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Graded Lie algebras and  
prehomogeneous vector spaces  
(次数つき Lie 代数と概均質ベクトル空間)

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# Graded Lie algebras and prehomogeneous vector spaces

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## Introduction

Let  $G$  be a connected algebraic group,  $\rho$  a representation of  $G$  on a finite-dimensional vector space  $V$  where all defined over  $\mathbb{C}$ . When there exists an element  $v \in V$  such that the orbit  $\rho(G)v$  is Zariski dense in  $V$ , we say that the triplet  $(G, \rho, V)$  is a prehomogeneous vector space (abbrev. PV) and  $v$  is called a generic point of  $(G, \rho, V)$ . For example, if we let  $\Lambda_1$  be the natural representation of the general linear group  $GL_n$  ( $n \geq 1$ ) on  $\mathbb{C}^n$ , then for any  $G, \rho$  and  $V$  such that  $\dim V \leq n$ , the triplet  $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$  is a PV. A PV which can be written in this form is called a trivial PV. Unfortunately, when  $\dim V > n$ , then a triplet  $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$  is not a PV in general. However, when  $m := \dim V > n$  and a triplet  $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$  is a PV, it is known that a triplet  $(G \times GL_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n})$  is also a PV. These two PVs are said to be **castling transforms** of each other (See Definition 11, p39, [S-K]).

The theory of prehomogeneous vector spaces is closely related to the theory of Lie algebras. We can express the prehomogeneity condition of a triplet  $(G, \rho, V)$  by the method of Lie algebras as follows.

We denote the Lie algebra of  $G$  by  $\text{Lie}(G)$ , the infinitesimal representation of  $\rho$  by  $d\rho$ . Then  $(G, \rho, V)$  is a PV if and only if there exists an element  $v \in V$  which satisfies  $d\rho(\text{Lie}(G))v = V$ . Such an element  $v$  is a generic point of  $(G, \rho, V)$ . This is an infinitesimal condition of the prehomogeneity.

Moreover, we have a class of prehomogeneous vector spaces called **prehomogeneous vector spaces of parabolic type**. These PVs can be obtained by graded semisimple Lie algebras in the following way.

Let  $\mathfrak{g}$  be a finite-dimensional semisimple Lie algebra,  $G$  the adjoint group of  $\mathfrak{g}$ ,  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ ,  $R$  the root system of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ ,  $\varphi$  a fundamental system of  $R$  all defined over  $\mathbb{C}$ . Let  $\theta$  be a subset of  $\varphi$  and  $H^\theta \in \mathfrak{h}$  the unique element which satisfies  $\alpha(H^\theta) = 0$  for all  $\alpha \in \theta$  and  $\alpha(H^\theta) = 2$  for all  $\alpha \in \varphi \setminus \theta$ . Put  $d_i(\theta) := \{X \in \mathfrak{g} \mid [H^\theta, X] = 2iX\}$  for each  $i \in \mathbb{Z}$ . Then we can obtain a  $\mathbb{Z}$ -grading of  $\mathfrak{g}$ :

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} d_i(\theta). \quad (0.1)$$

The graded Lie algebra (0.1) and the Killing form  $K$  of  $\mathfrak{g}$  satisfy the following conditions:

$$d_0(\theta) \text{ and } d_1(\theta) \text{ are finite-dimensional vector spaces,} \quad (0.2)$$

$$\text{the adjoint representation of } d_0(\theta) \text{ on } d_1(\theta) \text{ is completely reducible,} \quad (0.3)$$

$$\text{the restriction of } K \text{ to } d_i(\theta) \times d_{-i}(\theta) \text{ is non-degenerate for each } i \geq 0, \quad (0.4)$$

$$[d_1(\theta), d_i(\theta)] = d_{i+1}(\theta), \quad [d_{-1}(\theta), d_{-j}(\theta)] = d_{-j-1}(\theta) \text{ for each } i, j \geq 0. \quad (0.5)$$

Denote  $d_0(\theta)$  by  $\mathfrak{l}_\theta$  and the connected subgroup of  $G$  which corresponds to  $\mathfrak{l}_\theta$  by  $L_\theta$ , then it is known that  $(L_\theta, d_i(\theta))$  is a prehomogeneous vector space for any  $i \neq 0$ . We denote it by  $(\mathfrak{l}_\theta, d_i(\theta))$ . In other words, there exists an element  $v \in d_i(\theta)$  such that  $[\mathfrak{l}_\theta, v] = d_i(\theta)$ . In Rubenthaler [Ru-1], he showed that we can reduce to the case where  $i = 1$  and studied spaces of the form  $(\mathfrak{l}_\theta, d_1(\theta))$  and called them PVs of parabolic type. PVs of parabolic type can be described and classified by the weighted Dynkin diagrams.

For a PV of parabolic type  $(\mathfrak{l}_\theta, d_1(\theta))$ , we can recognize a space  $(\mathfrak{l}_\theta, d_{-1}(\theta))$  as the dual space of  $(\mathfrak{l}_\theta, d_1(\theta))$  via the Killing form  $K$  of  $\mathfrak{g}$  and assume that  $[d_1(\theta), d_{-1}(\theta)] = \mathfrak{l}_\theta$  without loss of generality. In other words, we can say that a PV of parabolic type is a triplet which consists of a reductive Lie algebra and its representation which can be embedded into a finite-dimensional graded Lie algebra with a non-degenerate symmetric invariant bilinear form. However, a casting transform of  $(\mathfrak{l}_\theta, d_1(\theta))$  is in general no longer of parabolic type, that is, it can not be embedded into a semisimple Lie algebra.

In this paper, we shall introduce a way to embed a given finite-dimensional reductive Lie algebra  $\mathfrak{g}$ , a finite-dimensional completely reducible representation  $(\rho, V)$  (do not assume the prehomogeneity condition) of  $\mathfrak{g}$  and its dual space  $(\rho^*, V^*)$  into a ‘‘large’’ graded Lie algebra with a non-degenerate symmetric invariant bilinear form. In particular, a prehomogeneous vector space of parabolic type can be embedded into a finite-dimensional semisimple Lie algebra by our construction. For this, by using a non-degenerate symmetric invariant bilinear form  $B_0$  on  $\mathfrak{g}$ , we shall construct a graded Lie algebra  $L(\mathfrak{g}, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$  which also has a non-degenerate symmetric invariant bilinear form  $B$  such that  $V_0$  and  $V_1$  are isomorphic to  $\mathfrak{g}$  and  $V$  as  $\mathfrak{g}$ -modules respectively and a restriction of  $B$  to  $V_0 \times V_0$  coincides with  $B_0$  (Theorem 2.11 and Proposition 3.2). In our construction, the bilinear form  $B_0$  plays important roles. In particular,  $B_0$  defines a linear map  $\hat{\Phi}_\rho$  from  $V \otimes V^*$  to  $\mathfrak{g}$  called  $\Phi$ -map and  $\hat{\Phi}_\rho$  induces a bracket product between  $V_1$  and  $V_{-1}$ . Moreover, we can prove that  $L(\mathfrak{g}, \rho, V, B_0)$  satisfies the following conditions:

$$V_0 \text{ and } V_1 \text{ are finite-dimensional vector spaces,} \quad (0.6)$$

$$\text{the adjoint representation of } V_0 \text{ on } V_1 \text{ is completely reducible,} \quad (0.7)$$

$$\text{the restriction of } B \text{ to } V_i \times V_{-i} \text{ is non-degenerate for each } i \geq 0, \quad (0.8)$$

$$[V_1, V_i] = V_{i+1}, \quad [V_{-1}, V_{-j}] = V_{-j-1} \text{ for each } i, j \geq 0, \quad (0.9)$$

$$[V_1, V_{-1}] = V_0. \quad (0.10)$$

Conversely, an arbitrary graded Lie algebra which satisfies the conditions from (0.6) to (0.10) can be obtained by our construction (Proposition 3.3). In particular, an arbitrary finite-dimensional semisimple Lie algebra can be obtained by a certain PV of parabolic type and a bilinear form (Theorem 3.14).

Now, the prehomogeneity condition of a triplet  $(G, \rho, V)$ , where  $G$  is a reductive algebraic group, is closely related to the  $\Phi$ -map of the quadruplet  $(\text{Lie}(G), d\rho, V, B_0)$  where  $B_0$  is a bilinear form on  $\text{Lie}(G)$ . In particular, the prehomogeneity condition of a triplet of the form  $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$  can be expressed by the  $\Phi$ -map of the quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$ , where  $T_n$  is a bilinear form on  $\mathfrak{gl}_n$  defined by:

$$T_n(a, a') = \text{Tr}(aa') \quad (a, a' \in \mathfrak{gl}_n = M(n; \mathbb{C})). \quad (0.11)$$

Furthermore, this condition is reduced to the condition of the  $\Phi$ -map  $\hat{\Phi}_\rho$  of  $(\mathfrak{g}, \rho, V, B_0)$ . Then, we can obtain another proof of casting transform.

This paper consists of four sections.

In section 1, first of all, we shall explain that a quadruplet  $(\mathfrak{g}, \rho, V, B_0)$  determines a linear map  $\hat{\Phi}_\rho$  from  $V \otimes V^*$  to  $\mathfrak{g}$ . Then  $\hat{\Phi}_\rho$  induces a  $\mathfrak{g}$ -submodule  $V_1$  in  $\text{Hom}(V^*, \mathfrak{g})$ . In order to have  $V_1 \simeq V$  as  $\mathfrak{g}$ -modules, we introduce the notion of **standard quadruplets**. Then, moreover, we can obtain  $\mathfrak{g}$ -modules  $V_n$  for all  $n \in \mathbb{Z}$  inductively and call them  **$n$ -graduations**. Each  $\mathfrak{g}$ -module  $V_n$  is finite-dimensional and the dual  $\mathfrak{g}$ -module of  $V_{-n}$ . In particular,  $V_0$  is isomorphic to  $\mathfrak{g}$  itself.

In section 2, we define a bilinear map  $[\cdot, \cdot]_m^n$  from  $V_n \times V_m$  to  $V_{n+m}$  for any  $n, m \in \mathbb{Z}$ . Denote a direct sum of  $n$ -graduations obtained in section 1 by  $L(\mathfrak{g}, \rho, V, B_0)$ . Then we have a bilinear map  $[\cdot, \cdot]: L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0) \rightarrow L(\mathfrak{g}, \rho, V, B_0)$  defined by

$$[x_n, y_m] := [x_n, y_m]_m^n \quad (x_n \in V_n, y_m \in V_m).$$

Our main result is that  $[\cdot, \cdot]$  satisfies the axioms of a Lie algebra, i.e.  $L(\mathfrak{g}, \rho, V, B_0)$  has a structure of a graded Lie algebra. We call the Lie algebra thus obtained a **Lie algebra associated to a standard quadruplet** (Theorem 2.11). For example, loop algebras and finite-dimensional semisimple Lie algebras are Lie algebras associated to some quadruplet. This will be proved in section 3.

In section 3, we will study some properties of  $L(\mathfrak{g}, \rho, V, B_0)$ . First, we shall construct a non-degenerate symmetric and invariant bilinear form on  $L(\mathfrak{g}, \rho, V, B_0)$ . Then we can prove that the Lie algebras associated to a standard quadruplet can be characterized by the existence of a non-degenerate symmetric and invariant bilinear form. It is described in Proposition 3.3. Finally, in Theorem 3.14, we will give a necessary and sufficient condition for  $(\mathfrak{g}, \rho, V, B_0)$  to generate a finite-dimensional Lie algebra  $L(\mathfrak{g}, \rho, V, B_0)$ . The condition is closely related to prehomogeneous vector spaces of parabolic type.

In section 4, we shall consider prehomogeneous vector spaces. We shall express the prehomogeneity condition of a given triplet  $(G, \rho, V)$  as a condition of the  $\Phi$ -map  $\hat{\Phi}_{d\rho}$  of the quadruplet

$(\text{Lie}(G), d\rho, V, B_0)$  for some bilinear form  $B_0$ . Moreover, we can express the prehomogeneity condition of a quadruplets of the form  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  as a condition of  $(\mathfrak{g}, \rho, V, B_0)$  (Lemma 4.5). By applying this lemma, we shall give another proof of castling transform (Theorem 4.7).

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Notation: We denote the transpose of a matrix  $X$  by  ${}^tX$ , the zero-matrix and the unit matrix of size  $n \times n$  by  $0_n$  and  $1_n$ , the Kronecker delta by  $\delta_{ij}$ . Put  $J_n := \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ . Throughout this paper, we shall always assume that all objects are defined over the complex number field  $\mathbb{C}$ .

## 1 A family of $n$ -graduations

### 1.1 Standard quadruplets

Let  $\mathfrak{g}$  be a finite-dimensional reductive Lie algebra,  $V$  a finite-dimensional vector space,  $\rho$  a representation of  $\mathfrak{g}$  on  $V$  all defined over  $\mathbb{C}$ . By the theory of Lie algebras, there exists a non-degenerate symmetric  $\mathfrak{g}$ -invariant bilinear form  $B_0$  on  $\mathfrak{g}$ . Let  $V^*$  be the dual module of  $V$  and  $\rho^*$  the dual representation of  $\rho$ . We denote the pairing between  $V$  and  $V^*$  by  $\langle \cdot, \cdot \rangle$ . By the assumption that  $B_0$  is non-degenerate, we can define the following linear maps.

**Definition 1.1.** We define a linear map  $\hat{\Phi}_\rho$  (resp.  $\hat{\Psi}_\rho$ ) from  $V \otimes V^*$  to  $\mathfrak{g}$  (resp. from  $V^* \otimes V$  to  $\mathfrak{g}$ ) by the following equation:

$$B_0(a, \hat{\Phi}_\rho(v \otimes \phi)) = \langle \rho(a)v, \phi \rangle = -\langle v, \rho^*(a)\phi \rangle \quad (1.1)$$

(resp.

$$B_0(a, \hat{\Psi}_\rho(\phi \otimes v)) = \langle v, \rho^*(a)\phi \rangle = -\langle \rho(a)v, \phi \rangle ) \quad (1.2)$$

for any  $a \in \mathfrak{g}$ ,  $v \in V$  and  $\phi \in V^*$ . We call this map  $\hat{\Phi}$  (resp.  $\hat{\Psi}$ ) **the  $\Phi$ -map** (resp. **the  $\Psi$ -map**) of the quadruplet  $(\mathfrak{g}, \rho, V, B_0)$ .

**Remark 1.2.** For any  $v \in V$  and  $\phi \in V^*$ , we have

$$\hat{\Phi}_\rho(v \otimes \phi) + \hat{\Psi}_\rho(\phi \otimes v) = 0. \quad (1.3)$$

**Example 1.3.** Let  $(\mathfrak{g}, \rho, V, B_0) = (\mathfrak{gl}_n, \Lambda_1, \mathbb{C}^n, T_n)$ , where  $\Lambda_1$  is the natural representation of  $\mathfrak{gl}_n$  on the space of column vectors  $\mathbb{C}^n = M(n, 1; \mathbb{C})$  and  $T_n$  is a non-degenerate symmetric invariant bilinear form on  $\mathfrak{gl}_n$  defined by

$$T_n(a, a') := \text{Tr}(aa') \quad (1.4)$$

where  $a, a' \in \mathfrak{g} = M(n, \mathbb{C})$ . Then the dual space  $V^*$  can be identified with  $V = \mathbb{C}^n$ . The representation  $\Lambda_1^*$  and the pairing  $\langle \cdot, \cdot \rangle$  are given as follows:

$$\Lambda_1^*(a)\phi := -{}^t a\phi, \quad (1.5)$$

$$\langle v, \phi \rangle := {}^t v\phi \quad (1.6)$$

where  $a \in \mathfrak{g}$ ,  $v \in V$  and  $\phi \in V^*$ . Then the  $\Phi$ -map is given as:

$$\hat{\Phi}_{\Lambda_1}(v \otimes \phi) = v^t \phi. \quad (1.7)$$

**Example 1.4.** Let  $(\mathfrak{g}, \rho, V, B_0) = (\mathfrak{so}_n, \Lambda_1, \mathbb{C}^n, T_n|_{\mathfrak{so}_n \times \mathfrak{so}_n})$ , where  $\Lambda_1$  is the natural representation of  $\mathfrak{so}_n$  on  $\mathbb{C}^n = M(n, 1; \mathbb{C})$ . Then the dual space  $V^*$  can be identified with  $V = \mathbb{C}^n$ . The representation  $\Lambda_1^*$  and the pairing  $\langle \cdot, \cdot \rangle$  are given as follows:

$$\Lambda_1^*(a)\phi := a\phi, \quad (1.8)$$

$$\langle v, \phi \rangle := {}^t v\phi \quad (1.9)$$

where  $a \in \mathfrak{g}$ ,  $v \in V$  and  $\phi \in V^*$ . Then the  $\Phi$ -map is given as:

$$\hat{\Phi}_{\Lambda_1}(v \otimes \phi) = \frac{1}{2}(v^t \phi - \phi^t v). \quad (1.10)$$

**Example 1.5.** Let  $(\mathfrak{g}, \rho, V, B_0) = (\mathfrak{sp}_n, \Lambda_1, \mathbb{C}^{2n}, T_{2n})$ , where  $\mathfrak{sp}_n = \{X \in \mathfrak{gl}_{2n} \mid {}^t X J_n + J_n X = 0\}$  and  $\Lambda_1$  is the natural representation of  $\mathfrak{sp}_n$  on  $\mathbb{C}^{2n} = M(2n, 1; \mathbb{C})$ . Then the dual space  $V^*$  can be identified with  $V = \mathbb{C}^{2n}$ . The representation  $\Lambda_1^*$  and the pairing  $\langle \cdot, \cdot \rangle$  are given as follows:

$$\Lambda_1^*(a)\phi := a\phi, \quad (1.11)$$

$$\langle v, \phi \rangle := {}^t v J_n \phi \quad (1.12)$$

where  $a \in \mathfrak{g}$ ,  $v \in V$  and  $\phi \in V^*$ . Then the  $\Phi$ -map is given as:

$$\hat{\Phi}_{\Lambda_1}(v \otimes \phi) = -\frac{1}{2}(v^t \phi J_n + \phi^t v J_n). \quad (1.13)$$

**Proposition 1.6.** Let  $\mathfrak{g}$  be any reductive Lie algebra,  $\rho^i, \sigma^i$  representations of  $\mathfrak{g}$  on  $V^i$  and  $U^i$  ( $i = 1, 2$ ),  $B_0$  a non-degenerate symmetric invariant bilinear form on  $\mathfrak{g}$ . For quadruplets  $(\mathfrak{g}, \rho^1 \oplus \rho^2, V^1 \oplus V^2, B_0)$  and  $(\mathfrak{g}, \sigma^1 \otimes \sigma^2, U^1 \otimes U^2, B_0)$ , the maps  $\hat{\Phi}_{\rho^1 \oplus \rho^2}$  and  $\hat{\Phi}_{\sigma^1 \otimes \sigma^2}$  are given as:

$$\hat{\Phi}_{\rho^1 \oplus \rho^2}((v^1, v^2) \otimes (\phi^1, \phi^2)) = \hat{\Phi}_{\rho^1}(v^1 \otimes \phi^1) + \hat{\Phi}_{\rho^2}(v^2 \otimes \phi^2), \quad (1.14)$$

$$\hat{\Phi}_{\sigma^1 \otimes \sigma^2}((u^1 \otimes u^2) \otimes (\psi^1 \otimes \psi^2)) = \langle u^2, \psi^2 \rangle \hat{\Phi}_{\sigma^1}(u^1 \otimes \psi^1) + \langle u^1, \psi^1 \rangle \hat{\Phi}_{\sigma^2}(u^2 \otimes \psi^2), \quad (1.15)$$

where  $v^i \in V^i, u^i \in U^i, \phi^i \in (V^i)^*, \psi^i \in (U^i)^*$  and  $\hat{\Phi}_{\rho^i}, \hat{\Phi}_{\sigma^i}$  are the  $\Phi$ -maps of quadruplets  $(\mathfrak{g}, \rho^i, V^i, B_0)$  and  $(\mathfrak{g}, \sigma^i, U^i, B_0)$  ( $i = 1, 2$ ).

