For (2) and (3), we remark that

$$\lim_{\alpha\to\infty} \frac{\alpha^{|\mathbf{k}|}}{(\alpha)_{\mathbf{k}}} = \lim_{\alpha\to\infty} \prod_{j=1}^{r} \frac{\alpha^{k_j}}{\left(\alpha - \frac{d}{2}(j-1)\right)_{k_j}} = 1,$$

$$\lim_{N\to\infty} \frac{N^{|\mathbf{k}|}}{(-N)_{\mathbf{k}}} = \lim_{N\to\infty} \prod_{j=1}^{r} \frac{N^{k_j}}{\left(-N - \frac{d}{2}(j-1)\right)_{k_j}} = (-1)^{|\mathbf{k}|}.$$

4.2 Generating functions

In this chapter, we put

$$z = u_1 \sum_{j=1}^{r} a_j c_j, \quad w = u_2 \sum_{j=1}^{r} b_j c_j \in V^{\mathbb{C}},$$

with $u_1, u_2 \in U$, $a_1 \geq \cdots \geq a_r \geq 0$, $b_1 \geq \cdots \geq b_r \geq 0$ unless otherwise specified.

To consider some generating functions of generating functions for the above polynomials, we need to prove their convergences.

Lemma 4.2.1. (1) *If* $1 > a_1 \geq \cdots \geq a_r \geq 0$, $b_1 \geq \cdots \geq b_r \geq 0$, *then*

$$\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x}; \alpha, c) \Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \leq e^{rb_1\left(1+\frac{a_1}{1-a_1}\left(\frac{1}{c}-1\right)\right)} (1-a_1)^{-r(|\alpha|+2n)}. \tag{4.2.1}$$

(2) *For any* $z, w \in V^{\mathbb{C}}$, *we have*

$$\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x}, a) \Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \leq e^{r\left(a_1+b_1+\frac{a_1 b_1}{a}\right)}. \tag{4.2.2}$$

Proof. (1) By Lemma 2.1.5, Lemma 2.1.10 and Lemma 2.1.15,

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \\
&\leq \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}} (|\alpha|+d(r-1))_{\mathbf{k}}} \left(\frac{1}{c}-1\right)^{|\mathbf{k}|} \\
&\quad \cdot \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(b_1).
\end{aligned}$$

Moreover, from (2.1.38) and (2.1.39) of Theorem 2.1.16,

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) &= (|\alpha|+d(r-1))_{\mathbf{k}}(1-a_1)^{-r|\alpha|-dr(r-1)} \left(\frac{a_1}{1-a_1}\right)^{|\mathbf{k}|}, \\
\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(b_1) &= e^{rb_1} {b_1}^{|\mathbf{k}|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \\
&\leq e^{rb_1}(1-a_1)^{-r(|\alpha|+d(r-1))} \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}} \left(\left(\frac{1}{c}-1\right) \frac{a_1 b_1}{1-a_1} \right) \\
&= e^{rb_1\left(1+\frac{a_1}{1-a_1}\left(\frac{1}{c}-1\right)\right)} (1-a_1)^{-r(|\alpha|+d(r-1))} < \infty.
\end{aligned}$$

(2) By a similar argument,

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x},a)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(w) \right| \\
&\leq \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} a^{-|\mathbf{k}|} \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(b_1) \\
&= e^{r(a_1+b_1)} \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}} \left(\frac{a_1 b_1}{a} \right) \\
&= e^{r\left(a_1+b_1+\frac{a_1 b_1}{a}\right)} < \infty.
\end{aligned}$$

□

The following theorem is the key result in our theory.

Theorem 4.2.2. (1) *For $z \in \mathcal{D}, w \in V^{\mathbb{C}}, \alpha \in \mathbb{C}, 0 < c < 1$, we obtain*

$$\sum_{\mathbf{m}\in\mathscr{P}} e^{\operatorname{tr} w} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}\left(\left(\frac{1}{c}-1\right)w\right)\Phi_{\mathbf{m}}(z) = \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \cdot d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(w) \tag{4.2.3}$$

$$= \Delta(e-z)^{-\alpha}\int_K e^{(kw|(e-\frac{1}{c}z)(e-z)^{-1})}\,dk. \tag{4.2.4}$$

(2) *For $w, z \in V^{\mathbb{C}}, a > 0$, we obtain*

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} e^{\operatorname{tr} w}\Phi_{\mathbf{m}}\left(e-\frac{1}{a}w\right)\Phi_{\mathbf{m}}(z) = \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x};a)\Phi_{\mathbf{m}}(z) \cdot d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(w) \tag{4.2.5}$$

$$= e^{\operatorname{tr}(w+z)}\int_K e^{-\frac{1}{a}(kw|z)}\,dk. \tag{4.2.6}$$

Proof. (1) By the above lemma, the series converges absolutely under the conditions. Therefore, we derive

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\Phi_{\mathbf{x}}(w)\\
&= \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(z)\sum_{\mathbf{k}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k}}\frac{1}{(\alpha)_{\mathbf{k}}}\left(1-\frac{1}{c}\right)^{|\mathbf{k}|}\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}\gamma_{\mathbf{k}}(\mathbf{x}-\rho)\Phi_{\mathbf{x}}(w)\\
&= \sum_{\mathbf{m}\in\mathscr{P}} e^{\operatorname{tr} w} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k}}\frac{(-1)^{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}\left(\left(\frac{1}{c}-1\right)w\right)\Phi_{\mathbf{m}}(z)\\
&= \sum_{\mathbf{m}\in\mathscr{P}} e^{\operatorname{tr} w} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}\left(\left(\frac{1}{c}-1\right)w\right)\Phi_{\mathbf{m}}(z).
\end{aligned}$$

(4.2.4) follows from (2.2.2).
(2) Put $c = \frac{a}{a+\alpha}, w \to \frac{w}{\alpha}, a, \alpha \in \mathbb{R}_{>0}$ in (1) of Theorem 4.2.2 and take the limit of $\alpha \to \infty$. □

The generating functions of our polynomials are a corollary of the above theorem.

Theorem 4.2.3. (1) *For $z \in \mathcal{D}, \mathbf{x} \in \mathscr{P}, \alpha \in \mathbb{C}, 0 < c < 1$, we have*

$$\Delta(e-z)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right) = \sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{n}}(z). \tag{4.2.7}$$

(2) *For $z \in \mathcal{D}, \mathbf{x} \in \mathscr{P}, a > 0$, we have*

$$e^{\operatorname{tr} z}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}z\right) = \sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{n}}} C_{\mathbf{n}}(\mathbf{x};a)\Phi_{\mathbf{n}}(z). \tag{4.2.8}$$

(3) *For* $z \in \mathcal{D}, \mathbf{x} \in \mathscr{P}, 0 < p < 1$, *we have*

$$\Delta(e+z)^N \Phi_{\mathbf{x}}\left(\left(e - \frac{1-p}{p}z\right)(e+z)^{-1}\right) = \sum_{\mathbf{n}\subset N} \binom{N}{\mathbf{n}} K_{\mathbf{n}}(\mathbf{x}; p, N)\Phi_{\mathbf{n}}(z). \tag{4.2.9}$$

Proof. (1) We evaluate the spherical Taylor expansion of (4.2.4) with respect to w:

$$\begin{aligned}
\Phi_{\mathbf{x}}(\partial_w)\Delta(e-z)^{-\alpha}\int_K e^{(kw|(e-\frac{1}{c}z)(e-z)^{-1})}\,dk\Big|_{w=0} &= \Delta(e-z)^{-\alpha}\int_K \Phi_{\mathbf{x}}(\partial_w)e^{(w|k(e-\frac{1}{c}z)(e-z)^{-1})}|_{w=0}\,dk \\
&= \Delta(e-z)^{-\alpha}\int_K \Phi_{\mathbf{x}}\left(k\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right)\right)\,dk \\
&= \Delta(e-z)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right).
\end{aligned}$$

