次数付きLie代数と概均質ベクトル空間

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Graded Lie algebras and prehomogeneous vector spaces

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Introduction

Let G be a connected algebraic group, ρ a representation of G on a finite-dimensional vector space V where all defined over $\mathbb{C}$. When there exists an element $v \in V$ such that the orbit $\rho(G)v$ is Zariski dense in V, we say that the triplet (G, ρ, V) is a prehomogeneous vector space (abbrev. PV) and v is called a generic point of (G, ρ, V). For example, if we let Λ_1 be the natural representation of the general linear group GL_n ($n \geq 1$) on $\mathbb{C}^n$, then for any G, ρ and V such that $\dim V \leq n$, the triplet $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$ is a PV. A PV which can be written in this form is called a trivial PV. Unfortunately, when $\dim V > n$, then a triplet $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$ is not a PV in general. However, when $m := \dim V > n$ and a triplet $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$ is a PV, it is known that a triplet $(G \times GL_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n})$ is also a PV. These two PVs are said to be **castling transforms** of each other (See Definition 11, p39, [S–K]).

The theory of prehomogeneous vector spaces is closely related to the theory of Lie algebras. We can express the prehomogeneity condition of a triplet (G, ρ, V) by the method of Lie algebras as follows.

We denote the Lie algebra of G by $\mathrm{Lie}(G)$, the infinitesimal representation of ρ by $d\rho$. Then (G, ρ, V) is a PV if and only if there exists an element $v \in V$ which satisfies $d\rho(\mathrm{Lie}(G))v = V$. Such an element v is a generic point of (G, ρ, V). This is an infinitesimal condition of the prehomogeneity.

Moreover, we have a class of prehomogneous vector spaces called **prehomogeneous vector spaces of parabolic type**. These PVs can be obtained by graded semisimple Lie algebras in the following way.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, G the adjoint group of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, R the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, φ a fundamental system of R all defined over $\mathbb{C}$. Let θ be a subset of φ and $H^\theta \in \mathfrak{h}$ the unique element which satisfies $\alpha(H^\theta) = 0$ for all $\alpha \in \theta$ and $\alpha(H^\theta) = 2$ for all $\alpha \in \varphi \setminus \theta$. Put $d_i(\theta) := \{X \in \mathfrak{g} \mid [H^\theta, X] = 2iX\}$ for each $i \in \mathbb{Z}$. Then we can obtain a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} d_i(\theta). \tag{0.1}$$

The graded Lie algebra (0.1) and the Killing form K of $\mathfrak{g}$ satisfy the following conditions:

$$d_0(\theta) \text{ and } d_1(\theta) \text{ are finite-dimensional vector spaces,} \tag{0.2}$$

$$\text{the adjoint representation of } d_0(\theta) \text{ on } d_1(\theta) \text{ is completely reducible,} \tag{0.3}$$

$$\text{the restriction of } K \text{ to } d_i(\theta) \times d_{-i}(\theta) \text{ is non-degenerate for each } i \geq 0, \tag{0.4}$$

$$[d_1(\theta), d_i(\theta)] = d_{i+1}(\theta),\ [d_{-1}(\theta), d_{-j}(\theta)] = d_{-j-1}(\theta) \text{ for each } i, j \geq 0. \tag{0.5}$$

Denote $d_0(\theta)$ by $\mathfrak{l}_\theta$ and the connected subgroup of G which corresponds to $\mathfrak{l}_\theta$ by L_θ, then it is known that $(L_\theta, d_i(\theta))$ is a prehomogeneous vector space for any $i \neq 0$. We denote it by $(\mathfrak{l}_\theta, d_i(\theta))$. In other words, there exists an element $v \in d_i(\theta)$ such that $[\mathfrak{l}_\theta, v] = d_i(\theta)$. In Rubenthaler [Ru–1], he showed that we can reduce to the case where $i = 1$ and studied spaces of the form $(\mathfrak{l}_\theta, d_1(\theta))$ and called them PVs of parabolic type. PVs of parabolic type can be described and classified by the weighted Dynkin diagrams.

For a PV of parabolic type $(\mathfrak{l}_\theta, d_1(\theta))$, we can recognize a space $(\mathfrak{l}_\theta, d_{-1}(\theta))$ as the dual space of $(\mathfrak{l}_\theta, d_1(\theta))$ via the Killing form K of $\mathfrak{g}$ and assume that $[d_1(\theta), d_{-1}(\theta)] = \mathfrak{l}_\theta$ without loss of generality. In other words, we can say that a PV of parabolic type is a triplet which consists of a reductive Lie algebra and its representation which can be embedded into a finite-dimensional graded Lie algebra with a non-degenerate symmetric invariant bilinear form. However, a castling transform of $(\mathfrak{l}_\theta, d_1(\theta))$ is in general no longer of parabolic type, that is, it can not be embedded into a semisimple Lie algebra.

In this paper, we shall introduce a way to embed a given finite-dimensional reductive Lie algebra $\mathfrak{g}$, a finite-dimensional completely reducible representation (ρ, V) (do not assume the prehomogeneity condition) of $\mathfrak{g}$ and its dual space (ρ^*, V^*) into a "large" graded Lie algebra with a non-degenerate symmetric invariant bilinear form. In particular, a prehomogeneous vector space of parabolic type can be embedded into a finite-dimensional semisimple Lie algebra by our construction. For this, by using a non-degenerate symmetric invariant bilinear form B_0 on $\mathfrak{g}$, we shall construct a graded Lie algebra $L(\mathfrak{g}, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$ which also has a non-degenerate symmetric invariant bilinear form B such that V_0 and V_1 are isomorphic to $\mathfrak{g}$ and V as $\mathfrak{g}$-modules respectively and a restriction of B to $V_0 \times V_0$ coincides with B_0 (Theorem 2.11 and Proposition 3.2). In our construction, the bilinear form B_0 plays important roles. In particular, B_0 defines a linear map $\hat{\Phi}_\rho$ from $V \otimes V^*$ to $\mathfrak{g}$ called Φ-map and $\hat{\Phi}_\rho$ induces a bracket product between V_1 and V_{-1}. Moreover, we can prove that $L(\mathfrak{g}, \rho, V, B_0)$ satisfies the following conditions:

$$V_0 \text{ and } V_1 \text{ are finite-dimensional vector spaces,} \tag{0.6}$$

$$\text{the adjoint representation of } V_0 \text{ on } V_1 \text{ is completely reducible,} \tag{0.7}$$

$$\text{the restriction of } B \text{ to } V_i \times V_{-i} \text{ is non-degenerate for each } i \geq 0, \tag{0.8}$$

$$[V_1, V_i] = V_{i+1},\ [V_{-1}, V_{-j}] = V_{-j-1} \text{ for each } i, j \geq 0, \tag{0.9}$$

$$[V_1, V_{-1}] = V_0. \tag{0.10}$$

Conversely, an arbitrary graded Lie algebra which satisfies the conditions from (0.6) to (0.10) can be obtained by our construction (Proposition 3.3). In particular, an arbitrary finite-dimensional semisimple Lie algebra can be obtained by a certain PV of parabolic type and a bilinear form (Theorem 3.14).

Now, the prehomogeneity condition of a triplet (G, ρ, V), where G is a reductive algebraic group, is closely related to the Φ-map of the quadruplet $(\mathrm{Lie}(G), d\rho, V, B_0)$ where B_0 is a bilinear form on $\mathrm{Lie}(G)$. In particular, the prehomogeneity condition of a triplet of the form $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$ can be expressed by the Φ-map of the quadruplet $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$, where T_n is a bilinear form on $\mathfrak{gl}_n$ defined by:

$$T_n(a, a') = \mathrm{Tr}\ (aa') \quad (a, a' \in \mathfrak{gl}_n = M(n; \mathbb{C})). \tag{0.11}$$

Furthermore, this condition is reduced to the condition of the Φ-map $\hat{\Phi}_\rho$ of $(\mathfrak{g}, \rho, V, B_0)$. Then, we can obtain another proof of castling transform.

This paper consists of four sections.

In section 1, first of all, we shall explain that a quadruplet $(\mathfrak{g}, \rho, V, B_0)$ determines a linear map $\hat{\Phi}_\rho$ from $V \otimes V^*$ to $\mathfrak{g}$. Then $\hat{\Phi}_\rho$ induces a $\mathfrak{g}$-submodule V_1 in $\mathrm{Hom}(V^*, \mathfrak{g})$. In order to have $V_1 \simeq V$ as $\mathfrak{g}$-modules, we introduce the notion of **standard quadruplets**. Then, moreover, we can obtain $\mathfrak{g}$-modules V_n for all $n \in \mathbb{Z}$ inductively and call them ***n*-graduations**. Each $\mathfrak{g}$-module V_n is finite-dimensional and the dual $\mathfrak{g}$-module of V_{-n}. In particular, V_0 is isomorphic to $\mathfrak{g}$ itself.

In section 2, we define a bilinear map $[\cdot, \cdot]_m^n$ from $V_n \times V_m$ to V_{n+m} for any $n, m \in \mathbb{Z}$. Denote a direct sum of n-graduations obtained in section 1 by $L(\mathfrak{g}, \rho, V, B_0)$. Then we have a bilinear map $[\cdot, \cdot] : L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0) \to L(\mathfrak{g}, \rho, V, B_0)$ defined by

$$[x_n, y_m] := [x_n, y_m]_m^n \ \ (x_n \in V_n, y_m \in V_m).$$

Our main result is that $[\cdot, \cdot]$ satisfies the axioms of a Lie algebra, i.e. $L(\mathfrak{g}, \rho, V, B_0)$ has a structure of a graded Lie algebra. We call the Lie algebra thus obtained a **Lie algebra associated to a standard quadruplet** (Theorem 2.11). For example, loop algebras and finite-dimensional semisimple Lie algebras are Lie algebras associated to some quadruplet. This will be proved in section 3.

In section 3, we will study some properties of $L(\mathfrak{g}, \rho, V, B_0)$. First, we shall construct a non-degenerate symmetric and invariant bilinear form on $L(\mathfrak{g}, \rho, V, B_0)$. Then we can prove that the Lie algebras associated to a standard quadruplet can be characterized by the existence of a non-degenerate symmetric and invariant bilinear form. It is described in Proposition 3.3. Finally, in Theorem 3.14, we will give a necessary and sufficient condition for $(\mathfrak{g}, \rho, V, B_0)$ to generate a finite-dimensional Lie algebra $L(\mathfrak{g}, \rho, V, B_0)$. The condition is closely related to prehomogeneous vector spaces of parabolic type.

In section 4, we shall consider prehomogeneous vector spaces. We shall express the prehomogeneity condition of a given triplet (G, ρ, V) as a condition of the Φ-map $\hat{\Phi}_{d\rho}$ of the quadruplet

$(\mathrm{Lie}(G), d\rho, V, B_0)$ for some bilinear form B_0. Moreover, we can express the prehomogeneity condition of a quadruplets of the form $(\mathfrak{g}\oplus\mathfrak{gl}_n, \rho\otimes\Lambda_1, V\otimes\mathbb{C}^n, B_0\oplus T_n)$ as a condition of $(\mathfrak{g}, \rho, V, B_0)$ (Lemma 4.5). By applying this lemma, we shall give another proof of castling transform (Theorem 4.7).

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Notation: We denote the transpose of a matrix X by ${}^tX$, the zero-matrix and the unit matrix of size $n\times n$ by 0_n and 1_n, the Kronecker delta by δ_{ij}. Put $J_n := \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$. Throughout this paper, we shall always assume that all objects are defined over the complex number field $\mathbb{C}$.

1 A family of n-graduations

1.1 Standard quadruplets

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra, V a finite-dimensional vector space, ρ a representation of $\mathfrak{g}$ on V all defined over $\mathbb{C}$. By the theory of Lie algebras, there exists a non-degenerate symmetric $\mathfrak{g}$-invariant bilinear form B_0 on $\mathfrak{g}$. Let V^* be the dual module of V and ρ^* the dual representation of ρ. We denote the pairing between V and V^* by $\langle\cdot,\cdot\rangle$. By the assumption that B_0 is non-degenerate, we can define the following linear maps.

Definition 1.1. We define a linear map $\hat{\Phi}_\rho$ (resp. $\hat{\Psi}_\rho$) from $V\otimes V^*$ to $\mathfrak{g}$ (resp. from $V^*\otimes V$ to $\mathfrak{g}$) by the following equation:

$$B_0(a, \hat{\Phi}_\rho(v\otimes\phi)) = \langle\rho(a)v, \phi\rangle = -\langle v, \rho^*(a)\phi\rangle \tag{1.1}$$

(resp.

$$B_0(a, \hat{\Psi}_\rho(\phi\otimes v)) = \langle v, \rho^*(a)\phi\rangle = -\langle\rho(a)v, \phi\rangle\ ) \tag{1.2}$$

for any $a\in\mathfrak{g}$, $v\in V$ and $\phi\in V^*$. We call this map $\hat{\Phi}$ (resp. $\hat{\Psi}$) **the Φ-map** (resp. the Ψ-map) of the quadruplet $(\mathfrak{g}, \rho, V, B_0)$.

Remark 1.2. For any $v\in V$ and $\phi\in V^*$, we have

$$\hat{\Phi}_\rho(v\otimes\phi) + \hat{\Psi}_\rho(\phi\otimes v) = 0. \tag{1.3}$$

Example 1.3. Let $(\mathfrak{g}, \rho, V, B_0) = (\mathfrak{gl}_n, \Lambda_1, \mathbb{C}^n, T_n)$, where Λ_1 is the natural representation of $\mathfrak{gl}_n$ on the space of column vectors $\mathbb{C}^n = M(n, 1; \mathbb{C})$ and T_n is a non-degenerate symmetric invariant bilinear form on $\mathfrak{gl}_n$ defined by

$$T_n(a, a') := \mathrm{Tr}\ (aa') \tag{1.4}$$

where $a, a' \in \mathfrak{g} = M(n, \mathbb{C})$. Then the dual space V^* can be identified with $V = \mathbb{C}^n$. The representation Λ_1^* and the pairing $\langle \cdot, \cdot \rangle$ are given as follows:

$$\Lambda_1^*(a)\phi := -{}^t a\phi, \tag{1.5}$$
$$\langle v, \phi \rangle := {}^t v\phi \tag{1.6}$$

where $a \in \mathfrak{g}$, $v \in V$ and $\phi \in V^*$. Then the Φ-map is given as:

$$\hat{\Phi}_{\Lambda_1}(v \otimes \phi) = v^t\phi. \tag{1.7}$$

Example 1.4. Let $(\mathfrak{g}, \rho, V, B_0) = (\mathfrak{so}_n, \Lambda_1, \mathbb{C}^n, T_n|_{\mathfrak{so}_n \times \mathfrak{so}_n})$, where Λ_1 is the natural representation of $\mathfrak{so}_n$ on $\mathbb{C}^n = M(n, 1; \mathbb{C})$. Then the dual space V^* can be identified with $V = \mathbb{C}^n$. The representation Λ_1^* and the pairing $\langle \cdot, \cdot \rangle$ are given as follows:

$$\Lambda_1^*(a)\phi := a\phi, \tag{1.8}$$
$$\langle v, \phi \rangle := {}^t v\phi \tag{1.9}$$

where $a \in \mathfrak{g}$, $v \in V$ and $\phi \in V^*$. Then the Φ-map is given as:

$$\hat{\Phi}_{\Lambda_1}(v \otimes \phi) = \frac{1}{2}(v^t\phi - \phi^t v). \tag{1.10}$$

Example 1.5. Let $(\mathfrak{g}, \rho, V, B_0) = (\mathfrak{sp}_n, \Lambda_1, \mathbb{C}^{2n}, T_{2n})$, where $\mathfrak{sp}_n = \{X \in \mathfrak{gl}_{2n} \mid {}^t X J_n + J_n X = 0\}$ and Λ_1 is the natural representation of $\mathfrak{sp}_n$ on $\mathbb{C}^{2n} = M(2n, 1; \mathbb{C})$. Then the dual space V^* can be identified with $V = \mathbb{C}^{2n}$. The representation Λ_1^* and the pairing $\langle \cdot, \cdot \rangle$ are given as follows:

$$\Lambda_1^*(a)\phi := a\phi, \tag{1.11}$$
$$\langle v, \phi \rangle := {}^t v J_n \phi \tag{1.12}$$

where $a \in \mathfrak{g}$, $v \in V$ and $\phi \in V^*$. Then the Φ-map is given as:

$$\hat{\Phi}_{\Lambda_1}(v \otimes \phi) = -\frac{1}{2}(v^t\phi J_n + \phi^t v J_n). \tag{1.13}$$

Proposition 1.6. *Let $\mathfrak{g}$ be any reductive Lie algebra, ρ^i, σ^i representations of $\mathfrak{g}$ on V^i and U^i $(i = 1, 2)$, B_0 a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. For quadruplets $(\mathfrak{g}, \rho^1 \oplus \rho^2, V^1 \oplus V^2, B_0)$ and $(\mathfrak{g}, \sigma^1 \otimes \sigma^2, U^1 \otimes U^2, B_0)$, the maps $\hat{\Phi}_{\rho^1 \oplus \rho^2}$ and $\hat{\Phi}_{\sigma^1 \otimes \sigma^2}$ are given as:*

$$\hat{\Phi}_{\rho^1 \oplus \rho^2}\big((v^1, v^2) \otimes (\phi^1, \phi^2)\big) = \hat{\Phi}_{\rho^1}(v^1 \otimes \phi^1) + \hat{\Phi}_{\rho^2}(v^2 \otimes \phi^2), \tag{1.14}$$
$$\hat{\Phi}_{\sigma^1 \otimes \sigma^2}\big((u^1 \otimes u^2) \otimes (\psi^1 \otimes \psi^2)\big) = \langle u^2, \psi^2 \rangle \hat{\Phi}_{\sigma^1}(u^1 \otimes \psi^1) + \langle u^1, \psi^1 \rangle \hat{\Phi}_{\sigma^2}(u^2 \otimes \psi^2), \tag{1.15}$$

where $v^i \in V^i, u^i \in U^i, \phi^i \in (V^i)^*, \psi^i \in (U^i)^*$ and $\hat{\Phi}_{\rho^i}, \hat{\Phi}_{\sigma^i}$ are the Φ-maps of quadruplets $(\mathfrak{g}, \rho^i, V^i, B_0)$ and $(\mathfrak{g}, \sigma^i, U^i, B_0)$ $(i = 1, 2)$.

