

## 多重ゼータ値と多重ポリベルヌーイ数

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# Multiple zeta values and multi-poly-Bernoulli numbers

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# 1 Introduction

Multiple zeta values have been studied by many authors since L. Euler. Recently, they appear not only in mathematics but also in physics. One of the most interesting problem in the theory of multiple zeta values is on the structure of the vector space spanned by multiple zeta values. On this problem, remarkable results are obtained by T. Terasoma, P. Deligne, A. Goncharov and F. Brown recently.

The “finite sum” version of multiple zeta values has also been investigated. The finite multiple zeta value is a collection of certain “finite sums” in the setting given by D. Zagier. As in the case of the classical multiple zeta values, we have the problem on the vector space spanned by finite multiple zeta values. However, we have obtained only a few results on this problem since we have less informations on finite multiple zeta values.

In this paper, we study linear relations of both real multiple zeta values and finite multiple zeta values. To study the latter, we introduce and investigate multi-poly-Bernoulli numbers.

In §2 and §3, we review the theory of multiple zeta values briefly. We give harmonic (stuffle) product and shuffle product among multiple zeta values which define algebra structure on the space of multiple zeta values. We introduce a non-commutative algebra, according to M. Hoffman, to study multiple zeta values algebraically. By using this setup, we recall several results on the special values of multiple zeta values, and give and prove our first main result (Theorem.3.1 in §3.2).

In §4, we present the definition and a few examples of finite multiple zeta values, and also describe the problem on the vector space spanned by finite multiple zeta values. At the end of this section, we introduce some conjectures concerning this problem.

In §5, we define multi-poly-Bernoulli numbers which generalize the Bernoulli numbers. Since finite multiple zeta values can be expressed in terms of multi-poly-Bernoulli numbers, the multi-poly-Bernoulli numbers is one of useful tools for studying finite multiple zeta values. We study fundamental properties of multi-poly-Bernoulli numbers, and give alternative proofs of some relations among finite multiple zeta values as corollaries. We also define in §6 multi-poly-Bernoulli polynomials and generalized Arakawa-Kaneko zeta function. We show that special values of this function at non-positive integers are multi-poly-Bernoulli polynomials.

Throughout the paper, the set of all natural numbers, integers and rational numbers are denoted by  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{Q}$  respectively. The real part of a complex number  $s$  is denoted by  $\Re(s)$ .

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## 2 Multiple Zeta Values

In this section, we review the theory of multiple zeta values.

### 2.1 Multiple zeta values

**Definition 2.1.** For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \geq 1$  and  $k_1 \geq 2$ , multiple zeta values (MZVs for short) and multiple zeta-star values (MZSVs for short) are defined by the convergent series

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

$$\zeta^*(k_1, \dots, k_r) = \sum_{m_1 \geq \dots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}$$

respectively.

The number  $k_1 + \dots + k_r$  is called weight and  $r$  is called depth. Multiple zeta value can be written as a linear combination of multiple zeta-star values and vice versa. For example,

$$\begin{aligned} \zeta(k_1, k_2) &= \zeta^*(k_1, k_2) - \zeta^*(k_1 + k_2), \\ \zeta^*(k_1, k_2) &= \zeta(k_1, k_2) + \zeta(k_1 + k_2), \\ \zeta(k_1, k_2, k_3) &= \zeta^*(k_1, k_2, k_3) - \zeta^*(k_1 + k_2, k_3) - \zeta^*(k_1, k_2 + k_3) + \zeta^*(k_1 + k_2 + k_3), \\ \zeta^*(k_1, k_2, k_3) &= \zeta(k_1, k_2, k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1 + k_2 + k_3), \\ &\dots \end{aligned}$$

Obviously, when  $r = 1$ , these numbers coincide with the special value of the Riemann zeta function at positive integers. In particular, Euler found the formula for the values of Riemann zeta function at even integers:

**Theorem 2.2.** For  $k \in \mathbb{N}$ , we have

$$\zeta(2k) = \frac{(-1)^{k-1} B_{2k}}{2} (2\pi)^{2k},$$

where  $B_{2k}$  is the Bernoulli number defined by the following generating function:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{te^t}{e^t - 1}.$$

We give an example of multiple zeta values whose depths are greater than 1.

**Example 2.3.** For  $n \in \mathbb{N}$ , we have

$$\zeta(\overbrace{2, \dots, 2}^n) = \frac{\pi^{2n}}{(2n+1)!}.$$

*Proof.* We consider the infinite product of  $\frac{\sin \pi x}{\pi x}$ .

$$\begin{aligned} \frac{\sin \pi x}{\pi x} &= \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2}\right) = \left(1 - \frac{x^2}{1^2}\right) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right) \cdots \\ &= 1 - \left(\sum_{m=1}^{\infty} \frac{1}{m^2}\right) x^2 + \left(\sum_{m_1 > m_2 > 0} \frac{1}{m_1^2 m_2^2}\right) x^4 - \left(\sum_{m_1 > m_2 > m_3 > 0} \frac{1}{m_1^2 m_2^2 m_3^2}\right) x^6 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\underbrace{2, \dots, 2}_n) x^{2n}. \end{aligned}$$

We also have the Taylor expansion of  $\frac{\sin \pi x}{\pi x}$  at 0:

$$\frac{\sin \pi x}{\pi x} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n+1)!} x^{2n}.$$

We complete the proof by comparing the coefficients. □

## 2.2 Vector space spanned by MZVs

In this section, we describe the vector space over  $\mathbb{Q}$  spanned by MZVs:

**Definition 2.4.**

$$\begin{aligned} \mathcal{Z}_0 &= \mathbb{Q}, \quad \mathcal{Z}_1 = \{0\}, \\ \mathcal{Z}_k &= \sum_{\text{weight}=k} \mathbb{Q} \cdot \zeta(k_1, \dots, k_r) \quad (k \geq 2), \\ \mathcal{Z} &= \sum_{k=0}^{\infty} \mathcal{Z}_k. \end{aligned}$$

Since MZVs with weight 2 is only  $\zeta(2)$ , we find that  $\mathcal{Z}_2 = \mathbb{Q} \cdot \zeta(2)$ . Furthermore, it is known that  $\mathcal{Z}_3 = \mathbb{Q} \cdot \zeta(3)$  and  $\mathcal{Z}_4 = \mathbb{Q} \cdot \zeta(4)$ , so that any MZVs with weight 3 and 4 can be written as a rational multiple of  $\zeta(3)$  and  $\zeta(4)$  respectively. For  $k \geq 5$ , the basis of  $\mathcal{Z}_k$  are not determined at all.

We introduce the following conjecture of D. Zagier on the dimension of  $\mathcal{Z}_k$ .

**Conjecture 2.5.** *We have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k,$$

where  $d_k$  is the non-negative integer satisfying the following recursion:

$$d_k = d_{k-2} + d_{k-3} \quad (k \geq 3), \quad d_0 = 1, \quad d_1 = 0, \quad d_2 = 1.$$

One can compute the asymptotic behavior of  $d_k$ :

$$d_k \approx 0.41149 \cdots \times (1.3247 \cdots)^k \quad (k \rightarrow \infty).$$

Therefore,  $d_k$  is considerably smaller than the number of MZVs with weight  $k$ , that is, the above conjecture means that there are many linear relations among MZVs. In fact, a large number of relation formulas are found by many authors.

The following theorems are remarkable results on the above conjecture.

**Theorem 2.6** (P. Deligne-A. Goncharov [7], T. Terasoma [27]). *The inequality*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$$

*holds.*

**Theorem 2.7** (F. Brown [4]). *The space  $\mathcal{Z}$  is generated by MZVs whose component is 2 or 3.*

Conjecture 2.5 is completely solved if all MZVs whose component is 2 or 3 are linearly independent over  $\mathbb{Q}$ . However, it is hard to solve the problem since even whether  $\zeta(3)/\zeta(2) \notin \mathbb{Q}$  holds or not is not solved.