On the other hand, by (4.2.3),

$$\Phi_{\mathbf{x}}(\partial_w)\Delta(e-z)^{-\alpha}\int_K e^{(kw|(e-\frac{1}{c}z)(e-z)^{-1})}\,dk\Big|_{w=0} = \sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}}M_{\mathbf{n}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{n}}(z).$$

Therefore, we obtain the conclusion.
(2) The result is proved by a similar argument as in (1). That is, by (2) of Theorem 4.2.2, we have

$$\begin{aligned}
\sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{n}}}C_{\mathbf{n}}(\mathbf{x};a)\Phi_{\mathbf{n}}(z) &= \Phi_{\mathbf{x}}(\partial_w)e^{\mathrm{tr}\,(w+z)}\int_K e^{-\frac{1}{a}(kw|z)}\,dk\Big|_{w=0} \\
&= e^{\mathrm{tr}\,z}\int_K \Phi_{\mathbf{x}}(\partial_w)e^{(w|k(e-\frac{1}{a}z))}|_{w=0}\,dk \\
&= e^{\mathrm{tr}\,z}\int_K \Phi_{\mathbf{x}}\left(k\left(e-\frac{1}{a}z\right)\right)\,dk \\
&= e^{\mathrm{tr}\,z}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}z\right).
\end{aligned}$$

(3) By putting $\alpha = -N$ in (4.2.3), we have

$$\begin{aligned}
\Delta(e-z)^N\Phi_{\mathbf{x}}\left(\left(e-\frac{1}{c}z\right)(e-z)^{-1}\right) &= \sum_{\mathbf{n}\subset N} d_{\mathbf{n}}\frac{(-N)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}}M_{\mathbf{n}}(\mathbf{x};-N,c)\Phi_{\mathbf{n}}(z) \\
&= \sum_{\mathbf{n}\subset N}\binom{N}{\mathbf{n}}M_{\mathbf{n}}(\mathbf{x};-N,c)\Phi_{\mathbf{n}}(-z).
\end{aligned}$$

Since this series is a finite sum, we can take $c = \frac{p}{p-1}$ in the above. Therefore, we obtain

$$\begin{aligned}
\Delta(e-z)^N\Phi_{\mathbf{x}}\left(\left(e+\frac{1-p}{p}z\right)(e-z)^{-1}\right) &= \sum_{\mathbf{n}\subset N}\binom{N}{\mathbf{n}}M_{\mathbf{n}}\left(\mathbf{x};-N,\frac{p}{p-1}\right)\Phi_{\mathbf{n}}(-z) \\
&= \sum_{\mathbf{n}\subset N}\binom{N}{\mathbf{n}}K_{\mathbf{n}}\left(\mathbf{x};p,N\right)\Phi_{\mathbf{n}}(-z).
\end{aligned}$$

□

Next, we apply the unitary transformations in (3.0.1) to Theorem 4.2.2. Here, we also check convergence.

Lemma 4.2.4. (1) *Fix* $0 < c < 1$ *and let* $0 < \varepsilon < 1$ *and* $w, z \in \mathcal{D}$ *satisfy that*

$$\begin{aligned} &\left(c + (1-c)\frac{\varepsilon}{1-\varepsilon}\right)\left(1 + (1-c)\frac{\varepsilon}{1-c\varepsilon}\right) < 1, \\ &|\Phi_{\mathbf{m}}(w)|, |\Phi_{\mathbf{m}}(z)| < \Phi_{\mathbf{m}}(\varepsilon) = \varepsilon^{|\mathbf{m}|}. \end{aligned} \tag{4.2.10}$$

Then,

$$\begin{aligned} &\sum_{\mathbf{x},\mathbf{m},\mathbf{n}\in\mathscr{P}} \left| d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{n}} \frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{n}}(cw) \right| \\ &\quad < ((1-c)(1-2(1+c)\varepsilon + (4c-1)\varepsilon^2))^{-r|\alpha|-dr(r-1)}. \end{aligned} \tag{4.2.11}$$

(2) *Let* $z \in \mathcal{D}$ *satisfy that*

$$\sqrt{c} + \frac{1-c}{\sqrt{c}} \frac{a_1}{1-a_1} < 1, \quad |\Phi_{\mathbf{m}}(z)| \le {a_1}^{|\mathbf{m}|}. \tag{4.2.12}$$

Under this condition, the series

$$\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{x}}\left(-i\mathbf{s} - \frac{\alpha}{2}\right)\left(\frac{c(1-e^{2i\theta})}{e^{2i\theta}-c}\right)^{|\mathbf{x}|} \tag{4.2.13}$$

and

$$\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{\left(\frac{1}{2}\left(\alpha + \frac{n}{r}\right) + i\nu\right)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \Phi_{\mathbf{x}}((e-\sigma)(e-c\sigma)^{-1}) \tag{4.2.14}$$

converge absolutely for any $\mathbf{s} \in \mathbb{R}^r$ *and* $\sigma \in \Sigma$ *respectively.*

Proof. (1) By Lemma 2.1.5 and Lemma 2.1.15, we have

$$\begin{aligned} (\text{LHS}) \le &\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(|\alpha| + d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \\ &\cdot \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \frac{1}{(|\alpha| + d(r-1))_{\mathbf{k}}} \left(\frac{1}{c} - 1\right)^{|\mathbf{k}|} \sum_{\mathbf{l}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{l}} \frac{1}{(|\alpha| + d(r-1))_{\mathbf{l}}} \left(\frac{1}{c} - 1\right)^{|\mathbf{l}|} \\ &\cdot \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha| + d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m} - \rho)\Phi_{\mathbf{m}}(\varepsilon) \sum_{\mathbf{n}\in\mathscr{P}} d_{\mathbf{n}} \frac{(|\alpha| + d(r-1))_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} \gamma_{\mathbf{l}}(\mathbf{n} - \rho)\Phi_{\mathbf{n}}(c\varepsilon). \end{aligned}$$

Furthermore, from Lemma 2.1.16 and the definition of the generalized binomial coefficients (2.1.24), we derive

$$
\begin{aligned}
(\mathrm{LHS}) &\le ((1-\varepsilon)(1-c\varepsilon))^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(|\alpha|+d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \\
&\quad \cdot \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \left(\frac{1}{c}-1\right)^{|\mathbf{k}|} \Phi_{\mathbf{k}}\left(\frac{\varepsilon}{1-\varepsilon}\right) \sum_{\mathbf{l}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{l}} \left(\frac{1}{c}-1\right)^{|\mathbf{l}|} \Phi_{\mathbf{l}}\left(\frac{c\varepsilon}{1-c\varepsilon}\right) \\
&= ((1-\varepsilon)(1-c\varepsilon))^{-r|\alpha|-dr(r-1)} \\
&\quad \cdot \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(|\alpha|+d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}\left(\left(c+(1-c)\frac{\varepsilon}{1-\varepsilon}\right)\left(1+(1-c)\frac{\varepsilon}{1-c\varepsilon}\right)\right).
\end{aligned}
$$

Finally, by using the assumption and Lemma 2.1.16, we obtain

$$
\begin{aligned}
(\mathrm{LHS}) &\le \left((1-\varepsilon)(1-c\varepsilon)\left(1-\left(c+(1-c)\frac{\varepsilon}{1-\varepsilon}\right)\left(1+(1-c)\frac{\varepsilon}{1-c\varepsilon}\right)\right)\right)^{-r|\alpha|-dr(r-1)} \\
&= ((1-c)(1-2(1+c)\varepsilon+(4c-1)\varepsilon^2))^{-r|\alpha|-dr(r-1)}.
\end{aligned}
$$