**Proof.** The dual spaces of  $V^1 \oplus V^2$  and  $U^1 \otimes U^2$  can be identified with  $(V^1)^* \oplus (V^2)^*$  and  $(U^1)^* \otimes (U^2)^*$  respectively. The pairings between them are given as:

$$\langle (v^1, v^2), (\phi^1, \phi^2) \rangle := \langle v^1, \phi^1 \rangle + \langle v^2, \phi^2 \rangle, \quad (1.16)$$

$$\langle u^1 \otimes u^2, \psi^1 \otimes \psi^2 \rangle := \langle u^1, \psi^1 \rangle \langle u^2, \psi^2 \rangle \quad (1.17)$$

where  $v^i \in V^i, u^i \in U^i, \phi^i \in (V^i)^*, \psi^i \in (U^i)^*$  ( $i = 1, 2$ ). Thus, our claim can be checked by a direct calculation.  $\blacksquare$

Here,  $V \otimes V^*$  and  $V^* \otimes V$  have canonical  $\mathfrak{g}$ -module structures and we have the following proposition.

**Proposition 1.7.** *The maps  $\hat{\Phi}_\rho$  and  $\hat{\Psi}_\rho$  are homomorphisms of  $\mathfrak{g}$ -modules.*

**Proof.** In fact, for any  $a, a' \in \mathfrak{g}, v \in V$  and  $\phi \in V^*$ , we have

$$\begin{aligned} & B_0(a', \hat{\Phi}_\rho((\rho(a)v) \otimes \phi + v \otimes (\rho^*(a)\phi))) \\ &= \langle \rho(a')\rho(a)v, \phi \rangle + \langle \rho(a')v, \rho^*(a)\phi \rangle \\ &= \langle \rho([a', a]v), \phi \rangle + \langle \rho(a)\rho(a')v, \phi \rangle + \langle \rho(a')v, \rho^*(a)\phi \rangle \\ &= B_0([a', a], \hat{\Phi}_\rho(v \otimes \phi)) - \langle \rho(a')v, \rho^*(a)\phi \rangle + \langle \rho(a')v, \rho^*(a)\phi \rangle \\ &= B_0(a', [a, \hat{\Phi}_\rho(v \otimes \phi)]). \end{aligned}$$

Hence,  $\hat{\Phi}_\rho((\rho(a)v) \otimes \phi + v \otimes (\rho^*(a)\phi)) = [a, \hat{\Phi}_\rho(v \otimes \phi)]$ . Similarly, we can obtain the equation  $\hat{\Psi}_\rho((\rho^*(a)\phi) \otimes v + \phi \otimes (\rho(a)v)) = [a, \hat{\Psi}_\rho(\phi \otimes v)]$  and thus  $\hat{\Phi}_\rho$  and  $\hat{\Psi}_\rho$  are homomorphisms of  $\mathfrak{g}$ -modules.  $\blacksquare$

By Proposition 1.7,  $\hat{\Phi}_\rho(V \otimes V^*)$  and  $\hat{\Phi}_\rho(V \otimes V^*)^\perp := \{a \in \mathfrak{g} \mid B_0(a, x) = 0 \text{ for any } x \in \hat{\Phi}_\rho(V \otimes V^*)\}$  are ideals of  $\mathfrak{g}$ . Then  $\hat{\Phi}_\rho(V \otimes V^*)^\perp$  coincides with  $\text{Ker } \rho$ . In fact, suppose that  $a \in \text{Ker } \rho$ . Then we have  $B_0(a, \hat{\Phi}_\rho(v \otimes \phi)) = \langle \rho(a)v, \phi \rangle = 0$  for any  $v \in V$  and  $\phi \in V^*$ . Thus we have  $\hat{\Phi}_\rho(V \otimes V^*)^\perp \supset \text{Ker } \rho$ . Similarly we have the converse inclusion. Hence we obtain the following proposition.

**Proposition 1.8.** *The maps  $\hat{\Phi}_\rho$  and  $\hat{\Psi}_\rho$  are surjective if and only if  $\rho$  is faithful.*

Next, let us construct  $\mathfrak{g}$ -modules which are isomorphic to  $V$  and  $V^*$  respectively.

**Definition 1.9.** For each element  $v \in V$  and  $\phi \in V^*$ , we define linear maps  $\Phi_{\rho, v} \in \text{Hom}(V^*, \mathfrak{g})$  and  $\Psi_{\rho, \phi} \in \text{Hom}(V, \mathfrak{g})$  by

$$\Phi_{\rho, v}(\psi) := \hat{\Phi}_\rho(v \otimes \psi), \quad (1.18)$$

$$\Psi_{\rho, \phi}(u) := \hat{\Psi}_\rho(\phi \otimes u) \quad (1.19)$$



where  $u \in V$  and  $\psi \in V^*$ . Then we have linear maps  $\Phi_\rho$  and  $\Psi_\rho$  as below:

$$\begin{aligned} \Phi_\rho : V &\rightarrow \text{Hom}(V^*, \mathfrak{g}) \\ v &\mapsto \Phi_{\rho,v}, \end{aligned} \tag{1.20}$$

$$\begin{aligned} \Psi_\rho : V^* &\rightarrow \text{Hom}(V, \mathfrak{g}) \\ \phi &\mapsto \Psi_{\rho,\phi}. \end{aligned} \tag{1.21}$$

We call these maps  $\Phi_{\rho,v}$  and  $\Psi_{\rho,\phi}$  the  $\Phi$ -**map at**  $v$  and  $\Psi$ -**map at**  $\phi$  of  $(\mathfrak{g}, \rho, V, B)$  respectively.

The spaces  $\text{Hom}(\mathfrak{g}, V)$  and  $\text{Hom}(\mathfrak{g}, V^*)$  have canonical  $\mathfrak{g}$ -module structures. Then we have the following proposition.

**Proposition 1.10.** *The maps  $\Phi_\rho$  and  $\Psi_\rho$  are homomorphisms of  $\mathfrak{g}$ -modules.*

*Proof.* By Proposition 1.7, we have the following equations:

$$\Phi_{\rho,\rho(a)v}(\phi) = [a, \Phi_{\rho,v}(\phi)] - \Phi_{\rho,v}(\rho^*(a)\phi), \tag{1.22}$$

$$\Psi_{\rho,\rho^*(b)\psi}(u) = [b, \Psi_{\rho,\psi}(u)] - \Psi_{\rho,\psi}(\rho(b)u) \tag{1.23}$$

for any  $a, b \in \mathfrak{g}$ ,  $v, u \in V$  and  $\phi, \psi \in V^*$ . Thus we have our result. ■

Put

$$V_0 := \mathfrak{g}, \tag{1.24}$$

$$V_1 := \text{Im}(\Phi_\rho), \tag{1.25}$$

$$V_{-1} := \text{Im}(\Psi_\rho). \tag{1.26}$$

Then  $V_0$ ,  $V_1$  and  $V_{-1}$  are  $\mathfrak{g}$ -modules. We denote the canonical representations of  $\mathfrak{g}$  on  $V_0$ ,  $V_1$  and  $V_{-1}$  by  $\rho_0$ ,  $\rho_1$  and  $\rho_{-1}$  respectively. Then we can define linear maps  $p_0$  and  $q_0$  by

$$\begin{aligned} p_0 : V_1 \otimes V_0 &\rightarrow V_1 \\ v_1 \otimes a &\mapsto -\rho_1(a)v_1, \end{aligned} \tag{1.27}$$

$$\begin{aligned} q_0 : V_{-1} \otimes V_0 &\rightarrow V_{-1} \\ \phi_{-1} \otimes a &\mapsto -\rho_{-1}(a)\phi_{-1}. \end{aligned} \tag{1.28}$$

We can easily show that  $p_0$  and  $q_0$  are homomorphisms of  $\mathfrak{g}$ -modules. Furthermore, if  $\rho$  is completely reducible and does not have a subrepresentation on a non-zero subspace of  $V$  which is isomorphic to a zero-representation, then  $p_0$  and  $q_0$  are surjective and  $\Phi_\rho$  and  $\Psi_\rho$  are injective. In fact, take an element  $v \in V$  which satisfies  $\Phi_{\rho,v} = 0$ . Then we have  $\langle \rho(a)v, \phi \rangle = B_0(a, \Phi_{\rho,v}(\phi)) = 0$  for any  $a \in \mathfrak{g}$  and  $\phi \in V^*$ , hence  $\rho(a)v = 0$  and  $v = 0$ . The same holds for  $\Psi_\rho$ . Therefore we obtain the following proposition.

**Proposition 1.11.** *If  $\rho$  is completely reducible and does not have a subrepresentation on a non-zero subspace of  $V$  which is isomorphic to a zero-representation, then  $\mathfrak{g}$ -modules  $V_1$  and  $V_{-1}$  are isomorphic to  $V$  and  $V^*$  respectively.*

**Remark 1.12.** In the situation of Proposition 1.11, we have

$$v_1(\Psi_\rho^{-1}(\phi_{-1})) + \phi_{-1}(\Phi_\rho^{-1}(v_1)) = 0 \quad (1.29)$$

for any  $v_1 \in V_1$  and  $\phi_{-1} \in V_{-1}$ .

Here, we give the following definition.

**Definition 1.13.** Let  $\mathfrak{g}$  be a finite-dimensional reductive Lie algebra,  $V$  a finite-dimensional vector space,  $\rho$  a representation of  $\mathfrak{g}$  on  $V$ ,  $B_0$  a non-degenerate symmetric invariant bilinear form on  $\mathfrak{g}$ . If a quadruplet  $(\mathfrak{g}, \rho, V, B_0)$  satisfies the following conditions, we call it a **standard quadruplet**:

$$\rho \text{ is completely reducible,} \quad (1.30)$$

$$\rho \text{ is faithful,} \quad (1.31)$$

$$\rho \text{ does not have a subrepresentation on a non-zero subspace of } V \text{ which is isomorphic to a zero-representation.} \quad (1.32)$$

In other words, if  $(\mathfrak{g}, \rho, V, B_0)$  is a standard quadruplet, the  $\mathfrak{g}$ -modules  $V_0$ ,  $V_1$  and  $V_{-1}$  are isomorphic to  $\hat{\Phi}_\rho(V \otimes V^*)$ ,  $V$  and  $V^*$  respectively. Next, let us construct  $\mathfrak{g}$ -modules from  $V_1$  and  $V_{-1}$ . For this, we will show the following proposition used later.

**Proposition 1.14.** *Let  $(\mathfrak{g}, \rho, V, B_0)$  be a standard quadruplet. Let  $W$  be a finite-dimensional  $\mathfrak{g}$ -module and  $\rho_W$  a representation of  $\mathfrak{g}$  on  $W$ . Assume that there exists a homomorphism of  $\mathfrak{g}$ -modules  $p$  from  $V_1 \otimes W$  to  $\text{Hom}(V^*, W)$ . We put*

$$\tilde{W} := \text{Im } p \quad (1.33)$$

and denote the canonical representation of  $\mathfrak{g}$  on  $\tilde{W}$  by  $\tilde{\rho}_W$ . Then the following linear map

$$\begin{aligned} \tilde{p} : V_1 \otimes \tilde{W} &\rightarrow \text{Hom}(V^*, \tilde{W}) \\ v_1 \otimes \tilde{w} &\mapsto (\phi \mapsto \tilde{\rho}_W(v_1(\phi))\tilde{w} + p(v_1 \otimes \tilde{w}(\phi))) \end{aligned} \quad (1.34)$$

is a homomorphism of  $\mathfrak{g}$ -modules.

**Proof.** For any  $a \in \mathfrak{g}$ ,  $v_1 \in V_1$ ,  $\tilde{w} \in \tilde{W}$  and  $\phi \in V^*$ , we have

$$\begin{aligned}
& \tilde{p}(\rho_1(a)v_1 \otimes \tilde{w})(\phi) + \tilde{p}(v_1 \otimes \tilde{\rho}_W(a)\tilde{w})(\phi) \\
&= \tilde{\rho}_W((\rho_1(a)v_1)(\phi))\tilde{w} + p(\rho_1(a)v_1 \otimes \tilde{w})(\phi) \\
&\quad + \tilde{\rho}_W(v_1(\phi))\tilde{\rho}_W(a)\tilde{w} + p(v_1 \otimes (\tilde{\rho}_W(a)\tilde{w})(\phi)) \\
&= \tilde{\rho}_W([a, v_1(\phi)])\tilde{w} - \tilde{\rho}_W(v_1(\rho^*(a)\phi))\tilde{w} \\
&\quad + p(\rho_1(a)v_1 \otimes \tilde{w})(\phi) + \tilde{\rho}_W(v_1(\phi))\tilde{\rho}_W(a)\tilde{w} \\
&\quad + p(v_1 \otimes \rho_W(a)(\tilde{w}(\phi))) - p(v_1 \otimes \tilde{w}(\rho^*(a)\phi)) \\
&= \tilde{\rho}_W(a)\tilde{\rho}_W(v_1(\phi))\tilde{w} + \tilde{\rho}_W(a)p(v_1 \otimes (\tilde{w}(\phi))) \\
&\quad - \tilde{p}(v_1 \otimes \tilde{w})(\rho^*(a)\phi) \\
&= \tilde{\rho}_W(a)(\tilde{p}(v_1 \otimes \tilde{w})(\phi)) - \tilde{p}(v_1 \otimes \tilde{w})(\rho^*(a)\phi).
\end{aligned}$$

Thus we have our result. ■

**Definition 1.15.** Let  $(\mathfrak{g}, \rho, V, B_0)$  be a standard quadruplet. Suppose that  $i \geq 1$  and there exist  $\mathfrak{g}$ -modules  $(\rho_{i-1}, V_{i-1})$  and  $(\rho_{-i+1}, V_{-i+1})$  and homomorphisms of  $\mathfrak{g}$ -modules  $p_{i-1} : V_1 \otimes V_{i-1} \rightarrow \text{Hom}(V^*, V_{i-1})$  and  $q_{-i+1} : V_{-1} \otimes V_{-i+1} \rightarrow \text{Hom}(V, V_{-i+1})$ . Put  $V_i := \text{Im } p_{i-1}$  and  $V_{-i} := \text{Im } q_{-i+1}$  and denote the canonical representations of  $\mathfrak{g}$  on them by  $\rho_i$  and  $\rho_{-i}$  respectively. We define linear maps  $p_i$  and  $q_{-i}$  by

$$\begin{aligned}
p_i : V_1 \otimes V_i &\rightarrow \text{Hom}(V^*, V_i) \\
v_1 \otimes u_i &\mapsto (\phi \mapsto \rho_i(v_1(\phi))u_i + p_{i-1}(v_1 \otimes u_i(\phi))), \tag{1.35}
\end{aligned}$$

$$\begin{aligned}
q_{-i} : V_{-1} \otimes V_{-i} &\rightarrow \text{Hom}(V, V_{-i}) \\
\phi_{-1} \otimes \psi_{-i} &\mapsto (v \mapsto \rho_{-i+1}(\phi_{-1}(v))\psi_{-i} + q_{-i+1}(\phi_{-1} \otimes \psi_{-i}(v))). \tag{1.36}
\end{aligned}$$

Then, by Proposition 1.14,  $p_i$  is a homomorphism of  $\mathfrak{g}$ -modules. Similarly,  $q_{-i}$  is also a homomorphism of  $\mathfrak{g}$ -modules. We denote by  $V_{i+1}$  and  $V_{-i-1}$  the images of  $p_i$  and  $q_{-i}$  and the canonical representations on  $V_{i+1}$  and  $V_{-i-1}$  by  $\rho_{i+1}$  and  $\rho_{-i-1}$  respectively. Thus, inductively, we obtain  $\mathfrak{g}$ -modules  $V_n$  and representations  $\rho_n$  of  $\mathfrak{g}$  on  $V_n$  for all  $n \in \mathbb{Z}$ . We call  $V_n$  **the  $n$ -graduation** of  $(\mathfrak{g}, \rho, V, B_0)$ . Throughout this paper, we use these notation.