The vector space  $\mathcal{Z}$  becomes a  $\mathbb{Q}$ -algebra. Moreover,  $\mathcal{Z}$  has two products, which are called the harmonic (stuffle) product and the shuffle product. These products create many linear relations, which are called (generalized) double shuffle relations, among MZVs. The former product is obtained by considering the product of two defining series of MZVs. The product of two MZVs is a linear combination of MZVs by decomposing the sum. For example,

$$\begin{aligned} \zeta(a)\zeta(b) &= \left( \sum_{m>0} \frac{1}{m^a} \right) \left( \sum_{n>0} \frac{1}{n^b} \right) \\ &= \sum_{m>n>0} \frac{1}{m^a n^b} + \sum_{n>m>0} \frac{1}{m^a n^b} + \sum_{m=n>0} \frac{1}{m^a n^b} \\ &= \zeta(a, b) + \zeta(b, a) + \zeta(a + b). \end{aligned}$$

In general, the product of MZV with weight  $k$  and MZV with weight  $k'$  is a linear combination of MZVs with weight  $k + k'$ , which implies that  $\mathcal{Z}_k \cdot \mathcal{Z}_{k'} \subset \mathcal{Z}_{k+k'}$ . The second product is described in the next section.

If we replace  $\zeta(k_1, \dots, k_r)$  by  $\zeta^*(k_1, \dots, k_r)$  in Definition 2.4, the resulting vector space is the same one since any MZSVs can be expressed as a linear combination of MZVs and vice versa.

## 2.3 Iterated integral expression of MZVs

We consider the following iterated integral: For  $|z| < 1$ ,

$$\begin{aligned}
I(\epsilon_1, \dots, \epsilon_k; z) &= \int_{z > t_1 > \dots > t_k > 0} \dots \int A_{\epsilon_1}(t_1) \dots A_{\epsilon_k}(t_k) dt_1 \dots dt_k \\
&= \int_0^z A_{\epsilon_1}(t_1) dt_1 \dots \int_0^{t_{k-2}} A_{\epsilon_{k-1}}(t_{k-1}) dt_{k-1} \int_0^{t_{k-1}} A_{\epsilon_k}(t_k) dt_k \\
&=: \int_0^z A_{\epsilon_1}(t_1) dt_1 \circ \dots \circ A_{\epsilon_k}(t_k) dt_k
\end{aligned}$$

where  $\epsilon_j \in \{0, 1\}$  for  $1 \leq j \leq k-1$ ,  $\epsilon_k = 1$  and

$$A_0(t) = \frac{1}{t}, \quad A_1(t) = \frac{1}{1-t}.$$

$I(\epsilon_1, \dots, \epsilon_k; z)$  has another expression in terms of series, which is obtained by expanding  $1/(1-t_i)$  and termwise integration.

**Definition 2.8.** For positive integers  $k_i \geq 1$  ( $1 \leq i \leq r$ ), we set

$$\begin{aligned}
Li_{k_1, \dots, k_r}(z) &:= \sum_{m_1 > \dots > m_r > 0} \frac{z^{m_1}}{m_1^{k_1} \dots m_r^{k_r}} \\
&= I(\underbrace{0, \dots, 0}_{k_1-1}, 1, 0, \dots, 1, \underbrace{0, \dots, 0}_{k_r-1}, 1; z).
\end{aligned}$$

We note that  $Li_{k_1, \dots, k_r}(z)$  is holomorphic on  $|z| < 1$  and that when  $k_1 > 1$  this function is convergent as  $z \rightarrow 1$  to the MZV:

$$Li_{k_1, \dots, k_r}(1) = \zeta(k_1, \dots, k_r).$$

The product of two iterated integrals can be expressed as a linear combination of iterated integrals ([23]). This product is called shuffle product.

**Proposition 2.9.** For  $\omega_i, \omega'_j \in \{dt/t, dt/(1-t)\}$  ( $1 \leq i \leq k, 1 \leq j \leq k'$ ) with  $\omega_k = \omega_{k'} = dt/(1-t)$ ,

$$\left( \int_0^z \omega_1 \circ \dots \circ \omega_k \right) \left( \int_0^z \omega'_1 \circ \dots \circ \omega'_{k'} \right) = \sum \left( \int_0^z \eta_1 \circ \dots \circ \eta_{k+k'} \right),$$

where the sum on the right hand side runs through all shuffles of  $(\omega_1, \dots, \omega_k)$  and  $(\omega'_1, \dots, \omega'_{k'})$ .

**Example 2.10.**

$$Li_2(z)^2 = 2Li_{2,2}(z) + 4Li_4(z).$$

Therefore, together with the harmonic product we obtain

$$4\zeta(3, 1) = \zeta(4).$$



*Proof.*

$$\begin{aligned}
Li_2(z)^2 &= \left( \int_0^z \frac{dt_1}{t_1} \circ \frac{dt_2}{1-t_2} \right) \left( \int_0^z \frac{du_1}{u_1} \circ \frac{du_2}{1-u_2} \right) \\
&= \int_0^z \frac{dt_1}{t_1} \circ \frac{dt_2}{1-t_2} \circ \frac{du_1}{u_1} \circ \frac{du_2}{1-u_2} + \int_0^z \frac{dt_1}{t_1} \circ \frac{du_1}{u_1} \circ \frac{dt_2}{1-t_2} \circ \frac{du_2}{1-u_2} \\
&+ \int_0^z \frac{dt_1}{t_1} \circ \frac{du_1}{u_1} \circ \frac{du_2}{1-u_2} \circ \frac{dt_2}{1-t_2} + \int_0^z \frac{du_1}{u_1} \circ \frac{dt_1}{t_1} \circ \frac{dt_2}{1-t_2} \circ \frac{du_2}{1-u_2} \\
&+ \int_0^z \frac{du_1}{u_1} \circ \frac{dt_1}{t_1} \circ \frac{du_2}{1-u_2} \circ \frac{dt_2}{1-t_2} + \int_0^z \frac{du_1}{u_1} \circ \frac{du_2}{1-u_2} \circ \frac{dt_1}{t_1} \circ \frac{dt_2}{1-t_2} \\
&= 2Li_{2,2}(z) + 4Li_4(z).
\end{aligned}$$

□

## 2.4 Algebraic setup

Let  $\mathfrak{H}$  be the noncommutative polynomial algebra in two indeterminates  $x$  and  $y$ , and  $\mathfrak{H}^1, \mathfrak{H}^0$  its subalgebras:

$$\mathfrak{H} := \mathbb{Q}\langle x, y \rangle \supset \mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y \supset \mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y.$$

The degree of a word is called its weight. We put  $z_l = x^{l-1}y$  for  $l \geq 1$ . The algebra  $\mathfrak{H}^1$  is the noncommutative polynomial algebra over  $\mathbb{Q}$  freely generated by  $\{z_1, z_2, z_3, \dots\}$ . We define two  $\mathbb{Q}$ -linear maps  $Z, \bar{Z} : \mathfrak{H}^0 \rightarrow \mathbb{R}$  respectively by

$$Z(z_{k_1}z_{k_2} \cdots z_{k_n}) = \zeta(k_1, k_2, \dots, k_n), \quad Z(1) = 1,$$

and

$$\bar{Z}(z_{k_1}z_{k_2} \cdots z_{k_n}) = \zeta^*(k_1, k_2, \dots, k_n), \quad \bar{Z}(1) = 1,$$

which are usually called the evaluation maps. The weight of a word is that of the corresponding MZV or MZSV.