(2) For (4.2.13), by a similar argument as that above, we obtain

$$
\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{x}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\left(\frac{c(1-e^{2i\theta})}{e^{2i\theta}-c}\right)^{|\mathbf{x}|} \right| \\
&\le \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left|\gamma_{\mathbf{x}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\right| \left(\frac{2c}{1+c}\right)^{|\mathbf{x}|} \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \frac{1}{(|\alpha|+d(r-1))_{\mathbf{k}}} \left(\frac{1}{c}-1\right)^{|\mathbf{k}|} \\
&\quad \cdot \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \\
&\le (1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left|\gamma_{\mathbf{x}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\right| c^{\frac{|\mathbf{x}|}{2}} \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \Phi_{\mathbf{k}}\left(\left(\frac{1}{c}-1\right)\frac{a_1}{1-a_1}\right) \\
&\le (1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left|\gamma_{\mathbf{x}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\right| \Phi_{\mathbf{x}}\left(\sqrt{c}+\frac{1-c}{\sqrt{c}}\frac{a_1}{1-a_1}\right).
\end{aligned}
$$

Finally, by Lemma 2.1.14,

$$
\begin{aligned}
&\le C_1 + C_2(1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(|\alpha|+d(r-1))_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}\left(\sqrt{c}+\frac{1-c}{\sqrt{c}}\frac{a_1}{1-a_1}\right) \\
&\le C_1 + C_2\left((1-\sqrt{c})\left(1-a_1-\left(1+\frac{1}{\sqrt{c}}\right)a_1\right)\right)^{-r|\alpha|-dr(r-1)}.
\end{aligned}
$$

For (4.2.14), by using $\sigma = \sum_{j=1}^{r} e^{i\theta_j} c_j$ and

$$
|(e-\sigma)(e-c\sigma)^{-1})| \le \sum_{j=1}^{r} \left|\frac{1-e^{i\theta_j}}{1-ce^{i\theta_j}}\right| c_j \le \sum_{j=1}^{r} \frac{2}{1+c} c_j = \frac{2}{1+c} \le \frac{1}{\sqrt{c}},
$$

we have

$$
\begin{aligned}
&\sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} \left| d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|}\Phi_{\mathbf{x}}((e-\sigma)(e-c\sigma)^{-1}) \right| \\
&\quad \le \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{\left(\frac{1}{2}\left(|\alpha|+\frac{n}{r}\right)+|\nu|+d(r-1)\right)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(\sqrt{c}) \\
&\qquad \cdot \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \frac{1}{(|\alpha|+d(r-1))_{\mathbf{k}}} \left(\frac{1}{c}-1\right)^{|\mathbf{k}|} \sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1) \\
&\quad = (1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{\left(\frac{1}{2}\left(|\alpha|+\frac{n}{r}\right)+|\nu|+d(r-1)\right)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}(\sqrt{c}) \\
&\qquad \cdot \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} \Phi_{\mathbf{k}}\left(\left(\frac{1}{c}-1\right)\frac{a_1}{1-a_1}\right) \\
&\quad = (1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{\left(\frac{1}{2}\left(|\alpha|+\frac{n}{r}\right)+|\nu|+d(r-1)\right)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \Phi_{\mathbf{x}}\left(\sqrt{c}+\frac{1-c}{\sqrt{c}}\frac{a_1}{1-a_1}\right) \\
&\quad = (1-a_1)^{-r|\alpha|-dr(r-1)} \left((1-\sqrt{c})\left(1-\left(1+\frac{1}{\sqrt{c}}\right)\frac{a_1}{1-a_1}\right)\right) < \infty.
\end{aligned}
$$

□

From this lemma, we can consider the following generating functions.

Theorem 4.2.5. (1) *For $z \in \mathcal{D}, u \in V^{\mathbb{C}}$, we obtain*

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} \psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z) &= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left(\frac{2c}{1-c}\right)^{|\mathbf{x}|} e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}}(u) \qquad (4.2.15) \\
&= \Delta(e-z)^{-\alpha} \int_K e^{-(ku|(e+z)(e-z)^{-1})}\, dk.
\end{aligned}
$$

(2) *Fixed $0 < c < 1$ and assume that $w, z \in \mathcal{D}$ satisfy the condition in* (1) *of Lemma 4.2.4. We obtain*

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= (1-c)^{r\alpha} \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|}\Delta(e-cw)^{-\alpha}\Phi_{\mathbf{x}}((e-w)(e-cw)^{-1}) \qquad (4.2.16) \\
&= \Delta(z)^{-\alpha} \int_K \Delta(kz^{-1}-w)^{-\alpha}\, dk.
\end{aligned}
$$

(3) *Fixed* $0 < c < 1$ *and assume that* $z \in \mathcal{D}$ *satisfy the condition in* (2) *of Lemma 4.2.4. For* $\mathbf{s} \in \mathbb{R}^r, 0 < \theta < 2\pi$, *we obtain*

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} q_{\mathbf{m}}^{(\alpha)}(\mathbf{s};\theta)\Phi_{\mathbf{m}}(z) &= \left(\frac{1-c}{1-ce^{-2i\theta}}\right)^{-i|\mathbf{s}|-\frac{\alpha}{2}r} \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\ &\quad \cdot d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{x}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\left(\frac{c(1-e^{2i\theta})}{e^{2i\theta}-c}\right)^{|\mathbf{x}|} \\ &= \Delta(e-z)^{-\alpha}\varphi_{i\mathbf{s}+\frac{\alpha}{2}}((e-e^{-2i\theta}z)^{-1}(e-z)).\end{aligned} \tag{4.2.17}$$

(4) *Fixed* $0 < c < 1$ *and assume that* $z \in \mathcal{D}$ *satisfy the condition in* (2) *of Lemma 4.2.4. For* $\sigma \in \Sigma$, *we obtain*

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)\Phi_{\mathbf{m}}(z) &= (1-c)^{\frac{r}{2}\left(\alpha+\frac{n}{r}\right)+ir\nu} \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\ &\quad \cdot d_{\mathbf{x}} \frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|}\Delta(e-c\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\Phi_{\mathbf{x}}((e-\sigma)(e-c\sigma)^{-1}) \\ &= \Delta(e-z)^{-\alpha}\int_K \Delta((e-z)^{-1}-(z(e-z)^{-1})k\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk.\end{aligned} \tag{4.2.18}$$

(5) *For* $w, z \in V^{\mathbb{C}}, a > 0$, *we obtain*

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= e^{-ra} \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{a^{|\mathbf{m}|}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x};a)\Phi_{\mathbf{m}}(z) \\ &\quad \cdot d_{\mathbf{x}} \frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} e^{\operatorname{tr} w}\Phi_{\mathbf{x}}\left(e-\frac{1}{a}w\right) \\ &= e^{\operatorname{tr} w}\int_K e^{-a(kw|e-z)}\,dk.\end{aligned} \tag{4.2.19}$$

Proof. As (1) and (5) follow immediately from Theorem 4.2.2, we only prove (2), (3) and (4).

First, we remark that the right hand sides of (4.2.16), (4.2.17) and (4.2.18) converge absolutely under the conditions in Lemma 4.2.4. Thus, by analytic continuation, it suffices to show these equations when $a_1 < \frac{1}{3}$. Moreover, we also remark that since (2.2.4)

$$\sum_{\mathbf{m}\in\mathscr{P}} |\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)| \le (1-a_1)^{-r|\alpha|-dr(r-1)}e^{-\frac{1-3a_1}{1-a_1}},$$

the exchange of unitary transformations $\mathcal{L}_\alpha$, $\mathcal{M}_{\alpha,\theta}$ and $\mathcal{F}_{\alpha,\nu}^{-1}$, and the summation are justified under these restrictions. Therefore, to obtain the results, we apply the unitary transforms to both sides of (4.2.15). We try to perform these calculations.