**Remark 1.16.** Especially, if  $i = 1$ , we have

$$\begin{aligned}
p_1(v_1 \otimes u_1)(\phi) &= -p_1(u_1 \otimes v_1)(\phi) \\
&= \rho_1(v_1(\phi))u_1 - \rho_1(u_1(\phi))v_1 \tag{1.37}
\end{aligned}$$

and

$$\begin{aligned}
q_{-1}(\phi_{-1} \otimes \psi_{-1})(v) &= -q_{-1}(\psi_{-1} \otimes \phi_{-1})(v) \\
&= \rho_{-1}(\phi_{-1}(v))\psi_{-1} - \rho_{-1}(\psi_{-1}(v))\phi_{-1}. \tag{1.38}
\end{aligned}$$

## 1.2 A bilinear form on $V_i \times V_{-i}$

In this section, we will consider the relation between  $\mathfrak{g}$ -modules  $V_i$  and  $V_{-i}$  for each  $i \geq 0$ . For this, we construct a bilinear form on  $V_i \times V_{-i}$ . First, let us show the following equation on  $B_0$  :

$$\begin{aligned} & B_0(p_0(v_1 \otimes a)(\Psi_\rho^{-1}(\phi_{-1})), b) \\ &= B_0(a, q_0(\phi_{-1} \otimes b)(\Phi_\rho^{-1}(v_1))) \end{aligned} \quad (1.39)$$

where  $a, b \in V_0$ ,  $v_1 \in V_1$  and  $\phi_{-1} \in V_{-1}$ .

Put  $v := \Phi_\rho^{-1}(v_1) \in V$  and  $\phi := \Psi_\rho^{-1}(\phi_{-1}) \in V^*$ . Then we have

$$\begin{aligned} & B_0(p_0(v_1 \otimes a)(\phi), b) \\ &= B_0((-\rho_1(a)v_1)(\phi), b) \\ &= B_0(v_1(\rho^*(a)\phi), b) - B_0([a, v_1(\phi)], b) \\ &= \langle \rho(b)v, \rho^*(a)\phi \rangle + B_0([a, \phi_{-1}(v)], b) \\ &= B_0(a, \phi_{-1}(\rho(b)v)) + B_0(a, [\phi_{-1}(v), b]) \\ &= B_0(a, q_0(\phi_{-1} \otimes b)(v)). \end{aligned} \quad (1.40)$$

Next, suppose that  $i \geq 1$ . Assume that there exists a  $\mathfrak{g}$ -invariant bilinear form  $B_{i-1}$  on  $V_{i-1} \times V_{-i+1}$  which satisfies the following equation:

$$\begin{aligned} & B_{i-1}(p_{i-1}(v_1 \otimes u_{i-1})(\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i+1}) \\ &= B_{i-1}(u_{i-1}, q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})(\Phi_\rho^{-1}(v_1))) \end{aligned} \quad (1.41)$$

for any  $v_1 \in V_1$ ,  $u_{i-1} \in V_{i-1}$ ,  $\phi_{-1} \in V_{-1}$  and  $\psi_{-i+1} \in V_{-i+1}$ . Then we can define a bilinear form  $B_i$  on  $V_i \times V_{-i}$  by

$$\begin{aligned} & B_i(p_{i-1}(v_1 \otimes u_{i-1}), q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})) \\ &:= B_{i-1}(p_{i-1}(v_1 \otimes u_{i-1})(\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i+1}) \end{aligned} \quad (1.42)$$

where  $v_1 \in V_1$ ,  $u_{i-1} \in V_{i-1}$ ,  $\phi_{-1} \in V_{-1}$  and  $\psi_{-i+1} \in V_{-i+1}$ . Indeed, by (1.41),  $B_i$  is well-defined. Then  $B_i$  is  $\mathfrak{g}$ -invariant. In fact, by the assumption that  $B_{i-1}$  is  $\mathfrak{g}$ -invariant, we have

$$\begin{aligned} & B_i(\rho_i(a)p_{i-1}(v_1 \otimes u_{i-1}), q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})) \\ &= B_{i-1}(\rho_i(a)(p_{i-1}(v_1 \otimes u_{i-1})(\Psi_\rho^{-1}(\phi_{-1}))), \psi_{-i+1}) \\ &\quad - B_{i-1}(p_{i-1}(v_1 \otimes u_{i-1})(\rho^*(a)\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i+1}) \\ &= -B_{i-1}(p_{i-1}(v_1 \otimes u_{i-1})(\Psi_\rho^{-1}(\phi_{-1})), \rho_{-i+1}(a)\psi_{-i+1}) \\ &\quad - B_{i-1}(p_{i-1}(v_1 \otimes u_{i-1})(\rho^*(a)\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i+1}) \\ &= -B_i(p_{i-1}(v_1 \otimes u_{i-1}), \rho_{-i}(a)q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})) \end{aligned} \quad (1.43)$$

for any  $a \in \mathfrak{g}$ ,  $v_1 \in V_1$ ,  $u_{i-1} \in V_{i-1}$ ,  $\phi_{-1} \in V_{-1}$  and  $\psi_{-i+1} \in V_{-i+1}$ . Moreover  $B_i$  satisfies the following equation:

$$\begin{aligned} & B_i(p_i(v_1 \otimes u_i)(\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i}) \\ &= B_i(u_i, q_{-i}(\phi_{-1} \otimes \psi_{-i})(\Phi_\rho^{-1}(v_1))) \end{aligned} \quad (1.44)$$

for any  $v_1 \in V_1$ ,  $u_i \in V_i$ ,  $\phi_{-1} \in V_{-1}$  and  $\psi_{-i} \in V_{-i}$ . In fact, put  $v := \Phi_\rho^{-1}(v_1)$  and  $\phi := \Psi_\rho^{-1}(\phi_{-1})$ . Then we have

$$\begin{aligned} & B_i(p_i(v_1 \otimes u_i)(\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i}) \\ &= B_i(\rho_i(v_1(\phi))u_i + p_{i-1}(v_1 \otimes u_i(\phi)), \psi_{-i}) \\ &= B_i(u_i, \rho_{-i}(\phi_{-1}(v))\psi_{-i}) + B_{i-1}(u_i(\phi), \psi_{-i}(v)) \\ &= B_i(u_i, \rho_{-i}(\phi_{-1}(v))\psi_{-i}) + B_i(u_i, q_{-i}(\phi_{-1} \otimes \psi_{-i}(v))) \\ &= B_i(u_i, q_{-i}(\phi_{-1} \otimes \psi_{-i})(\Phi_\rho^{-1}(v_1))). \end{aligned}$$

Thus, inductively, we obtain a bilinear form  $B_i$  on  $V_i \times V_{-i}$  for all  $i \geq 0$ .

**Proposition 1.17.** *For all  $i \geq 0$ ,  $V_{-i}$  is the dual  $\mathfrak{g}$ -module of  $V_i$ .*

**Proof.** It is sufficient to show that the bilinear map  $B_i : V_i \times V_{-i} \rightarrow \mathbb{C}$  is non-degenerate for each  $i \geq 0$ . We argue by induction on  $i$ . For  $i = 0$ , by the assumption,  $B_0 : V_0 \times V_0 \rightarrow \mathbb{C}$  is non-degenerate and  $\mathfrak{g}$ -invariant. Hence,  $V_0$  is the dual  $\mathfrak{g}$ -module of itself. Suppose that  $i \geq 1$ . Take an element  $u_i \in V_i$  such that  $B_i(u_i, q_{-i+1}(\phi_{-1} \otimes \psi_{-i+1})) = 0$  for any  $\phi_{-1} \in V_{-1}$  and  $\psi_{-i+1} \in V_{-i+1}$ . Then we have

$$B_{i-1}(u_i(\Psi_\rho^{-1}(\phi_{-1})), \psi_{-i+1}) = 0.$$

By the induction hypothesis,  $B_{i-1}$  is non-degenerate and thus we have  $u_i(\Psi_\rho^{-1}(\phi_{-1})) = 0$  for any  $\phi_{-1} \in V_{-1}$ . Thus  $u_i = 0$ . ■

### 1.3 Examples

In this section, we will give some examples.

**Example 1.18.** Let  $n \geq 2$  be a positive integer. Let  $\mathfrak{g} = \mathfrak{gl}_1 \oplus \mathfrak{sl}_n$  and  $V = \mathbb{C}^n$ . We define a representation  $\rho$  of  $\mathfrak{g}$  on  $V$  by:

$$\begin{aligned} \rho(a, A)v &= av + Av \\ (a \in \mathfrak{gl}_1, A \in \mathfrak{sl}_n, v \in V) \end{aligned}$$

where  $v \in V$  is considered as a column vector. The dual space  $V^*$  is also identified with  $\mathbb{C}^n$  and the pairing between  $V$  and  $V^*$  is given by:

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V^* &\rightarrow \mathbb{C} \\ (v, \phi) &\mapsto {}^t v \phi. \end{aligned} \quad (1.45)$$

Then the dual representation  $\rho^*$  of  $\mathfrak{g}$  on  $V^*$  is given by:

$$\begin{aligned} \rho(a, A)\phi &= -a\phi - {}^t A\phi \\ (a \in \mathfrak{gl}_1, A \in \mathfrak{sl}_n, \phi \in V^*) \end{aligned}$$

where  ${}^t A$  is a transpose of  $A$ . Let  $B_0^1$  be a bilinear form on  $\mathfrak{g}$  given by:

$$\begin{aligned} B_0^1((a, A), (a', A')) &:= \frac{n}{n+1}aa' + \text{Tr } AA' \\ (a, a' \in \mathfrak{gl}_1, A, A' \in \mathfrak{sl}_n). \end{aligned} \quad (1.46)$$

Then  $(\mathfrak{g}, \rho, V, B_0^1)$  is a standard quadruplet. Then its  $\Phi$ -map  $\hat{\Phi}_\rho^1$  from  $V \otimes V^*$  to  $\mathfrak{g}$  is derived by:

$$\hat{\Phi}_\rho^1(v \otimes \phi) = \left( \frac{n+1}{n} {}^t v \phi, v {}^t \phi - \frac{1}{n} {}^t v \phi 1_n \right). \quad (1.47)$$

In this situation, the  $m$ -graduation of  $(\mathfrak{g}, \rho, V, B_0^1)$  is  $\{0\}$  for all  $m \geq 2$ . In fact, we can identify  $V_1$  with  $V$ . Then, for any  $v_1, u_1 \in V_1$  and  $\phi \in V^*$ , we have

$$\begin{aligned} &p_1(v_1 \otimes u_1)(\phi) \\ &= \rho_1 \left( \frac{n+1}{n} {}^t v_1 \phi, v_1 {}^t \phi - \frac{1}{n} {}^t v_1 \phi 1_n \right) u_1 - \rho_1 \left( \frac{n+1}{n} {}^t u_1 \phi, u_1 {}^t \phi - \frac{1}{n} {}^t u_1 \phi 1_n \right) v_1 \\ &= \frac{n+1}{n} {}^t v_1 \phi u_1 + v_1 {}^t \phi u_1 - \frac{1}{n} {}^t v_1 \phi u_1 - \frac{n+1}{n} {}^t u_1 \phi v_1 - u_1 {}^t \phi v_1 + \frac{1}{n} {}^t u_1 \phi v_1 \\ &= \frac{n+1}{n} u_1 {}^t v_1 \phi + v_1 {}^t u_1 \phi - \frac{1}{n} u_1 {}^t v_1 \phi - \frac{n+1}{n} v_1 {}^t u_1 \phi - u_1 {}^t v_1 \phi + \frac{1}{n} v_1 {}^t u_1 \phi \\ &= 0. \end{aligned}$$

Thus our claim holds.

**Example 1.19.** We retain the notation  $\mathfrak{g}$ ,  $\rho$  and  $V$  of Example 1.18 and choose another bilinear form  $B_0^2$  on  $\mathfrak{g}$  given as follows:

$$\begin{aligned} B_0^2((a, A), (a', A')) &:= naa' + \text{Tr } AA' \\ (a, a' \in \mathfrak{gl}_1, A, A' \in \mathfrak{sl}_n). \end{aligned} \quad (1.48)$$

Then we have

$$\begin{aligned} \hat{\Phi}_\rho^2(v \otimes \phi) &= \left( \frac{1}{n} {}^t v \phi, v {}^t \phi - \frac{1}{n} {}^t v \phi 1_n \right) \\ (v \in V, \phi \in V^*) \end{aligned} \quad (1.49)$$

where  $\hat{\Phi}_\rho^2$  is the  $\Phi$ -map of  $(\mathfrak{g}, \rho, V, B_0^2)$ . Identifying  $V_1$  with  $V$ , we have

$$\begin{aligned} p_1(v_1 \otimes u_1)(\phi) &= (v_1 {}^t u_1 - u_1 {}^t v_1)\phi \\ &\quad (v_1, u_1 \in V, \phi \in V^*) \end{aligned} \quad (1.50)$$

from the argument in Example 1.18. Therefore, we can identify  $V_2$  with the totality of skew-symmetric matrices of size  $n$  and the representation  $\rho_2$  on  $V_2$  is given as follows:

$$\begin{aligned} \rho_2(a, A)u_2 &= 2au_2 + Au_2 + u_2 {}^t A \\ &\quad (a \in \mathfrak{gl}_1, A \in \mathfrak{sl}_n, u_2 \in V_2). \end{aligned} \quad (1.51)$$

In this situation, the  $m$ -graduation of  $(\mathfrak{g}, \rho, V, B_0^2)$  is  $\{0\}$  for all  $m \geq 3$ . In fact, for any  $v_1, u_1, w_1 \in V_1$  and  $\phi \in V^*$ , we have

$$\begin{aligned} & p_2(w_1 \otimes p_1(v_1 \otimes u_1))(\phi) \\ &= \rho_2\left(\frac{1}{n} {}^t w_1 \phi, w_1 {}^t \phi - \frac{1}{n} {}^t w_1 \phi 1_n\right) p_1(v_1 \otimes u_1) + p_1(w_1 \otimes p_1(v_1 \otimes u_1))(\phi) \\ &= p_1\left(\rho_1\left(\frac{1}{n} {}^t w_1 \phi, w_1 {}^t \phi - \frac{1}{n} {}^t w_1 \phi 1_n\right) v_1 \otimes u_1\right) + p_1(v_1 \otimes \rho_1\left(\frac{1}{n} {}^t w_1 \phi, w_1 {}^t \phi - \frac{1}{n} {}^t w_1 \phi 1_n\right) u_1) \\ &\quad + p_1(w_1 \otimes p_1(v_1 \otimes u_1))(\phi) \\ &= p_1(w_1 {}^t \phi v_1 \otimes u_1) + p_1(v_1 \otimes w_1 {}^t \phi u_1) + p_1(w_1 \otimes (v_1 {}^t u_1 - u_1 {}^t v_1)\phi) \\ &= w_1 {}^t \phi v_1 {}^t u_1 - u_1 {}^t v_1 \phi {}^t w_1 + v_1 {}^t u_1 \phi {}^t w_1 - w_1 {}^t \phi u_1 {}^t v_1 \\ &\quad + w_1 {}^t \phi u_1 {}^t v_1 - v_1 {}^t u_1 \phi {}^t w_1 - w_1 {}^t \phi v_1 {}^t u_1 + u_1 {}^t v_1 \phi {}^t w_1 \\ &= 0. \end{aligned}$$

Thus our claim holds.

## 2 A Lie algebra associated to a standard quadruplet

In the previous section, we constructed a family of  $n$ -graduations from a standard quadruplet  $(\mathfrak{g}, \rho, V, B_0)$ . In this section, we will consider the direct sum of  $n$ -graduations and denote it by  $L(\mathfrak{g}, \rho, V, B_0)$ . Then we can define a bilinear map from  $L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0)$  to  $L(\mathfrak{g}, \rho, V, B_0)$ . We will prove that this bilinear map satisfies the axioms of a Lie algebra. First of all, let us define the following bilinear maps.

**Definition 2.1.** For any  $n \in \mathbb{Z}$ , we define bilinear maps

$$\begin{aligned} [\cdot, \cdot]_n^0 &: V_0 \times V_n \rightarrow V_n, \\ [\cdot, \cdot]_n^1 &: V_1 \times V_n \rightarrow V_{n+1}, \\ [\cdot, \cdot]_n^{-1} &: V_{-1} \times V_n \rightarrow V_{n-1} \end{aligned}$$

by

$$[a, x_n]_n^0 := \rho_n(a)x_n, \quad (2.1)$$

$$[v_1, x_n]_n^1 := \begin{cases} p_n(v_1 \otimes x_n) & (n \geq 0) \\ -x_n(\Phi_\rho^{-1}(v_1)) & (n \leq -1) \end{cases}, \quad (2.2)$$

$$[\phi_{-1}, x_n]_n^{-1} := \begin{cases} -x_n(\Psi_\rho^{-1}(\phi_{-1})) & (n \geq 1) \\ q_n(\phi_{-1} \otimes x_n) & (n \leq 0) \end{cases} \quad (2.3)$$

where  $a \in V_0$ ,  $v_1 \in V_1$ ,  $\phi_{-1} \in V_{-1}$  and  $x_n \in V_n$ .

We have the following two propositions.