Let  $\gamma$  be the automorphism on  $\mathfrak{H}$  characterized by

$$\gamma(x) = x, \quad \gamma(y) = x + y.$$

We define the  $\mathbb{Q}$ -linear map  $d : \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$  by

$$d(wy) = \gamma(w)y$$

for any word  $w \in \mathfrak{H}$ . Then the linear transformation between MZV's and MZSV's is expressed as

$$\bar{Z} = Z \circ d.$$

Let  $*$ :  $\mathfrak{H}^1 \times \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$  be the  $\mathbb{Q}$ -bilinear map defined, for any words  $w, w' \in \mathfrak{H}^1$  and any positive integers  $k_1, k_2$ , by

$$1 * w = w * 1 = w$$

and the recursive rule

$$z_{k_1}w * z_{k_2}w' = z_{k_1}(w * z_{k_2}w') + z_{k_2}(z_{k_1}w * w') + z_{k_1+k_2}(w * w'). \quad (2.1)$$

It is known that the product  $*$  is commutative and associative ([12]). The product  $*$  is called the harmonic product on  $\mathfrak{H}^1$ . We find that  $\mathfrak{H}^0 * \mathfrak{H}^0 \subset \mathfrak{H}^0$  and the map  $Z$  is a homomorphism with respect to the harmonic product.

Let  $\mathfrak{m}: \mathfrak{H} \times \mathfrak{H} \rightarrow \mathfrak{H}$  be the  $\mathbb{Q}$ -bilinear map defined, for any words  $w, w' \in \mathfrak{H}$ , by

$$1\mathfrak{m}w = w\mathfrak{m}1 = w$$

and the recursive rule

$$uw\mathfrak{m}u'w' = u(w\mathfrak{m}u'w') + u'(uw\mathfrak{m}w'),$$

where  $u, u' \in \{x, y\}$ . It is also known that the product  $\mathfrak{m}$  is commutative and associative ([24]). The product  $\mathfrak{m}$  is called the shuffle product on  $\mathfrak{H}$ . We find that  $\mathfrak{H}^0\mathfrak{m}\mathfrak{H}^0 \subset \mathfrak{H}^0$  and the map  $Z$  is a homomorphism with respect to the shuffle product.

### 3 Special Values of MZ(S)V's

Special values of the Riemann zeta function have been investigated since Euler. Several results are found for example in [3, 8, 25, 30]. Special values of MZ(S)V's also have been studied by many authors. In this section, we introduce known facts of them and prove our result.

#### 3.1 Known facts

For MZV's, the following evaluations are known: For  $n > 0$ , we have

$$\zeta(\underbrace{2, \dots, 2}_n) = \frac{\pi^{2n}}{(2n+1)!},$$

$$\zeta(\underbrace{3, 1, \dots, 3, 1}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

The latter one was proved in [5] by using certain property of the iterated integral shuffle product rule. Kontsevich and Zagier gave another proof of the formula in connection with the Gauss hypergeometric function ([19]).

In [6, 22], the following more general identity is proved: For any non-negative integers  $n$  and  $m$  with  $n + m > 0$ , we have

$$\sum_{\substack{j_0 + j_1 + \dots + j_{2n} = m \\ j_0, j_1, \dots, j_{2n} \geq 0}} \zeta(\{2\}^{j_0}, 3, \{2\}^{j_1}, 1, \dots, 3, \{2\}^{j_{2n-1}}, 1, \{2\}^{j_{2n}})$$

$$= \binom{m+2n}{m} \frac{\pi^{2m+4n}}{(2n+1)(2m+4n+1)!}, \quad (3.1)$$

where  $\{2\}^j$  stands for the  $j$ -tuple of 2.

The situation for MZSV's is same as the one for MZV's. For  $m > 0$ , the property

$$\zeta^*(\underbrace{2, \dots, 2}_m) \in \mathbb{Q} \cdot \pi^{2m}$$

is proved for example in [10, 29, 1, 28]. In [21], formulas

$$\zeta^*(\underbrace{3, 1, \dots, 3, 1}_{2m}) \in \mathbb{Q} \cdot \pi^{4m}$$

for  $m > 0$  and

$$\sum_{\substack{j_0 + j_1 + \dots + j_{2n} = 1 \\ j_0, j_1, \dots, j_{2n} \geq 0}} \zeta^*(\{2\}^{j_0}, 3, \{2\}^{j_1}, 1, \dots, 3, \{2\}^{j_{2n-1}}, 1, \{2\}^{j_{2n}}) \in \mathbb{Q} \cdot \pi^{4n+2}$$

for  $n \geq 0$  are proved. Our result is the following:

**Theorem 3.1** (K. Imatomi, T. Tanaka, K. Tasaka and N. Wakabayashi [15]). *For any non-negative integer  $n$ , we have*

$$\sum_{\substack{j_0+j_1+\dots+j_{2n}=2 \\ j_0, j_1, \dots, j_{2n} \geq 0}} \zeta^*(\{2\}^{j_0}, 3, \{2\}^{j_1}, 1, \dots, 3, \{2\}^{j_{2n-1}}, 1, \{2\}^{j_{2n}}) \in \mathbb{Q} \cdot \pi^{4n+4}.$$

Further, the generalization of above results is proved in [17], that is: For any non-negative integer  $n$  and  $m$ , we have

$$\sum_{\substack{j_0+j_1+\dots+j_{2n}=m \\ j_0, j_1, \dots, j_{2n} \geq 0}} \zeta^*(\{2\}^{j_0}, 3, \{2\}^{j_1}, 1, \dots, 3, \{2\}^{j_{2n-1}}, 1, \{2\}^{j_{2n}}) \in \mathbb{Q} \cdot \pi^{4n+2m}.$$

### 3.2 Proof of Theorem 3.1

As is defined in [22], we introduce another  $\mathbb{Q}$ -bilinear map  $\tilde{\text{m}} : \mathfrak{H}^1 \times \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$  by

$$1 \tilde{\text{m}} w = w \tilde{\text{m}} 1 = w$$

and

$$z_{k_1} w \tilde{\text{m}} z_{k_2} w' = z_{k_1} (w \tilde{\text{m}} z_{k_2} w') + z_{k_2} (z_{k_1} w \tilde{\text{m}} w') \quad (3.2)$$

for any words  $w, w' \in \mathfrak{H}^1$  and any positive integers  $k_1, k_2$ . We see that the product  $\tilde{\text{m}}$  is commutative and associative. However, we notice that each of the evaluation maps  $Z$  and  $\bar{Z}$  can not be a homomorphism with respect to the product  $\tilde{\text{m}}$ .

To show Theorem.3.1, it suffices to prove the following:

**Theorem 3.2.** *Let  $a, b, c$  be positive integers. For any integer  $n \geq 0$ , we have*

$$\begin{aligned} (\alpha_n) \quad & d(z_c^2 \tilde{\text{m}} (z_a z_b)^n) \\ &= 2 \sum_{j+k=n} d(z_c \tilde{\text{m}} (z_a z_b)^j) * z_{(a+b)k+c} + \sum_{j+k=n} (z_c^2 \tilde{\text{m}} (z_a z_b)^j) * d(z_{a+b}^k) \\ &\quad - 4 \sum_{i+j+k=n} d((z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c} - \sum_{j+k=n} d((z_a z_b)^j) * z_{(a+b)k+2c}, \end{aligned}$$

$$\begin{aligned} (\beta_n) \quad & d(z_c^2 \tilde{\text{m}} z_b (z_a z_b)^n) \\ &= 2 \sum_{j+k=n} d(z_c \tilde{\text{m}} z_b (z_a z_b)^j) * z_{(a+b)k+c} + \sum_{j+k=n} (z_c^2 \tilde{\text{m}} z_b (z_a z_b)^j) * d(z_{a+b}^k) \\ &\quad - 4 \sum_{i+j+k=n} d(z_b (z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c} - \sum_{j+k=n} d(z_b (z_a z_b)^j) * z_{(a+b)k+2c}. \end{aligned}$$

Theorem 3.2 is the core property to prove our result (Theorem 3.1). We first prove Theorem 3.1 by assuming Theorem 3.2, the proof of which is given afterwards.