Proof. The dual spaces of $V^1 \oplus V^2$ and $U^1 \otimes U^2$ can be identified with $(V^1)^* \oplus (V^2)^*$ and $(U^1)^* \otimes (U^2)^*$ respectively. The pairings between them are given as:

$$\langle (v^1, v^2), (\phi^1, \phi^2) \rangle := \langle v^1, \phi^1 \rangle + \langle v^2, \phi^2 \rangle, \tag{1.16}$$
$$\langle u^1 \otimes u^2, \psi^1 \otimes \psi^2 \rangle := \langle u^1, \psi^1 \rangle \langle u^2, \psi^2 \rangle \tag{1.17}$$

where $v^i \in V^i, u^i \in U^i, \phi^i \in (V^i)^*, \psi^i \in (U^i)^*$ $(i = 1, 2)$. Thus, our claim can be checked by a direct calculation. ∎

Here, $V \otimes V^*$ and $V^* \otimes V$ have canonical $\mathfrak{g}$-module structures and we have the following proposition.

Proposition 1.7. *The maps $\hat{\Phi}_\rho$ and $\hat{\Psi}_\rho$ are homomorphisms of $\mathfrak{g}$-modules.*

Proof. In fact, for any $a, a' \in \mathfrak{g}$, $v \in V$ and $\phi \in V^*$, we have

$$\begin{aligned}
&B_0\big(a', \hat{\Phi}_\rho\big((\rho(a)v) \otimes \phi + v \otimes (\rho^*(a)\phi)\big)\big) \\
=&\langle \rho(a')\rho(a)v, \phi \rangle + \langle \rho(a')v, \rho^*(a)\phi \rangle \\
=&\langle \rho([a', a])v, \phi \rangle + \langle \rho(a)\rho(a')v, \phi \rangle + \langle \rho(a')v, \rho^*(a)\phi \rangle \\
=&B_0([a', a], \hat{\Phi}_\rho(v \otimes \phi)) - \langle \rho(a')v, \rho^*(a)\phi \rangle + \langle \rho(a')v, \rho^*(a)\phi \rangle \\
=&B_0(a', [a, \hat{\Phi}_\rho(v \otimes \phi)]).
\end{aligned}$$

Hence, $\hat{\Phi}_\rho\big((\rho(a)v) \otimes \phi + v \otimes (\rho^*(a)\phi)\big) = [a, \hat{\Phi}_\rho(v \otimes \phi)]$. Similarly, we can obtain the equation $\hat{\Psi}_\rho\big((\rho^*(a)\phi) \otimes v + \phi \otimes (\rho(a)v)\big) = [a, \hat{\Psi}_\rho(\phi \otimes v)]$ and thus $\hat{\Phi}_\rho$ and $\hat{\Psi}_\rho$ are homomorphisms of $\mathfrak{g}$-modules. ∎

By Proposition 1.7, $\hat{\Phi}_\rho(V \otimes V^*)$ and $\hat{\Phi}_\rho(V \otimes V^*)^\perp := \{a \in \mathfrak{g} \mid B_0(a, x) = 0 \text{ for any } x \in \hat{\Phi}_\rho(V \otimes V^*)\}$ are ideals of $\mathfrak{g}$. Then $\hat{\Phi}_\rho(V \otimes V^*)^\perp$ coincides with Ker ρ. In fact, suppose that $a \in$ Ker ρ. Then we have $B_0(a, \hat{\Phi}_\rho(v \otimes \phi)) = \langle \rho(a)v, \phi \rangle = 0$ for any $v \in V$ and $\phi \in V^*$. Thus we have $\hat{\Phi}_\rho(V \otimes V^*)^\perp \supset$ Ker ρ. Similarly we have the converse inclusion. Hence we obtain the following proposition.

Proposition 1.8. *The maps $\hat{\Phi}_\rho$ and $\hat{\Psi}_\rho$ are surjective if and only if ρ is faithful.*

Next, let us construct $\mathfrak{g}$-modules which are isomorphic to V and V^* respectively.

Definition 1.9. For each element $v \in V$ and $\phi \in V^*$, we define linear maps $\Phi_{\rho,v} \in \mathrm{Hom}(V^*, \mathfrak{g})$ and $\Psi_{\rho,\phi} \in \mathrm{Hom}(V, \mathfrak{g})$ by

$$\Phi_{\rho,v}(\psi) := \hat{\Phi}_\rho(v \otimes \psi), \tag{1.18}$$
$$\Psi_{\rho,\phi}(u) := \hat{\Psi}_\rho(\phi \otimes u) \tag{1.19}$$

where $u \in V$ and $\psi \in V^*$. Then we have linear maps Φ_ρ and Ψ_ρ as below:

$$\begin{aligned} \Phi_\rho : V &\to \mathrm{Hom}(V^*, \mathfrak{g}) \\ v &\mapsto \Phi_{\rho,v}, \end{aligned} \tag{1.20}$$

$$\begin{aligned} \Psi_\rho : V^* &\to \mathrm{Hom}(V, \mathfrak{g}) \\ \phi &\mapsto \Psi_{\rho,\phi}. \end{aligned} \tag{1.21}$$

We call these maps $\Phi_{\rho,v}$ and $\Psi_{\rho,\phi}$ the **Φ-map at** v and **Ψ-map at** ϕ of $(\mathfrak{g}, \rho, V, B)$ respectively.

The spaces $\mathrm{Hom}(\mathfrak{g}, V)$ and $\mathrm{Hom}(\mathfrak{g}, V^*)$ have canonical $\mathfrak{g}$-module structures. Then we have the following proposition.

Proposition 1.10. *The maps Φ_ρ and Ψ_ρ are homomorphisms of $\mathfrak{g}$-modules.*

Proof. By Proposition 1.7, we have the following equations:

$$\Phi_{\rho,\rho(a)v}(\phi) = [a, \Phi_{\rho,v}(\phi)] - \Phi_{\rho,v}(\rho^*(a)\phi), \tag{1.22}$$

$$\Psi_{\rho,\rho^*(b)\psi}(u) = [b, \Psi_{\rho,\psi}(u)] - \Psi_{\rho,\psi}(\rho(b)u) \tag{1.23}$$

for any $a, b \in \mathfrak{g}$, $v, u \in V$ and $\phi, \psi \in V^*$. Thus we have our result. ∎

Put

$$V_0 := \mathfrak{g}, \tag{1.24}$$

$$V_1 := \mathrm{Im}(\Phi_\rho), \tag{1.25}$$

$$V_{-1} := \mathrm{Im}(\Psi_\rho). \tag{1.26}$$

Then V_0, V_1 and V_{-1} are $\mathfrak{g}$-modules. We denote the canonical representations of $\mathfrak{g}$ on V_0, V_1 and V_{-1} by ρ_0, ρ_1 and ρ_{-1} respectively. Then we can define linear maps p_0 and q_0 by

$$\begin{aligned} p_0 : V_1 \otimes V_0 &\to V_1 \\ v_1 \otimes a &\mapsto -\rho_1(a)v_1, \end{aligned} \tag{1.27}$$

$$\begin{aligned} q_0 : V_{-1} \otimes V_0 &\to V_{-1} \\ \phi_{-1} \otimes a &\mapsto -\rho_{-1}(a)\phi_{-1}. \end{aligned} \tag{1.28}$$

We can easily show that p_0 and q_0 are homomorphisms of $\mathfrak{g}$-modules. Furthermore, if ρ is completely reducible and does not have a subrepresentation on a non-zero subspace of V which is isomorphic to a zero-representation, then p_0 and q_0 are surjective and Φ_ρ and Ψ_ρ are injective. In fact, take an element $v \in V$ which satisfies $\Phi_{\rho,v} = 0$. Then we have $\langle \rho(a)v, \phi \rangle = B_0\big(a, \Phi_{\rho,v}(\phi)\big) = 0$ for any $a \in \mathfrak{g}$ and $\phi \in V^*$, hence $\rho(a)v = 0$ and $v = 0$. The same holds for Ψ_ρ. Therefore we obtain the following proposition.

Proposition 1.11. *If ρ is completely reducible and does not have a subrepresentation on a non-zero subspace of V which is isomorphic to a zero-representation, then $\mathfrak{g}$-modules V_1 and V_{-1} are isomorphic to V and V^* respectively.*

Remark 1.12. In the situation of Proposition 1.11, we have

$$v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big) + \phi_{-1}\big(\Phi_\rho^{-1}(v_1)\big) = 0 \tag{1.29}$$

for any $v_1 \in V_1$ and $\phi_{-1} \in V_{-1}$.

Here, we give the following definition.

Definition 1.13. Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra, V a finite-dimensional vector space, ρ a representation of $\mathfrak{g}$ on V, B_0 a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. If a quadruplet $(\mathfrak{g}, \rho, V, B_0)$ satisfies the following conditions, we call it a **standard quadruplet**:

ρ is completely reducible, (1.30)

ρ is faithful, (1.31)

ρ does not have a subrepresentation on a non-zero subspace of V which is isomorphic to a zero-representation. (1.32)

In other words, if $(\mathfrak{g}, \rho, V, B_0)$ is a standard quadruplet, the $\mathfrak{g}$-modules V_0, V_1 and V_{-1} are isomorphic to $\hat{\Phi}_\rho(V \otimes V^*)$, V and V^* respectively. Next, let us construct $\mathfrak{g}$-modules from V_1 and V_{-1}. For this, we will show the following proposition used later.

Proposition 1.14. *Let $(\mathfrak{g}, \rho, V, B_0)$ be a standard quadruplet. Let W be a finite-dimensional $\mathfrak{g}$-module and ρ_W a representation of $\mathfrak{g}$ on W. Assume that there exists a homomorphism of $\mathfrak{g}$-modules p from $V_1 \otimes W$ to $\mathrm{Hom}(V^*, W)$. We put*

$$\tilde{W} := \mathrm{Im}\ p \tag{1.33}$$

and denote the canonical representation of $\mathfrak{g}$ on $\tilde{W}$ by $\tilde{\rho}_W$. Then the following linear map

$$\begin{aligned} \tilde{p} : V_1 \otimes \tilde{W} &\to \mathrm{Hom}(V^*, \tilde{W}) \\ v_1 \otimes \tilde{w} &\mapsto \big(\phi \mapsto \tilde{\rho}_W\big(v_1(\phi)\big)\tilde{w} + p\big(v_1 \otimes \tilde{w}(\phi)\big)\big) \end{aligned} \tag{1.34}$$

is a homomorphism of $\mathfrak{g}$-modules.

Proof. For any $a \in \mathfrak{g}$, $v_1 \in V_1$, $\tilde{w} \in \tilde{W}$ and $\phi \in V^*$, we have

$$\begin{aligned}
&\tilde{p}(\rho_1(a)v_1 \otimes \tilde{w})(\phi) + \tilde{p}(v_1 \otimes \tilde{\rho}_W(a)\tilde{w})(\phi) \\
=&\tilde{\rho}_W\big((\rho_1(a)v_1)(\phi)\big)\tilde{w} + p\big(\rho_1(a)v_1 \otimes \tilde{w}(\phi)\big) \\
&+ \tilde{\rho}_W\big(v_1(\phi)\big)\tilde{\rho}_W(a)\tilde{w} + p\big(v_1 \otimes \big(\tilde{\rho}_W(a)\tilde{w}\big)(\phi)\big) \\
=&\tilde{\rho}_W\big([a, v_1(\phi)]\big)\tilde{w} - \tilde{\rho}_W\big(v_1\big(\rho^*(a)\phi\big)\big)\tilde{w} \\
&+ p\big(\rho_1(a)v_1 \otimes \tilde{w}(\phi)\big) + \tilde{\rho}_W\big(v_1(\phi)\big)\tilde{\rho}_W(a)\tilde{w} \\
&+ p\big(v_1 \otimes \rho_W(a)(\tilde{w}(\phi))\big) - p\big(v_1 \otimes \tilde{w}(\rho^*(a)\phi)\big) \\
=&\tilde{\rho}_W(a)\tilde{\rho}_W\big(v_1(\phi)\big)\tilde{w} + \tilde{\rho}_W(a)p\big(v_1 \otimes (\tilde{w}(\phi))\big) \\
&- \tilde{p}\big(v_1 \otimes \tilde{w}\big)(\rho^*(a)\phi) \\
=&\tilde{\rho}_W(a)\big(\tilde{p}\big(v_1 \otimes \tilde{w}\big)(\phi)\big) - \tilde{p}\big(v_1 \otimes \tilde{w}\big)(\rho^*(a)\phi).
\end{aligned}$$

Thus we have our result. ∎

Definition 1.15. Let $(\mathfrak{g}, \rho, V, B_0)$ be a standard quadruplet. Suppose that $i \geq 1$ and there exist $\mathfrak{g}$-modules (ρ_{i-1}, V_{i-1}) and (ρ_{-i+1}, V_{-i+1}) and homomorphisms of $\mathfrak{g}$-modules $p_{i-1} : V_1 \otimes V_{i-1} \to \mathrm{Hom}(V^*, V_{i-1})$ and $q_{-i+1} : V_{-1} \otimes V_{-i+1} \to \mathrm{Hom}(V, V_{-i+1})$. Put $V_i := \mathrm{Im}\ p_{i-1}$ and $V_{-i} := \mathrm{Im}\ q_{-i+1}$ and denote the canonical representations of $\mathfrak{g}$ on them by ρ_i and ρ_{-i} respectively. We define linear maps p_i and q_{-i} by

$$\begin{aligned}
p_i :\ & V_1 \otimes V_i \to \mathrm{Hom}(V^*, V_i) \\
& v_1 \otimes u_i \mapsto \big(\phi \mapsto \rho_i\big(v_1(\phi)\big)u_i + p_{i-1}\big(v_1 \otimes u_i(\phi)\big)\big), \qquad (1.35) \\
q_{-i} :\ & V_{-1} \otimes V_{-i} \to \mathrm{Hom}(V, V_{-i}) \\
& \phi_{-1} \otimes \psi_{-i} \mapsto \big(v \mapsto \rho_{-i+1}\big(\phi_{-1}(v)\big)\psi_{-i} + q_{-i+1}\big(\phi_{-1} \otimes \psi_{-i}(v)\big)\big). \qquad (1.36)
\end{aligned}$$

Then, by Proposition 1.14, p_i is a homomorphism of $\mathfrak{g}$-modules. Similarly, q_{-i} is also a homomorphism of $\mathfrak{g}$-modules. We denote by V_{i+1} and V_{-i-1} the images of p_i and q_{-i} and the canonical representations on V_{i+1} and V_{-i-1} by ρ_{i+1} and ρ_{-i-1} respectively. Thus, inductively, we obtain $\mathfrak{g}$-modules V_n and representations ρ_n of $\mathfrak{g}$ on V_n for all $n \in \mathbb{Z}$. We call V_n **the n-graduation** of $(\mathfrak{g}, \rho, V, B_0)$. Throughout this paper, we use these notation.

Remark 1.16. Especially, if $i = 1$, we have

$$\begin{aligned}
p_1(v_1 \otimes u_1)(\phi) =& - p_1(u_1 \otimes v_1)(\phi) \\
=&\rho_1(v_1(\phi))u_1 - \rho_1(u_1(\phi))v_1 \qquad (1.37)
\end{aligned}$$

and

$$\begin{aligned}
q_{-1}(\phi_{-1} \otimes \psi_{-1})(v) =& - q_{-1}(\psi_{-1} \otimes \phi_{-1})(v) \\
=&\rho_{-1}(\phi_{-1}(v))\psi_{-1} - \rho_{-1}(\psi_{-1}(v))\phi_{-1}. \qquad (1.38)
\end{aligned}$$

1.2 A bilinear form on $V_i \times V_{-i}$

In this section, we will consider the relation between $\mathfrak{g}$-modules V_i and V_{-i} for each $i \geq 0$. For this, we construct a bilinear form on $V_i \times V_{-i}$. First, let us show the following equation on B_0 :

$$\begin{aligned}&B_0\big(p_0\big(v_1 \otimes a\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big), b\big)\\ =&B_0\big(a, q_0\big(\phi_{-1} \otimes b\big)\big(\Phi_\rho^{-1}(v_1)\big)\big)\end{aligned} \tag{1.39}$$

where $a, b \in V_0$, $v_1 \in V_1$ and $\phi_{-1} \in V_{-1}$.