**Proposition 2.2.** *For any  $n \in \mathbb{Z}$ ,  $a \in V_0$ ,  $v_1 \in V_1$ ,  $\phi_{-1} \in V_{-1}$  and  $x_n \in V_n$ , we have the following equations:*

$$[p_0(v_1 \otimes a), x_n]_n^1 = [v_1, [a, x_n]_n^0]_n^1 - [a, [v_1, x_n]_n^1]_{n+1}^0, \quad (2.4)$$

$$[q_0(\phi_{-1} \otimes a), x_n]_n^{-1} = [\phi_{-1}, [a, x_n]_n^0]_n^{-1} - [a, [\phi_{-1}, x_n]_n^{-1}]_{n+1}^0. \quad (2.5)$$

**Proof.** For  $n \geq 0$ , we have

$$\begin{aligned} [p_0(v_1 \otimes a), x_n]_n^1 &= [-\rho_1(a)v_1, x_n]_n^1 \\ &= p_n(-\rho_1(a)v_1 \otimes x_n) \\ &= p_n(v_1 \otimes \rho_n(a)x_n) - \rho_{n+1}(a)p_n(v_1 \otimes x_n) \\ &= [v_1, [a, x_n]_n^0]_n^1 - [a, [v_1, x_n]_n^1]_{n+1}^0. \end{aligned}$$

For  $n \leq -1$ , we have

$$\begin{aligned} [p_0(v_1 \otimes a), x_n]_n^1 &= [-\rho_1(a)v_1, x_n]_n^1 \\ &= x_n(\rho(a)(\Phi_\rho^{-1}(v_1))) \\ &= -(\rho_n(a)x_n)(\Phi_\rho^{-1}(v_1)) + \rho_{n+1}(a)(x_n(\Phi_\rho^{-1}(v_1))) \\ &= [v_1, [a, x_n]_n^0]_n^1 - [a, [v_1, x_n]_n^1]_{n+1}^0. \end{aligned}$$

Thus we have (2.4). Similarly we can obtain (2.5). ■

**Proposition 2.3.** *For any  $n \in \mathbb{Z}$ ,  $v_1 \in V_1$ ,  $\phi_{-1} \in V_{-1}$  and  $x_n \in V_n$ , we have the following equation:*

$$[\phi_{-1}, [v_1, x_n]_n^1]_{n+1}^{-1} = [[\phi_{-1}, v_1]_1^{-1}, x_n]_n^0 + [v_1, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^1. \quad (2.6)$$



**Proof.** For  $n \geq 1$ , we have

$$\begin{aligned} [\phi_{-1}, [v_1, x_n]_n^1]_{n+1}^{-1} &= [\phi_{-1}, p_n(v_1 \otimes x_n)]_{n+1}^{-1} \\ &= -p_n(v_1 \otimes x_n)(\Psi_\rho^{-1}(\phi_{-1})) \\ &= -\rho_n(v_1(\Psi_\rho^{-1}(\phi_{-1})))x_n - p_{n-1}(v_1 \otimes x_n(\Psi_\rho^{-1}(\phi_{-1}))) \\ &= [[\phi_{-1}, v_1]_1^{-1}, x_n]_n^0 + [v_1, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^1. \end{aligned}$$

For  $n = 0$ , we have

$$\begin{aligned} [\phi_{-1}, [v_1, x_0]_0^1]_1^{-1} &= -[\phi_{-1}, \rho_1(x_0)v_1]_1^{-1} \\ &= (\rho_1(x_0)v_1)(\Psi_\rho^{-1}(\phi_{-1})) \\ &= \rho_0(x_0)(v_1(\Psi_\rho^{-1}(\phi_{-1}))) - v_1(\Psi_\rho^{-1}(\rho_{-1}(x_0)\phi_{-1})) \\ &= [[\phi_{-1}, v_1]_1^{-1}, x_0]_0^0 + [v_1, [\phi_{-1}, x_0]_0^{-1}]_{-1}^1. \end{aligned}$$

For  $n \leq -1$ , we have

$$\begin{aligned} [\phi_{-1}, [v_1, x_n]_n^1]_{n+1}^{-1} &= -[\phi_{-1}, x_n(\Phi_\rho^{-1}(v_1))]_{n+1}^{-1} \\ &= -q_{n+1}(\phi_{-1} \otimes x_n(\Phi_\rho^{-1}(v_1))) \\ &= \rho_n(\phi_{-1}(\Phi_\rho^{-1}(v_1)))x_n - q_n(\phi_{-1} \otimes x_n)(\Phi_\rho^{-1}(v_1)) \\ &= [[\phi_{-1}, v_1]_1^{-1}, x_n]_n^0 + [v_1, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^1. \end{aligned}$$

Thus we have (2.6). ■

**Definition 2.4.** Assume that  $i \geq 0$ . For any  $n \in \mathbb{Z}$ , we define bilinear maps

$$[\cdot, \cdot]_n^{i+1} : V_{i+1} \times V_n \rightarrow V_{i+n+1}, \quad (2.7)$$

$$[\cdot, \cdot]_n^{-i-1} : V_{-i-1} \times V_n \rightarrow V_{-i+n-1} \quad (2.8)$$

by

$$\begin{aligned} [p_i(v_1 \otimes u_i), x_n]_n^{i+1} &:= [v_1, [u_i, x_n]_n^i]_{i+n}^1 - [u_i, [v_1, x_n]_n^1]_{n+1}^i \\ &\quad (v_1 \in V_1, u_i \in V_i, x_n \in V_n) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} [q_{-i}(\phi_{-1} \otimes \psi_{-i}), x_n]_n^{-i-1} &:= [\phi_{-1}, [\psi_{-i}, x_n]_n^{-i}]_{-i+n}^{-1} - [\psi_{-i}, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^{-i} \\ &\quad (\phi_{-1} \in V_{-1}, \psi_{-i} \in V_{-i}, x_n \in V_n) \end{aligned} \quad (2.10)$$

inductively.

We must prove the well-definedness of Definition 2.4. First, let us start with the case where  $i = 0$ . To check this, it is sufficient to show that the bilinear maps  $[\cdot, \cdot]_n^1$  and  $[\cdot, \cdot]_n^{-1}$  given in (2.9) and (2.10) coincide with (2.2) and (2.3) respectively. It follows from Proposition 2.2 immediately. Next, let us show the following proposition to prove the well-definedness of (2.9) for  $i \geq 1$ .

**Proposition 2.5.** *Let  $i \geq 0$  and assume that  $[\cdot, \cdot]_n^i$  is well-defined for any  $n \in \mathbb{Z}$ . Take elements  $v_1^1, \dots, v_1^l \in V_1$  and  $u_i^1, \dots, u_i^l \in V_i$  satisfying*

$$\sum_{s=1}^l p_i(v_1^s \otimes u_i^s) = 0. \quad (2.11)$$

Then for any  $x_n \in V_n$ , we have

$$\sum_{s=1}^l ([v_1^s, [u_i^s, x_n]_n]_{i+n}^1 - [u_i^s, [v_1^s, x_n]_n]_{n+1}^i) = 0 \quad (2.12)$$

and hence we obtain that (2.9) is well-defined.

**Proof.** We argue by induction on  $i$ . For  $i = 0$ . By Proposition 2.2, we have our result for any  $n \in \mathbb{Z}$ . Moreover, by definition and Proposition 2.3, we have the following equations:

$$[a, [v_1, x_n]_n]_{n+1}^1 = [[a, v_1]_1^0, x_n]_n^1 + [v_1, [a, x_n]_n^0]_n^1, \quad (2.13)$$

$$[\phi_{-1}, [v_1, x_n]_n]_{n+1}^{-1} = [[\phi_{-1}, v_1]_1^{-1}, x_n]_n^0 + [v_1, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^1, \quad (2.14)$$

$$[v_1, a]_0^1 = -[a, v_1]_1^0 = -\rho_1(a)v_1, \quad (2.15)$$

$$[u_1, v_1]_1^1 = -[v_1, u_1]_1^1 = -p_1(v_1 \otimes u_1), \quad (2.16)$$

$$[u_1, \phi_{-1}]_{-1}^1 = -[\phi_{-1}, u_1]_1^{-1} = u_1(\Psi_\rho^{-1}(\phi_{-1})) \quad (2.17)$$

where  $a \in V_0$ ,  $v_1, u_1 \in V_1$ ,  $\phi_{-1} \in V_{-1}$  and  $x_n \in V_n$ .

For  $i \geq 1$ . We assume that the bilinear maps  $[\cdot, \cdot]_n^i$  are well-defined for any  $n \in \mathbb{Z}$  and satisfy the following equations:

$$[a, [u_i, x_n]_n]_{i+n}^i = [[a, u_i]_i^0, x_n]_n^i + [u_i, [a, x_n]_n^0]_n^i, \quad (2.18)$$

$$[\phi_{-1}, [u_i, x_n]_n]_{i+n}^{-1} = [[\phi_{-1}, u_i]_i^{-1}, x_n]_n^{i-1} + [u_i, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^i, \quad (2.19)$$

$$[u_i, a]_0^i = -[a, u_i]_i^0 = -\rho_i(a)u_i, \quad (2.20)$$

$$[u_i, v_1]_1^i = -[v_1, u_i]_i^1 = -p_i(v_1 \otimes u_i), \quad (2.21)$$

$$[u_i, \phi_{-1}]_{-1}^i = -[\phi_{-1}, u_i]_i^{-1} = u_i(\Psi_\rho^{-1}(\phi_{-1})) \quad (2.22)$$

where  $a \in V_0$ ,  $v_1 \in V_1$ ,  $u_i \in V_i$ ,  $\phi_{-1} \in V_{-1}$  and  $x_n \in V_n$ .

We fix  $i$  and argue by induction on  $n$ .

(1) The case where  $n \geq 0$ .

For  $n = 0$ , by the induction hypothesis on  $i$ , we have

$$\begin{aligned}
& [v_1, [u_i, x_0]_0^i]_i^1 - [u_i, [v_1, x_0]_0^1]_1^i \\
&= -p_i(v_1 \otimes \rho_i(x_0)u_i) + [u_i, \rho_1(x_0)v_1]_1^i \\
&= -p_i(v_1 \otimes \rho_i(x_0)u_i) - p_i(\rho_1(x_0)v_1 \otimes u_i) \\
&= -\rho_{i+1}(x_0)p_i(v_1 \otimes u_i)
\end{aligned} \tag{2.23}$$

where  $v_1 \in V_1$ ,  $u_i \in V_i$  and  $x_0 \in V_0$ . Thus, we have our result immediately.

For  $n \geq 1$ . Note that by (2.19), we have

$$[u_i, y_m]_m^i(\phi) = [u_i(\phi), y_m]_m^{i-1} + [u_i, y_m(\phi)]_{m-1}^i \tag{2.24}$$

for any  $m \geq 1$ ,  $u_i \in V_i$ ,  $y_m \in V_m$  and  $\phi \in V^*$ . Hence, we have

$$\begin{aligned}
& [v_1, [u_i, x_n]_n^i]_{i+n}^1(\phi) - [u_i, [v_1, x_n]_n^1]_{n+1}^i(\phi) \\
&= p_{i+n}(v_1 \otimes [u_i, x_n]_n^i)(\phi) - [u_i, p_n(v_1 \otimes x_n)]_{n+1}^i(\phi) \\
&= \rho_{i+n}(v_1(\phi)) [u_i, x_n]_n^i \\
&\quad + p_{i+n-1}(v_1 \otimes ([u_i(\phi), x_n]_n^{i-1} + [u_i, x_n(\phi)]_{n-1}^i)) \\
&\quad - [u_i(\phi), p_n(v_1 \otimes x_n)]_{n+1}^{i-1} \\
&\quad - [u_i, \rho_n(v_1(\phi))x_n + p_{n-1}(v_1 \otimes x_n(\phi))]_n^i \\
&= [\rho_i(v_1(\phi))u_i, x_n]_n^i \\
&\quad + p_{i+n-1}(v_1 \otimes [u_i(\phi), x_n]_n^{i-1}) - [u_i(\phi), p_n(v_1 \otimes x_n)]_{n+1}^{i-1} \\
&\quad + p_{i+n-1}(v_1 \otimes [u_i, x_n(\phi)]_{n-1}^i) - [u_i, p_{n-1}(v_1 \otimes x_n(\phi))]_n^i \\
&= [\rho_i(v_1(\phi))u_i, x_n]_n^i + [p_{i-1}(v_1 \otimes u_i(\phi)), x_n]_n^i \\
&\quad + p_{i+n-1}(v_1 \otimes [u_i, x_n(\phi)]_{n-1}^i) - [u_i, p_n(v_1 \otimes x_n(\phi))]_n^i \\
&= [p_i(v_1 \otimes u_i)(\phi), x_n]_n^i \\
&\quad + [v_1, [u_i, x_n(\phi)]_{n-1}^i]_{i+n-1}^1 - [u_i, [v_1, x_n(\phi)]_{n-1}^1]_{n-1}^i
\end{aligned} \tag{2.25}$$

where  $v_1 \in V_1$ ,  $u_i \in V_i$ ,  $x_n \in V_n$  and  $\phi \in V^*$ . Here, by the induction hypotheses on  $i$  and  $n$ , if  $v_1^1, \dots, v_1^l \in V_1$  and  $u_i^1, \dots, u_i^l \in V_i$  satisfy

$$\sum_{s=1}^l p_i(v_1^s \otimes u_i^s) = 0,$$

then we have

$$\sum_{s=1}^l [p_i(v_1^s \otimes u_i^s)(\phi), x_n]_n^i = 0$$

and

$$\sum_{s=1}^l ([v_1^s, [u_i^s, x_n(\phi)]_{n-1}]_{n+i-1}^1 - [u_i^s, [v_1^s \otimes x_n(\phi)]_{n-1}]_n^i) = 0$$

respectively. Therefore we have our result for any  $n \geq 0$ .

(2) The case where  $n \leq -1$ .

For  $n = -1$ , by the induction hypothesis on  $i$ , we have

$$\begin{aligned} & [v_1, [u_i, x_{-1}]_{-1}]_{i-1}^1 - [u_i, [v_1, x_{-1}]_{-1}]_0^i \\ &= [v_1, u_i(\Psi_\rho^{-1}(x_{-1}))]_{i-1}^1 - [u_i, v_1(\Psi_\rho^{-1}(x_{-1}))]_0^i \\ &= p_{i-1}(v_1 \otimes u_i(\Psi_\rho^{-1}(x_{-1}))) + \rho_i(v_1(\Psi_\rho^{-1}(x_{-1})))u_i \\ &= p_i(v_1 \otimes u_i)(\Psi_\rho^{-1}(x_{-1})) \end{aligned} \quad (2.26)$$

where  $v_1 \in V_1$ ,  $u_i \in V_i$  and  $x_{-1} \in V_{-1}$ . Thus, we have our result immediately.

For  $n \leq -2$ . Note that by (2.19), we have

$$[u_i, q_{m+1}(\phi_{-1} \otimes y_{m+1})]_m^i = [u_i(\Psi_\rho^{-1}(\phi_{-1})), y_{m+1}]_{m+1}^{i-1} + [\phi_{-1}, [u_i, y_{m+1}]_{m+1}^i]_{i+m+1}^{-1} \quad (2.27)$$

for any  $m \leq -1$ ,  $u_i \in V_i$ ,  $\phi_{-1} \in V_{-1}$  and  $y_{m+1} \in V_{m+1}$ . Hence, by the induction hypotheses on  $i$  and  $n$  and Proposition 2.3, we have

$$\begin{aligned} & [v_1, [u_i, q_{n+1}(\phi_{-1} \otimes x_{n+1})]_n^i]_{i+n}^1 - [u_i, [v_1, q_{n+1}(\phi_{-1} \otimes x_{n+1})]_n^1]_{n+1}^i \\ &= [v_1, [u_i(\Psi_\rho^{-1}(\phi_{-1})), x_{n+1}]_{n+1}^{i-1}]_{i+n}^1 + [v_1, [\phi_{-1}, [u_i, x_{n+1}]_{n+1}^i]_{i+n+1}^{-1}]_{i+n}^1 \\ & \quad + [u_i, q_{n+1}(\phi_{-1} \otimes x_{n+1})(\Phi_\rho^{-1}(v_1))]_{n+1}^i \\ &= [p_{i-1}(v_1 \otimes u_i(\Psi_\rho^{-1}(\phi_{-1}))), x_{n+1}]_{n+1}^i - [u_i(\Psi_\rho^{-1}(\phi_{-1})), x_{n+1}(\Phi_\rho^{-1}(v_1))]_{n+2}^{i-1} \\ & \quad + [v_1(\Psi_\rho^{-1}(\phi_{-1})), [u_i, x_{n+1}]_{n+1}^i]_{i+n+1}^0 + [\phi_{-1}, [v_1, [u_i, x_{n+1}]_{n+1}^i]_{i+n+1}^1]_{i+n+2}^{-1} \\ & \quad + [u_i, \rho_{n+1}(\phi_{-1}(\Phi_\rho^{-1}(v_1)))x_{n+1}]_{n+1}^i + [u_i, q_{n+2}(\phi_{-1} \otimes x_{n+1}(\Phi_\rho^{-1}(v_1)))]_{n+1}^i \\ &= [p_{i-1}(v_1 \otimes u_i(\Psi_\rho^{-1}(\phi_{-1}))), x_{n+1}]_{n+1}^i - [u_i(\Psi_\rho^{-1}(\phi_{-1})), x_{n+1}(\Phi_\rho^{-1}(v_1))]_{n+2}^{i-1} \\ & \quad + [\rho_i(v_1(\Psi_\rho^{-1}(\phi_{-1})))u_i, x_{n+1}]_{n+1}^i + [u_i, \rho_{n+1}(v_1(\Psi_\rho^{-1}(\phi_{-1})))x_{n+1}]_{n+1}^i \\ & \quad + [\phi_{-1}, [v_1, [u_i, x_{n+1}]_{n+1}^i]_{i+n+1}^1]_{i+n+2}^{-1} + [u_i, \rho_{n+1}(\phi_{-1}(\Phi_\rho^{-1}(v_1)))x_{n+1}]_{n+1}^i \\ & \quad + [u_i(\Psi_\rho^{-1}(\phi_{-1})), x_{n+1}(\Phi_\rho^{-1}(v_1))]_{n+2}^{i-1} + [\phi_{-1}, [u_i, x_{n+1}(\Phi_\rho^{-1}(v_1))]_{n+2}^i]_{i+n+2}^{-1} \\ &= [p_{i-1}(v_1 \otimes u_i(\Psi_\rho^{-1}(\phi_{-1}))), x_{n+1}]_{n+1}^i + [\rho_i(v_1(\Psi_\rho^{-1}(\phi_{-1})))u_i, x_{n+1}]_{n+1}^i \\ & \quad + [\phi_{-1}, [v_1, [u_i, x_{n+1}]_{n+1}^i]_{i+n+1}^1]_{i+n+2}^{-1} - [\phi_{-1}, [u_i, [v_1, x_{n+1}]_{n+1}^1]_{n+2}^i]_{i+n+2}^{-1} \\ &= [p_i(v_1 \otimes u_i)(\Psi_\rho^{-1}(\phi_{-1})), x_{n+1}]_{n+1}^i \\ & \quad + [\phi_{-1}, ([v_1, [u_i, x_{n+1}]_{n+1}^i]_{i+n+1}^1 - [u_i, [v_1, x_{n+1}]_{n+1}^1]_{n+2}^i)]_{i+n+2}^{-1} \end{aligned} \quad (2.28)$$

where  $v_1 \in V_1$ ,  $u_i \in V_i$ ,  $x_{n+1} \in V_{n+1}$  and  $\phi_{-1} \in V_{-1}$ . Here, by the induction hypotheses on  $i$  and  $n$ , if  $v_1^1, \dots, v_1^l \in V_1$  and  $u_i^1, \dots, u_i^l \in V_i$  satisfy

$$\sum_{s=1}^l p_i(v_1^s \otimes u_i^s) = 0,$$

then we have

$$\sum_{s=1}^l [p_i(v_1^s \otimes u_i^s)(\Psi_\rho^{-1}(\phi_{-1})), x_{n+1}]_{n+1}^i = 0$$

and

$$\sum_{s=1}^l \left[ \phi_{-1}, \left( [v_1^s, [u_i^s, x_{n+1}]_{n+1}^i]_{i+n+1}^1 - [u_i^s, [v_1^s, x_{n+1}]_{n+1}^1]_{n+2}^i \right) \right]_{i+n+2}^{-1} = 0$$

respectively. Therefore we have our result for any  $n \leq -1$  and we obtain bilinear maps  $[\cdot, \cdot]_n^{i+1}$  for all  $n \in \mathbb{Z}$ .