For (2), we apply transform $C_\alpha^{-1} \circ \mathcal{L}_\alpha$ to both sides of (4.2.15). From Lemma 2.1.8, we have

$$\begin{aligned}\mathcal{L}_\alpha(e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}})(z) &= \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-\left(\frac{1+c}{1-c}e+z\middle|u\right)}\Phi_{\mathbf{x}}(u)\Delta(u)^{\alpha-\frac{n}{r}}\,du \\ &= 2^{r\alpha}(\alpha)_{\mathbf{x}}\Delta\left(\frac{1+c}{1-c}e+z\right)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(\frac{1+c}{1-c}e+z\right)^{-1}\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}C_\alpha^{-1}\circ\mathcal{L}_\alpha(e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}})(w) &= 2^{r\alpha}(\alpha)_{\mathbf{x}}C_\alpha^{-1}\left(\Delta\left(\frac{1+c}{1-c}e+z\right)^{-\alpha}\Phi_{\mathbf{x}}\left(\left(\frac{1+c}{1-c}e+z\right)^{-1}\right)\right)(w) \\ &= (\alpha)_{\mathbf{x}}\Delta(e-w)^{-\alpha}\Delta\left(\frac{1}{2}\left((e+w)(e-w)^{-1}+\frac{1+c}{1-c}e\right)\right)^{-\alpha} \\ &\quad\cdot\Phi_{\mathbf{x}}\left(\left((e+w)(e-w)^{-1}+\frac{1+c}{1-c}e\right)^{-1}\right) \\ &= (\alpha)_{\mathbf{x}}(1-c)^{r\alpha}\Delta(e-cw)^{-\alpha}\Phi_{\mathbf{x}}\left(\frac{1-c}{2}(e-w)(e-cw)^{-1}\right).\end{aligned}$$

Hence, the right-hand side of (4.2.15) becomes the right-hand side of (4.2.16). Therefore, since $C_\alpha^{-1}\circ\mathcal{L}_\alpha(\psi_{\mathbf{m}^{(\alpha)}})(w) = d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(w)$, we obtain the conclusion.

By similar arguments, for (3) and (4), we only need to evaluate the following calculations.

$$\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\mathcal{M}_{\alpha,\theta}(e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}})(\mathbf{s}) = \left(\frac{1-c}{1-ce^{-2i\theta}}\right)^{-i|\mathbf{s}|-\frac{\alpha}{2}r}\gamma_{\mathbf{x}}\left(-i\mathbf{s}-\frac{\alpha}{2}\right)\left(\frac{c(1-e^{2i\theta})}{e^{2i\theta}-c}\right)^{|\mathbf{x}|},$$

$$\begin{aligned}\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\mathcal{C}_{\alpha,\nu}^{-1}\circ\mathcal{F}_{\alpha,\nu}^{-1}(e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}})(\sigma) &= (1-c)^{\frac{r}{2}\left(\alpha+\frac{n}{r}\right)+ir\nu}\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{x}} \\ &\quad\cdot c^{|\mathbf{x}|}\Delta(e-c\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\Phi_{\mathbf{x}}((e-\sigma)(e-c\sigma)^{-1})\end{aligned}$$

These follow from Lemma 2.1.12 and Lemma 2.1.8. □

4.3 Orthogonality relations

We provide the orthogonality relations for our discrete orthogonal polynomials as a corollary of Theorem 4.2.5.

Theorem 4.3.1. (1) *For $\alpha > \frac{n}{r}-1, 0<c<1$, we obtain*

$$\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}}\frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}}c^{|\mathbf{x}|}M_{\mathbf{m}}(\mathbf{x};\alpha,c)M_{\mathbf{n}}(\mathbf{x};\alpha,c) = \frac{c^{-|\mathbf{m}|}}{(1-c)^{r\alpha}}\frac{1}{d_{\mathbf{m}}}\frac{\left(\frac{n}{r}\right)_{\mathbf{m}}}{(\alpha)_{\mathbf{m}}}\delta_{\mathbf{m},\mathbf{n}} \geq 0. \tag{4.3.1}$$

(2) *For $a > 0$, we obtain*

$$\sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} C_{\mathbf{m}}(\mathbf{x};a) C_{\mathbf{n}}(\mathbf{x};a) = a^{-|\mathbf{m}|} e^{ra} \frac{\left(\frac{n}{r}\right)_{\mathbf{m}}}{d_{\mathbf{m}}} \delta_{\mathbf{m},\mathbf{n}} \geq 0. \tag{4.3.2}$$

(3) *For $0 < p < 1$, we obtain*

$$\sum_{\mathbf{x}\subset N} \binom{N}{\mathbf{x}} p^{|\mathbf{x}|}(1-p)^{rN-|\mathbf{x}|} K_{\mathbf{m}}(\mathbf{x};p,N) K_{\mathbf{n}}(\mathbf{x};p,N) = \left(\frac{1-p}{p}\right)^{|\mathbf{m}|} \binom{N}{\mathbf{m}}^{-1} \delta_{\mathbf{m},\mathbf{n}} \geq 0. \tag{4.3.3}$$

Proof. (1) From (4.2.16) and (4.2.7), we have

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= (1-c)^{r\alpha} \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \Delta(e-cw)^{-\alpha} \Phi_{\mathbf{x}}((e-w)(e-cw)^{-1}) \\
&= \sum_{\mathbf{m},\mathbf{n}\in\mathscr{P}} (1-c)^{r\alpha} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} d_{\mathbf{n}} \frac{(\alpha)_{\mathbf{n}}}{\left(\frac{n}{r}\right)_{\mathbf{n}}} c^{|\mathbf{n}|} \\
&\quad \cdot \left\{ \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} M_{\mathbf{m}}(\mathbf{x};\alpha,c) M_{\mathbf{n}}(\mathbf{x};\alpha,c) \right\} \Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w).
\end{aligned}$$

Therefore, by comparing the coefficients of $\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w)$ on both sides of this equation, we obtain (4.3.1).
(2) From (4.2.19) and (4.2.8), we derive

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) &= e^{-ra} \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{a^{|\mathbf{m}|}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} C_{\mathbf{m}}(\mathbf{x};a)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot d_{\mathbf{x}} \frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} e^{\operatorname{tr} w} \Phi_{\mathbf{x}}\left(e - \frac{1}{a}w\right) \\
&= \sum_{\mathbf{m},\mathbf{n}\in\mathscr{P}} e^{-ra} d_{\mathbf{m}} \frac{a^{|\mathbf{m}|}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} d_{\mathbf{n}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{n}}} \\
&\quad \cdot \left\{ \sum_{\mathbf{x}\in\mathscr{P}} d_{\mathbf{x}} \frac{a^{|\mathbf{x}|}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} C_{\mathbf{m}}(\mathbf{x};a) C_{\mathbf{n}}(\mathbf{x};a) \right\} \Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w).
\end{aligned}$$

Then, by comparing the coefficients of $\Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w)$, we have the conclusion.