*Proof of Theorem 3.1.* By putting  $a = 3, b = 1$  and  $c = 2$  into  $(\alpha_n)$  of Theorem 3.2, we have

$$\begin{aligned} d(z_2^2 \tilde{\mathfrak{m}}(z_3 z_1)^n) &= 2 \sum_{j+k=n} d(z_2 \tilde{\mathfrak{m}}(z_3 z_1)^j) * z_{4k+2} + \sum_{j+k=n} (z_2^2 \tilde{\mathfrak{m}}(z_3 z_1)^j) * d(z_4^k) \\ &\quad - 4 \sum_{i+j+k=n} d((z_3 z_1)^i) * z_{4j+2} z_{4k+2} - \sum_{j+k=n} d((z_3 z_1)^j) * z_{4k+2}. \end{aligned} \quad (3.3)$$

By the harmonic product rule (2.1), the third term of the right-hand side of (3.3) can be written as

$$-2 \sum_{i+j+k=n} d((z_3 z_1)^i) * (z_{4j+2} * z_{4k+2} - z_{4j+4k+4})$$

Evaluating (3.3) via the map  $Z$ , we obtain

$$\begin{aligned} &\sum_{\substack{j_0+j_1+\dots+j_{2n}=2 \\ j_0, j_1, \dots, j_{2n} \geq 0}} \zeta^*({2}^{j_0}, 3, {2}^{j_1}, 1, \dots, 3, {2}^{j_{2n-1}}, 1, {2}^{j_{2n}}) \\ &= 2 \sum_{i=0}^n \sum_{\substack{j_0+j_1+\dots+j_{2i}=1 \\ j_0, j_1, \dots, j_{2i} \geq 0}} \zeta^*({2}^{j_0}, 3, {2}^{j_1}, 1, \dots, 3, {2}^{j_{2i-1}}, 1, {2}^{j_{2i}}) \zeta(4n - 4i + 2) \\ &\quad + \sum_{i=0}^n \sum_{\substack{j_0+j_1+\dots+j_{2i}=2 \\ j_0, j_1, \dots, j_{2i} \geq 0}} \zeta({2}^{j_0}, 3, {2}^{j_1}, 1, \dots, 3, {2}^{j_{2i-1}}, 1, {2}^{j_{2i}}) \zeta^*({4}^{n-i}) \\ &\quad - 2 \sum_{i+j+k=n} \zeta^*({3, 1}^i) \{ \zeta(4j+2) \zeta(4k+2) - \zeta(4j+4k+4) \} \\ &\quad - \sum_{j+k=n} \zeta^*({3, 1}^j) \zeta(4k+2), \end{aligned} \quad (3.4)$$

where  $\{3, 1\}^l$  stands for the string  $\underbrace{3, 1, \dots, 3, 1}_{2l}$  and  $\text{MZ}(\text{S})\text{V}$  of the empty index is regarded as 1. We know  $\zeta(2n) \in \mathbb{Q} \cdot \pi^{2n}$ ,  $\zeta^*({4}^{2l}) \in \mathbb{Q} \cdot \pi^{4n}$ , the formula (3.1) for  $m = 2$  and

$$\sum_{\substack{j_0+j_1+\dots+j_{2n}=m \\ j_0, j_1, \dots, j_{2n} \geq 0}} \zeta^*({2}^{j_0}, 3, {2}^{j_1}, 1, \dots, 3, {2}^{j_{2n-1}}, 1, {2}^{j_{2n}}) \in \mathbb{Q} \cdot \pi^{4n+2m}$$

for  $m = 0, 1$  (see [10, 6, 29, 1, 21] for example). Therefore the right-hand side of (3.4) is expressed as a rational multiple of  $\pi^{4n+4}$  and we conclude the theorem.  $\square$

Next we prove Theorem 3.2. For integers  $a, b, c > 0$  and  $i, j, k \geq 0$ , we put

$$\begin{aligned}
A_{i,j} &= (z_a z_b)^i z_c (z_a z_b)^j, \\
B_{i,j} &= (z_b z_a)^i z_c (z_b z_a)^j z_b, \\
C_{i,j,k} &= (z_a z_b)^i z_c (z_a z_b)^j z_c (z_a z_b)^k, \\
D_{i,j,k} &= (z_a z_b)^i z_c (z_a z_b)^j z_a z_c (z_b z_a)^k z_b, \\
E_{i,j,k} &= (z_b z_a)^i z_c (z_b z_a)^j z_b z_c (z_a z_b)^k, \\
F_{i,j,k} &= (z_b z_a)^i z_c (z_b z_a)^j z_c (z_b z_a)^k z_b,
\end{aligned}$$

where  $z_l = x^{l-1}y$  ( $l > 0$ ). By the definition of the product  $\tilde{\mathfrak{m}}$ , we obtain the following identities:

$$\left. \begin{aligned}
z_c \tilde{\mathfrak{m}} (z_a z_b)^n &= \sum_{i+j=n} A_{i,j} + \sum_{i+j=n-1} z_a B_{i,j}, \\
z_c \tilde{\mathfrak{m}} z_b (z_a z_b)^n &= \sum_{i+j=n} z_b A_{i,j} + \sum_{i+j=n} B_{i,j}, \\
z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^n &= \sum_{i+j+k=n} C_{i,j,k} + \sum_{i+j+k=n-1} D_{i,j,k} \\
&\quad + \sum_{i+j+k=n-1} z_a E_{i,j,k} + \sum_{i+j+k=n-1} z_a F_{i,j,k}, \\
z_c^2 \tilde{\mathfrak{m}} z_b (z_a z_b)^n &= \sum_{i+j+k=n} z_b C_{i,j,k} + \sum_{i+j+k=n-1} z_b D_{i,j,k} \\
&\quad + \sum_{i+j+k=n} E_{i,j,k} + \sum_{i+j+k=n} F_{i,j,k}.
\end{aligned} \right\} \quad (3.5)$$

for  $n \geq 0$ .

*Proof of Theorem 3.2.* The proof goes by induction on  $n$  such as  $(\alpha_0), (\beta_0) \Rightarrow (\alpha_1) \Rightarrow (\beta_1) \Rightarrow (\alpha_2) \Rightarrow \dots$ . We find that the identities  $(\alpha_0)$  and  $(\beta_0)$  hold by simple calculation. Assuming that it has been proved up to  $(\beta_{n-1})$ , we prove  $(\alpha_n)$ . The key identity is

$$d(z_{k_1} \cdots z_{k_n}) = \sum_{i=1}^n z_{k_1+\dots+k_i} d(z_{k_{i+1}} \cdots z_{k_n}), \quad (3.6)$$

where  $z_{k_{i+1}} \cdots z_{k_n} = 1$  if  $i = n$ . Using this key identity, we obtain

$$\begin{aligned}
&\sum_{i+j+k=n} d(C_{i,j,k}) \\
&= \sum_{i+j+k=n} \sum_{h=1}^i z_{(a+b)h} d(C_{i-h,j,k}) + \sum_{i+j+k=n} \sum_{h=0}^{i-1} z_{(a+b)h+a} d(z_b C_{i-h-1,j,k})
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i+j+k=n} \sum_{h=0}^j z_{(a+b)(h+i)+c} d(A_{j-h,k}) + \sum_{i+j+k=n} \sum_{h=0}^{j-1} z_{(a+b)(h+i)+c+a} d(z_b A_{j-h-1,k}) \\
& + \sum_{i+j+k=n} \sum_{h=0}^k z_{(a+b)(h+i+j)+2c} d((z_a z_b)^{k-h}) + \sum_{i+j+k=n} \sum_{h=0}^{k-1} z_{(a+b)(h+i+j)+2c+a} d(z_b (z_a z_b)^{k-h-1}) \\
& = \sum_{h+i+j+k=n-1} z_{(a+b)(h+1)} d(C_{i,j,k}) + \sum_{h+i+j+k=n-1} z_{(a+b)h+a} d(z_b C_{i,j,k}) \\
& + \sum_{h+i+j+k=n} z_{(a+b)(h+i)+c} d(A_{j,k}) + \sum_{h+i+j+k=n-1} z_{(a+b)(h+i)+a+c} d(z_b A_{j,k}) \\
& + \sum_{h+i+j+k=n} z_{(a+b)(h+i+j)+2c} d((z_a z_b)^k) + \sum_{h+i+j+k=n-1} z_{(a+b)(h+i+j)+a+2c} d(z_b (z_a z_b)^k) \\
& = \sum_{h+i+j+k=n-1} z_{(a+b)(h+1)} d(C_{i,j,k}) + \sum_{h+i+j+k=n-1} z_{(a+b)h+a} d(z_b C_{i,j,k}) \\
& + \sum_{i+j+k=n} (i+1) z_{(a+b)i+c} d(A_{j,k}) + \sum_{i+j+k=n-1} (i+1) z_{(a+b)i+a+c} d(z_b A_{j,k}) \\
& + \sum_{j+k=n} \binom{j+2}{2} z_{(a+b)j+2c} d((z_a z_b)^k) + \sum_{j+k=n-1} \binom{j+2}{2} z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k).
\end{aligned}$$