In order to complete the proof, let us show that the bilinear maps  $[\cdot, \cdot]_n^{i+1}$  ( $n \in \mathbb{Z}$ ) satisfy the following equations:

$$[a, [u_{i+1}, x_n]_n^{i+1}]_{i+n+1}^0 = [[a, u_{i+1}]_{i+1}^0, x_n]_n^{i+1} + [u_{i+1}, [a, x_n]_n^0]_n^{i+1}, \quad (2.29)$$

$$[\phi_{-1}, [u_{i+1}, x_n]_n^{i+1}]_{i+n+1}^{-1} = [[\phi_{-1}, u_{i+1}]_{i+1}^{-1}, x_n]_n^i + [u_{i+1}, [\phi_{-1}, x_n]_n^{-1}]_{n-1}^{i+1}, \quad (2.30)$$

$$[u_{i+1}, a]_0^{i+1} = -[a, u_{i+1}]_{i+1}^0 = -\rho_{i+1}(a)u_{i+1}, \quad (2.31)$$

$$[u_{i+1}, v_1]_1^{i+1} = -[v_1, u_{i+1}]_{i+1}^1 = -p_{i+1}(v_1 \otimes u_{i+1}), \quad (2.32)$$

$$[u_{i+1}, \phi_{-1}]_{-1}^{i+1} = -[\phi_{-1}, u_{i+1}]_{i+1}^{-1} = u_{i+1}(\Psi_\rho^{-1}(\phi_{-1})) \quad (2.33)$$

where  $a \in V_0$ ,  $v_1 \in V_1$ ,  $u_{i+1} \in V_{i+1}$ ,  $\phi_{-1} \in V_{-1}$  and  $x_n \in V_n$ .

(2.31) and (2.33) follow from (2.23) and (2.26) respectively.

Let us show (2.29). We can assume that  $u_{i+1} = p_i(v_1 \otimes u_i)$  for some  $v_1 \in V_1$  and  $u_i \in V_i$

without loss of generality. Then we have

$$\begin{aligned}
& [a, [p_i(v_1 \otimes u_i), x_n]_n^{i+1}]_{i+n+1}^0 \\
&= [a, [v_1, [u_i, x_n]_n^i]_{i+n+1}^1]_{i+n+1}^0 - [a, [u_i, [v_1, x_n]_n^1]_{i+n+1}^i]_{i+n+1}^0 \\
&= [\rho_1(a)v_1, [u_i, x_n]_n^i]_{i+n+1}^1 + [v_1, [\rho_i(a)u_i, x_n]_n^i]_{i+n+1}^1 + [v_1, [u_i, \rho_n(a)x_n]_n^i]_{i+n+1}^1 \\
&\quad - [\rho_i(a)u_i, [v_1, x_n]_n^1]_{i+n+1}^i - [u_i, [\rho_1(a)v_1, x_n]_n^1]_{i+n+1}^i - [u_i, [v_1, \rho_n(a)x_n]_n^1]_{i+n+1}^i \\
&= [\rho_1(a)v_1, [u_i, x_n]_n^i]_{i+n+1}^1 - [u_i, [\rho_1(a)v_1, x_n]_n^1]_{i+n+1}^i \\
&\quad + [v_1, [\rho_i(a)u_i, x_n]_n^i]_{i+n+1}^1 - [\rho_i(a)u_i, [v_1, x_n]_n^1]_{i+n+1}^i \\
&\quad + [v_1, [u_i, \rho_n(a)x_n]_n^i]_{i+n+1}^1 - [u_i, [v_1, \rho_n(a)x_n]_n^1]_{i+n+1}^i \\
&= [p_i(\rho_1(a)v_1 \otimes u_i), x_n]_n^{i+1} + [p_i(v_1 \otimes \rho_i(a)u_i), x_n]_n^{i+1} \\
&\quad + [p_i(v_1 \otimes u_i), \rho_n(a)x_n]_n^{i+1} \\
&= [[a, p_i(v_1 \otimes u_i)]_{i+1}^0, x_n]_n^{i+1} + [p_i(v_1 \otimes u_i), [a, x_n]_n^0]_{i+1}^{i+1}.
\end{aligned}$$

Thus (2.29) holds.

Let us show (2.30). If  $n \neq 0$ , (2.30) follows from (2.25) and (2.28). The case where  $n = 0$  is obtained from (2.31) and (2.33). Thus (2.30) holds.

Finally, let us show (2.32). By (2.30) and (2.31), for any  $\phi \in V^*$ , we have

$$\begin{aligned}
& [u_{i+1}, v_1]_1^{i+1}(\phi) \\
&= [u_{i+1}(\phi), v_1]_1^i + [u_{i+1}, v_1(\phi)]_0^{i+1} \\
&= -p_i(v_1 \otimes u_{i+1}(\phi)) - \rho_{i+1}(v_1(\phi))u_{i+1} \\
&= -p_{i+1}(v_1 \otimes u_{i+1})(\phi).
\end{aligned}$$

Thus (2.32) holds. This completes the proof.  $\blacksquare$

In the same way, we can prove the following proposition.

**Proposition 2.6.** *Assume that  $j \geq 0$  and  $[\cdot, \cdot]_n^{-j}$  is well-defined for any  $n \in \mathbb{Z}$ . Take elements  $\phi_{-1}^1, \dots, \phi_{-1}^l \in V_{-1}$  and  $\psi_{-j}^1, \dots, \psi_{-j}^l \in V_{-j}$  satisfying*

$$\sum_{s=1}^l q_{-j}(\phi_{-1}^s \otimes \psi_{-j}^s) = 0. \quad (2.34)$$

Then for any  $x_n \in V_n$ , we have

$$\sum_{s=1}^l ([\phi_{-1}^s, [\psi_{-j}^s, x_n]_n^{-j}]_{-j+n}^{-1} - [\psi_{-j}^s, [\phi_{-1}^s, x_n]_n^{-1}]_{n-1}^{-j}) = 0 \quad (2.35)$$

and hence we obtain a bilinear map

$$[\cdot, \cdot]_n^{-j-1} : V_{-j-1} \times V_n \rightarrow V_{-j+n-1}$$

satisfying the following equations:

$$[a, [\psi_{-j-1}, x_n]_n^{-j-1}]_{n-j-1}^0 = [[a, \psi_{-j-1}]_{-j-1}^0, x_n]_n^{-j-1} + [\psi_{-j-1}, [a, x_n]_n^0]_n^{-j-1}, \quad (2.36)$$

$$[v_1, [\psi_{-j-1}, x_n]_n^{-j-1}]_{-j+n-1}^1 = [[v_1, \psi_{-j-1}]_{-j-1}^1, x_n]_n^{-j} + [\psi_{-j-1}, [v_1, x_n]_n^1]_{n+1}^{-j-1}, \quad (2.37)$$

$$[\psi_{-j-1}, a]_0^{-j-1} = -[a, \psi_{-j-1}]_{-j-1}^0 = -\rho_{-j-1}(a)\psi_{-j-1}, \quad (2.38)$$

$$[\psi_{-j-1}, \phi_{-1}]_{-1}^{-j-1} = -[\phi_{-1}, \psi_{-j-1}]_{-j-1}^{-1} = -q_{-j-1}(\phi_{-1} \otimes \psi_{-j-1}), \quad (2.39)$$

$$[\psi_{-j-1}, v_1]_1^{-j-1} = -[v_1, \psi_{-j-1}]_{-j-1}^1 = \psi_{-j-1}(\Phi_\rho^{-1}(v_1)). \quad (2.40)$$

where  $a \in V_0$ ,  $v_1 \in V_1$ ,  $\phi_{-1} \in V_{-1}$ ,  $\psi_{-j-1} \in V_{-j-1}$  and  $x_n \in V_n$ .

Then Proposition 2.5 and Proposition 2.6 are summarized as follows.

**Definition 2.7.** We put

$$L(\mathfrak{g}, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n \quad (2.41)$$

and define a bilinear map

$$[\cdot, \cdot] : L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0) \rightarrow L(\mathfrak{g}, \rho, V, B_0)$$

by

$$[x_n, y_m] := [x_n, y_m]_m^n \quad (2.42)$$

where  $x_n \in V_n$  and  $y_m \in V_m$ .

**Proposition 2.8.** The bilinear map  $[\cdot, \cdot]$  satisfies the following equations:

$$[a, x] = -[x, a], \quad (2.43)$$

$$[v_1, x] = -[x, v_1], \quad (2.44)$$

$$[\phi_{-1}, x] = -[x, \phi_{-1}], \quad (2.45)$$

$$[a, [x, y]] = [[a, x], y] + [x, [a, y]], \quad (2.46)$$

$$[y, [v_1, x]] = [v_1, [y, x]] - [[v_1, y], x], \quad (2.47)$$

$$[y, [\phi_{-1}, x]] = [\phi_{-1}, [y, x]] - [[\phi_{-1}, y], x] \quad (2.48)$$

where  $a \in V_0$ ,  $v_1 \in V_1$ ,  $\phi_{-1} \in V_{-1}$  and  $x, y \in L(\mathfrak{g}, \rho, V, B_0)$ .

Let us show that  $[\cdot, \cdot]$  satisfies the axioms of a Lie algebra.

**Proposition 2.9.** *For any  $x, y \in L(\mathfrak{g}, \rho, V, B_0)$ , we have*

$$[x, y] + [y, x] = 0. \quad (2.49)$$

*Namely,  $[\cdot, \cdot]$  is skew-symmetric.*

**Proof.** Without loss of generality, it is sufficient to show in the cases where  $x = x_n \in V_n$  for each  $n \in \mathbb{Z}$ . We assume that  $n \geq 0$  and argue by induction on  $n$ .

For  $n = 0$ , our claim follows from Proposition 2.8.

For  $n \geq 1$ . Without loss of generality, we can assume that  $x_n = p_{n-1}(v_1 \otimes x_{n-1})$  for some  $v_1 \in V_1$  and  $x_{n-1} \in V_{n-1}$ . Then, by (2.9) and (2.47), we have

$$\begin{aligned} & [p_{n-1}(v_1 \otimes x_{n-1}), y] + [y, p_{n-1}(v_1 \otimes x_{n-1})] \\ &= [v_1, [x_{n-1}, y]] - [x_{n-1}, [v_1, y]] \\ & \quad + [v_1, [y, x_{n-1}]] - [[v_1, y], x_{n-1}]. \end{aligned}$$

By the induction hypothesis, we have

$$[x_{n-1}, y] + [y, x_{n-1}] = 0$$

and

$$[x_{n-1}, [v_1, y]] + [[v_1, y], x_{n-1}] = 0.$$

Hence our claim holds. Similarly we can obtain the proof of the cases where  $n \leq -1$ . This completes the proof.  $\blacksquare$

**Proposition 2.10.** *For any  $x, y, z \in L(\mathfrak{g}, \rho, V, B_0)$ , we have*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \quad (2.50)$$

*Namely,  $[\cdot, \cdot]$  satisfies the Jacobi identity.*

**Proof.** Without loss of generality, it is sufficient to show in the cases where  $x = x_n \in V_n$  for each  $n \in \mathbb{Z}$ . We assume  $n \geq 0$  and argue by induction on  $n$ .

For  $n = 0$  and 1, our claim follows from Proposition 2.8.

For  $n \geq 2$ . Without loss of generality, we can assume that  $x_n = p_{n-1}(v_1 \otimes x_{n-1})$  for some



$v_1 \in V_1$  and  $x_{n-1} \in V_{n-1}$ . Then, by the induction hypothesis, we have

$$\begin{aligned}
& [p_{n-1}(v_1 \otimes x_{n-1}), [y, z]] \\
&= [v_1, [x_{n-1}, [y, z]]] - [x_{n-1}, [v_1, [y, z]]] \\
&= [v_1, [[x_{n-1}, y], z]] + [v_1, [y, [x_{n-1}, z]]] \\
&\quad - [x_{n-1}, [[v_1, y], z]] - [x_{n-1}, [y, [v_1, z]]] \\
&= [v_1, [[x_{n-1}, y], z]] + [y, [v_1, [x_{n-1}, z]]] + [[v_1, y], [x_{n-1}, z]] \\
&\quad - [x_{n-1}, [[v_1, y], z]] - [y, [x_{n-1}, [v_1, z]]] - [[x_{n-1}, y], [v_1, z]] \\
&= [[v_1, [x_{n-1}, y]], z] - [[x_{n-1}, [v_1, y]], z] + [y, [p_{n-1}(v_1 \otimes x_{n-1}), z]] \\
&= [[p_{n-1}(v_1 \otimes x_{n-1}), y], z] + [y, [p_{n-1}(v_1 \otimes x_{n-1}), z]].
\end{aligned}$$

Thus we have our claim. We can also obtain the proof of the cases where  $n \leq -1$  by the same argument. This completes the proof.  $\blacksquare$

By Propositions 2.9 and 2.10, we have the following theorem.

**Theorem 2.11.**  *$L(\mathfrak{g}, \rho, V, B_0)$  is a Lie algebra with the bracket  $[\cdot, \cdot]$ . We call the Lie algebra thus obtained the Lie algebra associated to  $(\mathfrak{g}, \rho, V, B_0)$ .*

In particular, for a given reductive Lie algebra  $\mathfrak{g}$  and its finite-dimensional completely reducible representation  $(\rho, V)$ , there exists a graded Lie algebra  $\sum_{n \in \mathbb{Z}} \mathfrak{g}_n$  such that its Lie subalgebra  $\mathfrak{g}_0$  is isomorphic to  $\mathfrak{g}$  and the adjoint action of  $\mathfrak{g}_0$  on  $\mathfrak{g}_1$  is isomorphic to  $(\rho, V)$ .

In the next section, we will construct a bilinear form on  $L(\mathfrak{g}, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$  such that the restriction to  $V_0 \times V_0$  coincides with  $B_0$ .

### 3 Some properties of $L(\mathfrak{g}, \rho, V, B_0)$

#### 3.1 A bilinear form on $L(\mathfrak{g}, \rho, V, B_0)$

In this section, we will consider the structure of the Lie algebra  $L(\mathfrak{g}, \rho, V, B_0)$  associated to a standard quadruplet  $(\mathfrak{g}, \rho, V, B_0)$ . First of all, we construct a non-degenerate symmetric  $L(\mathfrak{g}, \rho, V, B_0)$ -invariant bilinear form on  $L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0)$ .

**Definition 3.1.** We define a bilinear form  $B$  on  $L(\mathfrak{g}, \rho, V, B_0) \times L(\mathfrak{g}, \rho, V, B_0)$  by

$$B\left(\sum_{n \in \mathbb{Z}} x_n, \sum_{m \in \mathbb{Z}} y_m\right) = \sum_{s \geq 0} B_s(x_s, y_{-s}) + \sum_{t \leq -1} B_{-t}(y_{-t}, x_t) \quad (3.1)$$

where  $x_n \in V_n$  and  $y_m \in V_m$ .

**Proposition 3.2.** *The bilinear form  $B$  is non-degenerate symmetric and  $L(\mathfrak{g}, \rho, V, B_0)$ -invariant.*

**Proof.** It can be easily proved that  $B$  is non-degenerate and symmetric. Let us prove that  $B$  is  $L(\mathfrak{g}, \rho, V, B_0)$ -invariant. For this, it is enough to show that we have

$$B([u_i, x_n], y_m) = B(x_n, [y_m, u_i]) \quad (3.2)$$

where  $i \geq 0$ ,  $n, m \in \mathbb{Z}$ ,  $u_i \in V_i$ ,  $x_n \in V_n$ ,  $y_m \in V_m$  and  $i + n + m = 0$ . We argue it by induction on  $i$ .

For  $i = 0, 1$ , our result follows from (1.43) and (1.44) immediately.

For  $i \geq 2$ . We can assume that  $u_i = p_{i-1}(v_1 \otimes u_{i-1})$  for some  $v_1 \in V_1$  and  $u_{i-1} \in V_{i-1}$  without loss of generality. Then, by the induction hypothesis, we have

$$\begin{aligned} & B([p_{i-1}(v_1 \otimes u_{i-1}), x_n], y_m) \\ &= B([v_1, [u_{i-1}, x_n]] - [u_{i-1}, [v_1, x_n]], y_m) \\ &= B([u_{i-1}, x_n], [y_m, v_1]) - B([v_1, x_n], [y_m, u_{i-1}]) \\ &= B(x_n, [[y_m, v_1], u_{i-1}]) - B(x_n, [[y_m, u_{i-1}], v_1]) \\ &= B(x_n, [y_m, p_{i-1}(v_1 \otimes u_{i-1})]). \end{aligned}$$

This completes the proof. ■

We prove the “universality” of the Lie algebras associated to a standard quadruplet.