(3) In (4.2.16), taking $\alpha = -N$, one has

$$\begin{aligned}\sum_{\mathbf{m}\subset N} d_{\mathbf{m}} \frac{(-N)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(-z) &= \sum_{\mathbf{m}\subset N} \binom{N}{\mathbf{m}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) \\ &= (1-c)^{-rN} \sum_{\mathbf{x},\mathbf{m}\subset N} d_{\mathbf{m}} \frac{(-N)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x}; -N, c)\Phi_{\mathbf{m}}(-z) \\ &\quad \cdot d_{\mathbf{x}} \frac{(-N)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} c^{|\mathbf{x}|} \Delta(e-cw)^N \Phi_{\mathbf{x}}((e-w)(e-cw)^{-1}).\end{aligned}$$

The first equality follows from (2.1.31). Since the above sum is finite, we can put $c = \frac{p}{p-1}$, $(0 < p < 1)$. Hence,

$$\begin{aligned}\sum_{\mathbf{m}\subset N} \binom{N}{\mathbf{m}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) = (1-p)^{rN} \sum_{\mathbf{x},\mathbf{m}\subset N} \binom{N}{\mathbf{m}} K_{\mathbf{m}}(\mathbf{x}; p, N)\Phi_{\mathbf{m}}(z) \binom{N}{\mathbf{x}} \left(\frac{p}{1-p}\right)^{|\mathbf{x}|} \\ \cdot \Delta\left(e + \frac{p}{1-p}w\right)^N \Phi_{\mathbf{x}}\left((e-w)\left(e+\frac{p}{1-p}w\right)^{-1}\right).\end{aligned}$$

From (4.2.9), we have

$$\Delta\left(e + \frac{p}{1-p}w\right)^N \Phi_{\mathbf{x}}\left((e-w)\left(e+\frac{p}{1-p}w\right)^{-1}\right) = \sum_{\mathbf{n}\subset N} \binom{N}{\mathbf{n}} K_{\mathbf{n}}(\mathbf{x}; p, N) \left(\frac{p}{1-p}\right)^{|\mathbf{n}|} \Phi_{\mathbf{n}}(w).$$

Therefore,

$$\begin{aligned}\sum_{\mathbf{m}\subset N} \binom{N}{\mathbf{m}} \Phi_{\mathbf{m}}(w)\Phi_{\mathbf{m}}(z) = \sum_{\mathbf{m},\mathbf{n}\subset N} \binom{N}{\mathbf{m}} \left(\frac{p}{1-p}\right)^{|\mathbf{n}|} \binom{N}{\mathbf{n}} \\ \cdot \left\{\sum_{\mathbf{x}\subset N} \binom{N}{\mathbf{x}} p^{|\mathbf{x}|}(1-p)^{rN-|\mathbf{x}|} K_{\mathbf{m}}(\mathbf{x}; p, N) K_{\mathbf{n}}(\mathbf{x}; p, N)\right\} \Phi_{\mathbf{m}}(z)\Phi_{\mathbf{n}}(w).\end{aligned}$$

□

4.4 Difference equations and recurrence relations

In this section, we derive the difference equations and recurrence formulas for our polynomials from (2.2.6), Lemma 2.2.3 and (1) of Theorem 4.2.5.

Theorem 4.4.1. (1) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}, \alpha \in \mathbb{C}, c \in \mathbb{C}^*$, *we have*

$$\begin{aligned} d_{\mathbf{x}}(c-1)|\mathbf{m}|M_{\mathbf{m}}(\mathbf{x};\alpha,c) = &\sum_{j=1}^{r} d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)\left(x_j+\alpha-\frac{d}{2}(j-1)\right)cM_{\mathbf{m}}(\mathbf{x}+\epsilon_j;\alpha,c) \\ &-\sum_{j=1}^{r} d_{\mathbf{x}}(x_j+(x_j+\alpha)c)M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\ &+\sum_{j=1}^{r} d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\left(x_j+\frac{d}{2}(r-j)\right)M_{\mathbf{m}}(\mathbf{x}-\epsilon_j;\alpha,c). \quad (4.4.1) \end{aligned}$$

(2) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}, a \in \mathbb{C}^*$, *we have*

$$\begin{aligned} -d_{\mathbf{x}}|\mathbf{m}|C_{\mathbf{m}}(\mathbf{x};a) = &\sum_{j=1}^{r} d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)aC_{\mathbf{m}}(\mathbf{x}+\epsilon_j;a) \\ &-\sum_{j=1}^{r} d_{\mathbf{x}}(x_j+a)C_{\mathbf{m}}(\mathbf{x};a) \\ &+\sum_{j=1}^{r} d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\left(x_j+\frac{d}{2}(r-j)\right)C_{\mathbf{m}}(\mathbf{x}-\epsilon_j;a). \quad (4.4.2) \end{aligned}$$

(3) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}, p \in \mathbb{C}^*$, *we have*

$$\begin{aligned} -d_{\mathbf{x}}|\mathbf{m}|K_{\mathbf{m}}(\mathbf{x};p,N) = &\sum_{j=1}^{r} d_{\mathbf{x}+\epsilon_j}\tilde{a}_j(-\mathbf{x}-\epsilon_j)\left(N-x_j+\frac{d}{2}(j-1)\right)pK_{\mathbf{m}}(\mathbf{x}+\epsilon_j;p,N) \\ &-\sum_{j=1}^{r} d_{\mathbf{x}}(p(N-x_j)+x_j(1-p))K_{\mathbf{m}}(\mathbf{x};p,N) \\ &+\sum_{j=1}^{r} d_{\mathbf{x}-\epsilon_j}\tilde{a}_j(\mathbf{x}-\epsilon_j)\left(x_j+\frac{d}{2}(r-j)\right)(1-p)K_{\mathbf{m}}(\mathbf{x}-\epsilon_j;p,N). \end{aligned} \quad (4.4.3)$$

Proof. (1) Let us apply operator $\frac{c-1}{2}e^{\frac{1+c}{1-c}\operatorname{tr} u}D_{\alpha}^{(3)}$ to both sides of (4.2.15). Since $D_{\alpha}^{(3)}\psi_{\mathbf{m}}^{(\alpha)}(u) = 2|\mathbf{m}|\psi_{\mathbf{m}}^{(\alpha)}(u)$, we have

$$\begin{aligned} \frac{c-1}{2}e^{\frac{1+c}{1-c}\operatorname{tr} u}D_{\alpha}^{(3)}\left(\sum_{\mathbf{m}\in\mathscr{P}}\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z)\right) &= \sum_{\mathbf{m}\in\mathscr{P}}(c-1)e^{\frac{1+c}{1-c}\operatorname{tr} u}|\mathbf{m}|\psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z) \\ &= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\left(\frac{2c}{1-c}\right)^{|\mathbf{x}|}\Phi_{\mathbf{x}}(u)\Phi_{\mathbf{m}}(z) \\ &\quad \cdot d_{\mathbf{x}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}}(c-1)|\mathbf{m}|M_{\mathbf{m}}(\mathbf{x};\alpha,c). \end{aligned}$$