Put $v := \Phi_\rho^{-1}(v_1) \in V$ and $\phi := \Psi_\rho^{-1}(\phi_{-1}) \in V^*$. Then we have

$$\begin{aligned}&B_0\big(p_0\big(v_1 \otimes a\big)\big(\phi\big), b\big)\\ =&B_0\big(\big(-\rho_1(a)v_1\big)\big(\phi\big), b\big)\\ =&B_0\big(v_1\big(\rho^*(a)\phi\big), b\big) - B_0\big(\big[a, v_1\big(\phi\big)\big], b\big)\\ =&\langle\rho(b)v, \rho^*(a)\phi\rangle + B_0\big(\big[a, \phi_{-1}\big(v\big)\big], b\big)\\ =&B_0\big(a, \phi_{-1}\big(\rho(b)v\big)\big) + B_0\big(a, \big[\phi_{-1}\big(v\big), b\big]\big)\\ =&B_0\big(a, q_0\big(\phi_{-1} \otimes b\big)\big(v\big)\big).\end{aligned} \tag{1.40}$$

Next, suppose that $i \geq 1$. Assume that there exists a $\mathfrak{g}$-invariant bilinear form B_{i-1} on $V_{i-1} \times V_{-i+1}$ which satisfies the following equation:

$$\begin{aligned}&B_{i-1}\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i+1}\big)\\ =&B_{i-1}\big(u_{i-1}, q_{-i+1}\big(\phi_{-1} \otimes \psi_{-i+1}\big)\big(\Phi_\rho^{-1}(v_1)\big)\big)\end{aligned} \tag{1.41}$$

for any $v_1 \in V_1$, $u_{i-1} \in V_{i-1}$, $\phi_{-1} \in V_{-1}$ and $\psi_{-i+1} \in V_{-i+1}$. Then we can define a bilinear form B_i on $V_i \times V_{-i}$ by

$$\begin{aligned}&B_i\big(p_{i-1}(v_1 \otimes u_{i-1}), q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})\big)\\ :=&B_{i-1}\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i+1}\big)\end{aligned} \tag{1.42}$$

where $v_1 \in V_1$, $u_{i-1} \in V_{i-1}$, $\phi_{-1} \in V_{-1}$ and $\psi_{-i+1} \in V_{-i+1}$. Indeed, by (1.41), B_i is well-defined. Then B_i is $\mathfrak{g}$-invariant. In fact, by the assumption that B_{i-1} is $\mathfrak{g}$-invariant, we have

$$\begin{aligned}&B_i\big(\rho_i\big(a\big)p_{i-1}\big(v_1 \otimes u_{i-1}\big), q_{-i+1}\big(\phi_{-1} \otimes \psi_{-i+1}\big)\big)\\ =&B_{i-1}\big(\rho_i\big(a\big)\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big), \psi_{-i+1}\big)\\ &- B_{i-1}\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big)\big(\rho^*\big(a\big)\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i+1}\big)\\ =&- B_{i-1}\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big), \rho_{-i+1}\big(a\big)\psi_{-i+1}\big)\\ &- B_{i-1}\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big)\big(\rho^*\big(a\big)\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i+1}\big)\\ =&- B_i\big(p_{i-1}\big(v_1 \otimes u_{i-1}\big), \rho_{-i}\big(a\big)q_{-i+1}\big(\phi_{-1} \otimes \psi_{-i+1}\big)\big)\end{aligned} \tag{1.43}$$

for any $a \in \mathfrak{g}$, $v_1 \in V_1$, $u_{i-1} \in V_{i-1}$, $\phi_{-1} \in V_{-1}$ and $\psi_{-i+1} \in V_{-i+1}$. Moreover B_i satisfies the following equation:

$$\begin{aligned} &B_i\big(p_i\big(v_1 \otimes u_i\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i}\big) \\ =&B_i\big(u_i, q_{-i}\big(\phi_{-1} \otimes \psi_{-i}\big)\big(\Phi_\rho^{-1}(v_1)\big)\big) \end{aligned} \tag{1.44}$$

for any $v_1 \in V_1$, $u_i \in V_i$, $\phi_{-1} \in V_{-1}$ and $\psi_{-i} \in V_{-i}$. In fact, put $v := \Phi_\rho^{-1}(v_1)$ and $\phi := \Psi_\rho^{-1}(\phi_{-1})$. Then we have

$$\begin{aligned} &B_i\big(p_i\big(v_1 \otimes u_i\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i}\big) \\ =&B_i\big(\rho_i\big(v_1(\phi)\big)u_i + p_{i-1}\big(v_1 \otimes u_i(\phi)\big), \psi_{-i}\big) \\ =&B_i\big(u_i, \rho_{-i}\big(\phi_{-1}(v)\big)\psi_{-i}\big) + B_{i-1}\big(u_i(\phi), \psi_{-i}(v)\big) \\ =&B_i\big(u_i, \rho_{-i}\big(\phi_{-1}(v)\big)\psi_{-i}\big) + B_i\big(u_i, q_{-i}\big(\phi_{-1} \otimes \psi_{-i}(v)\big)\big) \\ =&B_i\big(u_i, q_{-i}\big(\phi_{-1} \otimes \psi_{-i}\big)\big(\Phi_\rho^{-1}(v_1)\big)\big). \end{aligned}$$

Thus, inductively, we obtain a bilinear form B_i on $V_i \times V_{-i}$ for all $i \geq 0$.

Proposition 1.17. *For all $i \geq 0$, V_{-i} is the dual $\mathfrak{g}$-module of V_i.*

Proof. It is sufficient to show that the bilinear map $B_i : V_i \times V_{-i} \to \mathbb{C}$ is non-degenerate for each $i \geq 0$. We argue by induction on i. For $i = 0$, by the assumption, $B_0 : V_0 \times V_0 \to \mathbb{C}$ is non-degenerate and $\mathfrak{g}$-invariant. Hence, V_0 is the dual $\mathfrak{g}$-module of itself. Suppose that $i \geq 1$. Take an element $u_i \in V_i$ such that $B_i(u_i, q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})) = 0$ for any $\phi_{-1} \in V_{-1}$ and $\psi_{-i+1} \in V_{-i+1}$. Then we have

$$B_{i-1}\big(u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big), \psi_{-i+1}\big) = 0.$$

By the induction hypothesis, B_{i-1} is non-degenerate and thus we have $u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big) = 0$ for any $\phi_{-1} \in V_{-1}$. Thus $u_i = 0$. ∎

1.3 Examples

In this section, we will give some examples.

Example 1.18. Let $n \geq 2$ be a positive integer. Let $\mathfrak{g} = \mathfrak{gl}_1 \oplus \mathfrak{sl}_n$ and $V = \mathbb{C}^n$. We define a representation ρ of $\mathfrak{g}$ on V by:

$$\rho(a, A)v = av + Av \qquad (a \in \mathfrak{gl}_1,\ A \in \mathfrak{sl}_n,\ v \in V)$$

where $v \in V$ is considered as a column vector. The dual space V^* is also identified with $\mathbb{C}^n$ and the pairing between V and V^* is given by:

$$\begin{aligned}\langle \cdot, \cdot \rangle : V \times V^* &\to \mathbb{C} \\ (v, \phi) &\mapsto {}^t v \phi.\end{aligned} \tag{1.45}$$

Then the dual representation ρ^* of $\mathfrak{g}$ on V^* is given by:

$$\rho(a, A)\phi = -a\phi - {}^t A\phi \qquad (a \in \mathfrak{gl}_1,\ A \in \mathfrak{sl}_n,\ \phi \in V^*)$$

where ${}^t A$ is a transpose of A. Let B_0^1 be a bilinear form on $\mathfrak{g}$ given by:

$$B_0^1\big((a, A), (a', A')\big) := \frac{n}{n+1} aa' + \operatorname{Tr} AA' \qquad (a, a' \in \mathfrak{gl}_1, A, A' \in \mathfrak{sl}_n). \tag{1.46}$$

Then $(\mathfrak{g}, \rho, V, B_0^1)$ is a standard quadruplet. Then its Φ-map $\hat{\Phi}_\rho^1$ from $V \otimes V^*$ to $\mathfrak{g}$ is derived by:

$$\hat{\Phi}_\rho^1\big(v \otimes \phi\big) = \big(\frac{n+1}{n} {}^t v\phi, v {}^t\phi - \frac{1}{n} {}^t v \phi 1_n\big). \tag{1.47}$$

In this situation, the m-graduation of $(\mathfrak{g}, \rho, V, B_0^1)$ is $\{0\}$ for all $m \geq 2$. In fact, we can identify V_1 with V. Then, for any $v_1, u_1 \in V_1$ and $\phi \in V^*$, we have

$$\begin{aligned}&p_1\big(v_1 \otimes u_1\big)\big(\phi\big) \\ =&\rho_1\big(\frac{n+1}{n} {}^t v_1 \phi, v_1 {}^t \phi - \frac{1}{n} {}^t v_1 \phi 1_n\big) u_1 - \rho_1\big(\frac{n+1}{n} {}^t u_1 \phi, u_1 {}^t \phi - \frac{1}{n} {}^t u_1 \phi 1_n\big) v_1 \\ =&\frac{n+1}{n} {}^t v_1 \phi u_1 + v_1 {}^t \phi u_1 - \frac{1}{n} {}^t v_1 \phi u_1 - \frac{n+1}{n} {}^t u_1 \phi v_1 - u_1 {}^t \phi v_1 + \frac{1}{n} {}^t u_1 \phi v_1 \\ =&\frac{n+1}{n} u_1 {}^t v_1 \phi + v_1 {}^t u_1 \phi - \frac{1}{n} u_1 {}^t v_1 \phi - \frac{n+1}{n} v_1 {}^t u_1 \phi - u_1 {}^t v_1 \phi + \frac{1}{n} v_1 {}^t u_1 \phi \\ =&0.\end{aligned}$$

Thus our claim holds.

Example 1.19. We retain the notation $\mathfrak{g}$, ρ and V of Example 1.18 and choose another bilinear form B_0^2 on $\mathfrak{g}$ given as follows:

$$B_0^2\big((a, A), (a', A')\big) := naa' + \operatorname{Tr} AA' \qquad (a, a' \in \mathfrak{gl}_1, A, A' \in \mathfrak{sl}_n). \tag{1.48}$$

Then we have

$$\hat{\Phi}_\rho^2\big(v \otimes \phi\big) = \big(\frac{1}{n} {}^t v\phi, v {}^t\phi - \frac{1}{n} {}^t v \phi 1_n\big) \qquad (v \in V, \phi \in V^*) \tag{1.49}$$

where $\hat{\Phi}^2_\rho$ is the Φ-map of $(\mathfrak{g}, \rho, V, B_0^2)$. Identifying V_1 with V, we have

$$p_1\big(v_1 \otimes u_1\big)\big(\phi\big) = \big(v_1{}^t u_1 - u_1{}^t v_1\big)\phi \qquad (v_1, u_1 \in V, \phi \in V^*) \tag{1.50}$$

from the argument in Example 1.18. Therefore, we can identify V_2 with the totality of skew-symmetric matrices of size n and the representation ρ_2 on V_2 is given as follows:

$$\rho_2(a, A)u_2 = 2au_2 + Au_2 + u_2{}^t A \qquad (a \in \mathfrak{gl}_1,\ A \in \mathfrak{sl}_n,\ u_2 \in V_2). \tag{1.51}$$

In this situation, the m-graduation of $(\mathfrak{g}, \rho, V, B_0^2)$ is $\{0\}$ for all $m \geq 3$. In fact, for any $v_1, u_1, w_1 \in V_1$ and $\phi \in V^*$, we have

$$\begin{aligned}
&p_2\big(w_1 \otimes p_1(v_1 \otimes u_1)\big)\big(\phi\big)\\
=&\rho_2\big(\frac{1}{n}{}^t w_1\phi, w_1{}^t\phi - \frac{1}{n}{}^t w_1\phi 1_n\big)p_1\big(v_1 \otimes u_1\big) + p_1\big(w_1 \otimes p_1(v_1 \otimes u_1)(\phi)\big)\\
=&p_1\big(\rho_1\big(\frac{1}{n}{}^t w_1\phi, w_1{}^t\phi - \frac{1}{n}{}^t w_1\phi 1_n\big)v_1 \otimes u_1\big) + p_1\big(v_1 \otimes \rho_1\big(\frac{1}{n}{}^t w_1\phi, w_1{}^t\phi - \frac{1}{n}{}^t w_1\phi 1_n\big)u_1\big)\\
&+ p_1\big(w_1 \otimes p_1(v_1 \otimes u_1)(\phi)\big)\\
=&p_1\big(w_1{}^t\phi v_1 \otimes u_1\big) + p_1\big(v_1 \otimes w_1{}^t\phi u_1\big) + p_1\big(w_1 \otimes \big(v_1{}^t u_1 - u_1{}^t v_1\big)\phi\big)\\
=&w_1{}^t\phi v_1{}^t u_1 - u_1{}^t v_1\phi^t w_1 + v_1{}^t u_1\phi^t w_1 - w_1{}^t\phi u_1{}^t v_1\\
&+ w_1{}^t\phi u_1{}^t v_1 - v_1{}^t u_1\phi^t w_1 - w_1{}^t\phi v_1{}^t u_1 + u_1{}^t v_1\phi^t w_1\\
=&0.
\end{aligned}$$

Thus our claim holds.

2 A Lie algebra associated to a standard quadruplet

In the previous section, we constructed a family of n-graduations from a standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$. In this section, we will consider the direct sum of n-graduations and denote it by $L(\mathfrak{g}, \rho, V, B_0)$. Then we can define a bilinear map from $L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0)$ to $L(\mathfrak{g}, \rho, V, B_0)$. We will prove that this bilinear map satisfies the axioms of a Lie algebra. First of all, let us define the following bilinear maps.

Definition 2.1. For any $n \in \mathbb{Z}$, we define bilinear maps

$$\begin{aligned}
&[\cdot,\cdot]^0_n : V_0 \times V_n \to V_n,\\
&[\cdot,\cdot]^1_n : V_1 \times V_n \to V_{n+1},\\
&[\cdot,\cdot]^{-1}_n : V_{-1} \times V_n \to V_{n-1}
\end{aligned}$$

by

$$[a, x_n]_n^0 := \rho_n(a)x_n, \tag{2.1}$$

$$[v_1, x_n]_n^1 := \begin{cases} p_n(v_1 \otimes x_n) & (n \geq 0) \\ -x_n\big(\Phi_\rho^{-1}(v_1)\big) & (n \leq -1) \end{cases}, \tag{2.2}$$

$$[\phi_{-1}, x_n]_n^{-1} := \begin{cases} -x_n\big(\Psi_\rho^{-1}(\phi_{-1})\big) & (n \geq 1) \\ q_n(\phi_{-1} \otimes x_n) & (n \leq 0) \end{cases} \tag{2.3}$$

where $a \in V_0$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$ and $x_n \in V_n$.

We have the following two propositions.

Proposition 2.2. *For any $n \in \mathbb{Z}$, $a \in V_0$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$ and $x_n \in V_n$, we have the following equations*:

$$[p_0(v_1 \otimes a), x_n]_n^1 = \big[v_1, [a, x_n]_n^0\big]_n^1 - \big[a, [v_1, x_n]_n^1\big]_{n+1}^0, \tag{2.4}$$

$$[q_0(\phi_{-1} \otimes a), x_n]_n^{-1} = \big[\phi_{-1}, [a, x_n]_n^0\big]_n^{-1} - \big[a, [\phi_{-1}, x_n]_n^{-1}\big]_{n+1}^0. \tag{2.5}$$

Proof. For $n \geq 0$, we have

$$\begin{aligned} [p_0(v_1 \otimes a), x_n]_n^1 =& [-\rho_1(a)v_1, x_n]_n^1 \\ =& p_n\big(-\rho_1(a)v_1 \otimes x_n\big) \\ =& p_n\big(v_1 \otimes \rho_n(a)x_n\big) - \rho_{n+1}(a)p_n\big(v_1 \otimes x_n\big) \\ =& \big[v_1, [a, x_n]_n^0\big]_n^1 - \big[a, [v_1, x_n]_n^1\big]_{n+1}^0. \end{aligned}$$

For $n \leq -1$, we have

$$\begin{aligned} [p_0(v_1 \otimes a), x_n]_n^1 =& [-\rho_1(a)v_1, x_n]_n^1 \\ =& x_n\big(\rho(a)\big(\Phi_\rho^{-1}(v_1)\big)\big) \\ =& -\big(\rho_n(a)x_n\big)\big(\Phi_\rho^{-1}(v_1)\big) + \rho_{n+1}(a)\big(x_n\big(\Phi_\rho^{-1}(v_1)\big)\big) \\ =& \big[v_1, [a, x_n]_n^0\big]_n^1 - \big[a, [v_1, x_n]_n^1\big]_{n+1}^0. \end{aligned}$$

Thus we have (2.4). Similarly we can obtain (2.5). ∎

Proposition 2.3. *For any $n \in \mathbb{Z}$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$ and $x_n \in V_n$, we have the following equation*:

$$\big[\phi_{-1}, [v_1, x_n]_n^1\big]_{n+1}^{-1} = \big[[\phi_{-1}, v_1]_1^{-1}, x_n\big]_n^0 + \big[v_1, [\phi_{-1}, x_n]_n^{-1}\big]_{n-1}^1. \tag{2.6}$$

Proof. For $n \geq 1$, we have

$$\begin{aligned}
\big[\phi_{-1}, [v_1, x_n]_n^1\big]_{n+1}^{-1} =& \big[\phi_{-1}, p_n(v_1 \otimes x_n)\big]_{n+1}^{-1} \\
=& - p_n\big(v_1 \otimes x_n\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big) \\
=& - \rho_n\big(v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big)x_n - p_{n-1}\big(v_1 \otimes x_n\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big) \\
=& \big[[\phi_{-1}, v_1]_1^{-1}, x_n\big]_n^0 + \big[v_1, [\phi_{-1}, x_n]_n^{-1}\big]_{n-1}^1.
\end{aligned}$$