In the same way, we find

$$\begin{aligned}
& \sum_{i+j+k=n-1} d(D_{i,j,k}) \\
& = \sum_{h+i+j+k=n-2} z_{(a+b)(h+1)} d(D_{i,j,k}) + \sum_{h+i+j+k=n-2} z_{(a+b)h+a} d(z_b D_{i,j,k}) \\
& + \sum_{i+j+k=n-1} (i+1) z_{(a+b)i+c} d(z_a B_{j,k}) + \sum_{i+j+k=n-1} (i+1) z_{(a+b)i+a+c} d(B_{j,k}) \\
& + \sum_{j+k=n} \binom{j+1}{2} z_{(a+b)j+2c} d((z_a z_b)^k) + \sum_{j+k=n-1} \binom{j+2}{2} z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k),
\end{aligned}$$

$$\begin{aligned}
& \sum_{i+j+k=n-1} d(z_a E_{i,j,k}) \\
& = \sum_{h+i+j+k=n-2} z_{(a+b)(h+1)} d(z_a E_{i,j,k}) + \sum_{h+i+j+k=n-1} z_{(a+b)h+a} d(E_{i,j,k}) \\
& + \sum_{i+j+k=n} i z_{(a+b)i+c} d(A_{j,k}) + \sum_{i+j+k=n-1} (i+1) z_{(a+b)i+a+c} d(z_b A_{j,k}) \\
& + \sum_{j+k=n} \binom{j+1}{2} z_{(a+b)j+2c} d((z_a z_b)^k) + \sum_{j+k=n-1} \binom{j+1}{2} z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i+j+k=n-1} d(z_a F_{i,j,k}) \\
&= \sum_{h+i+j+k=n-2} z_{(a+b)(h+1)} d(z_a F_{i,j,k}) + \sum_{h+i+j+k=n-1} z_{(a+b)h+a} d(F_{i,j,k}) \\
&+ \sum_{i+j+k=n-1} i z_{(a+b)i+c} d(z_a B_{j,k}) + \sum_{i+j+k=n-1} (i+1) z_{(a+b)i+a+c} d(B_{j,k}) \\
&+ \sum_{j+k=n} \binom{j+1}{2} z_{(a+b)j+2c} d((z_a z_b)^k) + \sum_{j+k=n-1} \binom{j+2}{2} z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k).
\end{aligned}$$

These four identities add up to the left-hand side of  $(\alpha_n)$  because of (3.5). Therefore, again using (3.5), we obtain

$$(\text{LHS of } (\alpha_n)) = \sum_{j+k=n-1} z_{(a+b)(j+1)} d(z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^k) \quad (3.7)$$

$$+ \sum_{j+k=n-1} z_{(a+b)j+a} d(z_c^2 \tilde{\mathfrak{m}} z_b (z_a z_b)^k) \quad (3.8)$$

$$+ \sum_{j+k=n} (2j+1) z_{(a+b)j+c} d(z_c \tilde{\mathfrak{m}} (z_a z_b)^k) \quad (3.9)$$

$$+ \sum_{j+k=n-1} (2j+2) z_{(a+b)j+a+c} d(z_c \tilde{\mathfrak{m}} z_b (z_a z_b)^k) \quad (3.10)$$

$$+ \sum_{j+k=n} \binom{2j+2}{2} z_{(a+b)j+2c} d((z_a z_b)^k) \quad (3.11)$$

$$+ \sum_{j+k=n-1} \binom{2j+3}{2} z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k). \quad (3.12)$$

So, it is sufficient to show that the right-hand side of the above identity equals the right-hand side of  $(\alpha_n)$ .

First we have

$$\begin{aligned}
& \sum_{j+k=n} d(A_{j,k}) \\
&= \sum_{j+k=n} \sum_{i=1}^j z_{(a+b)i} d(A_{j-i,k}) + \sum_{j+k=n} \sum_{i=0}^{j-1} z_{(a+b)i+a} d(z_b A_{j-i-1,k}) \\
&+ \sum_{j+k=n} \sum_{i=0}^k z_{(a+b)(i+j)+c} d((z_a z_b)^{k-j}) + \sum_{j+k=n} \sum_{i=0}^{k-1} z_{(a+b)(i+j)+a+c} d(z_b (z_a z_b)^{k-i-1})
\end{aligned}$$



$$\begin{aligned}
&= \sum_{i+j+k=n-1} z_{(a+b)(i+1)} d(A_{j,k}) + \sum_{i+j+k=n-1} z_{(a+b)i+a} d(z_b A_{j,k}) \\
&+ \sum_{i+j+k=n} z_{(a+b)(i+j)+c} d((z_a z_b)^k) + \sum_{i+j+k=n-1} z_{(a+b)(i+j)+a+c} d(z_b (z_a z_b)^k), \\
&= \sum_{i+j+k=n-1} z_{(a+b)(i+1)} d(A_{j,k}) + \sum_{i+j+k=n-1} z_{(a+b)i+a} d(z_b A_{j,k}) \\
&+ \sum_{i+j=n} (i+1) z_{(a+b)i+c} d((z_a z_b)^j) + \sum_{i+j=n-1} (i+1) z_{(a+b)i+a+c} d(z_b (z_a z_b)^j).
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\sum_{j+k=n-1} d(z_a B_{j,k}) \\
&= \sum_{i+j+k=n-2} z_{(a+b)(i+1)} d(z_a B_{j,k}) + \sum_{i+j+k=n-1} z_{(a+b)i+a} d(B_{j,k}) \\
&+ \sum_{i+j=n} i z_{(a+b)i+c} d((z_a z_b)^j) + \sum_{i+j=n-1} (i+1) z_{(a+b)i+a+c} d(z_b (z_a z_b)^j).
\end{aligned}$$

Using the above two identities and (3.5), we obtain

$$\begin{aligned}
&d(z_c \tilde{\mathfrak{m}} (z_a z_b)^n) \\
&= \sum_{j+k=n} d(A_{j,k}) + \sum_{j+k=n-1} d(z_a B_{j,k}) \\
&= \sum_{i+j=n-1} z_{(a+b)(i+1)} d(z_c \tilde{\mathfrak{m}} (z_a z_b)^j) + \sum_{i+j=n-1} z_{(a+b)i+a} d(z_c \tilde{\mathfrak{m}} z_b (z_a z_b)^j) \\
&+ \sum_{i+j=n} (2i+1) z_{(a+b)i+c} d((z_a z_b)^j) + \sum_{i+j=n-1} (2i+2) z_{(a+b)i+a+c} d(z_b (z_a z_b)^j)
\end{aligned}$$

for  $n \geq 0$ . By this identity and the harmonic product rule (2.1), we write the first term of the right-hand side of  $(\alpha_n)$  (divided by the coefficient 2) as

$$\begin{aligned}
&\sum_{j+k=n} d(z_c \tilde{\mathfrak{m}} (z_a z_b)^j) * z_{(a+b)k+c} \\
&= \sum_{i+j+k=n-1} z_{(a+b)(i+1)} \{d(z_c \tilde{\mathfrak{m}} (z_a z_b)^j) * z_{(a+b)k+c}\} \tag{3.13}
\end{aligned}$$