On the other hand, by (2.2.12), we have

$$
\begin{aligned}
&\frac{c-1}{2} e^{\frac{1+c}{1-c}\operatorname{tr} u} D_\alpha^{(3)} \left(\sum_{\mathbf{m}\in\mathscr{P}} \psi_{\mathbf{m}}^{(\alpha)}(u)\Phi_{\mathbf{m}}(z) \right) \\
&= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left(\frac{2c}{1-c}\right)^{|\mathbf{x}|} \frac{c-1}{2} e^{\frac{1+c}{1-c}\operatorname{tr} u} D_\alpha^{(3)} (e^{-\frac{1+c}{1-c}\operatorname{tr} u}\Phi_{\mathbf{x}}(u)) \\
&= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} M_{\mathbf{m}}(\mathbf{x};\alpha,c)\Phi_{\mathbf{m}}(z) d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \left(\frac{2c}{1-c}\right)^{|\mathbf{x}|} \\
&\quad \cdot \left\{ \frac{2c}{1-c} \sum_{j=1}^r \tilde{a}_j(\mathbf{x})\Phi_{\mathbf{x}+\epsilon_j}(u) - \sum_{j=1}^r (x_j + (x_j+\alpha)c)\Phi_{\mathbf{x}}(u) \right. \\
&\quad \left. + \frac{1-c}{2} \sum_{j=1}^r \left(x_j + \frac{d}{2}(r-j)\right)\left(x_j+\alpha-1-\frac{d}{2}(j-1)\right) \tilde{a}_j(-\mathbf{x})\Phi_{\mathbf{x}-\epsilon_j}(u) \right\} \\
&= \sum_{\mathbf{x},\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \left(\frac{2c}{1-c}\right)^{|\mathbf{x}|} \Phi_{\mathbf{x}}(u)\Phi_{\mathbf{m}}(z) \\
&\quad \cdot \left\{ \sum_{j=1}^r d_{\mathbf{x}+\epsilon_j} \tilde{a}_j(-\mathbf{x}-\epsilon_j) \frac{x_j+1+\frac{d}{2}(r-j)}{\left(\frac{n}{r}\right)_{\mathbf{x}+\epsilon_j}} \left(x_j+\alpha-\frac{d}{2}(j-1)\right) M_{\mathbf{m}}(\mathbf{x}+\epsilon_j;\alpha,c) \right. \\
&\quad - \sum_{j=1}^r d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} (x_j+(x_j+\alpha)c) M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\
&\quad \left. + \sum_{j=1}^r d_{\mathbf{x}-\epsilon_j} \tilde{a}_j(\mathbf{x}-\epsilon_j) \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}-\epsilon_j}} M_{\mathbf{m}}(\mathbf{x}-\epsilon_j;\alpha,c) \right\}.
\end{aligned}
$$

Finally, the conclusion is obtained by

$$
\left(\frac{n}{r}\right)_{\mathbf{x}+\epsilon_j} = \left(x_j + 1 + \frac{d}{2}(r-j)\right)\left(\frac{n}{r}\right)_{\mathbf{x}}
$$

and comparing the coefficients in the above.
(2) Put $c = \frac{a}{a+\alpha}$ in (4.4.1) and take the limit as $\alpha \to \infty$. Then, by (4.1.14), we have the conclusion.
(3) Put $c = \frac{p}{p-1}, \alpha = -N$ and multiply $1-p$ in (4.4.1). Then, by (4.1.13), we have the conclusion. □

The recurrence formulas follow immediately from Theorem 4.4.1 and Proposition 4.1.2.

Theorem 4.4.2. (1) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}, \alpha \in \mathbb{C}, c \in \mathbb{C}^*$, *we have*

$$\begin{aligned} d_{\mathbf{m}}(c-1)|\mathbf{x}|M_{\mathbf{m}}(\mathbf{x};\alpha,c) = &\sum_{j=1}^{r} d_{\mathbf{m}+\epsilon_j}\tilde{a}_j(-\mathbf{m}-\epsilon_j)\left(m_j+\alpha-\frac{d}{2}(j-1)\right)cM_{\mathbf{m}+\epsilon_j}(\mathbf{x};\alpha,c) \\ &-\sum_{j=1}^{r} d_{\mathbf{m}}(m_j+(m_j+\alpha)c)M_{\mathbf{m}}(\mathbf{x};\alpha,c) \\ &+\sum_{j=1}^{r} d_{\mathbf{m}-\epsilon_j}\tilde{a}_j(\mathbf{m}-\epsilon_j)\left(m_j+\frac{d}{2}(r-j)\right)M_{\mathbf{m}-\epsilon_j}(\mathbf{x};\alpha,c). \end{aligned} \tag{4.4.4}$$

(2) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}, a \in \mathbb{C}^*$, *we have*

$$\begin{aligned} -d_{\mathbf{m}}|\mathbf{x}|C_{\mathbf{m}}(\mathbf{x};a) = &\sum_{j=1}^{r} d_{\mathbf{m}+\epsilon_j}\tilde{a}_j(-\mathbf{m}-\epsilon_j)aC_{\mathbf{m}+\epsilon_j}(\mathbf{x};a) \\ &-\sum_{j=1}^{r} d_{\mathbf{m}}(m_j+a)C_{\mathbf{m}}(\mathbf{x};a) \\ &+\sum_{j=1}^{r} d_{\mathbf{m}-\epsilon_j}\tilde{a}_j(\mathbf{m}-\epsilon_j)\left(m_j+\frac{d}{2}(r-j)\right)C_{\mathbf{m}-\epsilon_j}(\mathbf{x};a). \end{aligned} \tag{4.4.5}$$

(3) *For* $\mathbf{x}, \mathbf{m} \in \mathscr{P}, p \in \mathbb{C}^*$, *we have*

$$\begin{aligned} -d_{\mathbf{m}}|\mathbf{x}|K_{\mathbf{m}}(\mathbf{x};p,N) = &\sum_{j=1}^{r} d_{\mathbf{m}+\epsilon_j}\tilde{a}_j(-\mathbf{m}-\epsilon_j)\left(N-m_j+\frac{d}{2}(j-1)\right)pK_{\mathbf{m}+\epsilon_j}(\mathbf{x};p,N) \\ &-\sum_{j=1}^{r} d_{\mathbf{m}}(p(N-m_j)+m_j(1-p))K_{\mathbf{m}}(\mathbf{x};p,N) \\ &+\sum_{j=1}^{r} d_{\mathbf{m}-\epsilon_j}\tilde{a}_j(\mathbf{m}-\epsilon_j)\left(m_j+\frac{d}{2}(r-j)\right)(1-p)K_{\mathbf{m}-\epsilon_j}(\mathbf{x};p,N). \end{aligned} \tag{4.4.6}$$

4.5 Concluding remarks

Interesting problems remain that are related to multivariate Meixner, Charlier and Krawtchouk polynomials. First, we may consider a generalization of our discrete orthogonal polynomials for an arbitrary real value of multiplicity d, which is similar to Conjecture 3.5.2.

Definition 4.5.1. Using the notations of Conjecture 3.5.2, we define the generalized multi-

variate Meixner, Charlier and Krawtchouk polynomials by

$$M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \binom{\mathbf{x}}{\mathbf{k}}_{\frac{d}{2}} \left(1-\frac{1}{c}\right)^{|\mathbf{k}|}, \tag{4.5.1}$$

$$C_{\mathbf{m}}^{(d)}(\mathbf{x};a) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \left(\frac{n}{r}\right)_{\mathbf{k}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \binom{\mathbf{x}}{\mathbf{k}}_{\frac{d}{2}} \left(-\frac{1}{a}\right)^{|\mathbf{k}|}, \tag{4.5.2}$$

$$K_{\mathbf{m}}^{(d)}(\mathbf{x};p,N) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{d_{\mathbf{k}}} \frac{\left(\frac{n}{r}\right)_{\mathbf{k}}}{(-N)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \binom{\mathbf{x}}{\mathbf{k}}_{\frac{d}{2}} \left(\frac{1}{p}\right)^{|\mathbf{k}|} \quad (\mathbf{m}\subset N=(N,\dots,N)). \tag{4.5.3}$$

By the definitions, Proposition 4.1.2 and 4.1.3 also hold for the generalized multivariate Meixner, Charlier and Krawtchouk polynomials. Therefore, we think the following conjecture is natural.

Conjecture 4.5.2. *Generating functions, orthogonality, difference equations and recurrence formulas also hold for the generalized multivariate Meixner, Charlier and Krawtchouk polynomials, as in Theorems 4.2.3, 4.3.1, 4.4.1 and 4.4.2 respectively. Here, we consider* $\Delta(e-z)=(1-z_1)\cdots(1-z_r)$.