For $n = 0$, we have

$$\begin{aligned}
\big[\phi_{-1}, [v_1, x_0]_0^1\big]_1^{-1} =& - \big[\phi_{-1}, \rho_1(x_0)v_1\big]_1^{-1} \\
=& \big(\rho_1(x_0)v_1\big)\big(\Psi_\rho^{-1}(\phi_{-1})\big) \\
=& \rho_0(x_0)\big(v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big) - v_1\big(\Psi_\rho^{-1}\big(\rho_{-1}(x_0)\phi_{-1}\big)\big) \\
=& \big[[\phi_{-1}, v_1]_1^{-1}, x_0\big]_0^0 + \big[v_1, [\phi_{-1}, x_0]_0^{-1}\big]_{-1}^1.
\end{aligned}$$

For $n \leq -1$, we have

$$\begin{aligned}
\big[\phi_{-1}, [v_1, x_n]_n^1\big]_{n+1}^{-1} =& - \big[\phi_{-1}, x_n\big(\Phi_\rho^{-1}(v_1)\big)\big]_{n+1}^{-1} \\
=& - q_{n+1}\big(\phi_{-1} \otimes x_n\big(\Phi_\rho^{-1}(v_1)\big)\big) \\
=& \rho_n\big(\phi_{-1}\big(\Phi_\rho^{-1}(v_1)\big)\big)x_n - q_n\big(\phi_{-1} \otimes x_n\big)\big(\Phi_\rho^{-1}(v_1)\big) \\
=& \big[[\phi_{-1}, v_1]_1^{-1}, x_n\big]_n^0 + \big[v_1, [\phi_{-1}, x_n]_n^{-1}\big]_{n-1}^1.
\end{aligned}$$

Thus we have (2.6). ∎

Definition 2.4. Assume that $i \geq 0$. For any $n \in \mathbb{Z}$, we define bilinear maps

$$[\cdot, \cdot]_n^{i+1} : V_{i+1} \times V_n \to V_{i+n+1}, \tag{2.7}$$

$$[\cdot, \cdot]_n^{-i-1} : V_{-i-1} \times V_n \to V_{-i+n-1} \tag{2.8}$$

by

$$\big[p_i(v_1 \otimes u_i), x_n\big]_n^{i+1} := \big[v_1, [u_i, x_n]_n^i\big]_{i+n}^1 - \big[u_i, [v_1, x_n]_n^1\big]_{n+1}^i \tag{2.9}$$

$$(v_1 \in V_1, u_i \in V_i, x_n \in V_n)$$

and

$$\big[q_{-i}(\phi_{-1} \otimes \psi_{-i}), x_n\big]_n^{-i-1} := \big[\phi_{-1}, [\psi_{-i}, x_n]_n^{-i}\big]_{-i+n}^{-1} - \big[\psi_{-i}, [\phi_{-1}, x_n]_n^{-1}\big]_{n-1}^{-i} \tag{2.10}$$

$$(\phi_{-1} \in V_{-1}, \psi_{-i} \in V_{-i}, x_n \in V_n)$$

inductively.

We must prove the well-definedness of Definition 2.4. First, let us start with the case where $i = 0$. To check this, it is sufficient to show that the bilinear maps $\left[\cdot,\cdot\right]_n^1$ and $\left[\cdot,\cdot\right]_n^{-1}$ given in (2.9) and (2.10) coincide with (2.2) and (2.3) respectively. It follows from Proposition 2.2 immediately. Next, let us show the following proposition to prove the well-definedness of (2.9) for $i \geq 1$.

Proposition 2.5. *Let $i \geq 0$ and assume that $\left[\cdot,\cdot\right]_n^i$ is well-defined for any $n \in \mathbb{Z}$. Take elements $v_1^1, \ldots, v_1^l \in V_1$ and $u_i^1, \ldots, u_i^l \in V_i$ satisfying*

$$\sum_{s=1}^{l} p_i(v_1^s \otimes u_i^s) = 0. \tag{2.11}$$

Then for any $x_n \in V_n$, we have

$$\sum_{s=1}^{l} \left(\left[v_1^s, \left[u_i^s, x_n\right]_n^i\right]_{i+n}^1 - \left[u_i^s, \left[v_1^s, x_n\right]_n^1\right]_{n+1}^i \right) = 0 \tag{2.12}$$

and hence we obtain that (2.9) is well-defined.

Proof. We argue by induction on i. For $i = 0$. By Proposition 2.2, we have our result for any $n \in \mathbb{Z}$. Moreover, by definition and Proposition 2.3, we have the following equations:

$$\left[a, \left[v_1, x_n\right]_n^1\right]_{n+1}^0 = \left[\left[a, v_1\right]_1^0, x_n\right]_n^1 + \left[v_1, \left[a, x_n\right]_n^0\right]_n^1, \tag{2.13}$$

$$\left[\phi_{-1}, \left[v_1, x_n\right]_n^1\right]_{n+1}^{-1} = \left[\left[\phi_{-1}, v_1\right]_1^{-1}, x_n\right]_n^0 + \left[v_1, \left[\phi_{-1}, x_n\right]_n^{-1}\right]_{n-1}^1, \tag{2.14}$$

$$\left[v_1, a\right]_0^1 = -\left[a, v_1\right]_1^0 = -\rho_1(a)v_1, \tag{2.15}$$

$$\left[u_1, v_1\right]_1^1 = -\left[v_1, u_1\right]_1^1 = -p_1(v_1 \otimes u_1), \tag{2.16}$$

$$\left[u_1, \phi_{-1}\right]_{-1}^1 = -\left[\phi_{-1}, u_1\right]_1^{-1} = u_1\left(\Psi_\rho^{-1}(\phi_{-1})\right) \tag{2.17}$$

where $a \in V_0$, $v_1, u_1 \in V_1$, $\phi_{-1} \in V_{-1}$ and $x_n \in V_n$.

For $i \geq 1$. We assume that the bilinear maps $[\cdot,\cdot]_n^i$ are well-defined for any $n \in \mathbb{Z}$ and satisfy the following equations:

$$\left[a, \left[u_i, x_n\right]_n^i\right]_{i+n}^0 = \left[\left[a, u_i\right]_i^0, x_n\right]_n^i + \left[u_i, \left[a, x_n\right]_n^0\right]_n^i, \tag{2.18}$$

$$\left[\phi_{-1}, \left[u_i, x_n\right]_n^i\right]_{i+n}^{-1} = \left[\left[\phi_{-1}, u_i\right]_i^{-1}, x_n\right]_n^{i-1} + \left[u_i, \left[\phi_{-1}, x_n\right]_n^{-1}\right]_{n-1}^i, \tag{2.19}$$

$$\left[u_i, a\right]_0^i = -\left[a, u_i\right]_i^0 = -\rho_i(a)u_i, \tag{2.20}$$

$$\left[u_i, v_1\right]_1^i = -\left[v_1, u_i\right]_i^1 = -p_i(v_1 \otimes u_i), \tag{2.21}$$

$$\left[u_i, \phi_{-1}\right]_{-1}^i = -\left[\phi_{-1}, u_i\right]_i^{-1} = u_i\left(\Psi_\rho^{-1}(\phi_{-1})\right) \tag{2.22}$$

where $a \in V_0$, $v_1 \in V_1$, $u_i \in V_i$, $\phi_{-1} \in V_{-1}$ and $x_n \in V_n$.

We fix i and argue by induction on n.

(1) The case where $n \geq 0$.

For $n=0$, by the induction hypothesis on i, we have

$$\begin{aligned}&\big[v_1, \big[u_i, x_0\big]_0^i\big]_i^1 - \big[u_i, \big[v_1, x_0\big]_0^1\big]_1^i \\ =& - p_i\big(v_1 \otimes \rho_i(x_0)u_i\big) + \big[u_i, \rho_1(x_0)v_1\big]_1^i \\ =& - p_i\big(v_1 \otimes \rho_i(x_0)u_i\big) - p_i\big(\rho_1(x_0)v_1 \otimes u_i\big) \\ =& - \rho_{i+1}(x_0)p_i(v_1 \otimes u_i)\end{aligned} \tag{2.23}$$

where $v_1 \in V_1$, $u_i \in V_i$ and $x_0 \in V_0$. Thus, we have our result immediately.

For $n \geq 1$. Note that by (2.19), we have

$$\big[u_i, y_m\big]_m^i(\phi) = \big[u_i(\phi), y_m\big]_m^{i-1} + \big[u_i, y_m(\phi)\big]_{m-1}^i \tag{2.24}$$

for any $m \geq 1$, $u_i \in V_i$, $y_m \in V_m$ and $\phi \in V^*$. Hence, we have

$$\begin{aligned}&\big[v_1, \big[u_i, x_n\big]_n^i\big]_{i+n}^1(\phi) - \big[u_i, \big[v_1, x_n\big]_n^1\big]_{n+1}^i(\phi) \\ =& p_{i+n}\big(v_1 \otimes \big[u_i, x_n\big]_n^i\big)(\phi) - \big[u_i, p_n\big(v_1 \otimes x_n\big)\big]_{n+1}^i(\phi) \\ =& \rho_{i+n}\big(v_1(\phi)\big)\big[u_i, x_n\big]_n^i \\ &+ p_{i+n-1}\big(v_1 \otimes \big(\big[u_i(\phi), x_n\big]_n^{i-1} + \big[u_i, x_n(\phi)\big]_{n-1}^i\big)\big) \\ &- \big[u_i(\phi), p_n(v_1 \otimes x_n)\big]_{n+1}^{i-1} \\ &- \big[u_i, \rho_n\big(v_1(\phi)\big)x_n + p_{n-1}\big(v_1 \otimes x_n(\phi)\big)\big]_n^i \\ =& \big[\rho_i\big(v_1(\phi)\big)u_i, x_n\big]_n^i \\ &+ p_{i+n-1}\big(v_1 \otimes \big[u_i(\phi), x_n\big]_n^{i-1}\big) - \big[u_i(\phi), p_n(v_1 \otimes x_n)\big]_{n+1}^{i-1} \\ &+ p_{i+n-1}\big(v_1 \otimes \big[u_i, x_n(\phi)\big]_{n-1}^i\big) - \big[u_i, p_{n-1}\big(v_1 \otimes x_n(\phi)\big)\big]_n^i \\ =& \big[\rho_i\big(v_1(\phi)\big)u_i, x_n\big]_n^i + \big[p_{i-1}\big(v_1 \otimes u_i(\phi)\big), x_n\big]_n^i \\ &+ p_{i+n-1}\big(v_1 \otimes \big[u_i, x_n(\phi)\big]_{n-1}^i\big) - \big[u_i, p_n\big(v_1 \otimes x_n(\phi)\big)\big]_n^i \\ =& \big[p_i\big(v_1 \otimes u_i\big)(\phi), x_n\big]_n^i \\ &+ \big[v_1, \big[u_i, x_n(\phi)\big]_{n-1}^i\big]_{i+n-1}^1 - \big[u_i, \big[v_1, x_n(\phi)\big]_{n-1}^1\big]_n^i\end{aligned} \tag{2.25}$$

where $v_1 \in V_1$, $u_i \in V_i$, $x_n \in V_n$ and $\phi \in V^*$. Here, by the induction hypotheses on i and n, if $v_1^1, \ldots, v_1^l \in V_1$ and $u_i^1, \ldots, u_i^l \in V_i$ satisfy

$$\sum_{s=1}^{l} p_i(v_1^s \otimes u_i^s) = 0,$$

then we have

$$\sum_{s=1}^{l} \big[p_i\big(v_1^s \otimes u_i^s\big)(\phi), x_n\big]_n^i = 0$$

and

$$\sum_{s=1}^{l}\big(\big[v_1^s,\big[u_i^s,x_n(\phi)\big]_{n-1}^{i}\big]_{n+i-1}^{1}-\big[u_i^s,\big[v_1^s\otimes x_n(\phi)\big]_{n-1}^{1}\big]_{n}^{i}\big)=0$$

respectively. Therefore we have our result for any $n\geq 0$.

(2) The case where $n\leq -1$.

For $n=-1$, by the induction hypothesis on i, we have

$$\begin{aligned}
&\big[v_1,\big[u_i,x_{-1}\big]_{-1}^{i}\big]_{i-1}^{1}-\big[u_i,\big[v_1,x_{-1}\big]_{-1}^{1}\big]_{0}^{i}\\
=&\big[v_1,u_i\big(\Psi_\rho^{-1}(x_{-1})\big)\big]_{i-1}^{1}-\big[u_i,v_1\big(\Psi_\rho^{-1}(x_{-1})\big)\big]_{0}^{i}\\
=&p_{i-1}\big(v_1\otimes u_i\big(\Psi_\rho^{-1}(x_{-1})\big)\big)+\rho_i\big(v_1\big(\Psi_\rho^{-1}(x_{-1})\big)\big)u_i\\
=&p_i(v_1\otimes u_i)\big(\Psi_\rho^{-1}\big(x_{-1}\big)\big)
\end{aligned}\tag{2.26}$$

where $v_1\in V_1$, $u_i\in V_i$ and $x_{-1}\in V_{-1}$. Thus, we have our result immediately.

For $n\leq -2$. Note that by (2.19), we have

$$\big[u_i,q_{m+1}(\phi_{-1}\otimes y_{m+1})\big]_{m}^{i}=\big[u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big),y_{m+1}\big]_{m+1}^{i-1}+\big[\phi_{-1},\big[u_i,y_{m+1}\big]_{m+1}^{i}\big]_{i+m+1}^{-1}\tag{2.27}$$

for any $m\leq -1$, $u_i\in V_i$, $\phi_{-1}\in V_{-1}$ and $y_{m+1}\in V_{m+1}$. Hence, by the induction hypotheses on i and n and Proposition 2.3, we have

$$\begin{aligned}
&\big[v_1,\big[u_i,q_{n+1}(\phi_{-1}\otimes x_{n+1})\big]_{n}^{i}\big]_{i+n}^{1}-\big[u_i,\big[v_1,q_{n+1}(\phi_{-1}\otimes x_{n+1})\big]_{n}^{1}\big]_{n+1}^{i}\\
=&\big[v_1,\big[u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big),x_{n+1}\big]_{n+1}^{i-1}\big]_{i+n}^{1}+\big[v_1,\big[\phi_{-1},\big[u_i,x_{n+1}\big]_{n+1}^{i}\big]_{i+n+1}^{-1}\big]_{i+n}^{1}\\
&+\big[u_i,q_{n+1}(\phi_{-1}\otimes x_{n+1})\big(\Phi_\rho^{-1}(v_1)\big)\big]_{n+1}^{i}\\
=&\big[p_{i-1}\big(v_1\otimes u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big),x_{n+1}\big]_{n+1}^{i}-\big[u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big),x_{n+1}\big(\Phi_\rho^{-1}(v_1)\big)\big]_{n+2}^{i-1}\\
&+\big[v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big),\big[u_i,x_{n+1}\big]_{n+1}^{i}\big]_{i+n+1}^{0}+\big[\phi_{-1},\big[v_1,\big[u_i,x_{n+1}\big]_{n+1}^{i}\big]_{i+n+1}^{1}\big]_{i+n+2}^{-1}\\
&+\big[u_i,\rho_{n+1}\big(\phi_{-1}\big(\Phi_\rho^{-1}(v_1)\big)\big)x_{n+1}\big]_{n+1}^{i}+\big[u_i,q_{n+2}\big(\phi_{-1}\otimes x_{n+1}\big(\Phi_\rho^{-1}(v_1)\big)\big)\big]_{n+1}^{i}\\
=&\big[p_{i-1}\big(v_1\otimes u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big),x_{n+1}\big]_{n+1}^{i}-\big[u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big),x_{n+1}\big(\Phi_\rho^{-1}(v_1)\big)\big]_{n+2}^{i-1}\\
&+\big[\rho_i\big(v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big)u_i,x_{n+1}\big]_{n+1}^{i}+\big[u_i,\rho_{n+1}\big(v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big)x_{n+1}\big]_{n+1}^{i}\\
&+\big[\phi_{-1},\big[v_1,\big[u_i,x_{n+1}\big]_{n+1}^{i}\big]_{i+n+1}^{1}\big]_{i+n+2}^{-1}+\big[u_i,\rho_{n+1}\big(\phi_{-1}\big(\Phi_\rho^{-1}(v_1)\big)\big)x_{n+1}\big]_{n+1}^{i}\\
&+\big[u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big),x_{n+1}\big(\Phi_\rho^{-1}(v_1)\big)\big]_{n+2}^{i-1}+\big[\phi_{-1},\big[u_i,x_{n+1}\big(\Phi_\rho^{-1}(v_1)\big)\big]_{n+2}^{i}\big]_{i+n+2}^{-1}\\
=&\big[p_{i-1}\big(v_1\otimes u_i\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big),x_{n+1}\big]_{n+1}^{i}+\big[\rho_i\big(v_1\big(\Psi_\rho^{-1}(\phi_{-1})\big)\big)u_i,x_{n+1}\big]_{n+1}^{i}\\
&+\big[\phi_{-1},\big[v_1,\big[u_i,x_{n+1}\big]_{n+1}^{i}\big]_{i+n+1}^{1}\big]_{i+n+2}^{-1}-\big[\phi_{-1},\big[u_i,\big[v_1,x_{n+1}\big]_{n+1}^{1}\big]_{n+2}^{i}\big]_{i+n+2}^{-1}\\
=&\big[p_i(v_1\otimes u_i)\big(\Psi_\rho^{-1}(\phi_{-1})\big),x_{n+1}\big]_{n+1}^{i}\\
&+\Big[\phi_{-1},\Big(\big[v_1,\big[u_i,x_{n+1}\big]_{n+1}^{i}\big]_{i+n+1}^{1}-\big[u_i,[v_1,x_{n+1}]_{n+1}^{1}\big]_{n+2}^{i}\Big)\Big]_{i+n+2}^{-1}
\end{aligned}\tag{2.28}$$

where $v_1 \in V_1$, $u_i \in V_i$, $x_{n+1} \in V_{n+1}$ and $\phi_{-1} \in V_{-1}$. Here, by the induction hypotheses on i and n, if $v_1^1, \ldots, v_1^l \in V_1$ and $u_i^1, \ldots, u_i^l \in V_i$ satisfy

$$\sum_{s=1}^{l} p_i(v_1^s \otimes u_i^s) = 0,$$

then we have

$$\sum_{s=1}^{l} \left[p_i(v_1^s \otimes u_i^s)\big(\Psi_\rho^{-1}(\phi_{-1})\big), x_{n+1}\right]_{n+1}^{i} = 0$$

and

$$\sum_{s=1}^{l} \Big[\phi_{-1}, \Big(\big[v_1^s, [u_i^s, x_{n+1}]_{n+1}^{i}\big]_{i+n+1}^{1} - \big[u_i^s, [v_1^s, x_{n+1}]_{n+1}^{1}\big]_{n+2}^{i}\Big)\Big]_{i+n+2}^{-1} = 0$$

respectively. Therefore we have our result for any $n \leq -1$ and we obtain bilinear maps $[\cdot,\cdot]_n^{i+1}$ for all $n \in \mathbb{Z}$.