$$+ \sum_{j+k=n} (j+1) z_{(a+b)j+c} d(z_c \tilde{\mathfrak{m}} (z_a z_b)^k) \tag{3.14}$$

$$+ \sum_{i+j+k=n-1} z_{(a+b)i+a} \{d(z_c \tilde{\mathfrak{m}} z_b (z_a z_b)^j) * z_{(a+b)k+c}\} \tag{3.15}$$

$$+ \sum_{j+k=n-1} (j+1) z_{(a+b)j+a+c} d(z_c \tilde{\mathfrak{m}} z_b (z_a z_b)^k) \quad (3.16)$$

$$+ \sum_{i+j+k=n} (2i+1) z_{(a+b)i+c} \{d((z_a z_b)^j) * z_{(a+b)k+c}\} \quad (3.17)$$

$$+ \sum_{j+k=n} (j+1)^2 z_{(a+b)j+2c} d((z_a z_b)^k) \quad (3.18)$$

$$+ \sum_{i+j+k=n-1} (2i+2) z_{(a+b)i+a+c} \{d(z_b (z_a z_b)^j) * z_{(a+b)k+c}\} \quad (3.19)$$

$$+ \sum_{j+k=n-1} (j+1)(j+2) z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k). \quad (3.20)$$

Note that the key identity (3.6) shows

$$d(z_{a+b}^l) = \sum_{i+k=l-1} z_{(a+b)(i+1)} d(z_{a+b}^k) \quad (3.21)$$

and

$$d((z_a z_b)^l) = \sum_{i+j=l-1} z_{(a+b)i+a} d(z_b (z_a z_b)^j) + \sum_{i+j=l-1} z_{(a+b)(i+1)} d((z_a z_b)^j) \quad (3.22)$$

for  $l \geq 1$ . By using the  $\tilde{\mathfrak{m}}$ -product rule (3.2) and the identity (3.21), the second term of the right-hand side of  $(\alpha_n)$  is calculated as

$$\sum_{j+k=n} (z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^j) * d(z_{a+b}^k) \quad (3.23)$$

$$= z_c^2 * d(z_{a+b}^n) + \sum_{\substack{j+k=n \\ j,k \geq 1}} (z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^j) * d(z_{a+b}^k) + z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^n$$

$$= \sum_{j+k=n-1} z_c^2 * z_{(a+b)(j+1)} d(z_{a+b}^k)$$

$$+ \sum_{\substack{i+j+k=n-1 \\ k \geq 1}} \{z_c(z_c \tilde{\mathfrak{m}} (z_a z_b)^k) + z_a(z_c^2 \tilde{\mathfrak{m}} z_b (z_a z_b)^{k-1})\} * z_{(a+b)(i+1)} d(z_{a+b}^j)$$

$$+ z_c(z_c \tilde{\mathfrak{m}} (z_a z_b)^n) \quad (3.24)$$

$$+ z_a(z_c^2 \tilde{\mathfrak{m}} z_b (z_a z_b)^{n-1}). \quad (3.25)$$

Expanding the first and the second terms of the right-hand side by the harmonic product rule (2.1), we have

$$\begin{aligned} & \sum_{j+k=n-1} z_c^2 * z_{(a+b)(j+1)} d(z_{a+b}^k) \\ &= z_c (z_c * d(z_{a+b}^n)) \end{aligned} \quad (3.26)$$

$$+ \sum_{j+k=n-1} z_{(a+b)(j+1)} \{z_c^2 * d(z_{a+b}^k)\} \quad (3.27)$$

$$+ \sum_{j+k=n-1} z_{(a+b)(j+1)+c} \{z_c * d(z_{a+b}^k)\} \quad (3.28)$$

and

$$\begin{aligned} & \sum_{\substack{i+j+k=n-1 \\ k \geq 1}} \{z_c(z_c \tilde{\mathfrak{m}} (z_a z_b)^k) + z_a(z_c^2 \tilde{\mathfrak{m}} z_b(z_a z_b)^{k-1})\} * z_{(a+b)(i+1)} d(z_{a+b}^j) \\ &= \sum_{\substack{i+j+k=n-1 \\ k \geq 1}} z_{(a+b)(i+1)} \{(z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^k) * d(z_{a+b}^j)\} \end{aligned} \quad (3.29)$$

$$+ \sum_{\substack{j+k=n \\ j, k \geq 1}} z_c \{(z_c \tilde{\mathfrak{m}} (z_a z_b)^j) * d(z_{a+b}^k)\} \quad (3.30)$$

$$+ \sum_{\substack{i+j+k=n-1 \\ k \geq 1}} z_{(a+b)(i+1)+c} \{(z_c \tilde{\mathfrak{m}} (z_a z_b)^k) * d(z_{a+b}^j)\} \quad (3.31)$$

$$+ \sum_{\substack{j+k=n \\ j, k \geq 1}} z_a \{(z_c^2 \tilde{\mathfrak{m}} z_b(z_a z_b)^{j-1}) * d(z_{a+b}^k)\} \quad (3.32)$$

$$+ \sum_{\substack{i+j+k=n-1 \\ k \geq 1}} z_{(a+b)(i+1)+a} \{(z_c^2 \tilde{\mathfrak{m}} z_b(z_a z_b)^{k-1}) * d(z_{a+b}^j)\}. \quad (3.33)$$

We see that identities

$$(3.27) + (3.29) = \sum_{i+j+k=n-1} z_{(a+b)(i+1)} \{(z_c^2 \tilde{\mathfrak{m}} (z_a z_b)^j) * d(z_{a+b}^k)\},$$

$$(3.24) + (3.26) + (3.30) = \sum_{j+k=n} z_c \{(z_c \tilde{\mathfrak{m}} (z_a z_b)^j) * d(z_{a+b}^k)\},$$

$$(3.25) + (3.32) = \sum_{j+k=n-1} z_a \{(z_c^2 \tilde{\mathfrak{m}} z_b(z_a z_b)^j) * d(z_{a+b}^k)\},$$

$$(3.28) + (3.31) = \sum_{i+j+k=n-1} z_{(a+b)(i+1)+c} \{(z_c \tilde{\mathfrak{m}} (z_a z_b)^j) * d(z_{a+b}^k)\},$$

$$(3.33) = \sum_{\substack{i+j+k=n-1 \\ i \geq 1}} z_{(a+b)i+a} \{(z_c^2 \tilde{\mathfrak{m}} z_b(z_a z_b)^j) * d(z_{a+b}^k)\}$$

hold. Therefore we have

$$(3.23) = \sum_{i+j+k=n-1} z_{(a+b)(i+1)} \left\{ (z_c^2 \tilde{\text{III}} (z_a z_b)^j) * d(z_{a+b}^k) \right\} \quad (3.34)$$

$$+ \sum_{i+j+k=n} z_{(a+b)i+c} \left\{ (z_c \tilde{\text{III}} (z_a z_b)^j) * d(z_{a+b}^k) \right\} \quad (3.35)$$

$$+ \sum_{i+j+k=n-1} z_{(a+b)i+a} \left\{ (z_c^2 \tilde{\text{III}} z_b (z_a z_b)^j) * d(z_{a+b}^k) \right\}. \quad (3.36)$$