We remark that when $d=1,2,4$ or $r=2, d\in\mathbb{Z}_{>0}$ or $r=3, d=8$, this conjecture is proved in the same way as Conjecture 3.5.2. However, it may be necessary to consider some algebraic treatment to prove the general case. In particular, since the difference equation for the multivariate Meixner polynomials is equivalent to the differential equation for the multivariate Laguerre polynomials which is explained by the degenerate double affine Hecke algebra [Ka], we expect the existence of a particular algebraic structure related to this algebra for our polynomials. Once we obtain such an interpretation, we may not only succeed in proving the above conjecture but also in providing further generalizations of our polynomials associated with root systems.

It is also valuable to give a group theoretic picture of our multivariate discrete orthogonal polynomials. In the one variable case, there are many geometric interpretations for these polynomials [VK1], [VK2]. Moreover, for the multivariate case for the Aomoto-Gelfand hypergeometric series, such group theoretic interpretations have recently been studied [GVZ], [GMVZ]. On the other hand, since our multivariate discrete orthogonal polynomials have many rich properties which are generalizations of the one variable case, they are considered to be a good multivariate analogue of the Meixner, Charlier and Krawtchouk polynomials. Hence, for our multivariate discrete orthogonal polynomials, it seems that there is some group theoretic interpretation as some matrix elements or some spherical functions etc. We are also interested in a connection between our multivariate discrete orthogonal polynomials and the Aomoto-Gelfand type.

We are interested in whether we can apply our method to other discrete orthogonal polynomials, for example, the Hahn polynomial which is a special orthogonal polynomial in

the Askey scheme [KLS],

$$
\begin{aligned}
Q_m(x;\alpha,\beta,N) &= {}_3F_2\left(\begin{matrix}-m, m+\alpha+\beta+1, -x\\ \alpha+1, -N\end{matrix};1\right)\\
&= \sum_{k=0}^{m} \frac{k!}{(-N)_k}\frac{(m+\alpha+\beta+1)_k}{(\alpha+1)_k}\binom{m}{k}\binom{x}{k} \quad (m=0,1,\cdots,N).
\end{aligned}
$$

Namely, by considering "some generating functions of the generating functions" for these discrete orthogonal polynomials, we expect to obtain correspondence between the Hahn polynomials and other orthogonal polynomials, for example, the Jacobi polynomials.

Finally, we would like to raise the issue of applications of our multivariate Meixner, Charlier and Krawtchouk polynomials. The standard Meixner, Charlier and Krawtchouk polynomials of single discrete variable have found numerous applications in combinatorics, stochastic processes, probability theory and mathematical physics (for their reference, see the introduction in [GMVZ]). Hence, we hope that our multivariate polynomials can be applied to various situations and we intend to investigate these in research tasks in the future.

Appendix A

Operator orderings and Meixner-Pollaczek polynomials

The aim of this chapter is to give some identities which are generalizations of the formulas given by Koornwinder [Ko] and Hamdi-Zeng [HZ]. Our proofs are much simpler than and different from the previous investigations. This chapter is based on [Shi].

A.1 Introduction

Let W be the Weyl algebra generated by p and q with the relation $[p,q] := pq - qp = 1$. In this paper, we prove the following theorems.

Theorem A.1.1. *We put $T := pq + qp$. We obtain*

$$\begin{aligned} 2^n \sum_{k=0}^{m} \binom{m}{k} p^k q^n p^{m-k} &= 2^m \sum_{k=0}^{n} \binom{n}{k} q^k p^m q^{n-k} \\ &= \begin{cases} 2^m n! i^{-n} P_n^{(\frac{1+m-n}{2})} \left(\frac{i(T+m-n)}{2}; \frac{\pi}{2} \right) p^{m-n} & (m \geq n) \\ 2^n m! i^{-m} q^{n-m} P_m^{(\frac{1+n-m}{2})} \left(\frac{i(T+n-m)}{2}; \frac{\pi}{2} \right) & (n \geq m) \end{cases}. \end{aligned} \tag{A.1.1}$$

In particular, we have([HZ])

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^n p^{n-k} = \sum_{k=0}^{n} \binom{n}{k} q^k p^n q^{n-k} = n! i^{-n} P_n^{(\frac{1}{2})} \left(\frac{iT}{2}; \frac{\pi}{2} \right). \tag{A.1.2}$$

Here $P_n^{(\alpha)}(x;\phi)$ is the Meixner-Pollaczek polynomial given by the hypergeometric series

$$P_n^{(\alpha)}(x;\phi) := \frac{(2\alpha)_n}{n!} e^{in\phi} {}_2F_1 \left(\begin{matrix} -n, \alpha + ix \\ 2\alpha \end{matrix}; 1 - e^{-2i\phi} \right). \tag{A.1.3}$$

Theorem A.1.2. *Let $T_{m,n}$ be the sum of all possible terms containing m factors of p and n factors of q. We have*

$$T_{m,n} = \begin{cases} \frac{n!}{2^n}\binom{m+n}{n} i^{-n} P_n^{(\frac{1+m-n}{2})}\left(\frac{i(T+m-n)}{2}; \frac{\pi}{2}\right) p^{m-n} & (m \geq n) \\ \frac{m!}{2^m}\binom{m+n}{m} i^{-m} q^{n-m} P_m^{(\frac{1+n-m}{2})}\left(\frac{i(T+n-m)}{2}; \frac{\pi}{2}\right) & (n \geq m) \end{cases}. \tag{A.1.4}$$

In particular, we have([Ko],[HZ],[FW2])

$$T_n := T_{n,n} = \frac{n!}{2^n}\binom{2n}{n} i^{-n} P_n^{(\frac{1}{2})}\left(\frac{iT}{2}; \frac{\pi}{2}\right). \tag{A.1.5}$$

The formula (A.1.5) for T_n was first observed by Bender, Mead and Pinsky([BMP]), and proved by Koorwinder([Ko]). The idea of the proof in [Ko] is to consider the irreducible unitary representations of the Heisenberg group and some analysis for special functions. Moreover, a combinatorial proof was given by Hamdi and Zeng([HZ]). They used the rook placement interpretation of the normal ordering of the Weyl algebra and gave also a proof of (A.1.2), which was first observed by [BD]. Our results extend these to general m and n.

The proofs given in this paper are much simpler than the investigations([Ko], [BD]). Actually, we only use some basic properties of the Weyl algebra and a certain transformation formula of the hypergeometric function. Our proofs clarify the reason why (A.1.2) and (A.1.5) are equal up to constant, which is not explained in [HZ].

A.2 Proof of Theorem A.1.1

The operations $L_A, R_A \in \mathrm{End}_{\mathbb{C}}(W)$ are respectively left and right multiplications, that is,

$$L_A.X := AX, \quad R_A.X := XA, \quad (A, X \in W). \tag{A.2.1}$$

We introduce some useful operators([W]).

$$\check{\mathrm{ad}}(A) := L_A + R_A. \tag{A.2.2}$$

We remark that $L, R : W \to \mathrm{End}_{\mathbb{C}}(W)$ are linear, hence $\check{\mathrm{ad}}$ is also linear. In addition, since $\check{\mathrm{ad}}(A)^N.1 = 2^N A^N$, we obtain the following lemma immediately.