**Proposition 3.3.** *Assume that a  $\mathbb{Z}$ -graded Lie algebra  $\hat{\mathfrak{g}} = \sum_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_n$  with a symmetric  $\hat{\mathfrak{g}}$ -invariant bilinear form  $\hat{B}$  satisfies the following conditions:*

$$\hat{\mathfrak{g}}_0 \text{ and } \hat{\mathfrak{g}}_1 \text{ are finite-dimensional vector spaces,} \quad (3.3)$$

$$\text{the adjoint representation of } \hat{\mathfrak{g}}_0 \text{ on } \hat{\mathfrak{g}}_1 \text{ is completely reducible,} \quad (3.4)$$

$$\text{the restriction of } \hat{B} \text{ to } \hat{\mathfrak{g}}_i \times \hat{\mathfrak{g}}_{-i} \text{ is non-degenerate for each } i \geq 0, \quad (3.5)$$

$$[\hat{\mathfrak{g}}_1, \hat{\mathfrak{g}}_i] = \hat{\mathfrak{g}}_{i+1}, \quad [\hat{\mathfrak{g}}_{-1}, \hat{\mathfrak{g}}_{-j}] = \hat{\mathfrak{g}}_{-j-1} \text{ for each } i, j \geq 0, \quad (3.6)$$

$$[\hat{\mathfrak{g}}_1, \hat{\mathfrak{g}}_{-1}] = \hat{\mathfrak{g}}_0. \quad (3.7)$$

Then  $(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  is a standard quadruplet and the Lie algebra  $L(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  is isomorphic to  $\hat{\mathfrak{g}}$  where  $\text{ad}$  stands for the adjoint representation of  $\hat{\mathfrak{g}}_0$  on  $\hat{\mathfrak{g}}_1$ .

Obviously,  $L(\mathfrak{g}, \rho, V, B_0)$  satisfies these conditions, i.e.  $L(\mathfrak{g}, \rho, V, B_0)$  is, although it is defined over  $\mathbb{C}$ , of type  $\alpha_0$  (See [K-A]). In order to prove Proposition 3.3, we prove the following lemma needed later.

**Lemma 3.4.** *Under the notation in Proposition 3.3, we have*

$$\{X_i \in \hat{\mathfrak{g}}_i \mid [X_i, Y_{-1}] = 0 \text{ for any } Y_{-1} \in \hat{\mathfrak{g}}_{-1}\} = \{0\}$$

for each  $i \geq 0$ . In particular, a quadruplet  $(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  satisfies (1.31).

**Proof.** Suppose that  $X_i \in \hat{\mathfrak{g}}_i$  satisfies  $[X_i, Y_{-1}] = 0$  for any  $Y_{-1} \in \hat{\mathfrak{g}}_{-1}$ . Then, for any  $Z_{-i+1} \in \hat{\mathfrak{g}}_{-i+1}$ , we have

$$\begin{aligned} & \hat{B}(X_i, [Y_{-1}, Z_{-i+1}]) \\ &= \hat{B}([X_i, Y_{-1}], Z_{-i+1}) = 0. \end{aligned}$$

Thus by (3.5), (3.6) and (3.7), we obtain  $X_i = 0$ . ■

By (3.6),  $(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  satisfies (1.32). Hence it is a standard quadruplet.

Let us prove Proposition 3.3. For this, we construct an isomorphism of Lie algebras from  $L(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  to  $\hat{\mathfrak{g}}$ . We denote the  $n$ -graduation of  $(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  by  $(\hat{\mathfrak{g}})_n$  and the bilinear form on  $L(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  defined in Definition 3.1 by  $B$ . Let  $\tau_0$  be the identity map on  $\hat{\mathfrak{g}}_0$  and  $\tau_1 := \Phi_{\text{ad}}^{-1}$ . Then we obtain a linear isomorphism  $\tau_{-1} : (\hat{\mathfrak{g}})_{-1} \rightarrow \hat{\mathfrak{g}}_{-1}$  defined by  $B(\phi_{-1}, v_1) = \hat{B}(\tau_{-1}(\phi_{-1}), \tau_1(v_1))$  where  $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$  and  $v_1 \in (\hat{\mathfrak{g}})_1$ . These linear isomorphisms  $\tau_i : (\hat{\mathfrak{g}})_i \rightarrow \hat{\mathfrak{g}}_i$  ( $i = 0, \pm 1$ ) satisfy the following equations:

$$[\tau_0(a), \tau_1(v_1)] = \tau_1([a, v_1]), \quad (3.8)$$

$$[\tau_{-1}(\phi_{-1}), \tau_1(v_1)] = \tau_0([\phi_{-1}, v_1]), \quad (3.9)$$

$$[\tau_0(a), \tau_{-1}(\phi_{-1})] = \tau_{-1}([a, \phi_{-1}]) \quad (3.10)$$

where  $a \in (\hat{\mathfrak{g}})_0$ ,  $v_1 \in (\hat{\mathfrak{g}})_1$  and  $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$ . In fact, (3.8) obviously holds. Let us check (3.9). For any  $b \in (\hat{\mathfrak{g}})_0$ , we have

$$\begin{aligned} \hat{B}(\tau_0(b), \tau_0([\phi_{-1}, v_1])) &= B(b, [\phi_{-1}, v_1]) \\ &= B([b, \phi_{-1}], v_1) \\ &= \hat{B}([\tau_0(b), \tau_{-1}(\phi_{-1})], \tau_1(v_1)) \\ &= \hat{B}(\tau_0(b), [\tau_{-1}(\phi_{-1}), \tau_1(v_1)]) \end{aligned}$$

and hence (3.9) holds. Similarly, we can obtain (3.10).

Next, suppose that  $i \geq 1$  and assume that there exist linear isomorphisms  $\tau_{i-1} : (\hat{\mathfrak{g}})_{i-1} \rightarrow \hat{\mathfrak{g}}_{i-1}$  and  $\tau_i : (\hat{\mathfrak{g}})_i \rightarrow \hat{\mathfrak{g}}_i$  satisfying the equations:

$$[\tau_0(a), \tau_i(u_i)] = \tau_i([a, u_i]), \quad (3.11)$$

$$[\tau_1(v_1), \tau_{i-1}(u_{i-1})] = \tau_i([v_1, u_{i-1}]), \quad (3.12)$$

$$[\tau_{-1}(\phi_{-1}), \tau_i(u_i)] = \tau_{i-1}([\phi_{-1}, u_i]) \quad (3.13)$$

where  $a \in (\hat{\mathfrak{g}})_0$ ,  $v_1 \in (\hat{\mathfrak{g}})_1$ ,  $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$ ,  $u_{i-1} \in (\hat{\mathfrak{g}})_{i-1}$  and  $u_i \in (\hat{\mathfrak{g}})_i$ . Then we have

$$\tau_i([\phi_{-1}, [v_1, u_i]]) = [\tau_{-1}(\phi_{-1}), [\tau_1(v_1), \tau_i(u_i)]]. \quad (3.14)$$

In fact,

$$\begin{aligned}
& \tau_i([\phi_{-1}, [v_1, u_i]]) \\
&= \tau_i([\phi_{-1}, v_1], u_i) + \tau_i([v_1, [\phi_{-1}, u_i]]) \\
&= [\tau_0([\phi_{-1}, v_1]), \tau_i(u_i)] + [\tau_1(v_1), \tau_{i-1}([\phi_{-1}, u_i])] \\
&= [[\tau_{-1}(\phi_{-1}), \tau_1(v_1)], \tau_i(u_i)] + [\tau_1(v_1), [\tau_{-1}(\phi_{-1}), \tau_i(u_i)]] \\
&= [\tau_{-1}(\phi_{-1}), [\tau_1(v_1), \tau_i(u_i)]].
\end{aligned}$$

Thus, by Lemma 3.4, we can define a linear map  $\tau_{i+1} : (\hat{\mathfrak{g}})_{i+1} \rightarrow \hat{\mathfrak{g}}_{i+1}$  by

$$\tau_{i+1}(p_i(v_1 \otimes u_i)) := [\tau_1(v_1), \tau_i(u_i)]. \quad (3.15)$$

Then, by (3.6) and (3.14), we obtain that  $\tau_{i+1}$  is bijective. Moreover,  $\tau_{i+1}$  satisfies the following equations:

$$[\tau_0(a), \tau_{i+1}(u_{i+1})] = \tau_{i+1}([a, u_{i+1}]), \quad (3.16)$$

$$[\tau_1(v_1), \tau_i(u_i)] = \tau_{i+1}([v_1, u_i]), \quad (3.17)$$

$$[\tau_{-1}(\phi_{-1}), \tau_{i+1}(u_{i+1})] = \tau_i([\phi_{-1}, u_{i+1}]) \quad (3.18)$$

where  $a \in (\hat{\mathfrak{g}})_0$ ,  $v_1 \in (\hat{\mathfrak{g}})_1$ ,  $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$ ,  $u_i \in (\hat{\mathfrak{g}})_i$  and  $u_{i+1} \in (\hat{\mathfrak{g}})_{i+1}$ . Indeed, (3.16) follows from a similar argument to the proof of (3.14). Equations (3.17) and (3.18) have already been shown. Thus, inductively, we obtain linear isomorphisms  $\tau_n$  from  $(\hat{\mathfrak{g}})_n$  to  $\hat{\mathfrak{g}}_n$  for all  $n \geq 0$ . Moreover,  $\tau_n$  induces a linear isomorphism  $\tau_{-n}$  from  $(\hat{\mathfrak{g}})_{-n}$  to  $\hat{\mathfrak{g}}_{-n}$  which satisfies the following equation:

$$B(\psi_{-n}, u_n) = \hat{B}(\tau_{-n}(\psi_{-n}), \tau_n(u_n)) \quad (3.19)$$

where  $u_n \in (\hat{\mathfrak{g}})_n$  and  $\psi_{-n} \in (\hat{\mathfrak{g}})_{-n}$ . Then  $\tau_{-n}$  satisfies the following equations:

$$[\tau_1(v_1), \tau_{-n}(\psi_{-n})] = \tau_{-n+1}([v_1, \psi_{-n}]), \quad (3.20)$$

$$[\tau_{-1}(\phi_{-1}), \tau_{-n}(\psi_{-n})] = \tau_{-n-1}([\phi_{-1}, \psi_{-n}]) \quad (3.21)$$

where  $v_1 \in (\hat{\mathfrak{g}})_1$ ,  $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$  and  $\psi_{-n} \in (\hat{\mathfrak{g}})_{-n}$ . In fact, we can show (3.20) by induction on  $n$ . For  $n = 0$ , our claim has already been shown. Suppose that  $n \geq 1$ , then by the induction hypothesis, we have

$$\begin{aligned}
\hat{B}(\tau_{n-1}(u_{n-1}), [\tau_1(v_1), \tau_{-n}(\psi_{-n})]) &= \hat{B}([\tau_{n-1}(u_{n-1}), \tau_1(v_1)], \tau_{-n}(\psi_{-n})) \\
&= \hat{B}(\tau_n([u_{n-1}, v_1]), \tau_{-n}(\psi_{-n})) \\
&= B([u_{n-1}, v_1], \psi_{-n}) \\
&= B(u_{n-1}, [v_1, \psi_{-n}]) \\
&= \hat{B}(\tau_{n-1}(u_{n-1}), \tau_{-n+1}([v_1, \psi_{-n}]))
\end{aligned}$$

for any  $u_{n-1} \in (\hat{\mathfrak{g}})_{n-1}$ . Hence (3.20) holds for all  $n \geq 0$ . Similarly we can show (3.21). Summarizing the above argument, we can define a linear isomorphism  $\tau$  from  $L(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$  to  $\hat{\mathfrak{g}}$  by

$$\tau\left(\sum_{n \in \mathbb{Z}} x_n\right) := \sum_{n \in \mathbb{Z}} \tau_n(x_n) \quad (3.22)$$

where  $x_n \in (\hat{\mathfrak{g}})_n$ . Then  $\tau$  satisfies the following equations:

$$[\tau(v_1), \tau(x)] = \tau([v_1, x]), \quad (3.23)$$

$$[\tau(\phi_{-1}), \tau(x)] = \tau([\phi_{-1}, x]) \quad (3.24)$$

where  $v_1 \in (\hat{\mathfrak{g}})_1$ ,  $\phi_{-1} \in (\hat{\mathfrak{g}})_{-1}$  and  $x \in L(\hat{\mathfrak{g}}_0, \text{ad}, \hat{\mathfrak{g}}_1, \hat{B}|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0})$ . Therefore, by the definition of a Lie algebra associated to a standard quadruplet,  $\tau$  is an isomorphism of Lie algebras. This completes the proof of Proposition 3.3.

### 3.2 Equivalent standard quadruplets

In this section, we shall introduce a notion of equivalence between standard quadruplets.

**Definition 3.5.** Let  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  be standard quadruplets. We call these quadruplets are equivalent if and only if there exists an isomorphism of Lie algebras  $\sigma : \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$  and a linear isomorphism  $\tau : V^1 \rightarrow V^2$  such that

$$\tau(\rho^1(a^1)v^1) = \rho^2(\sigma(a^1))(\tau(v^1)), \quad (3.25)$$

$$B_0^1(a^1, b^1) = B_0^2(\sigma(a^1), \sigma(b^1)) \quad (3.26)$$

for any  $a^1, b^1 \in \mathfrak{g}^1$  and  $v^1 \in V^1$ . We will denote this equivalence relation by

$$(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \simeq (\mathfrak{g}^2, \rho^2, V^2, B_0^2). \quad (3.27)$$

In particular, for a standard quadruplet  $(\mathfrak{g}, \rho, V, B_0)$  and a non-zero element  $c \in \mathbb{C}$ ,  $(\mathfrak{g}, \rho, V, cB_0)$  and  $(\mathfrak{g}, \rho_1, V_1, B_0)$  are equivalent to  $(\mathfrak{g}, \rho, V, B_0)$  where  $(\rho_1, V_1)$  is the 1-graduation of  $(\mathfrak{g}, \rho, V, B_0)$ . Then we have the following proposition.

**Proposition 3.6.** *If standard quadruplets  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  are equivalent, then the Lie algebras  $L(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $L(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  are isomorphic.*

**Proof.** By the same argument as proof of Proposition 3.3, we can construct a linear isomorphism from  $V_n^1$  to  $V_n^2$  for each  $n \in \mathbb{Z}$ , where  $V_n^i$  denotes the  $n$ -graduation of  $(\mathfrak{g}^i, \rho^i, V^i, B_0^i)$  for  $i = 1, 2$ , and an isomorphism of Lie algebras from  $L(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  to  $L(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ . ■

However, in general, the converse is not true. For example, let  $\mathfrak{g} = \mathfrak{sl}_4$ . Put

$$H^1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H^2 := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and

$$\mathfrak{g}_j^1 := \{X \in \mathfrak{g} \mid [H^1, X] = 2jX\}, \quad \mathfrak{g}_j^2 := \{X \in \mathfrak{g} \mid [H^2, X] = 2jX\}$$

for each  $j \in \mathbb{Z}$ . Then we obtain two gradings:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_{-1}^1 \oplus \mathfrak{g}_0^1 \oplus \mathfrak{g}_1^1 \\ &= \mathfrak{g}_{-2}^2 \oplus \mathfrak{g}_{-1}^2 \oplus \mathfrak{g}_0^2 \oplus \mathfrak{g}_1^2 \oplus \mathfrak{g}_2^2. \end{aligned}$$

These gradings and the Killing form  $K$  on  $\mathfrak{g}$  satisfy the assumptions of Proposition 3.3. Therefore, by applying Proposition 3.3, it follows that the Lie algebras  $L(\mathfrak{g}_0^i, \text{ad}, \mathfrak{g}_1^i, K|_{\mathfrak{g}_0^i \times \mathfrak{g}_0^i})$  ( $i = 1, 2$ ) are isomorphic to  $\mathfrak{g}$ . However, by a direct calculation, we can obtain that  $\mathfrak{g}_0^1$  and  $\mathfrak{g}_0^2$  are isomorphic to  $\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$  and  $\mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_2$  respectively. In particular, the quadruplets  $(\mathfrak{g}_0^i, \text{ad}, \mathfrak{g}_1^i, K|_{\mathfrak{g}_0^i \times \mathfrak{g}_0^i})$  ( $i = 1, 2$ ) are not equivalent.

### 3.3 Loop algebras

In this section, we shall give a well-known example of infinite-dimensional Lie algebra which can be written in the form  $L(\mathfrak{g}, \rho, V, B_0)$ .

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra,  $\mathbb{C}[t, t^{-1}]$  be the algebra of Laurent polynomials in  $t$ . Set  $\mathcal{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$  and define a bilinear map  $[\cdot, \cdot]_0 : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow \mathcal{L}(\mathfrak{g})$  by:

$$[t^n \otimes X, t^m \otimes Y]_0 := t^{n+m} \otimes [X, Y] \quad (3.28)$$

where  $n, m \in \mathbb{Z}$ ,  $X, Y \in \mathfrak{g}$  and  $[\cdot, \cdot]$  stands for the bracket product of  $\mathfrak{g}$ . This bilinear map satisfies the axioms of a Lie algebra. The infinite-dimensional Lie algebra  $\mathcal{L}(\mathfrak{g})$  with the bracket  $[\cdot, \cdot]_0$  is called the loop algebra and we have a  $\mathbb{Z}$ -grading  $\mathcal{L}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}} t^n \otimes \mathfrak{g}$  (See [Ka]).

Now, we denote by  $K$  the Killing form on  $\mathfrak{g}$ . Then we can define a bilinear form  $K_0$  on  $\mathcal{L}(\mathfrak{g})$  by  $K_0(t^n \otimes X, t^m \otimes Y) := \delta_{n+m, 0} K(X, Y)$ , where  $\delta_{n+m, 0}$  is the Kronecker delta. By applying Proposition 3.3, we have the following proposition.