In order to complete the proof, let us show that the bilinear maps $[\cdot,\cdot]_n^{i+1}$ $(n \in \mathbb{Z})$ satisfy the following equations:

$$\big[a, [u_{i+1}, x_n]_n^{i+1}\big]_{i+n+1}^{0} = \big[[a, u_{i+1}]_{i+1}^{0}, x_n\big]_n^{i+1} + \big[u_{i+1}, [a, x_n]_n^{0}\big]_n^{i+1}, \tag{2.29}$$

$$\big[\phi_{-1}, [u_{i+1}, x_n]_n^{i+1}\big]_{i+n+1}^{-1} = \big[[\phi_{-1}, u_{i+1}]_{i+1}^{-1}, x_n\big]_n^{i} + \big[u_{i+1}, [\phi_{-1}, x_n]_n^{-1}\big]_{n-1}^{i+1}, \tag{2.30}$$

$$[u_{i+1}, a]_0^{i+1} = -[a, u_{i+1}]_{i+1}^{0} = -\rho_{i+1}(a)u_{i+1}, \tag{2.31}$$

$$[u_{i+1}, v_1]_1^{i+1} = -[v_1, u_{i+1}]_{i+1}^{1} = -p_{i+1}(v_1 \otimes u_{i+1}), \tag{2.32}$$

$$[u_{i+1}, \phi_{-1}]_{-1}^{i+1} = -[\phi_{-1}, u_{i+1}]_{i+1}^{-1} = u_{i+1}\big(\Psi_\rho^{-1}(\phi_{-1})\big) \tag{2.33}$$

where $a \in V_0$, $v_1 \in V_1$, $u_{i+1} \in V_{i+1}$, $\phi_{-1} \in V_{-1}$ and $x_n \in V_n$.

(2.31) and (2.33) follow from (2.23) and (2.26) respectively.

Let us show (2.29). We can assume that $u_{i+1} = p_i(v_1 \otimes u_i)$ for some $v_1 \in V_1$ and $u_i \in V_i$

without loss of generality. Then we have

$$\begin{aligned}
&\big[a, \big[p_i(v_1 \otimes u_i), x_n\big]_n^{i+1}\big]_{i+n+1}^0 \\
=&\big[a, \big[v_1, \big[u_i, x_n\big]_n^i\big]_{i+n}^1\big]_{i+n+1}^0 - \big[a, \big[u_i, \big[v_1, x_n\big]_n^1\big]_{n+1}^i\big]_{i+n+1}^0 \\
=&\big[\rho_1(a)v_1, \big[u_i, x_n\big]_n^i\big]_{i+n}^1 + \big[v_1, \big[\rho_i(a)u_i, x_n\big]_n^i\big]_{i+n}^1 + \big[v_1, \big[u_i, \rho_n(a)x_n\big]_n^i\big]_{i+n}^1 \\
&- \big[\rho_i(a)u_i, \big[v_1, x_n\big]_n^1\big]_{n+1}^i - \big[u_i, \big[\rho_1(a)v_1, x_n\big]_n^1\big]_{n+1}^i - \big[u_i, \big[v_1, \rho_n(a)x_n\big]_n^1\big]_{n+1}^i \\
=&\big[\rho_1(a)v_1, \big[u_i, x_n\big]_n^i\big]_{i+n}^1 - \big[u_i, \big[\rho_1(a)v_1, x_n\big]_n^1\big]_{n+1}^i \\
&+ \big[v_1, \big[\rho_i(a)u_i, x_n\big]_n^i\big]_{i+n}^1 - \big[\rho_i(a)u_i, \big[v_1, x_n\big]_n^1\big]_{n+1}^i \\
&+ \big[v_1, \big[u_i, \rho_n(a)x_n\big]_n^i\big]_{i+n}^1 - \big[u_i, \big[v_1, \rho_n(a)x_n\big]_n^1\big]_{n+1}^i \\
=&\big[p_i(\rho_1(a)v_1 \otimes u_i), x_n\big]_n^{i+1} + \big[p_i(v_1 \otimes \rho_i(a)u_i), x_n\big]_n^{i+1} \\
&+ \big[p_i(v_1 \otimes u_i), \rho_n(a)x_n\big]_n^{i+1} \\
=&\big[\big[a, p_i(v_1 \otimes u_i)\big]_{i+1}^0, x_n\big]_n^{i+1} + \big[p_i(v_1 \otimes u_i), \big[a, x_n\big]_n^0\big]_n^{i+1}.
\end{aligned}$$

Thus (2.29) holds.

Let us show (2.30). If $n \neq 0$, (2.30) follows from (2.25) and (2.28). The case where $n = 0$ is obtained from (2.31) and (2.33). Thus (2.30) holds.

Finally, let us show (2.32). By (2.30) and (2.31), for any $\phi \in V^*$, we have

$$\begin{aligned}
&\big[u_{i+1}, v_1\big]_1^{i+1}(\phi) \\
=&\big[u_{i+1}(\phi), v_1\big]_1^i + \big[u_{i+1}, v_1(\phi)\big]_0^{i+1} \\
=&- p_i\big(v_1 \otimes u_{i+1}(\phi)\big) - \rho_{i+1}\big(v_1(\phi)\big)u_{i+1} \\
=&- p_{i+1}\big(v_1 \otimes u_{i+1}\big)(\phi).
\end{aligned}$$

Thus (2.32) holds. This completes the proof. ∎

In the same way, we can prove the following proposition.

Proposition 2.6. *Assume that $j \geq 0$ and $[\cdot,\cdot]_n^{-j}$ is well-defined for any $n \in \mathbb{Z}$. Take elements $\phi_{-1}^1, \ldots, \phi_{-1}^l \in V_{-1}$ and $\psi_{-j}^1, \ldots, \psi_{-j}^l \in V_{-j}$ satisfying*

$$\sum_{s=1}^{l} q_{-j}(\phi_{-1}^s \otimes \psi_{-j}^s) = 0. \tag{2.34}$$

Then for any $x_n \in V_n$, we have

$$\sum_{s=1}^{l}\big(\big[\phi_{-1}^s, \big[\psi_{-j}^s, x_n\big]_n^{-j}\big]_{-j+n}^{-1} - \big[\psi_{-j}^s, \big[\phi_{-1}^s, x_n\big]_n^{-1}\big]_{n-1}^{-j}\big) = 0 \tag{2.35}$$

and hence we obtain a bilinear map

$$[\cdot,\cdot]_n^{-j-1} : V_{-j-1} \times V_n \to V_{-j+n-1}$$

satisfying the following equations:

$$\begin{aligned}
&[a, [\psi_{-j-1}, x_n]_n^{-j-1}]_{n-j-1}^0 = [[a, \psi_{-j-1}]_{-j-1}^0, x_n]_n^{-j-1} + [\psi_{-j-1}, [a, x_n]_n^0]_n^{-j-1}, && (2.36)\\
&[v_1, [\psi_{-j-1}, x_n]_n^{-j-1}]_{-j+n-1}^1 = [[v_1, \psi_{-j-1}]_{-j-1}^1, x_n]_n^{-j} + [\psi_{-j-1}, [v_1, x_n]_n^1]_{n+1}^{-j-1}, && (2.37)\\
&[\psi_{-j-1}, a]_0^{-j-1} = -[a, \psi_{-j-1}]_{-j-1}^0 = -\rho_{-j-1}(a)\psi_{-j-1}, && (2.38)\\
&[\psi_{-j-1}, \phi_{-1}]_{-1}^{-j-1} = -[\phi_{-1}, \psi_{-j-1}]_{-j-1}^{-1} = -q_{-j-1}(\phi_{-1} \otimes \psi_{-j-1}), && (2.39)\\
&[\psi_{-j-1}, v_1]_1^{-j-1} = -[v_1, \psi_{-j-1}]_{-j-1}^1 = \psi_{-j-1}(\Phi_\rho^{-1}(v_1)). && (2.40)
\end{aligned}$$

where $a \in V_0$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$, $\psi_{-j-1} \in V_{-j-1}$ *and* $x_n \in V_n$.

Then Proposition 2.5 and Proposition 2.6 are summarized as follows.

Definition 2.7. We put

$$L(\mathfrak{g}, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n \tag{2.41}$$

and define a bilinear map

$$[\cdot,\cdot] : L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0) \to L(\mathfrak{g}, \rho, V, B_0)$$

by

$$[x_n, y_m] := [x_n, y_m]_m^n \tag{2.42}$$

where $x_n \in V_n$ and $y_m \in V_m$.

Proposition 2.8. *The bilinear map* $[\cdot,\cdot]$ *satisfies the following equations*:

$$\begin{aligned}
&[a, x] = -[x, a], && (2.43)\\
&[v_1, x] = -[x, v_1], && (2.44)\\
&[\phi_{-1}, x] = -[x, \phi_{-1}], && (2.45)\\
&[a, [x, y]] = [[a, x], y] + [x, [a, y]], && (2.46)\\
&[y, [v_1, x]] = [v_1, [y, x]] - [[v_1, y], x], && (2.47)\\
&[y, [\phi_{-1}, x]] = [\phi_{-1}, [y, x]] - [[\phi_{-1}, y], x] && (2.48)
\end{aligned}$$

where $a \in V_0$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$ *and* $x, y \in L(\mathfrak{g}, \rho, V, B_0)$.

Let us show that $[\cdot,\cdot]$ satisfies the axioms of a Lie algebra.

Proposition 2.9. *For any $x, y \in L(\mathfrak{g}, \rho, V, B_0)$, we have*

$$[x, y] + [y, x] = 0. \tag{2.49}$$

Namely, $[\cdot, \cdot]$ is skew-symmetric.

Proof. Without loss of generality, it is sufficient to show in the cases where $x = x_n \in V_n$ for each $n \in \mathbb{Z}$. We assume that $n \geq 0$ and argue by induction on n.

For $n = 0$, our claim follows from Proposition 2.8.

For $n \geq 1$. Without loss of generality, we can assume that $x_n = p_{n-1}(v_1 \otimes x_{n-1})$ for some $v_1 \in V_1$ and $x_{n-1} \in V_{n-1}$. Then, by (2.9) and (2.47), we have

$$\begin{aligned}
&[p_{n-1}(v_1 \otimes x_{n-1}), y] + [y, p_{n-1}(v_1 \otimes x_{n-1})] \\
=&[v_1, [x_{n-1}, y]] - [x_{n-1}, [v_1, y]] \\
&+ [v_1, [y, x_{n-1}]] - [[v_1, y], x_{n-1}].
\end{aligned}$$

By the induction hypothesis, we have

$$[x_{n-1}, y] + [y, x_{n-1}] = 0$$

and

$$[x_{n-1}, [v_1, y]] + [[v_1, y], x_{n-1}] = 0.$$

Hence our claim holds. Similarly we can obtain the proof of the cases where $n \leq -1$. This completes the proof. ∎

Proposition 2.10. *For any $x, y, z \in L(\mathfrak{g}, \rho, V, B_0)$, we have*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{2.50}$$

Namely, $[\cdot, \cdot]$ satisfies the Jacobi identity.

Proof. Without loss of generality, it is sufficient to show in the cases where $x = x_n \in V_n$ for each $n \in \mathbb{Z}$. We assume $n \geq 0$ and argue by induction on n.

For $n = 0$ and 1, our claim follows from Proposition 2.8.

For $n \geq 2$. Without loss of generality, we can assume that $x_n = p_{n-1}(v_1 \otimes x_{n-1})$ for some

$v_1 \in V_1$ and $x_{n-1} \in V_{n-1}$. Then, by the induction hypothesis, we have

$$
\begin{aligned}
&[p_{n-1}(v_1 \otimes x_{n-1}), [y, z]] \\
=&[v_1, [x_{n-1}, [y, z]]] - [x_{n-1}, [v_1, [y, z]]] \\
=&[v_1, [[x_{n-1}, y], z]] + [v_1, [y, [x_{n-1}, z]]] \\
&- [x_{n-1}, [[v_1, y], z]] - [x_{n-1}, [y, [v_1, z]]] \\
=&[v_1, [[x_{n-1}, y], z]] + [y, [v_1, [x_{n-1}, z]]] + [[v_1, y], [x_{n-1}, z]] \\
&- [x_{n-1}, [[v_1, y], z]] - [y, [x_{n-1}, [v_1, z]]] - [[x_{n-1}, y], [v_1, z]] \\
=&[[v_1, [x_{n-1}, y]], z] - [[x_{n-1}, [v_1, y]], z] + [y, [p_{n-1}(v_1 \otimes x_{n-1}), z]] \\
=&[[p_{n-1}(v_1 \otimes x_{n-1}), y], z] + [y, [p_{n-1}(v_1 \otimes x_{n-1}), z]].
\end{aligned}
$$

Thus we have our claim. We can also obtain the proof of the cases where $n \leq -1$ by the same argument. This completes the proof. ∎

By Propositions 2.9 and 2.10, we have the following theorem.

Theorem 2.11. *$L(\mathfrak{g}, \rho, V, B_0)$ is a Lie algebra with the bracket $[\cdot, \cdot]$. We call the Lie algebra thus obtained* **the Lie algebra associated to** $(\mathfrak{g}, \rho, V, B_0)$.

In particular, for a given reductive Lie algebra $\mathfrak{g}$ and its finite-dimensional completely reducible representation (ρ, V), there exists a graded Lie algebra $\sum_{n\in\mathbb{Z}} \mathfrak{g}_n$ such that its Lie subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}$ and the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is isomorphic to (ρ, V).

In the next section, we will construct a bilinear form on $L(\mathfrak{g}, \rho, V, B_0) = \bigoplus_{n\in\mathbb{Z}} V_n$ such that the restriction to $V_0 \times V_0$ coincides with B_0.

3 Some properties of $L(\mathfrak{g}, \rho, V, B_0)$

3.1 A bilinear form on $L(\mathfrak{g}, \rho, V, B_0)$

In this section, we will consider the structure of the Lie algebra $L(\mathfrak{g}, \rho, V, B_0)$ associated to a standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$. First of all, we construct a non-degenerate symmetric $L(\mathfrak{g}, \rho, V, B_0)$-invariant bilinear form on $L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0)$.

Definition 3.1. We define a bilinear form B on $L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0)$ by

$$B\Big(\sum_{n\in\mathbb{Z}} x_n, \sum_{m\in\mathbb{Z}} y_m\Big) = \sum_{s\geq 0} B_s\big(x_s, y_{-s}\big) + \sum_{t\leq -1} B_{-t}\big(y_{-t}, x_t\big) \tag{3.1}$$

where $x_n \in V_n$ and $y_m \in V_m$.

Proposition 3.2. *The bilinear form B is non-degenerate symmetric and $L(\mathfrak{g}, \rho, V, B_0)$-invariant.*

Proof. It can be easily proved that B is non-degenerate and symmetric. Let us prove that B is $L(\mathfrak{g}, \rho, V, B_0)$-invariant. For this, it is enough to show that we have

$$B([u_i, x_n], y_m) = B(x_n, [y_m, u_i]) \tag{3.2}$$

where $i \geq 0$, $n, m \in \mathbb{Z}$, $u_i \in V_i$, $x_n \in V_n$, $y_m \in V_m$ and $i + n + m = 0$. We argue it by induction on i.

For $i = 0, 1$, our result follows from (1.43) and (1.44) immediately.

For $i \geq 2$. We can assume that $u_i = p_{i-1}(v_1 \otimes u_{i-1})$ for some $v_1 \in V_1$ and $u_{i-1} \in V_{i-1}$ without loss of generality. Then, by the induction hypothesis, we have

$$\begin{aligned}
&B\big(\big[p_{i-1}(v_1 \otimes u_{i-1}), x_n\big], y_m\big) \\
=&B\big(\big[v_1, [u_{i-1}, x_n]\big] - \big[u_{i-1}, [v_1, x_n]\big], y_m\big) \\
=&B\big([u_{i-1}, x_n], [y_m, v_1]\big) - B\big([v_1, x_n], [y_m, u_{i-1}]\big) \\
=&B\big(x_n, \big[[y_m, v_1], u_{i-1}\big]\big) - B\big(x_n, \big[[y_m, u_{i-1}], v_1\big]\big) \\
=&B\big(x_n, \big[y_m, p_{i-1}(v_1 \otimes u_{i-1})\big]\big).
\end{aligned}$$

This completes the proof. ∎

We prove the "universality" of the Lie algebras associated to a standard quadruplet.