Lemma A.2.1. *Let $t_1, \cdots, t_n$ be indeterminates. For any $N \in \mathbb{Z}_{\geq 0}$, we obtain*

$$\left\{\sum_{k=1}^{n} t_k \check{\mathrm{ad}}(A_k)\right\}^N .1 = 2^N \left\{\sum_{k=1}^{n} t_k A_k\right\}^N. \tag{A.2.3}$$

In particular, we have

$$(t_1\check{\mathrm{ad}}(p) + t_2\check{\mathrm{ad}}(q))^N.1 = 2^N(t_1 p + t_2 q)^N. \tag{A.2.4}$$

Remark A.2.2. When $N = n$ in Lemma A.2.1, comparing the coefficients of $t_1 \cdots t_n$ on both sides of the (A.2.3), we obtain the following formula immediately.

$$F(\check{\mathrm{ad}}(\underline{A}_n)).1 = 2^n F(\underline{A}_n). \tag{A.2.5}$$

Here $\underline{A}_n := (A_1, \ldots, A_n)$, $\check{\mathrm{ad}}(\underline{A}_n) := (\check{\mathrm{ad}}(A_1), \ldots, \check{\mathrm{ad}}(A_n))$ and

$$F(\underline{A}_n) := \sum_{\sigma \in \mathfrak{S}_n} A_{\sigma(1)} \cdots A_{\sigma(n)}, \quad F(\check{\mathrm{ad}}(\underline{A}_n)) := \sum_{\sigma \in \mathfrak{S}_n} \check{\mathrm{ad}}(A_{\sigma(1)}) \cdots \check{\mathrm{ad}}(A_{\sigma(n)}). \tag{A.2.6}$$

Lemma A.2.3. *The operators* $\check{\mathrm{ad}}(p)$ *and* $\check{\mathrm{ad}}(q)$ *are commutative.*

Proof. Obviously L_A and R_B are commutative. Since L is a homomorphism and R is an anti-homomorphism, we have

$$[\check{\mathrm{ad}}(p), \check{\mathrm{ad}}(q)] = [L_p + R_p, L_q + R_q] = [L_p, L_q] + [R_p, R_q] = L_{pq-qp} - R_{pq-qp} = 0.$$

□

Proposition A.2.4.

$$\check{\mathrm{ad}}(p)^m \check{\mathrm{ad}}(q)^n.1 = 2^n \sum_{k=0}^{m} \binom{m}{k} p^k q^n p^{m-k} = 2^m \sum_{k=0}^{n} \binom{n}{k} q^k p^m q^{n-k}. \tag{A.2.7}$$

Proof. Since L_A and R_B are commutative, L is a homomorphism and R is an anti-homomorphism, we obtain

$$\check{\mathrm{ad}}(p)^m \check{\mathrm{ad}}(q)^n.1 = (L_p + R_p)^m.2^n q^n = 2^n \sum_{k=0}^{m} \binom{m}{k} L_{p^k} R_{p^{m-k}}.q^n = 2^n \sum_{k=0}^{m} \binom{m}{k} p^k q^n q^{m-k}.$$

On the other hand, since $\check{\mathrm{ad}}(p)$ and $\check{\mathrm{ad}}(q)$ are commutative, we have

$$\check{\mathrm{ad}}(p)^m \check{\mathrm{ad}}(q)^n = \check{\mathrm{ad}}(q)^n \check{\mathrm{ad}}(p)^m.$$

Hence, the second equality of (A.2.7) can be proved in the same way. □

Remark A.2.5. Wakayama([W]) has constructed the oscillator representation of the simple Lie algebra $\mathfrak{sl}_2$ by $\check{\mathrm{ad}}$ and ad in $\mathrm{End}_{\mathbb{C}}(W)$ and then, proves that $\check{\mathrm{ad}}(p)^n \check{\mathrm{ad}}(q)^n.1$ satisfies the difference equation of the Meixner-Pollaczek polynomials.

Since $T = pq + qp$ and $pq - qp = 1$, we have

$$pq = \frac{T+1}{2}, \quad qp = \frac{T-1}{2}. \tag{A.2.8}$$

The proof of the following lemma is straightforward.

Lemma A.2.6. (1) *Let $f(T) \in \mathbb{C}[T], l \in \mathbb{Z}_{\geq 0}$. We have*

$$p^l f(T) = f(T+2l)p^l, \quad q^l f(T) = f(T-2l)q^l. \tag{A.2.9}$$

(2) *For any $l \in \mathbb{Z}_{\geq 0}$, we have*

$$p^l q^l = \left(\frac{1+T}{2}\right)_l, \quad q^l p^l = (-1)^l \left(\frac{1-T}{2}\right)_l. \tag{A.2.10}$$

Here, $(x)_l := x(x+1)\cdots(x+l-1)$, $(x)_0 := 1$.

Proposition A.2.7.

$$n! i^{-n} P_n^{(\alpha)}\left(\frac{ix}{2}; \frac{\pi}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\alpha - \frac{x}{2}\right)_k \left(\alpha + \frac{x}{2}\right)_{n-k}. \tag{A.2.11}$$

Proof. It follows from the formula (2.3.14) in [AAR] that

$$(LHS) = (2\alpha)_n {}_2F_1\left(\begin{matrix} -n, \alpha - \frac{x}{2} \\ 2\alpha \end{matrix}; 2\right) = \left(\alpha + \frac{x}{2}\right)_n {}_2F_1\left(\begin{matrix} -n, \alpha - \frac{x}{2} \\ -n - \alpha - \frac{x}{2} + 1 \end{matrix}; -1\right) = (RHS).$$

Remark A.2.8. One may also prove this proposition using the generating function for Meixner-Pollaczek polynomials.

□

We now prove Theorem A.1.1 as follows. If $m \geq n$,

$$\begin{aligned}
2^m \sum_{k=0}^{n} \binom{n}{k} q^k p^m q^{n-k} &= 2^m \sum_{k=0}^{n} \binom{n}{k} q^k p^k p^{m-n} p^{n-k} q^{n-k} \\
&= 2^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1-T}{2}\right)_k p^{m-n} \left(\frac{1+T}{2}\right)_{n-k} \\
&= 2^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1-T}{2}\right)_k \left(\frac{1+T}{2} + m - n\right)_{n-k} p^{m-n} \\
&= 2^m n! i^{-n} P_n^{\left(\frac{1+m-n}{2}\right)} \left(\frac{i(T+m-n)}{2}; \frac{\pi}{2}\right) p^{m-n}.
\end{aligned}$$

The second equality follows from (A.2.10), the third from (A.2.9) and the fourth from (A.2.11). By Proposition A.2.4, the case of $n \geq m$ can be proved in the same way.

A.3 Proof of Theorem A.1.2

Comparing the coefficients of $t_1^m t_2^n$ on both sides in (A.2.4) for $N = m + n$, one obtain the key proposition.

Proposition A.3.1. *For any $m, n \in \mathbb{N}$, we have*

$$T_{m,n} = \frac{1}{2^{m+n}} \frac{(m+n)!}{m!n!} \check{\mathrm{ad}}(p)^m \check{\mathrm{ad}}(q)^n.1. \tag{A.3.1}$$

Theorem A.1.2 follows immediately from (A.3.1), (A.2.7) and (A.1.1).

Remark A.3.2. (1) If $m \geq n$, then we have the following result immediately by Theorem A.1.2 and (A.2.10).

$$T_{m,n} q^{m-n} = \frac{n!}{2^n} \binom{m+n}{n} i^{-n} \left(\frac{1+T}{2}\right)_{m-n} P_n^{(\frac{1+m-n}{2})} \left(\frac{i(T+m-n)}{2}; \frac{\pi}{2}\right). \tag{A.3.2}$$

The case of $n \geq m$ is similar.
(2) If $m \geq n$, then a explicit expression of the Poincare-Birkhoff-Witt theorem for $T_{m,n}$ follows from (A.1.4), (A.1.3) and (A.2.10).

$$T_{m,n} = \frac{1}{2^n} \frac{m!}{(m-n)!} \binom{m+n}{n} \sum_{k \geq 0} \binom{n}{k} \frac{2^k}{(1+m-n)_k} q^k p^{k+m-n}. \tag{A.3.3}$$

The case of $n \geq m$ is similar.

Recently, a generalization of Theorem A.1.2 using the multivariate Meixner-Pollaczek polynomials in the framework of the Gelfand pair has been established in [FW2]. Another proof of [FW2] in our current approach would be desirable.

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