**Proposition 3.7.** *Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra and  $K$  the Killing form on  $\mathfrak{g}$ . Then the Lie algebra  $L(\mathfrak{g}, \text{ad}, \mathfrak{g}, K)$  is isomorphic to the loop algebra  $\mathcal{L}(\mathfrak{g})$ .*

### 3.4 A direct sum of standard quadruplets

**Definition 3.8.** Let  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  be standard quadruplets. Let  $\rho^1 \boxplus \rho^2$  be a representation of  $\mathfrak{g}^1 \oplus \mathfrak{g}^2$  on  $V^1 \oplus V^2$  defined by:

$$((\rho^1 \boxplus \rho^2)(a^1, a^2))(v^1, v^2) := (\rho^1(a^1)v^1, \rho^2(a^2)v^2) \quad (3.29)$$

where  $a^i \in \mathfrak{g}^i$  and  $v^i \in \mathfrak{g}^i$  ( $i = 1, 2$ ). Let  $B_0^1 \oplus B_0^2$  be a bilinear form on  $\mathfrak{g}^1 \oplus \mathfrak{g}^2$  defined by:

$$(B_0^1 \oplus B_0^2)((a^1, a^2), (b^1, b^2)) := B_0^1(a^1, b^1) + B_0^2(a^2, b^2) \quad (3.30)$$

where  $a^i, b^i \in \mathfrak{g}^i$  ( $i = 1, 2$ ). Then a quadruplet  $(\mathfrak{g}^1 \oplus \mathfrak{g}^2, \rho^1 \boxplus \rho^2, V^1 \oplus V^2, B_0^1 \oplus B_0^2)$  is a standard quadruplet. We call it a **direct sum** of  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  and denote by  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus (\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ .

**Remark 3.9.** Let  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$ ,  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  and  $(\mathfrak{g}, \rho, V, B_0)$  be standard quadruplets. Assume that  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  are equivalent. Then we have

$$\begin{aligned} & (\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus (\mathfrak{g}, \rho, V, B_0) \\ & \simeq (\mathfrak{g}^2, \rho^2, V^2, B_0^2) \oplus (\mathfrak{g}, \rho, V, B_0). \end{aligned}$$

By applying Proposition 3.3, we have the following proposition immediately.

**Proposition 3.10.** *Let  $(\mathfrak{g}^1, \rho^1, V^1, B_0^1)$  and  $(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$  be standard quadruplets. Then the Lie algebra  $L((\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus (\mathfrak{g}^2, \rho^2, V^2, B_0^2))$  is isomorphic to  $L(\mathfrak{g}^1, \rho^1, V^1, B_0^1) \oplus L(\mathfrak{g}^2, \rho^2, V^2, B_0^2)$ .*

**Definition 3.11.** A standard quadruplet  $(\mathfrak{g}, \rho, V, B_0)$  is said to be **decomposable** if and only if there exist non-trivial ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  of  $\mathfrak{g}$  and non-trivial subspaces  $U$  and  $W$  of  $V$  which satisfy

$$\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{g}, \quad (3.31)$$

$$U \oplus W = V, \quad (3.32)$$

$$\rho(a)w = 0, \quad (3.33)$$

$$\rho(b)u = 0, \quad (3.34)$$

$$B_0(a, b) = 0 \quad (3.35)$$

for any  $a \in \mathfrak{a}$ ,  $b \in \mathfrak{b}$ ,  $u \in U$  and  $w \in W$ . Then  $(\mathfrak{a}, \rho|_{\mathfrak{a}}, U, B_0|_{\mathfrak{a} \times \mathfrak{a}})$  and  $(\mathfrak{b}, \rho|_{\mathfrak{b}}, W, B_0|_{\mathfrak{b} \times \mathfrak{b}})$  are standard quadruplets and  $(\mathfrak{g}, \rho, V, B_0) \simeq (\mathfrak{a}, \rho|_{\mathfrak{a}}, U, B_0|_{\mathfrak{a} \times \mathfrak{a}}) \oplus (\mathfrak{b}, \rho|_{\mathfrak{b}}, W, B_0|_{\mathfrak{b} \times \mathfrak{b}})$ . If  $(\mathfrak{g}, \rho, V, B_0)$  is not decomposable, it is said to be **indecomposable**. Any standard quadruplet can be written as a direct sum of indecomposable standard quadruplets.

**Proposition 3.12.** *Let  $(\mathfrak{g}, \rho, V, B_0)$  be an indecomposable standard quadruplet. Let  $n$  be an integer such that  $V_{n+1}$  is not  $\{0\}$ . Then for any non-zero element  $x_n \in V_n$ , there exists an element  $v_1 \in V_1$  such that  $[v_1, x_n] \neq 0$ .*

**Proof.** If  $n \leq 0$ , we have our result immediately. Suppose that  $n \geq 1$ . Put

$$U_n := \{x_n \in V_n \mid [v_1, x_n] = 0 \text{ for any } v_1 \in V_1\}. \quad (3.36)$$

Then  $U_n$  is a  $V_0$ -submodule of  $V_n$ . Moreover, for each  $m = 0, \dots, n-1$ , we can define a  $V_0$ -submodule  $U_m$  of  $V_m$  by

$$U_m := \{x_m \in V_m \mid [v_1, x_m] \in U_{m+1} \text{ for any } v_1 \in V_1\} \quad (3.37)$$

inductively. Then we can easily show that

$$[V_{-1}, U_m] \subset U_{m-1} \quad (3.38)$$

for  $m = 1, \dots, n$  by induction. Moreover, we have

$$V_0 \neq U_0. \quad (3.39)$$

In fact, since  $V_{n+1} \neq \{0\}$ , we can deduce that  $U_n \neq V_n$ . Similarly, we have  $U_m \neq V_m$  for  $m = 0, \dots, n-1$  by induction.

Now, since  $V_1$  is a completely reducible  $V_0$ -module, we can take a  $V_0$ -submodule  $W_1 \subset V_1$  which satisfies  $V_1 = U_1 \oplus W_1$ . Then by the definition of  $U_0$ , we have  $[U_0, W_1] = \{0\}$ . Put  $W_0 := \{a \in V_0 \mid B(a, b) = 0 \text{ for any } b \in U_0\}$ . Then we have  $[W_0, U_1] = \{0\}$ . In fact, suppose that  $a \in W_0$  and  $u_1 \in U_1$ . Then for any  $\phi_{-1} \in V_{-1}$ , we have  $[u_1, \phi_{-1}] \in U_0$  and  $B([a, u_1], \phi_{-1}) = B(a, [u_1, \phi_{-1}]) = 0$ , hence  $[a, u_1] = 0$ . Since  $\rho$  is faithful, we have  $V_0 = U_0 \oplus W_0$ . Therefore, by the assumption that  $(\mathfrak{g}, \rho, V, B_0)$  is an indecomposable standard quadruplet, we have  $U_0 = \{0\}$ . Therefore, by Lemma 3.4 and (3.38), we can obtain  $U_m = \{0\}$  for  $m = 1, \dots, n$  by induction. In particular,  $U_n = \{0\}$  and thus we have our result.  $\blacksquare$

**Proposition 3.13.** *Let  $(\mathfrak{g}, \rho, V, B_0)$  be a standard quadruplet. If the Lie algebra  $L(\mathfrak{g}, \rho, V, B_0)$  is finite-dimensional, then  $L(\mathfrak{g}, \rho, V, B_0)$  is semisimple. In particular, if  $(\mathfrak{g}, \rho, V, B_0)$  is indecomposable, then  $L(\mathfrak{g}, \rho, V, B_0)$  is simple.*

**Proof.** It is sufficient to show the case where  $(\mathfrak{g}, \rho, V, B_0)$  is indecomposable. Let  $\mathfrak{a}$  be a non-zero ideal of  $L(\mathfrak{g}, \rho, V, B_0)$ . Let  $k$  be a positive integer such that  $V_k \neq \{0\}$  and  $V_{k+1} = \{0\}$ . Then there exist integers  $-k \leq n_1 < \dots < n_l \leq k$  and non-zero elements  $x_{n_1} \in V_{n_1}, \dots, x_{n_l} \in V_{n_l}$  such that

$$x := x_{n_1} + \dots + x_{n_l} \in \mathfrak{a}. \quad (3.40)$$

Then by Proposition 3.12, there exist non-zero elements  $v_1^1, \dots, v_1^{k-n_1} \in V_1$  such that

$$[v_1^{k-n_1}, [v_1^{k-n_1-1}, \dots, [v_1^1, x] \dots]] \in V_k \setminus \{0\}. \quad (3.41)$$

Thus we can obtain  $\mathfrak{a}_k := \mathfrak{a} \cap V_k \neq \{0\}$  and  $\mathfrak{a}_m := \mathfrak{a} \cap V_m \neq \{0\}$  for  $m = 0, \dots, k-1$  by induction. By a similar argument to the argument in the proof of Proposition 3.12, we have  $\mathfrak{a}_0 = V_0$ , i.e.  $V_0 \subset \mathfrak{a}$ . Therefore, we have  $V_1 = [V_1, V_0]$ ,  $V_{-1} = [V_{-1}, V_0] \subset \mathfrak{a}$  and thus  $\mathfrak{a} = L(\mathfrak{g}, \rho, V, B_0)$ . Therefore,  $L(\mathfrak{g}, \rho, V, B_0)$  is simple and thus we have our result.  $\blacksquare$

### 3.5 A quadruplet of parabolic type

Let  $\mathfrak{g}$  be a finite-dimensional semisimple Lie algebra,  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ ,  $R$  the root system of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Let  $K$  be the Killing form on  $\mathfrak{g}$ . We fix a fundamental root system  $\varphi$  of  $R$ .



Let  $\theta$  be a subset of  $\varphi$ . Then there exists the unique element  $H^\theta$  such that  $\alpha(H^\theta) = 0$  for  $\alpha \in \theta$  and  $\alpha(H^\theta) = 2$  for  $\alpha \in \varphi \setminus \theta$ . We call  $H^\theta$  the grading element.

For each  $n \in \mathbb{Z}$ , we put  $d_n(\theta) := \{X \in \mathfrak{g} \mid [H^\theta, X] = 2nX\}$  and denote  $d_0(\theta)$  by  $\mathfrak{l}_\theta$ . Then  $\mathfrak{l}_\theta$  is a reductive subalgebra of  $\mathfrak{g}$  and acts on each  $d_n(\theta)$  by adjoint representation. Let  $G$  be the adjoint group of  $\mathfrak{g}$  and  $L_\theta$  be the connected subgroup of  $G$  corresponding to  $\mathfrak{l}_\theta$ . Then it is known that the representation of  $L_\theta$  on  $d_1(\theta)$  is completely reducible and  $(L_\theta, d_1(\theta))$  is a prehomogeneous vector space. Such prehomogeneous vector spaces are called prehomogeneous vector spaces of parabolic type.

We can easily show that the  $\mathbb{Z}$ -graded Lie algebra  $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} d_n(\theta)$  and the Killing form  $K$  on  $\mathfrak{g}$  satisfy the conditions from (3.3) to (3.6). If  $(\mathfrak{l}_\theta, \text{ad}, d_1(\theta), K|_{\mathfrak{l}_\theta \times \mathfrak{l}_\theta})$  is a standard quadruplet, i.e.  $\mathfrak{l}_\theta = [d_{-1}(\theta), d_1(\theta)]$  holds, we call it a **quadruplet of parabolic type**. Note that a direct sum of quadruplets of parabolic type is again a quadruplet of parabolic type. Then we have the following theorem.

**Theorem 3.14.** *Let  $(\mathfrak{g}, \rho, V, B_0)$  be a standard quadruplet. Then the Lie algebra  $L(\mathfrak{g}, \rho, V, B_0)$  is finite-dimensional if and only if  $(\mathfrak{g}, \rho, V, B_0)$  is equivalent to some quadruplet of parabolic type.*

**Proof.** By applying Proposition 3.3, we can obtain that the Lie algebra associated to a quadruplet of parabolic type is finite-dimensional immediately. Let us prove the converse. Assume that  $L(\mathfrak{g}, \rho, V, B_0)$  is finite-dimensional. Without loss of generality, we can assume that  $(\mathfrak{g}, \rho, V, B_0)$  is indecomposable. Then, by Proposition 3.13,  $L(\mathfrak{g}, \rho, V, B_0)$  is simple. Hence, the bilinear form  $B$  defined in Definition 3.1 is a scalar multiple of the Killing form  $K_L$  on  $L(\mathfrak{g}, \rho, V, B_0)$ , that is, there exists a non-zero element  $c \in \mathbb{C}$  such that  $K_L = cB$ . We define a linear endomorphism  $\delta$  on  $L(\mathfrak{g}, \rho, V, B_0)$  by  $\delta(\sum_{n \in \mathbb{Z}} x_n) := \sum_{n \in \mathbb{Z}} 2nx_n$  where  $x_n \in V_n$ . Then, by the theory of Lie algebras,  $\delta$  is an inner derivation, i.e., there exists an element  $H \in L(\mathfrak{g}, \rho, V, B_0)$  such that  $\delta = \text{ad}(H)$ . Since  $\text{ad}(H)$  is diagonalizable, there exists a Cartan subalgebra  $\mathfrak{h}$  of  $L(\mathfrak{g}, \rho, V, B_0)$  such that  $H \in \mathfrak{h}$ . Let  $R$  be the root system of  $L(\mathfrak{g}, \rho, V, B_0)$  with respect to  $\mathfrak{h}$ . Let us construct a fundamental system of  $R$ .

Suppose that  $\mathfrak{g}$  is not commutative, i.e.,  $[V_0, V_0] \neq \{0\}$ . By the definition of  $H$ ,  $\mathfrak{h}$  is also a Cartan subalgebra of  $V_0$ . Put  $\bar{\mathfrak{h}} := \mathfrak{h} \cap [V_0, V_0]$  and denote the restriction of  $\gamma \in \mathfrak{h}^*$  to  $\bar{\mathfrak{h}}$  by  $\bar{\gamma}$ . Then there exists a subset  $\theta = \{\alpha^1, \dots, \alpha^r\}$  of  $R$  such that  $\{\bar{\alpha}^1, \dots, \bar{\alpha}^r\}$  is a fundamental root system of the root system of  $[V_0, V_0]$  with respect to  $\bar{\mathfrak{h}}$ . Put

$$\begin{aligned} \varphi' &:= \{\beta \in R \mid \beta(H) = 2, \beta - \alpha^s \notin R \text{ for } s = 1, \dots, r\} \\ &= \{\beta^1, \dots, \beta^l\}. \end{aligned}$$

Then  $\theta \cup \varphi' = \{\alpha^1, \dots, \alpha^r, \beta^1, \dots, \beta^l\}$  is a fundamental root system of  $R$ . In fact, we can easily show that any root can be written as  $n(1)\alpha^1 + \dots + n(r)\alpha^r + m(1)\beta^1 + \dots + m(l)\beta^l$  where all  $n(i)$  and  $m(j)$  are non-negative integers or all  $n(i)$  and  $m(j)$  are non-positive integers. Thus it is enough

to show that  $\theta \cup \varphi'$  is linearly independent over  $\mathbb{R}$ . Suppose that  $c(1), \dots, c(r), d(1), \dots, d(l) \in \mathbb{R}$  satisfy

$$c(1)\alpha^1 + \dots + c(r)\alpha^r + d(1)\beta^1 + \dots + d(l)\beta^l = 0.$$

By renumbering the elements of  $\theta \cup \varphi'$ , we can suppose that

$$\begin{aligned} c(1), \dots, c(s) &\geq 0, & c(s+1), \dots, c(r) &< 0, \\ d(1), \dots, d(t) &\geq 0, & d(t+1), \dots, d(l) &< 0 \end{aligned}$$

without loss of generality. Put

$$\begin{aligned} \lambda &:= c(1)\alpha^1 + \dots + c(s)\alpha^s + d(1)\beta^1 + \dots + d(t)\beta^t \\ &= -c(s+1)\alpha^{s+1} - \dots - c(r)\alpha^r - d(t+1)\beta^{t+1} - \dots - d(l)\beta^l. \end{aligned}$$

Then we have

$$\begin{aligned} 0 \leq K_L(H_\lambda, H_\lambda) &= - \sum_{1 \leq i \leq s, s+1 \leq j \leq r} c(i)c(j)K_L(H_{\alpha^i}, H_{\alpha^j}) - \sum_{1 \leq i \leq t, t+1 \leq j \leq l} d(i)d(j)K_L(H_{\beta^i}, H_{\beta^j}) \\ &\quad - \sum_{1 \leq s \leq r, t+1 \leq j \leq l} c(i)d(j)K_L(H_{\alpha^i}, H_{\beta^j}) - \sum_{1 \leq i \leq t, s+1 \leq j \leq r} d(i)c(j)K_L(H_{\beta^i}, H_{\alpha^j}) \end{aligned} \quad (3.42)$$

where  $H_\gamma$  denotes the element in  $\mathfrak{h}$  corresponding to  $\gamma \in \mathfrak{h}^*$  which satisfies  $K_L(H_\gamma, h) = \gamma(h)$  for any  $h \in \mathfrak{h}$ . By the definition of  $\theta \cup \varphi'$ , it follows that  $\gamma^1 - \gamma^2 \notin R$  for any  $\gamma^1, \gamma^2 \in \theta \cup \varphi'$  and thus  $K_L(H_{\gamma^1}, H_{\gamma^2}) \leq 0$ . Therefore, we have  $K_L(H_\lambda, H_\lambda) = 0$  and  $\lambda = 0$ . Therefore, we have

$$\begin{aligned} c(1)\alpha^1(H) + \dots + c(s)\alpha^s(H) + d(1)\beta^1(H) + \dots + d(t)\beta^t(H) \\ = 2d(1) + \dots + 2d(t) = 0 \end{aligned}$$

and thus  $d(1) = \dots = d(t) = 0$ . Moreover, it follows that  $c(1) = \dots = c(s) = 0$  from the assumption that  $\theta$  is linearly independent. Similarly, we have  $c(s+1) = \dots = c(r) = d(t+1) = \dots = d(l) = 0$ . Therefore  $\theta \cup \varphi'$  is a fundamental root system of  $R$ .

If  $[V_0, V_0] = \{0\}$ , that is,  $V_0$  is commutative, then a set  $\varphi := \{\beta \in R \mid \beta(H) = 2\}$  is a fundamental root system of  $R$ . Indeed, we can show it by a similar argument.