Proposition 3.3. *Assume that a $\mathbb{Z}$-graded Lie algebra $\hat{\mathfrak{g}} = \sum_{n\in\mathbb{Z}} \hat{\mathfrak{g}}_n$ with a symmetric $\hat{\mathfrak{g}}$-invariant bilinear form $\hat{B}$ satisfies the following conditions*:

$$\hat{\mathfrak{g}}_0 \text{ and } \hat{\mathfrak{g}}_1 \text{ are finite-dimensional vector spaces,} \tag{3.3}$$

$$\text{the adjoint representation of } \hat{\mathfrak{g}}_0 \text{ on } \hat{\mathfrak{g}}_1 \text{ is completely reducible,} \tag{3.4}$$

$$\text{the restriction of } \hat{B} \text{ to } \hat{\mathfrak{g}}_i \times \hat{\mathfrak{g}}_{-i} \text{ is non-degenerate for each } i \geq 0, \tag{3.5}$$

$$[\hat{\mathfrak{g}}_1, \hat{\mathfrak{g}}_i] = \hat{\mathfrak{g}}_{i+1},\ [\hat{\mathfrak{g}}_{-1}, \hat{\mathfrak{g}}_{-j}] = \hat{\mathfrak{g}}_{-j-1} \text{ for each } i, j \geq 0, \tag{3.6}$$

$$[\hat{\mathfrak{g}}_1, \hat{\mathfrak{g}}_{-1}] = \hat{\mathfrak{g}}_0. \tag{3.7}$$

Then $(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0\times\hat{\mathfrak{g}}_0})$ is a standard quadruplet and the Lie algebra $L(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0\times\hat{\mathfrak{g}}_0})$ is isomorphic to $\hat{\mathfrak{g}}$ where ad *stands for the adjoint representation of $\hat{\mathfrak{g}}_0$ on $\hat{\mathfrak{g}}_1$.*

Obviously, $L(\mathfrak{g}, \rho, V, B_0)$ satisfies these conditions, i.e. $L(\mathfrak{g}, \rho, V, B_0)$ is, although it is defined over $\mathbb{C}$, of type α_0 (See [K–A]). In order to prove Preposition 3.3, we prove the following lemma needed later.

Lemma 3.4. *Under the notation in Proposition 3.3, we have*

$$\{X_i \in \hat{\mathfrak{g}}_i \mid [X_i, Y_{-1}] = 0 \textit{ for any } Y_{-1} \in \hat{\mathfrak{g}}_{-1}\} = \{0\}$$

for each $i \geq 0$. In particular, a quadruplet $(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0\times\hat{\mathfrak{g}}_0})$ satisfies (1.31).

Proof. Suppose that $X_i \in \hat{\mathfrak{g}}_i$ satisfies $[X_i, Y_{-1}] = 0$ for any $Y_{-1} \in \hat{\mathfrak{g}}_{-1}$. Then, for any $Z_{-i+1} \in \hat{\mathfrak{g}}_{-i+1}$, we have

$$\begin{aligned}&\hat{B}(X_i, [Y_{-1}, Z_{-i+1}])\\ =&\hat{B}([X_i, Y_{-1}], Z_{-i+1}) = 0.\end{aligned}$$

Thus by (3.5), (3.6) and (3.7), we obtain $X_i = 0$. ∎

By (3.6), $(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$ satisfies (1.32). Hence it is a standard quadruplet.

Let us prove Proposition 3.3. For this, we construct an isomorphism of Lie algebras from $L(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$ to $\hat{\mathfrak{g}}$. We denote the n-graduation of $(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$ by $(\hat{\mathfrak{g}})_n$ and the bilinear form on $L(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$ defined in Definition 3.1 by B. Let τ_0 be the identity map on $\hat{\mathfrak{g}}_0$ and $\tau_1 := \Phi_{\mathrm{ad}}^{-1}$. Then we obtain a linear isomorphism $\tau_{-1} : (\hat{\mathfrak{g}})_{-1} \to \hat{\mathfrak{g}}_{-1}$ defined by $B(\phi_{-1}, v_1) = \hat{B}(\tau_{-1}(\phi_{-1}), \tau_1(v_1))$ where $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$ and $v_1 \in (\hat{\mathfrak{g}})_1$. These linear isomorphisms $\tau_i : (\hat{\mathfrak{g}})_i \to \hat{\mathfrak{g}}_i$ $(i = 0, \pm 1)$ satisfy the following equations:

$$[\tau_0(a), \tau_1(v_1)] = \tau_1([a, v_1]), \tag{3.8}$$
$$[\tau_{-1}(\phi_{-1}), \tau_1(v_1)] = \tau_0([\phi_{-1}, v_1]), \tag{3.9}$$
$$[\tau_0(a), \tau_{-1}(\phi_{-1})] = \tau_{-1}([a, \phi_{-1}]) \tag{3.10}$$

where $a \in (\hat{\mathfrak{g}})_0$, $v_1 \in (\hat{\mathfrak{g}})_1$ and $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$. In fact, (3.8) obviously holds. Let us check (3.9). For any $b \in (\hat{\mathfrak{g}})_0$, we have

$$\begin{aligned}\hat{B}(\tau_0(b), \tau_0([\phi_{-1}, v_1])) &= B(b, [\phi_{-1}, v_1])\\ &= B([b, \phi_{-1}], v_1)\\ &= \hat{B}([\tau_0(b), \tau_{-1}(\phi_{-1})], \tau_1(v_1))\\ &= \hat{B}(\tau_0(b), [\tau_{-1}(\phi_{-1}), \tau_1(v_1)])\end{aligned}$$

and hence (3.9) holds. Similarly, we can obtain (3.10).

Next, suppose that $i \geq 1$ and assume that there exist linear isomorphisms $\tau_{i-1} : (\hat{\mathfrak{g}})_{i-1} \to \hat{\mathfrak{g}}_{i-1}$ and $\tau_i : (\hat{\mathfrak{g}})_i \to \hat{\mathfrak{g}}_i$ satisfying the equations:

$$\big[\tau_0(a), \tau_i(u_i)\big] = \tau_i\big([a, u_i]\big), \tag{3.11}$$
$$\big[\tau_1(v_1), \tau_{i-1}(u_{i-1})\big] = \tau_i\big([v_1, u_{i-1}]\big), \tag{3.12}$$
$$\big[\tau_{-1}(\phi_{-1}), \tau_i(u_i)\big] = \tau_{i-1}\big([\phi_{-1}, u_i]\big) \tag{3.13}$$

where $a \in (\hat{\mathfrak{g}})_0$, $v_1 \in (\hat{\mathfrak{g}})_1$, $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$, $u_{i-1} \in (\hat{\mathfrak{g}})_{i-1}$ and $u_i \in (\hat{\mathfrak{g}})_i$. Then we have

$$\tau_i\big(\big[\phi_{-1}, [v_1, u_i]\big]\big) = \big[\tau_{-1}(\phi_{-1}), \big[\tau_1(v_1), \tau_i(u_i)\big]\big]. \tag{3.14}$$

In fact,

$$\begin{aligned}
&\tau_i\big([\phi_{-1},[v_1,u_i]]\big)\\
=&\tau_i\big([[\phi_{-1},v_1],u_i]\big)+\tau_i\big([v_1,[\phi_{-1},u_i]]\big)\\
=&\big[\tau_0\big([\phi_{-1},v_1]\big),\tau_i(u_i)\big]+\big[\tau_1(v_1),\tau_{i-1}\big([\phi_{-1},u_i]\big)\big]\\
=&\big[[\tau_{-1}(\phi_{-1}),\tau_1(v_1)],\tau_i(u_i)\big]+\big[\tau_1(v_1),[\tau_{-1}(\phi_{-1}),\tau_i(u_i)]\big]\\
=&\big[\tau_{-1}(\phi_{-1}),[\tau_1(v_1),\tau_i(u_i)]\big].
\end{aligned}$$

Thus, by Lemma 3.4, we can define a linear map $\tau_{i+1}:(\hat{\mathfrak{g}})_{i+1}\to\hat{\mathfrak{g}}_{i+1}$ by

$$\tau_{i+1}\big(p_i(v_1\otimes u_i)\big):=\big[\tau_1(v_1),\tau_i(u_i)\big]. \tag{3.15}$$

Then, by (3.6) and (3.14), we obtain that τ_{i+1} is bijective. Moreover, τ_{i+1} satisfies the following equations:

$$\begin{aligned}
\big[\tau_0(a),\tau_{i+1}(u_{i+1})\big]&=\tau_{i+1}\big([a,u_{i+1}]\big), &(3.16)\\
\big[\tau_1(v_1),\tau_i(u_i)\big]&=\tau_{i+1}\big([v_1,u_i]\big), &(3.17)\\
\big[\tau_{-1}(\phi_{-1}),\tau_{i+1}(u_{i+1})\big]&=\tau_i\big([\phi_{-1},u_{i+1}]\big) &(3.18)
\end{aligned}$$

where $a\in(\hat{\mathfrak{g}})_0$, $v_1\in(\hat{\mathfrak{g}})_1$, $\phi_{-1}\in(\hat{\mathfrak{g}})_{-1}$, $u_i\in(\hat{\mathfrak{g}})_i$ and $u_{i+1}\in(\hat{\mathfrak{g}})_{i+1}$. Indeed, (3.16) follows from a similar argument to the proof of (3.14). Equations (3.17) and (3.18) have already been shown. Thus, inductively, we obtain linear isomorphisms τ_n from $(\hat{\mathfrak{g}})_n$ to $\hat{\mathfrak{g}}_n$ for all $n\geq 0$. Moreover, τ_n induces a linear isomorphism τ_{-n} from $(\hat{\mathfrak{g}})_{-n}$ to $\hat{\mathfrak{g}}_{-n}$ which satisfies the following equation:

$$B(\psi_{-n},u_n)=\hat{B}(\tau_{-n}(\psi_{-n}),\tau_n(u_n)) \tag{3.19}$$

where $u_n\in(\hat{\mathfrak{g}})_n$ and $\psi_{-n}\in(\hat{\mathfrak{g}})_{-n}$. Then τ_{-n} satisfies the following equations:

$$\begin{aligned}
\big[\tau_1(v_1),\tau_{-n}\big(\psi_{-n}\big)\big]&=\tau_{-n+1}\big([v_1,\psi_{-n}]\big), &(3.20)\\
\big[\tau_{-1}(\phi_{-1}),\tau_{-n}\big(\psi_{-n}\big)\big]&=\tau_{-n-1}\big([\phi_{-1},\psi_{-n}]\big) &(3.21)
\end{aligned}$$

where $v_1\in(\hat{\mathfrak{g}})_1$, $\phi_{-1}\in(\hat{\mathfrak{g}})_{-1}$ and $\psi_{-n}\in(\hat{\mathfrak{g}})_{-n}$. In fact, we can show (3.20) by induction on n. For $n=0$, our claim has already been shown. Suppose that $n\geq 1$, then by the induction hypothesis, we have

$$\begin{aligned}
\hat{B}\big(\tau_{n-1}(u_{n-1}),\big[\tau_1(v_1),\tau_{-n}\big(\psi_{-n}\big)\big]\big)&=\hat{B}\big(\big[\tau_{n-1}(u_{n-1}),\tau_1(v_1)\big],\tau_{-n}(\psi_{-n})\big)\\
&=\hat{B}\big(\tau_n\big([u_{n-1},v_1]\big),\tau_{-n}(\psi_{-n})\big)\\
&=B\big([u_{n-1},v_1],\psi_{-n}\big)\\
&=B\big(u_{n-1},[v_1,\psi_{-n}]\big)\\
&=\hat{B}\big(\tau_{n-1}(u_{n-1}),\tau_{-n+1}\big([v_1,\psi_{-n}]\big)\big)
\end{aligned}$$

for any $u_{n-1} \in (\hat{\mathfrak{g}})_{n-1}$. Hence (3.20) holds for all $n \geq 0$. Similarly we can show (3.21). Summarizing the above argument, we can define a linear isomorphism τ from $L(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$ to $\hat{\mathfrak{g}}$ by

$$\tau\big(\sum_{n\in\mathbb{Z}} x_n\big) := \sum_{n\in\mathbb{Z}} \tau_n(x_n) \tag{3.22}$$

where $x_n \in (\hat{\mathfrak{g}})_n$. Then τ satisfies the following equations:

$$\big[\tau(v_1), \tau(x)\big] = \tau\big([v_1, x]\big), \tag{3.23}$$

$$\big[\tau(\phi_{-1}), \tau(x)\big] = \tau\big([\phi_{-1}, x]\big) \tag{3.24}$$

where $v_1 \in (\hat{\mathfrak{g}})_1$, $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$ and $x \in L(\hat{\mathfrak{g}}_0, \mathrm{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$. Therefore, by the definition of a Lie algebra associated to a standard quadruplet, τ is an isomorphism of Lie algebras. This completes the proof of Proposition 3.3.

3.2 Equivalent standard quadruplets

In this section, we shall introduce a notion of equivalence between standard quadruplets.

Definition 3.5. Let $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ be standard quadruplets. We call these quadruplets are equivalent if and only if there exists an isomorphism of Lie algebras $\sigma : \mathfrak{g}^1 \to \mathfrak{g}^2$ and a linear isomorphism $\tau : V^1 \to V^2$ such that

$$\tau\big(\rho^1(a^1)v^1\big) = \rho^2\big(\sigma(a^1)\big)\big(\tau(v^1)\big), \tag{3.25}$$

$$B_0^1(a^1, b^1) = B_0^2(\sigma(a^1), \sigma(b^1)) \tag{3.26}$$

for any $a^1, b^1 \in \mathfrak{g}^1$ and $v^1 \in V^1$. We will denote this equivalence relation by

$$(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \simeq (\mathfrak{g}^2, \rho^2, V^2, B_0^2). \tag{3.27}$$

In particular, for a standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$ and a non-zero element $c \in \mathbb{C}$, $(\mathfrak{g}, \rho, V, cB_0)$ and $(\mathfrak{g}, \rho_1, V_1, B_0)$ are equivalent to $(\mathfrak{g}, \rho, V, B_0)$ where (ρ_1, V_1) is the 1-graduation of $(\mathfrak{g}, \rho, V, B_0)$. Then we have the following proposition.

Proposition 3.6. *If standard quadruplets $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ are equivalent, then the Lie algebras $L(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ and $L(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ are isomorphic.*

Proof. By the same argument as proof of Proposition 3.3, we can construct a linear isomorphism from V_n^1 to V_n^2 for each $n \in \mathbb{Z}$, where V_n^i denotes the n-graduation of $(\mathfrak{g}^i, \rho^i, V^i, B_0^i)$ for $i = 1, 2$, and an isomorphism of Lie algebras from $L(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ to $L(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$. ∎

However, in general, the converse is not true. For example, let $\mathfrak{g} = \mathfrak{sl}_4$. Put

$$H^1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ H^2 := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and

$$\mathfrak{g}^1_j := \{X \in \mathfrak{g} \mid [H^1, X] = 2jX\},\ \mathfrak{g}^2_j := \{X \in \mathfrak{g} \mid [H^2, X] = 2jX\}$$

for each $j \in \mathbb{Z}$. Then we obtain two gradings:

$$\begin{aligned}\mathfrak{g} =& \mathfrak{g}^1_{-1} \oplus \mathfrak{g}^1_0 \oplus \mathfrak{g}^1_1 \\ =& \mathfrak{g}^2_{-2} \oplus \mathfrak{g}^2_{-1} \oplus \mathfrak{g}^2_0 \oplus \mathfrak{g}^2_1 \oplus \mathfrak{g}^2_2.\end{aligned}$$

These gradings and the Killing form K on $\mathfrak{g}$ satisfy the assumptions of Proposition 3.3. Therefore, by applying Proposition 3.3, it follows that the Lie algebras $L(\mathfrak{g}^i_0, \mathrm{ad}, \mathfrak{g}^i_1, K|_{\mathfrak{g}^i_0 \times \mathfrak{g}^i_0})$ $(i = 1, 2)$ are isomorphic to $\mathfrak{g}$. However, by a direct calculation, we can obtain that $\mathfrak{g}^1_0$ and $\mathfrak{g}^2_0$ are isomorphic to $\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_2$ respectively. In particular, the quadruplets $(\mathfrak{g}^i_0, \mathrm{ad}, \mathfrak{g}^i_1, K|_{\mathfrak{g}^i_0 \times \mathfrak{g}^i_0})$ $(i = 1, 2)$ are not equivalent.

3.3 Loop algebras

In this section, we shall give a well-known example of infinite-dimensional Lie algebra which can be written in the form $L(\mathfrak{g}, \rho, V, B_0)$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t. Set $\mathcal{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ and define a bilinear map $[\cdot,\cdot]_0 : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})$ by:

$$[t^n \otimes X, t^m \otimes Y]_0 := t^{n+m} \otimes [X, Y] \tag{3.28}$$

where $n, m \in \mathbb{Z}$, $X, Y \in \mathfrak{g}$ and $[\cdot,\cdot]$ stands for the bracket product of $\mathfrak{g}$. This bilinear map satisfies the axioms of a Lie algebra. The infinite-dimensional Lie algebra $\mathcal{L}(\mathfrak{g})$ with the bracket $[\cdot,\cdot]_0$ is called the loop algebra and we have a $\mathbb{Z}$-grading $\mathcal{L}(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} t^n \otimes \mathfrak{g}$ (See [Ka]).