By (3.22) and the harmonic product rule (2.1), the third term of the right-hand side of  $(\alpha_n)$  (divided by the coefficient  $-4$ ) is calculated as

$$\sum_{i+j+k=n} d((z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c} \quad (3.37)$$

$$= \sum_{j+k=n} z_{(a+b)j+c} z_{(a+b)k+c} + \sum_{\substack{i+j+k=n \\ i \geq 1}} d((z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c}$$

$$= \sum_{j+k=n} z_{(a+b)j+c} z_{(a+b)k+c}$$

$$+ \sum_{h+i+j+k=n-1} z_{(a+b)h+a} d(z_b (z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c}$$

$$+ \sum_{h+i+j+k=n-1} z_{(a+b)(h+1)} d((z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c}$$

$$= \sum_{j+k=n} z_{(a+b)j+c} z_{(a+b)k+c} \quad (3.38)$$

$$+ \sum_{h+i+j+k=n-1} \left\{ z_{(a+b)h+a} \left( d(z_b (z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c} \right) \right.$$

$$\left. + z_{(a+b)j+c} \left( z_{(a+b)h+c} d(z_b (z_a z_b)^i) * z_{(a+b)k+c} \right) \right. \quad (3.39)$$

$$\left. + z_{(a+b)(h+j)+a+c} \left( d(z_b (z_a z_b)^i) * z_{(a+b)k+c} \right) \right. \quad (3.40)$$

$$\left. + z_{(a+b)(h+1)} \left( d((z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c} \right) \right. \quad (3.41)$$

$$\left. + z_{(a+b)j+c} \left( z_{(a+b)(h+1)} d((z_a z_b)^i) * z_{(a+b)k+c} \right) \right. \quad (3.41)$$

$$\left. + z_{(a+b)(h+j+1)+c} \left( d((z_a z_b)^i) * z_{(a+b)k+c} \right) \right\}. \quad (3.42)$$

By (3.22),

$$(3.39) + (3.41) = \sum_{\substack{i+j+k=n \\ j \geq 1}} z_{(a+b)i+c} \left( d((z_a z_b)^j) * z_{(a+b)k+c} \right).$$

Also we have

$$(3.42) = \sum_{i+j+k=n} i z_{(a+b)i+c} \left( d((z_a z_b)^j) * z_{(a+b)k+c} \right).$$

Hence we find

$$(3.38) + (3.39) + (3.41) + (3.42) = \sum_{i+j+k=n} (i+1)z_{(a+b)i+c} \{d((z_a z_b)^j) * z_{(a+b)k+c}\}.$$

Since

$$(3.40) = \sum_{i+j+k=n-1} (i+1)z_{(a+b)i+a+c} \{d(z_b(z_a z_b)^j) * z_{(a+b)k+c}\},$$

we have

$$(3.37) = \sum_{h+i+j+k=n-1} z_{(a+b)(h+1)} \{d((z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c}\} \quad (3.43)$$

$$+ \sum_{h+i+j+k=n-1} z_{(a+b)h+a} \{d(z_b(z_a z_b)^i) * z_{(a+b)j+c} z_{(a+b)k+c}\} \quad (3.44)$$

$$+ \sum_{i+j+k=n} (i+1)z_{(a+b)i+c} \{d((z_a z_b)^j) * z_{(a+b)k+c}\} \quad (3.45)$$

$$+ \sum_{i+j+k=n-1} (i+1)z_{(a+b)i+a+c} \{d(z_b(z_a z_b)^j) * z_{(a+b)k+c}\}. \quad (3.46)$$

By (3.22) and the harmonic product rule (2.1), the fourth term of the right-hand side of  $(\alpha_n)$  (divided by the coefficient  $-1$ ) is calculated as

$$\sum_{j+k=n} d((z_a z_b)^j) * z_{(a+b)k+2c} \quad (3.47)$$

$$= z_{(a+b)n+2c} + \sum_{\substack{j+k=n \\ j \geq 1}} d((z_a z_b)^j) * z_{(a+b)k+2c}$$

$$= z_{(a+b)n+2c} + \sum_{i+j+k=n-1} z_{(a+b)i+a} d(z_b(z_a z_b)^j) * z_{(a+b)k+2c}$$

$$+ \sum_{i+j+k=n-1} z_{(a+b)(i+1)} d((z_a z_b)^j) * z_{(a+b)k+2c}$$

$$= z_{(a+b)n+2c} \quad (3.48)$$

$$+ \sum_{i+j+k=n-1} \left\{ z_{(a+b)i+a} \left( d(z_b(z_a z_b)^j) * z_{(a+b)k+2c} \right) \right.$$

$$\left. + z_{(a+b)k+2c} z_{(a+b)i+a} d(z_b(z_a z_b)^j) \right. \quad (3.49)$$

$$\left. + z_{(a+b)(i+k)+a+2c} d(z_b(z_a z_b)^j) \right. \quad (3.50)$$

$$\left. + z_{(a+b)(i+1)} \left( d((z_a z_b)^j) * z_{(a+b)k+2c} \right) \right.$$

$$\left. + z_{(a+b)k+2c} z_{(a+b)(i+1)} d((z_a z_b)^j) \right. \quad (3.51)$$

$$\left. + z_{(a+b)(i+k+1)+2c} d((z_a z_b)^j) \right\}. \quad (3.52)$$

By (3.22),

$$(3.49) + (3.51) = \sum_{\substack{j+k=n \\ j \geq 1}} z_{(a+b)k+2c} d((z_a z_b)^j).$$

Also we have

$$(3.52) = \sum_{j+k=n} k z_{(a+b)k+2c} d((z_a z_b)^j).$$

Hence we find

$$(3.48) + (3.49) + (3.51) + (3.52) = \sum_{j+k=n} (j+1) z_{(a+b)j+2c} d((z_a z_b)^k).$$

Since

$$(3.50) = \sum_{j+k=n-1} (j+1) z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k),$$

we have

$$(3.47) = \sum_{i+j+k=n-1} z_{(a+b)(i+1)} \{d((z_a z_b)^j) * z_{(a+b)k+2c}\} \quad (3.53)$$

$$+ \sum_{i+j+k=n-1} z_{(a+b)i+a} \{d(z_b (z_a z_b)^j) * z_{(a+b)k+2c}\} \quad (3.54)$$

$$+ \sum_{j+k=n} (j+1) z_{(a+b)j+2c} d((z_a z_b)^k) \quad (3.55)$$

$$+ \sum_{j+k=n-1} (j+1) z_{(a+b)j+a+2c} d(z_b (z_a z_b)^k). \quad (3.56)$$

The following identity is shown by [21, Theorem 7 (5)] (where the map  $d$  is denoted by  $S$ ) and (3.5):

$$d(z_c \tilde{\mathfrak{m}}(z_a z_b)^n) = 2 \sum_{j+k=n} d((z_a z_b)^j) * z_{(a+b)k+c} - \sum_{j+k=n} (z_c \tilde{\mathfrak{m}}(z_a z_b)^j) * d(z_{a+b}^k) \quad (3.57)$$

for  $n \geq 0$ . By induction hypothesis and (3.57), we have

$$(3.7) = 2 \times (3.13) + (3.34) - 4 \times (3.43) - (3.53) \quad (\text{by } (\alpha_0), \dots, (\alpha_{n-1})),$$

$$(3.8) = 2 \times (3.15) + (3.36) - 4 \times (3.44) - (3.54) \quad (\text{by } (\beta_0), \dots, (\beta_{n-1})),$$

$$(3.9) = 2 \times (3.14) + 2 \times (3.17) + (3.35) - 4 \times (3.45) \quad (\text{by (3.57)}).$$

Also we immediately find that

$$(3.10) = 2 \times (3.16),$$

$$(3.11) = 2 \times (3.18) - (3.55),$$

$$(3.12) = 2 \times (3.20) - (3.56),$$

$$0 = 2 \times (3.19) - 4 \times (3.46).$$

We have already observed that the left (resp. right)-hand sides of these seven equations add up to the left (resp. right)-hand side of the identity  $(\alpha_n)$ . Therefore we prove  $(\alpha_n)$ . In the same way,  $(\beta_n)$  can be proved by using the induction hypothesis and  $(\alpha_n)$  for  $n$ . This completes the proof of Theorem 3.2.  $\square$

## 4 Finite Multiple Zeta Values

“Mod  $p$  multiple zeta values” have been investigated as with study of MZVs. D. Zagier suggests a natural setting to consider them. Finite multiple zeta values are “mod  $p$  multiple zeta values” in this setting.