Summarizing the above argument, it follows that  $H$  is a grading element and thus  $(\mathfrak{g}, \rho_1, V_1, cB_0)$  is a quadruplet of parabolic type and  $(\mathfrak{g}, \rho, V, B_0)$  is equivalent to it. This completes the proof.  $\blacksquare$

It follows from Theorem 3.14 that any semisimple Lie algebra can be obtained from some standard quadruplet. In particular, for a positive integer  $m \geq 2$ , we can obtain  $\mathfrak{sl}_{m+1}$  and  $\mathfrak{so}_{2m+1}$  from standard quadruplets given in Example 1.18 and 1.19 respectively.

## 4 Prehomogeneous vector spaces and quadruplets

In this section, we shall discuss about the prehomogeneity condition of a triplet  $(G, \rho, V)$  under the assumption that  $G$  is reductive by using the  $\Phi$ -map of  $(\text{Lie}(G), d\rho, V, B_0)$  where  $B_0$  is a bilinear form on  $\text{Lie}(G)$ . The main result of this section is to give another proof of castling transform (Theorem 4.7).

### 4.1 Prehomogeneous quadruplets

A prehomogeneous vector space is a triplet which consists of a connected algebraic group  $G$  and its representation  $\rho$  on a finite-dimensional vector space  $V$  such that there exists a Zariski dense orbit  $\rho(G)v \subset V$  ( $v \in V$ ). Such an element  $v$  is called a generic point of  $(G, \rho, V)$ . In the case where  $G$  is reductive, the prehomogeneity of a triplet  $(G, \rho, V)$  can be described by using a  $\Phi$ -map.

**Proposition 4.1.** *Let  $G$  be a connected reductive algebraic group,  $V$  a finite-dimensional vector space and  $\rho$  a representation of  $G$  on  $V$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and  $d\rho$  the derived representation of  $\rho$  on  $V$ . Then the triplet  $(G, \rho, V)$  is prehomogeneous if and only if there exists a non-degenerate symmetric invariant form  $B_0$  and an element  $v \in V$  such that the  $\Phi$ -map at  $v$  of a quadruplet  $(\mathfrak{g}, d\rho, V, B_0)$ ,  $\Phi_{d\rho, v} : V^* \rightarrow \mathfrak{g}$  is injective.*

**Proof.** Take an arbitrary non-degenerate symmetric invariant bilinear form and denote it by  $B_0$ .

Assume that  $(G, \rho, V)$  is a prehomogeneous vector space and  $v$  is its generic point. Then it is known that a vector subspace  $d\rho(\mathfrak{g})v$  of  $V$  coincides with  $V$ . Then  $\Phi_{d\rho, v}$  is injective. If not, there exists a non-zero element  $\phi$  such that  $\Phi_{d\rho, v}(\phi) = 0$  and we have

$$\begin{aligned} 0 &= B_0(a, \Phi_{d\rho, v}(\phi)) \\ &= \langle d\rho(a)v, \phi \rangle \end{aligned}$$

for any  $a \in \mathfrak{g}$ . This is a contradiction to the assumption that  $d\rho(\mathfrak{g})v$  coincides with  $V$ . Thus  $\Phi_{d\rho, v}$  is injective.

Conversely, suppose that there exists an element  $v \in V$  such that  $\Phi_{d\rho, v}$  of  $(\mathfrak{g}, d\rho, V, B_0)$  is injective. Then  $(G, \rho, V)$  is a PV and  $v$  is its generic point. Indeed, if  $v$  is not a generic point, then  $d\rho(\mathfrak{g})v$  is a proper subspace of  $V$  and there exists a non-zero element  $\phi$  such that  $0 = \langle d\rho(a)v, \phi \rangle = B_0(a, \Phi_{d\rho, v}(\phi))$  for any  $a \in \mathfrak{g}$ . Since  $B_0$  is non-degenerate, we have  $\Phi_{d\rho, v}(\phi) = 0$  and a contradiction to the assumption that  $\Phi_{d\rho, v}$  is injective. Therefore  $v$  is a generic point.  $\blacksquare$

**Definition 4.2.** A standard quadruplet  $(\mathfrak{g}, \rho, V, B_0)$  is called a **prehomogeneous quadruplet** if and only if there exists an element  $v \in V$  such that  $\Phi_{\rho, v}$  is injective.

**Example 4.3.** For  $n \geq 3$ , a triplet  $(SO_n, \Lambda_1, \mathbb{C}^n)$  is not a PV, where  $\Lambda_1$  is the natural representation of  $SO_n$  on  $\mathbb{C}^n = M(n, 1; \mathbb{C})$ . To check this, under the notation of Example 1.4, let us show

that a quadruplet  $(\mathfrak{so}_n, \Lambda_1, \mathbb{C}^n, T_n|_{\mathfrak{so}_n \times \mathfrak{so}_n})$  is not prehomogeneous. In fact, for any column vector  $v \in \mathbb{C}^n$  we have

$$\Phi_{\Lambda_1, v}(v) = \frac{1}{2}(v^t v - v^t v) = 0. \quad (4.1)$$

Thus we have our claim.

However, a quadruplet  $(\mathfrak{gl}_1 \oplus \mathfrak{so}_n, \square \otimes \Lambda_1, \mathbb{C} \otimes \mathbb{C}^n, B_0)$  is prehomogeneous, where  $\square$  is a scalar multiplication of  $\mathfrak{gl}_1 = \mathbb{C}$  and  $B_0$  is a bilinear form defined by:

$$B_0((a, A), (a', A')) := -aa' + 2\text{Tr } AA' \quad (4.2)$$

where  $a, a' \in \mathfrak{gl}_1 = \mathbb{C}$  and  $A, A' \in \mathfrak{so}_n = \text{Alt}_n$ . The representation space can be identified with  $\mathbb{C}^n$  and the action of  $\mathfrak{gl}_1 \oplus \mathfrak{so}_n$  can be given as:

$$(\square \otimes \Lambda_1)(a, A)v = av + Av \quad (4.3)$$

where  $a \in \mathfrak{gl}_1$ ,  $A \in \mathfrak{so}_n$  and  $v \in \mathbb{C}^n$ . Then, the map  $\hat{\Phi}_{\square \otimes \Lambda_1}$  is given as follows:

$$\hat{\Phi}_{\square \otimes \Lambda_1}(v \otimes \phi) = (-{}^t v \phi, v^t \phi - \phi^t v) \quad (4.4)$$

where  $v, \phi \in \mathbb{C}^n$ . Put  $v_0 := {}^t(1, 0, \dots, 0) \in \mathbb{C}^n$ . Then we have

$$(\square \otimes \Lambda_1)(\Phi_{\square \otimes \Lambda_1, v_0}(\phi)) \cdot v_0 = -\phi. \quad (4.5)$$

Thus the map  $\hat{\Phi}_{\square \otimes \Lambda_1, v_0}$  is injective. Therefore  $(\mathfrak{gl}_1 \oplus \mathfrak{so}_n, \square \otimes \Lambda_1, \mathbb{C} \otimes \mathbb{C}^n, B_0)$  is a prehomogeneous quadruplet.

## 4.2 A triplet of the form $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n)$

In this section, we will consider an important theorem in the theory of prehomogeneous vector spaces, castling transform (See Proposition 7, p37, [S-K]).

**Definition 4.4.** Let  $\mathfrak{g}$  be a reductive Lie algebra,  $\rho$  a representation of  $\mathfrak{g}$  on a finite-dimensional vector space  $V$ ,  $B_0$  a non-degenerate symmetric invariant bilinear form on  $\mathfrak{g}$ . For any  $n \in \mathbb{N}$  and  $n$ -vectors  $v^1, \dots, v^n \in V$  (resp.  $\phi^1, \dots, \phi^n \in V^*$ ), we define a vector subspace  $S_{(v^1, \dots, v^n)} \subset (V^*)^n$  (resp.  $S_{(\phi^1, \dots, \phi^n)}^* \subset V^n$ ) by:

$$S_{(v^1, \dots, v^n)} := \{(\psi^1, \dots, \psi^n) \in (V^*)^n \mid \sum_{k=1}^n \hat{\Phi}_\rho(v^k \otimes \phi^k) = 0, \langle v^i, \psi^j \rangle = 0 \text{ for } 1 \leq i, j \leq n\} \quad (4.6)$$

(resp.

$$S_{(\phi^1, \dots, \phi^n)}^* := \{(u^1, \dots, u^n) \in V^n \mid \sum_{k=1}^n \hat{\Phi}_\rho(u^k \otimes \phi^k) = 0, \langle u^i, \phi^j \rangle = 0 \text{ for } 1 \leq i, j \leq n\} \quad (4.7)$$

where  $\hat{\Phi}_\rho$  is the  $\Phi$ -map of  $(\mathfrak{g}, \rho, V, B_0)$ .

**Lemma 4.5.** *We continue to use the notation of Definition 4.4. Then a standard quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  (resp.  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^n, B_0 \oplus T_n)$ ) is prehomogeneous if and only if there exists  $n$ -vectors  $v^1, \dots, v^n \in V$  (resp.  $\phi^1, \dots, \phi^n \in V^*$ ) such that  $S_{(v^1, \dots, v^n)} = \{(0, \dots, 0)\}$  (resp.  $S_{(\phi^1, \dots, \phi^n)} = \{(0, \dots, 0)\}$ ).*

**Proof.** We prove for a quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$ . For a quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^n, B_0 \oplus T_n)$ , our claim can be proved by the same way.

Let  $e_i \in \mathbb{C}^n$  be a column vector whose coefficients are all zero except the  $i$ -th one which is equal to 1 and  $E_{ij} \in \mathfrak{gl}_n$  be an  $n \times n$  matrix whose coefficients are all zero except the  $(i, j)$ -th one which is equal to 1. Then for any  $v \in V$  and  $\phi \in V^*$ , we have

$$\hat{\Phi}_{\rho \otimes \Lambda_1}((v \otimes e_i) \otimes (\phi \otimes e_j)) = (\delta_{ij} \hat{\Phi}_{\rho}(v \otimes \phi), \langle v, \phi \rangle E_{ij}) \quad (4.8)$$

where  $\delta_{ij}$  is the Kronecker delta.

Suppose that  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  is a prehomogeneous quadruplet and  $v^1 \otimes e_1 + \dots + v^n \otimes e_n \in V \otimes \mathbb{C}^n$  ( $v^1, \dots, v^n \in V$ ) is its generic point. Then we have  $S_{(v^1, \dots, v^n)} = \{(0, \dots, 0)\}$ . In fact, take an arbitrary element  $(\psi^1, \dots, \psi^n) \in S_{(v^1, \dots, v^n)}$ , then we have:

$$\begin{aligned} & \hat{\Phi}_{\rho \otimes \Lambda_1}((v^1 \otimes e_1 + \dots + v^n \otimes e_n) \otimes (\psi^1 \otimes e_1 + \dots + \psi^n \otimes e_n)) \\ &= \left( \hat{\Phi}_{\rho}(v^1 \otimes \psi^1 + \dots + v^n \otimes \psi^n), \sum_{1 \leq i, j \leq n} \langle v^i, \psi^j \rangle E_{ij} \right) \\ &= 0. \end{aligned} \quad (4.9)$$

Therefore, we have  $\psi^1 \otimes e_1 + \dots + \psi^n \otimes e_n = 0$  and thus  $(\psi^1, \dots, \psi^n) = (0, \dots, 0)$ .

Conversely, assume that  $n$ -vectors  $v^1, \dots, v^n \in V$  satisfy  $S_{(v^1, \dots, v^n)} = \{(0, \dots, 0)\}$ . Then  $v^1 \otimes e_1 + \dots + v^n \otimes e_n$  is a generic point of  $(\mathfrak{g}, \rho, V, B_0)$ . In fact, suppose that  $\phi^1 \otimes e_1 + \dots + \phi^n \otimes e_n$  satisfies

$$\hat{\Phi}_{\rho \otimes \Lambda_1}((v^1 \otimes e_1 + \dots + v^n \otimes e_n) \otimes (\phi^1 \otimes e_1 + \dots + \phi^n \otimes e_n)) = 0, \quad (4.10)$$

then we can obtain that  $(\phi^1, \dots, \phi^n) \in S_{(v^1, \dots, v^n)}$  and thus  $\phi^1 \otimes e_1 + \dots + \phi^n \otimes e_n = 0$ . This completes the proof.  $\blacksquare$

The following corollary is immediate.

**Corollary 4.6.** *Under the notation of Definition 4.4, if  $n \geq m = \dim V$ , then a quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  is prehomogeneous.*

**Theorem 4.7.** *(castling transform) Under the notation of Definition 4.4, we let  $n < m = \dim V$ . Then a quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  is prehomogeneous if and only if a quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n}, B_0 \oplus T_{m-n})$  is prehomogeneous. Moreover, if  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  is prehomogeneous (thus  $(\mathfrak{g} \oplus \mathfrak{gl}_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n}, B_0 \oplus T_{m-n})$  is also prehomogeneous),*

then for any generic point  $x \in V \otimes \mathbb{C}^n$  there exists a generic point  $y \in V^* \otimes \mathbb{C}^{m-n}$  such that the  $\mathfrak{g}$ -part of the isotropy subalgebra at  $x$  denoted by  $\mathfrak{g}_x$  coincides with the  $\mathfrak{g}$ -part of the isotropy subalgebra at  $y$  denoted by  $\mathfrak{g}_y$ .

**Proof.** Suppose that a quadruplet  $(\mathfrak{g} \oplus \mathfrak{gl}_n, \rho \otimes \Lambda_1, V \otimes \mathbb{C}^n, B_0 \oplus T_n)$  is prehomogeneous and  $n$ -vectors  $v^1, \dots, v^n$  satisfy  $S_{(v^1, \dots, v^n)} = \{(0, \dots, 0)\}$ . Then vectors  $v^1, \dots, v^n \in V$  are linearly independent. In fact, if there exists scalars  $(c(1), \dots, c(n)) \neq (0, \dots, 0)$  such that  $c(1)v^1 + \dots + c(n)v^n = 0$ , then we have a non-zero element  $\phi \in V^*$  such that  $\langle v^i, \phi \rangle = 0$  for  $i = 1, \dots, n$  and  $(c(1)\phi, \dots, c(n)\phi)$  is a non-zero element of  $S_{(v^1, \dots, v^n)}$ . It is a contradiction to the assumption that  $S_{(v^1, \dots, v^n)} = \{(0, \dots, 0)\}$ .

Put  $U := \mathbb{C}v^1 + \dots + \mathbb{C}v^n$  and denote the orthogonal space of  $U$  in  $V^*$  by  $U^\perp$ . Then  $U^\perp$  is a  $(m-n)$ -dimensional vector subspace of  $V^*$ . Take an arbitrary basis of  $U^\perp$ , denoted by  $\phi^1, \dots, \phi^{m-n} \in V^*$ . Then  $\phi^1, \dots, \phi^{m-n} \in V^*$  satisfy  $S_{(\phi^1, \dots, \phi^{m-n})} = \{(0, \dots, 0)\}$ . In fact, suppose that  $(u^1, \dots, u^{m-n}) \in S_{(\phi^1, \dots, \phi^{m-n})}$ . Then for any  $i, j$  ( $1 \leq i \leq n, 1 \leq j \leq m-n$ ), we have

$$\langle u^i, \phi^j \rangle = 0, \quad (4.11)$$

$$\hat{\Phi}_\rho(u^1 \otimes \phi^1 + \dots + u^{m-n} \otimes \phi^{m-n}) = 0. \quad (4.12)$$

It follows from (4.11) that  $u^1, \dots, u^{m-n} \in U$  and thus there exist scalars  $c(k, l) \in \mathbb{C}$  ( $1 \leq k \leq m-n, 1 \leq l \leq n$ ) which satisfy

$$u^k = c(k, 1)v^1 + \dots + c(k, n)v^n \quad (1 \leq k \leq m-n). \quad (4.13)$$

Then it follows from (4.12) that

$$\sum_{1 \leq l \leq n} \hat{\Phi}_\rho \left( v^l \otimes \left( \sum_{1 \leq k \leq m-n} c(k, l) \phi^k \right) \right) = 0. \quad (4.14)$$

Therefore  $((\sum_{1 \leq k \leq m-n} c(k, 1)\phi^k), \dots, (\sum_{1 \leq k \leq m-n} c(k, n)\phi^k)) \in S_{(v^1, \dots, v^n)} = \{(0, \dots, 0)\}$ . Therefore we have  $c(k, l) = 0$  for any  $k, l$  and  $(u^1, \dots, u^{m-n}) = (0, \dots, 0)$ , that is,  $(\mathfrak{g} \oplus \mathfrak{gl}_{m-n}, \rho^* \otimes \Lambda_1, V^* \otimes \mathbb{C}^{m-n}, B_0 \oplus T_{m-n})$  is also prehomogeneous.

Next, we put  $x := v^1 \otimes e_1 + \dots + v^n \otimes e_n \in V \otimes \mathbb{C}^n$  and  $y := \phi^1 \otimes e_1 + \dots + \phi^{m-n} \otimes e_{m-n} \in V^* \otimes \mathbb{C}^{m-n}$ . Let us show that we have  $\mathfrak{g}_x = \mathfrak{g}_y$ . A necessary and sufficient condition for an element  $a \in \mathfrak{g}$  to belong to  $\mathfrak{g}_x$  is that there exists an element  $A = \sum_{1 \leq i, j \leq n} b(i, j)E_{i, j} \in \mathfrak{gl}_n$  ( $b(i, j) \in \mathbb{C}$ )

which satisfies

$$\begin{aligned} 0 &= (\rho \otimes \Lambda_1)(a, A) \cdot x \\ &= (\rho \otimes \Lambda_1)(a, A) \cdot \sum_{1 \leq i \leq n} v^i \otimes e_i \\ &= \sum_{1 \leq i \leq n} \left( \rho(a)v^i + \sum_{1 \leq j \leq n} b(i, j)v^j \right) \otimes e_i. \end{aligned}$$

Therefore we have

$$\mathfrak{g}_x = \{a \in \mathfrak{g} \mid \rho(a)U \subset U\}. \quad (4.15)$$

Similarly we have

$$\mathfrak{g}_y = \{a \in \mathfrak{g} \mid \rho^*(a)U^\perp \subset U^\perp\}. \quad (4.16)$$

By an easy calculation, we can obtain  $\mathfrak{g}_x = \mathfrak{g}_y$ . This completes the proof. ■

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