Now, we denote by K the Killing form on $\mathfrak{g}$. Then we can define a bilinear form K_0 on $\mathcal{L}(\mathfrak{g})$ by $K_0(t^n \otimes X, t^m \otimes Y) := \delta_{n+m,0}K(X, Y)$, where $\delta_{n+m,0}$ is the Kronecker delta. By applying Proposition 3.3, we have the following proposition.

Proposition 3.7. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and K the Killing form on $\mathfrak{g}$. Then the Lie algebra $L(\mathfrak{g}, \mathrm{ad}, \mathfrak{g}, K)$ is isomorphic to the loop algebra $\mathcal{L}(\mathfrak{g})$.*

3.4 A direct sum of standard quadruplets

Definition 3.8. Let $(\mathfrak{g}^1, \rho^1, V^1, B^1_0)$ and $(\mathfrak{g}^2, \rho^2, V^2, B^2_0)$ be standard quadruplets. Let $\rho^1 \boxplus \rho^2$ be a representation of $\mathfrak{g}^1 \oplus \mathfrak{g}^2$ on $V^1 \oplus V^2$ defined by:

$$\big((\rho^1 \boxplus \rho^2)(a^1, a^2)\big)(v^1, v^2) := (\rho^1(a^1)v^1, \rho^2(a^2)v^2) \tag{3.29}$$

where $a^i \in \mathfrak{g}^i$ and $v^i \in \mathfrak{g}^i$ $(i = 1, 2)$. Let $B_0^1 \oplus B_0^2$ be a bilinear form on $\mathfrak{g}^1 \oplus \mathfrak{g}^2$ defined by:

$$(B_0^1 \oplus B_0^2)((a^1, a^2), (b^1, b^2)) := B_0^1(a^1, b^1) + B_0^2(a^2, b^2) \tag{3.30}$$

where $a^i, b^i \in \mathfrak{g}^i$ $(i = 1, 2)$. Then a quadruplet $(\mathfrak{g}^1 \oplus \mathfrak{g}^2, \rho^1 \boxplus \rho^2, V^1 \oplus V^2, B_0^1 \oplus B_0^2)$ is a standard quadruplet. We call it a **direct sum** of $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ and denote by $(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus (\mathfrak{g}^2, \rho^2, V^2, B_0^2)$.

Remark 3.9. Let $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$, $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ and $(\mathfrak{g}, \rho, V, B_0)$ be standard quadruplets. Assume that $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ are equivalent. Then we have

$$\begin{aligned} &(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus (\mathfrak{g}, \rho, V, B_0) \\ \simeq &(\mathfrak{g}^2, \rho^2, V^2, B_0^2) \oplus (\mathfrak{g}, \rho, V, B_0). \end{aligned}$$

By applying Proposition 3.3, we have the following proposition immediately.

Proposition 3.10. *Let $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ be standard quadruplets. Then the Lie algebra $L\big((\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus (\mathfrak{g}^2, \rho^2, V^2, B_0^2)\big)$ is isomorphic to $L(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus L(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$.*

Definition 3.11. A standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$ is said to be **decomposable** if and only if there exist non-trivial ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ and non-trivial subspaces U and W of V which satisfy

$$\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{g}, \tag{3.31}$$

$$U \oplus W = V, \tag{3.32}$$

$$\rho(a)w = 0, \tag{3.33}$$

$$\rho(b)u = 0, \tag{3.34}$$

$$B_0(a, b) = 0 \tag{3.35}$$

for any $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, $u \in U$ and $w \in W$. Then $(\mathfrak{a}, \rho|_{\mathfrak{a}}, U, B_0|_{\mathfrak{a}\times\mathfrak{a}})$ and $(\mathfrak{b}, \rho|_{\mathfrak{b}}, W, B_0|_{\mathfrak{b}\times\mathfrak{b}})$ are standard quadruplets and $(\mathfrak{g}, \rho, V, B_0) \simeq (\mathfrak{a}, \rho|_{\mathfrak{a}}, U, B_0|_{\mathfrak{a}\times\mathfrak{a}}) \oplus (\mathfrak{b}, \rho|_{\mathfrak{b}}, W, B_0|_{\mathfrak{b}\times\mathfrak{b}})$. If $(\mathfrak{g}, \rho, V, B_0)$ is not decomposable, it is said to be **indecomposable**. Any standard quadruplet can be written as a direct sum of indecomposable standard quadruplets.

Proposition 3.12. *Let $(\mathfrak{g}, \rho, V, B_0)$ be an indecomposable standard quadruplet. Let n be an integer such that V_{n+1} is not $\{0\}$. Then for any non-zero element $x_n \in V_n$, there exists an element $v_1 \in V_1$ such that $[v_1, x_n] \neq 0$.*

Proof. If $n \leq 0$, we have our result immediately. Suppose that $n \geq 1$. Put

$$U_n := \{x_n \in V_n \mid [v_1, x_n] = 0 \text{ for any } v_1 \in V_1\}. \tag{3.36}$$

Then U_n is a V_0-submodule of V_n. Moreover, for each $m = 0, \ldots, n-1$, we can define a V_0-submodule U_m of V_m by

$$U_m := \{x_m \in V_m \mid [v_1, x_m] \in U_{m+1} \text{ for any } v_1 \in V_1\} \tag{3.37}$$

inductively. Then we can easily show that

$$[V_{-1}, U_m] \subset U_{m-1} \tag{3.38}$$

for $m = 1, \ldots, n$ by induction. Moreover, we have

$$V_0 \neq U_0. \tag{3.39}$$

In fact, since $V_{n+1} \neq \{0\}$, we can deduce that $U_n \neq V_n$. Similarly, we have $U_m \neq V_m$ for $m = 0, \ldots, n-1$ by induction.

Now, since V_1 is a completely reducible V_0-module, we can take a V_0-submodule $W_1 \subset V_1$ which satisfies $V_1 = U_1 \oplus W_1$. Then by the definition of U_0, we have $[U_0, W_1] = \{0\}$. Put $W_0 := \{a \in V_0 \mid B(a, b) = 0 \text{ for any } b \in U_0\}$. Then we have $[W_0, U_1] = \{0\}$. In fact, suppose that $a \in W_0$ and $u_1 \in U_1$. Then for any $\phi_{-1} \in V_{-1}$, we have $[u_1, \phi_{-1}] \in U_0$ and $B([a, u_1], \phi_{-1}) = B(a, [u_1, \phi_{-1}]) = 0$, hence $[a, u_1] = 0$. Since ρ is faithful, we have $V_0 = U_0 \oplus W_0$. Therefore, by the assumption that $(\mathfrak{g}, \rho, V, B_0)$ is an indecomposable standard quadruplet, we have $U_0 = \{0\}$. Therefore, by Lemma 3.4 and (3.38), we can obtain $U_m = \{0\}$ for $m = 1, \ldots, n$ by induction. In particular, $U_n = \{0\}$ and thus we have our result. ∎

Proposition 3.13. *Let $(\mathfrak{g}, \rho, V, B_0)$ be a standard quadruplet. If the Lie algebra $L(\mathfrak{g}, \rho, V, B_0)$ is finite-dimensional, then $L(\mathfrak{g}, \rho, V, B_0)$ is semisimple. In particular, if $(\mathfrak{g}, \rho, V, B_0)$ is indecomposable, then $L(\mathfrak{g}, \rho, V, B_0)$ is simple.*

Proof. It is sufficient to show the case where $(\mathfrak{g}, \rho, V, B_0)$ is indecomposable. Let $\mathfrak{a}$ be a non-zero ideal of $L(\mathfrak{g}, \rho, V, B_0)$. Let k be a positive integer such that $V_k \neq \{0\}$ and $V_{k+1} = \{0\}$. Then there exist integers $-k \leq n_1 < \ldots < n_l \leq k$ and non-zero elements $x_{n_1} \in V_{n_1}, \ldots, x_{n_l} \in V_{n_l}$ such that

$$x := x_{n_1} + \cdots + x_{n_l} \in \mathfrak{a}. \tag{3.40}$$

Then by Proposition 3.12, there exist non-zero elements $v_1^1, \ldots, v_1^{k-n_1} \in V_1$ such that

$$[v_1^{k-n_1}, [v_1^{k-n_1-1}, \cdots, [v_1^1, x] \cdots]] \in V_k \setminus \{0\}. \tag{3.41}$$

Thus we can obtain $\mathfrak{a}_k := \mathfrak{a} \cap V_k \neq \{0\}$ and $\mathfrak{a}_m := \mathfrak{a} \cap V_m \neq \{0\}$ for $m = 0, \ldots, k-1$ by induction. By a similar argument to the argument in the proof of Proposition 3.12, we have $\mathfrak{a}_0 = V_0$, i.e. $V_0 \subset \mathfrak{a}$. Therefore, we have $V_1 = [V_1, V_0]$, $V_{-1} = [V_{-1}, V_0] \subset \mathfrak{a}$ and thus $\mathfrak{a} = L(\mathfrak{g}, \rho, V, B_0)$. Therefore, $L(\mathfrak{g}, \rho, V, B_0)$ is simple and thus we have our result. ∎

3.5 A quadruplet of parabolic type

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, R the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let K be the Killing form on $\mathfrak{g}$. We fix a fundamental root system φ of R.

Let θ be a subset of φ. Then there exists the unique element H^θ such that $\alpha(H^\theta) = 0$ for $\alpha \in \theta$ and $\alpha(H^\theta) = 2$ for $\alpha \in \varphi \setminus \theta$. We call H^θ the grading element.

For each $n \in \mathbb{Z}$, we put $d_n(\theta) := \{X \in \mathfrak{g} \mid [H^\theta, X] = 2nX\}$ and denote $d_0(\theta)$ by $\mathfrak{l}_\theta$. Then $\mathfrak{l}_\theta$ is a reductive subalgebra of $\mathfrak{g}$ and acts on each $d_n(\theta)$ by adjoint representation. Let G be the adjoint group of $\mathfrak{g}$ and L_θ be the connected subgroup of G corresponding to $\mathfrak{l}_\theta$. Then it is known that the representation of L_θ on $d_1(\theta)$ is completely reducible and $(L_\theta, d_1(\theta))$ is a prehomogeneous vector space. Such prehomogeneous vector spaces are called prehomogeneous vector spaces of parabolic type.

We can easily show that the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} d_n(\theta)$ and the Killing form K on $\mathfrak{g}$ satisfy the conditions from (3.3) to (3.6). If $(\mathfrak{l}_\theta, \mathrm{ad}, d_1(\theta), K|_{\mathfrak{l}_\theta\times\mathfrak{l}_\theta})$ is a standard quadruplet, i.e. $\mathfrak{l}_\theta = [d_{-1}(\theta), d_1(\theta)]$ holds, we call it a **quadruplet of parabolic type**. Note that a direct sum of quadruplets of parabolic type is again a quadruplet of parabolic type. Then we have the following theorem.

Theorem 3.14. *Let $(\mathfrak{g}, \rho, V, B_0)$ be a standard quadruplet. Then the Lie algebra $L(\mathfrak{g}, \rho, V, B_0)$ is finite-dimensional if and only if $(\mathfrak{g}, \rho, V, B_0)$ is equivalent to some quadruplet of parabolic type.*

Proof. By applying Proposition 3.3, we can obtain that the Lie algebra associated to a quadruplet of parabolic type is finite-dimensional immediately. Let us prove the converse. Assume that $L(\mathfrak{g}, \rho, V, B_0)$ is finite-dimensional. Without loss of generality, we can assume that $(\mathfrak{g}, \rho, V, B_0)$ is indecomposable. Then, by Proposition 3.13, $L(\mathfrak{g}, \rho, V, B_0)$ is simple. Hence, the bilinear form B defined in Definition 3.1 is a scalar multiple of the Killing form K_L on $L(\mathfrak{g}, \rho, V, B_0)$, that is, there exists a non-zero element $c \in \mathbb{C}$ such that $K_L = cB$. We define a linear endomorphism δ on $L(\mathfrak{g}, \rho, V, B_0)$ by $\delta(\sum_{n\in\mathbb{Z}} x_n) := \sum_{n\in\mathbb{Z}} 2nx_n$ where $x_n \in V_n$. Then, by the theory of Lie algebras, δ is an inner derivation, i.e., there exists an element $H \in L(\mathfrak{g}, \rho, V, B_0)$ such that $\delta = \mathrm{ad}(H)$. Since $\mathrm{ad}(H)$ is diagonalizable, there exists a Cartan subalgebra $\mathfrak{h}$ of $L(\mathfrak{g}, \rho, V, B_0)$ such that $H \in \mathfrak{h}$. Let R be the root system of $L(\mathfrak{g}, \rho, V, B_0)$ with respect to $\mathfrak{h}$. Let us construct a fundamental system of R.

Suppose that $\mathfrak{g}$ is not commutative, i.e., $[V_0, V_0] \neq \{0\}$. By the definition of H, $\mathfrak{h}$ is also a Cartan subalgebra of V_0. Put $\overline{\mathfrak{h}} := \mathfrak{h} \cap [V_0, V_0]$ and denote the restriction of $\gamma \in \mathfrak{h}^*$ to $\overline{\mathfrak{h}}$ by $\overline{\gamma}$. Then there exists a subset $\theta = \{\alpha^1, \ldots, \alpha^r\}$ of R such that $\{\overline{\alpha^1}, \ldots, \overline{\alpha^r}\}$ is a fundamental root system of the root system of $[V_0, V_0]$ with respect to $\overline{\mathfrak{h}}$. Put

$$\begin{aligned}\varphi' :=& \{\beta \in R \mid \beta(H) = 2,\ \beta - \alpha^s \notin R \text{ for } s = 1, \ldots, r\}\\ =& \{\beta^1, \ldots, \beta^l\}.\end{aligned}$$

Then $\theta \cup \varphi' = \{\alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^l\}$ is a fundamental root system of R. In fact, we can easily show that any root can be written as $n(1)\alpha^1 + \cdots + n(r)\alpha^r + m(1)\beta^1 + \cdots + m(l)\beta^l$ where all $n(i)$ and $m(j)$ are non-negative integers or all $n(i)$ and $m(j)$ are non-positive integers. Thus it is enough to show that $\theta \cup \varphi'$ is linearly independent over $\mathbb{R}$. Suppose that $c(1), \dots, c(r), d(1), \dots, d(l) \in \mathbb{R}$ satisfy

$$c(1)\alpha^1 + \cdots + c(r)\alpha^r + d(1)\beta^1 + \cdots + d(l)\beta^l = 0.$$

By renumbering the elements of $\theta \cup \varphi'$, we can suppose that

$$\begin{aligned} &c(1), \dots, c(s) \geq 0, \quad c(s+1), \dots, c(r) < 0, \\ &d(1), \dots, d(t) \geq 0, \quad d(t+1), \dots, d(l) < 0 \end{aligned}$$

without loss of generality. Put

$$\begin{aligned} \lambda &:= c(1)\alpha^1 + \cdots + c(s)\alpha^s + d(1)\beta^1 + \cdots + d(t)\beta^t \\ &= -c(s+1)\alpha^{s+1} - \cdots - c(r)\alpha^r - d(t+1)\beta^{t+1} - \cdots - d(l)\beta^l. \end{aligned}$$

Then we have

$$\begin{aligned} 0 \leq K_L(H_\lambda, H_\lambda) = &- \sum_{1 \leq i \leq s, s+1 \leq j \leq r} c(i)c(j)K_L(H_{\alpha^i}, H_{\alpha^j}) - \sum_{1 \leq i \leq t, t+1 \leq j \leq l} d(i)d(j)K_L(H_{\beta^i}, H_{\beta^j}) \\ &- \sum_{1 \leq s \leq r, t+1 \leq j \leq l} c(i)d(j)K_L(H_{\alpha^i}, H_{\beta^j}) - \sum_{1 \leq i \leq t, s+1 \leq j \leq r} d(i)c(j)K_L(H_{\beta^i}, H_{\alpha^j}) \end{aligned} \tag{3.42}$$

where H_γ denotes the element in $\mathfrak{h}$ corresponding to $\gamma \in \mathfrak{h}^*$ which satisfies $K_L(H_\gamma, h) = \gamma(h)$ for any $h \in \mathfrak{h}$. By the definition of $\theta \cup \varphi'$, it follows that $\gamma^1 - \gamma^2 \notin R$ for any $\gamma^1, \gamma^2 \in \theta \cup \varphi'$ and thus $K_L(H_{\gamma^1}, H_{\gamma^2}) \leq 0$. Therefore, we have $K_L(H_\lambda, H_\lambda) = 0$ and $\lambda = 0$. Therefore, we have

$$\begin{aligned} &c(1)\alpha^1(H) + \cdots + c(s)\alpha^s(H) + d(1)\beta^1(H) + \cdots + d(t)\beta^t(H) \\ =&2d(1) + \cdots + 2d(t) = 0 \end{aligned}$$

and thus $d(1) = \cdots = d(t) = 0$. Moreover, it follows that $c(1) = \cdots = c(s) = 0$ from the assumption that θ is linearly independent. Similarly, we have $c(s+1) = \cdots = c(r) = d(t+1) = \cdots = d(l) = 0$. Therefore $\theta \cup \varphi'$ is a fundamental root system of R.

If $[V_0, V_0] = \{0\}$, that is, V_0 is commutative, then a set $\varphi := \{\beta \in R \mid \beta(H) = 2\}$ is a fundamental root system of R. Indeed, we can show it by a similar argument.

Summarizing the above argument, it follows that H is a grading element and thus $(\mathfrak{g}, \rho_1, V_1, cB_0)$ is a quadruplet of parabolic type and $(\mathfrak{g}, \rho, V, B_0)$ is equivalent to it. This completes the proof. ∎

It follows from Theorem 3.14 that any semisimple Lie algebra can be obtained from some standard quadruplet. In particular, for a positive integer $m \geq 2$, we can obtain $\mathfrak{sl}_{m+1}$ and $\mathfrak{so}_{2m+1}$ from standard quadruplets given in Example 1.18 and 1.19 respectively.

4 Prehomogeneous vector spaces and quadruplets

In this section, we shall discuss about the prehomogeneity condition of a triplet (G, ρ, V) under the assumption that G is reductive by using the Φ-map of $(\mathrm{Lie}(G), d\rho, V, B_0)$ where B_0 is a bilinear form on $\mathrm{Lie}(G)$. The main result of this section is to give another proof of castling transform (Theorem 4.7).

4.1 Prehomogeneous quadruplets

A prehomogeneous vector space is a triplet which consists of a connected algebraic group G and its representation ρ on a finite-dimensional vector space V such that there exists a Zariski dense orbit $\rho(G)v \subset V$ $(v \in V)$. Such an element v is called a generic point of (G, ρ, V). In the case where G is reductive, the prehomogeneity of a triplet (G, ρ, V) can be described by using a Φ-map.

Proposition 4.1. *Let G be a connected reductive algebraic group, V a finite-dimensional vector space and ρ a representation of G on V. Let $\mathfrak{g}$ be the Lie algebra of G and $d\rho$ the derived representation of ρ on V. Then the triplet (G, ρ, V) is prehomogeneous if and only if there exists a non-degenerate symmetric invariant form B_0 and an element $v \in V$ such that the Φ-map at v of a quadruplet $(\mathfrak{g}, d\rho, V, B_0)$, $\Phi_{d\rho,v} : V^* \to \mathfrak{g}$ is injective.*

Proof. Take an arbitrary non-degenerate symmetric invariant bilinear form and denote it by B_0.

Assume that (G, ρ, V) is a prehomogeneous vector space and v is its generic point. Then it is known that a vector subspace $d\rho(\mathfrak{g})v$ of V coincides with V. Then $\Phi_{d\rho,v}$ is injective. If not, there exists a non-zero element ϕ such that $\Phi_{d\rho,v}(\phi) = 0$ and we have

$$\begin{aligned} 0 &= B_0(a, \Phi_{d\rho,v}(\phi)) \\ &= \langle d\rho(a)v, \phi \rangle \end{aligned}$$

for any $a \in \mathfrak{g}$. This is a contradiction to the assumption that $d\rho(\mathfrak{g})v$ coincides with V. Thus $\Phi_{d\rho,v}$ is injective.

Conversely, suppose that there exists an element $v \in V$ such that $\Phi_{d\rho,v}$ of $(\mathfrak{g}, d\rho, V, B_0)$ is injective. Then (G, ρ, V) is a PV and v is its generic point. Indeed, if v is not a generic point, then $d\rho(\mathfrak{g})v$ is a proper subspace of V and there exists a non-zero element ϕ such that $0 = \langle d\rho(a)v, \phi \rangle = B_0(a, \Phi_{d\rho,v}(\phi))$ for any $a \in \mathfrak{g}$. Since B_0 is non-degenerate, we have $\Phi_{d\rho,v}(\phi) = 0$ and a contradiction to the assumption that $\Phi_{d\rho,v}$ is injective. Therefore v is a generic point. ∎

Definition 4.2. A standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$ is called **a prehomogeneous quadruplet** if and only if there exists an element $v \in V$ such that $\Phi_{\rho,v}$ is injective.

Example 4.3. For $n \geq 3$, a triplet $(SO_n, \Lambda_1, \mathbb{C}^n)$ is not a PV, where Λ_1 is the natural representation of SO_n on $\mathbb{C}^n = M(n, 1; \mathbb{C})$. To check this, under the notation of Example 1.4, let us show that a quadruplet $(\mathfrak{so}_n, \Lambda_1, \mathbb{C}^n, T_n|_{\mathfrak{so}_n \times \mathfrak{so}_n})$ is not prehomogeneous. In fact, for any column vector $v \in \mathbb{C}^n$ we have

$$\Phi_{\Lambda_1, v}(v) = \frac{1}{2}(v^t v - v^t v) = 0. \tag{4.1}$$

Thus we have our claim.

However, a quadruplet $(\mathfrak{gl}_1 \oplus \mathfrak{so}_n, \Box \otimes \Lambda_1, \mathbb{C} \otimes \mathbb{C}^n, B_0)$ is prehomogeneous, where $\Box$ is a scalar multiplication of $\mathfrak{gl}_1 = \mathbb{C}$ and B_0 is a bilinear form defined by:

$$B_0((a, A), (a', A')) := -aa' + 2\mathrm{Tr}\ AA' \tag{4.2}$$

where $a, a' \in \mathfrak{gl}_1 = \mathbb{C}$ and $A, A' \in \mathfrak{so}_n = Alt_n$. The representation space can be identified with $\mathbb{C}^n$ and the action of $\mathfrak{gl}_1 \oplus \mathfrak{so}_n$ can be given as:

$$(\Box \otimes \Lambda_1)(a, A)v = av + Av \tag{4.3}$$

where $a \in \mathfrak{gl}_1$, $A \in \mathfrak{so}_n$ and $v \in \mathbb{C}^n$. Then, the map $\hat{\Phi}_{\Box \otimes \Lambda_1}$ is given as follows:

$$\hat{\Phi}_{\Box \otimes \Lambda_1}(v \otimes \phi) = (-{}^t v\phi, v^t\phi - \phi^t v) \tag{4.4}$$

where $v, \phi \in \mathbb{C}^n$. Put $v_0 := {}^t(1, 0, \ldots, 0) \in \mathbb{C}^n$. Then we have

$$(\Box \otimes \Lambda_1)(\Phi_{\Box \otimes \Lambda_1, v_0}(\phi)) \cdot v_0 = -\phi. \tag{4.5}$$

Thus the map $\hat{\Phi}_{\Box \otimes \Lambda_1, v_0}$ is injective. Therefore $(\mathfrak{gl}_1 \oplus \mathfrak{so}_n, \Box \otimes \Lambda_1, \mathbb{C} \otimes \mathbb{C}^n, B_0)$ is a prehomogeneous quadruplet.

4.2 A triplet of the form $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$

In this section, we will consider an important theorem in the theory of prehomogeneous vector spaces, castling transform (See Proposition 7, p37, [S–K]).

Definition 4.4. Let $\mathfrak{g}$ be a reductive Lie algebra, ρ a representation of $\mathfrak{g}$ on a finite-dimensional vector space V, B_0 a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. For any $n \in \mathbb{N}$ and n-vectors $v^1, \ldots, v^n \in V$ (*resp.* $\phi^1, \ldots, \phi^n \in V^*$), we define a vector subspace $S_{(v^1, \ldots, v^n)} \subset (V^*)^n$ (*resp.* $S^*_{(\phi^1, \ldots, \phi^n)} \subset V^n$) by:

$$S_{(v^1, \ldots, v^n)} := \{(\psi^1, \ldots, \psi^n) \in (V^*)^n |\ \sum_{k=1}^{n} \hat{\Phi}_\rho(v^k \otimes \phi^k) = 0, \langle v^i, \psi^j \rangle = 0 \text{ for } 1 \leq i, j \leq n\} \tag{4.6}$$

(*resp.*

$$S^*_{(\phi^1, \ldots, \phi^n)} := \{(u^1, \ldots, u^n) \in V^n |\ \sum_{k=1}^{n} \hat{\Phi}_\rho(u^k \otimes \phi^k) = 0, \langle u^i, \phi^j \rangle = 0 \text{ for } 1 \leq i, j \leq n\}) \tag{4.7}$$

where $\hat{\Phi}_\rho$ is the Φ-map of $(\mathfrak{g}, \rho, V, B_0)$.

Lemma 4.5. *We continue to use the notation of Definition 4.4. Then a standard quadruplet* $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$ *(resp.* $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^n, B_0 \oplus T_n)$*) is prehomogeneous if and only if there exists n-vectors* $v^1, \ldots, v^n \in V$ *(resp.* $\phi^1, \ldots, \phi^n \in V^*$*) such that* $S_{(v^1,\ldots,v^n)} = \{(0, \ldots, 0)\}$ *(resp.* $S^*_{(\phi^1,\ldots,\phi^n)} = \{(0, \ldots, 0)\}$*).*

Proof. We prove for a quadruplet $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$. For a quadruplet $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^n, B_0 \oplus T_n)$, our claim can be proved by the same way.

Let $e_i \in \mathbb{C}^n$ be a column vector whose coefficients are all zero except the i-th one which is equal to 1 and $E_{ij} \in \mathfrak{gl}_n$ be an $n \times n$ matrix whose coefficients are all zero except the (i, j)-th one which is equal to 1. Then for any $v \in V$ and $\phi \in V^*$, we have

$$\hat{\Phi}_{\rho\otimes\Lambda_1}\big((v \otimes e_i) \otimes (\phi \otimes e_j)\big) = (\delta_{ij}\hat{\Phi}_\rho(v \otimes \phi), \langle v, \phi\rangle E_{ij}) \tag{4.8}$$

where δ_{ij} is the Kronecker delta.

Suppose that $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$ is a prehomogeneous quadruplet and $v^1 \otimes e_1 + \cdots + v^n \otimes e_n \in V \otimes \mathbb{C}^n$ $(v^1, \ldots, v^n \in V)$ is its generic point. Then we have $S_{(v^1,\ldots,v^n)} = \{(0, \ldots, 0)\}$. In fact, take an arbitrary element $(\psi^1, \ldots, \psi^n) \in S_{(v^1,\ldots,v^n)}$, then we have:

$$\begin{aligned}
&\hat{\Phi}_{\rho\otimes\Lambda_1}\big((v^1 \otimes e_1 + \cdots + v^n \otimes e_n) \otimes (\psi^1 \otimes e_1 + \cdots + \psi^n \otimes e_n)\big)\\
=&\left(\hat{\Phi}_\rho(v^1 \otimes \psi^1 + \cdots + v^n \otimes \psi^n), \sum_{1\le i,j\le n} \langle v^i, \psi^j\rangle E_{ij}\right)\\
=&0.
\end{aligned} \tag{4.9}$$

Therefore, we have $\psi^1 \otimes e_1 + \cdots + \psi^n \otimes e_n = 0$ and thus $(\psi^1, \ldots, \psi^n) = (0, \ldots, 0)$.

Conversely, assume that n-vectors $v^1, \ldots, v^n \in V$ satisfy $S_{(v^1,\ldots,v^n)} = \{(0, \ldots, 0)\}$. Then $v^1 \otimes e_1 + \cdots + v^n \otimes e_n$ is a generic point of $(\mathfrak{g}, \rho, V, B_0)$. In fact, suppose that $\phi^1 \otimes e_1 + \cdots + \phi^n \otimes e_n$ satisfies

$$\hat{\Phi}_{\rho\otimes\Lambda_1}\big((v^1 \otimes e_1 + \cdots + v^n \otimes e_n) \otimes (\phi^1 \otimes e_1 + \cdots + \phi^n \otimes e_n)\big) = 0, \tag{4.10}$$

then we can obtain that $(\phi^1, \ldots, \phi^n) \in S_{(v^1,\ldots,v^n)}$ and thus $\phi^1 \otimes e_1 + \cdots + \phi^n \otimes e_n = 0$. This completes the proof. ∎

The following corollary is immediate.

Corollary 4.6. *Under the notation of Definition 4.4, if* $n \geq m = \dim V$*, then a quadruplet* $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$ *is prehomogeneous.*

Theorem 4.7. *(castling transform) Under the notation of Definition 4.4, we let* $n < m = \dim V$*. Then a quadruplet* $(\mathfrak{g}\oplus\mathfrak{gl}_n, \rho\otimes\Lambda_1, V\otimes\mathbb{C}^n, B_0\oplus T_n)$ *is prehomogeneous if and only if a quadruplet* $(\mathfrak{g}\oplus \mathfrak{gl}_{m-n}, \rho^*\otimes\Lambda_1, V^*\otimes\mathbb{C}^{m-n}, B_0\oplus T_{m-n})$ *is prehomogeneous. Moreover, if* $(\mathfrak{g}\oplus\mathfrak{gl}_n, \rho\otimes\Lambda_1, V\otimes\mathbb{C}^n, B_0\oplus T_n)$ *is prehomogeneous (thus* $(\mathfrak{g} \oplus \mathfrak{gl}_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n}, B_0 \oplus T_{m-n})$ *is also prehomogeneous),* *then for any generic point $x \in V \otimes \mathbb{C}^n$ there exists a generic point $y \in V^* \otimes \mathbb{C}^{m-n}$ such that the $\mathfrak{g}$-part of the isotropy subalgebra at x denoted by $\mathfrak{g}_x$ coincides with the $\mathfrak{g}$-part of the isotropy subalgebra at y denoted by $\mathfrak{g}_y$.*

Proof. Suppose that a quadruplet $(\mathfrak{g}\oplus\mathfrak{gl}_n, \rho\otimes\Lambda_1, V\otimes\mathbb{C}^n, B_0\oplus T_n)$ is prehomogeneous and n-vectors $v^1, \ldots, v^n$ satisfy $S_{(v^1,\ldots,v^n)} = \{(0,\ldots,0)\}$. Then vectors $v^1, \ldots, v^n \in V$ are linearly independent. In fact, if there exists scalars $(c(1), \ldots, c(n)) \neq (0, \ldots, 0)$ such that $c(1)v^1+\cdots+c(n)v^n = 0$, then we have a non-zero element $\phi \in V^*$ such that $\langle v^i, \phi\rangle = 0$ for $i = 1, \ldots, n$ and $(c(1)\phi, \ldots, c(n)\phi)$ is a non-zero element of $S_{(v^1,\ldots,v^n)}$. It is a contradiction to the assumption that $S_{(v^1,\ldots,v^n)} = \{(0, \ldots, 0)\}$.

Put $U := \mathbb{C}v^1 + \cdots + \mathbb{C}v^n$ and denote the orthogonal space of U in V^* by $U^\perp$. Then $U^\perp$ is a $(m-n)$-dimensional vector subspace of V^*. Take an arbitrary basis of $U^\perp$, denoted by $\phi^1, \ldots, \phi^{m-n} \in V^*$. Then $\phi^1, \ldots, \phi^{m-n} \in V^*$ satisfy $S^*_{(\phi^1,\ldots,\phi^{m-n})} = \{(0, \ldots, 0)\}$. In fact, suppose that $(u^1, \ldots, u^{m-n}) \in S^*_{(\phi^1,\ldots,\phi^{m-n})}$. Then for any i, j $(1 \le i \le n, 1 \le j \le m-n)$, we have

$$\langle u^i, \phi^j\rangle = 0, \tag{4.11}$$

$$\hat{\Phi}_\rho(u^1 \otimes \phi^1 + \cdots + u^{m-n} \otimes \phi^{m-n}) = 0. \tag{4.12}$$

It follows from (4.11) that $u^1, \ldots, u^{m-n} \in U$ and thus there exist scalars $c(k,l) \in \mathbb{C}$ $(1 \le k \le m-n, 1 \le l \le n)$ which satisfy

$$u^k = c(k,1)v^1 + \cdots + c(k,n)v^n \quad (1 \le k \le m-n). \tag{4.13}$$

Then it follows from (4.12) that

$$\sum_{1\le l\le n} \hat{\Phi}_\rho\left(v^l \otimes (\sum_{1\le k\le m-n} c(k,l)\phi^k)\right) = 0. \tag{4.14}$$

Therefore $\left((\sum_{1\le k\le m-n} c(k,1)\phi^k), \ldots, (\sum_{1\le k\le m-n} c(k,n)\phi^k)\right) \in S_{(v^1,\ldots,v^n)} = \{(0, \ldots, 0)\}$. Therefore we have $c(k,l) = 0$ for any k, l and $(u^1, \ldots, u^n) = (0, \ldots, 0)$, that is, $(\mathfrak{g} \oplus \mathfrak{gl}_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n}, B_0 \oplus T_{m-n})$ is also prehomogeneous.

Next, we put $x := v^1 \otimes e_1 + \cdots + v^n \otimes e_n \in V \otimes \mathbb{C}^n$ and $y := \phi^1 \otimes e_1 + \cdots + \phi^{m-n} \otimes e_{m-n} \in V^* \otimes \mathbb{C}^{m-n}$. Let us show that we have $\mathfrak{g}_x = \mathfrak{g}_y$. A necessary and sufficient condition for an element $a \in \mathfrak{g}$ to belong to $\mathfrak{g}_x$ is that there exists an element $A = \sum\limits_{1\le i,j\le n} b(i,j)E_{i,j} \in \mathfrak{gl}_n$ $(b(i,j) \in \mathbb{C})$ which satisfies

$$\begin{aligned} 0 =& (\rho \otimes \Lambda_1)(a, A) \cdot x \\ =& (\rho \otimes \Lambda_1)(a, A) \cdot \sum_{1\le i\le n} v^i \otimes e_i \\ =& \sum_{1\le i\le n} \left(\rho(a)v^i + \sum_{1\le j\le n} b(i,j)v^j\right) \otimes e_i. \end{aligned}$$

Therefore we have

$$\mathfrak{g}_x = \{a \in \mathfrak{g} \mid \rho(a)U \subset U\}. \tag{4.15}$$

Similarly we have

$$\mathfrak{g}_y = \{a \in \mathfrak{g} \mid \rho^*(a)U^\perp \subset U^\perp\}. \tag{4.16}$$

By an easy calculation, we can obtain $\mathfrak{g}_x = \mathfrak{g}_y$. This completes the proof. ∎

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