### 4.1 Finite multiple zeta values

Let

$$\mathcal{A} := \frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p\mathbb{Z}} = \{(a_p)_p; a_p \in \mathbb{Z}/p\mathbb{Z}\} / \sim,$$

where  $(a_p)_p \sim (b_p)_p$  is equivalent to the equalities  $a_p = b_p$  for all but finitely many primes  $p$ . The set  $\mathcal{A}$  naturally becomes a ring with component-wise addition and multiplication, and further, a  $\mathbb{Q}$ -algebra by identifying  $\mathbb{Q}$  as a subring of  $\mathcal{A}$  by the diagonal embedding.

**Definition 4.1.** For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \in \mathbb{N}$ , finite multiple zeta values (FMZVs for short) and finite multiple zeta-star values (FMZSVs for short) are defined by the following:

$$\begin{aligned} \zeta_{\mathcal{A}}(k_1, \dots, k_r) &:= (H_p(k_1, \dots, k_r) \bmod p)_p \in \mathcal{A}, \\ \zeta_{\mathcal{A}}^*(k_1, \dots, k_r) &:= (H_p^*(k_1, \dots, k_r) \bmod p)_p \in \mathcal{A}, \end{aligned}$$

where  $H_n(k_1, \dots, k_r)$  and  $H_n^*(k_1, \dots, k_r)$  are the multiple harmonic sum defined by

$$\begin{aligned} H_n(k_1, \dots, k_r) &= \sum_{n > m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}, \\ H_n^*(k_1, \dots, k_r) &= \sum_{n-1 \geq m_1 \geq \dots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}. \end{aligned}$$

We use the terms weight and depth for FMZ(S)V in the same manner. By definition, we easily find that each value can be expressed as the same linear combination in §.2.1. We give a few relation formulas for FMZ(S)V.

**Example 4.2.** For any positive integer  $k$ , we have

$$\zeta_{\mathcal{A}}(k) = \zeta_{\mathcal{A}}^*(k) = (0)_p.$$

*Proof.* Let  $p$  prime such that  $p > k$ . By Fermat’s theorem and Seki-Bernoulli’s formula, we obtain

$$\begin{aligned} H_p(k) &= \sum_{m=1}^{p-1} \frac{1}{m^k} \equiv \sum_{m=1}^{p-1} m^{p-1-k} \\ &= \frac{1}{p-k} \sum_{j=0}^{p-1-k} (p-1)^{p-k-j} \binom{p-k}{j} B_j \\ &\equiv \frac{(-1)^{p-k}}{p-k} \sum_{j=0}^{p-1-k} (-1)^j \binom{p-k}{j} B_j = 0 \pmod{p}. \end{aligned}$$

The last equality holds by the recurrence relation for the Bernoulli number.  $\square$

**Example 4.3** (M. Hoffman [10]). *For any positive integers  $k_1, k_2$ , we have*

$$\zeta_{\mathcal{A}}(k_1, k_2) = \zeta_{\mathcal{A}}^*(k_1, k_2) = \left( \frac{(-1)^{k_1}}{k_1 + k_2} \binom{k_1 + k_2}{k_1} B_{p-k_1-k_2} \right)_p.$$

*Proof.* By Example 4.2, we find

$$\zeta_{\mathcal{A}}^*(k_1, k_2) = \zeta_{\mathcal{A}}(k_1, k_2) + \zeta_{\mathcal{A}}(k_1 + k_2) = \zeta_{\mathcal{A}}(k_1, k_2).$$

Therefore we prove the identity only for FMZVs. Let  $p$  prime such that  $p > k_1 + k_2$ . Using Fermat's theorem and Seki-Bernoulli's formula, we have

$$\begin{aligned} H_p(k_1, k_2) &= \sum_{i=1}^{p-1} \frac{1}{i^{k_1}} \sum_{j=1}^{k_1-1} \frac{1}{j^{k_2}} \equiv \sum_{i=1}^{p-1} \frac{1}{i^{k_1}} \sum_{j=1}^{k_1-1} j^{p-1-k_2} \\ &= \sum_{i=1}^{p-1} \frac{1}{i^{k_1}} \frac{1}{p-k_2} \sum_{l=0}^{p-1-k_2} \binom{p-k_2}{l} B_l i^{p-k_2-l} \\ &= \frac{1}{p-k_2} \sum_{l=0}^{p-1-k_2} \binom{p-k_2}{l} B_l \sum_{i=1}^{p-1} i^{p-k_1-k_2-l} \\ &\equiv \frac{1}{p-k_2} \binom{p-k_2}{p-k_1-k_2} B_{p-k_1-k_2} (p-1) \\ &\equiv \frac{1}{k_2} \binom{p-k_2}{k_1} B_{p-k_1-k_2} \\ &\equiv \frac{(-1)^{k_1}}{k_1 + k_2} \binom{k_1 + k_2}{k_1} B_{p-k_1-k_2} \pmod{p}. \end{aligned}$$

$\square$

**Example 4.4.** *For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \geq 1$ , we have*

$$\begin{aligned} \zeta_{\mathcal{A}}(k_1, \dots, k_r) &= (-1)^{k_1 + \dots + k_r} \zeta_{\mathcal{A}}(k_r, \dots, k_1), \\ \zeta_{\mathcal{A}}^*(k_1, \dots, k_r) &= (-1)^{k_1 + \dots + k_r} \zeta_{\mathcal{A}}^*(k_r, \dots, k_1). \end{aligned}$$

*Proof.* Replacing  $m_i$  by  $p - m_i$  in Definition 4.1, the following holds.

$$\begin{aligned} H_p(k_1, \dots, k_r) &= \sum_{p > m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \\ &= \sum_{p > m_r > \dots > m_1 > 0} \frac{1}{(p - m_r)^{k_r} \dots (p - m_1)^{k_1}} \\ &\equiv \sum_{p > m_r > \dots > m_1 > 0} \frac{(-1)^{k_1 + \dots + k_r}}{m_r^{k_r} \dots m_1^{k_1}} \pmod{p}. \end{aligned}$$

For FMZSVs, the required formula is proved in the same manner.  $\square$



## 4.2 Vector space spanned by FMZVs

We introduce the following vector space spanned by finite multiple zeta values:

**Definition 4.5.**

$$\begin{aligned}\mathcal{Z}_{0,\mathcal{A}} &= \mathbb{Q}, \\ \mathcal{Z}_{k,\mathcal{A}} &= \sum_{\text{weight}=k} \mathbb{Q} \cdot \zeta_{\mathcal{A}}(k_1, \dots, k_r) \quad (k \geq 1), \\ \mathcal{Z}_{\mathcal{A}} &= \sum_{k=0}^{\infty} \mathcal{Z}_{k,\mathcal{A}}.\end{aligned}$$

For example, we easily find that  $\mathcal{Z}_{1,\mathcal{A}} = \{0\}$  by Example 4.2. Next, FMZVs with weight 2 are  $\zeta_{\mathcal{A}}(2)$  and  $\zeta_{\mathcal{A}}(1, 1)$ .  $\zeta_{\mathcal{A}}(2) = (0)_p$  and  $\zeta_{\mathcal{A}}(1, 1) = (-B_{p-2})_p$  hold by Example 4.2 and Example 4.3 respectively. Moreover we find  $B_{p-2} = 0$  unless  $p = 2$ . Hence we conclude  $\mathcal{Z}_{2,\mathcal{A}} = \{0\}$ . For  $k = 3$ ,  $\zeta_{\mathcal{A}}(3) = \zeta_{\mathcal{A}}(1, 1, 1) = (0)_p$  and  $\zeta_{\mathcal{A}}(2, 1) = -\zeta_{\mathcal{A}}(1, 2)$  are obtained by Example 4.2 and Example 4.4, which implies  $\mathcal{Z}_{3,\mathcal{A}}$  is generated by one element.

It is known that if we add the condition  $k_1 > 1$  to the definition of  $\mathcal{Z}_{k,\mathcal{A}}$ , the space remains same, and that if we replace FMZVs by FMZSVs in the definition, the resulting space is same.

D. Zagier conjectures the dimension of the vector space  $\mathcal{Z}_{k,\mathcal{A}}$ .

**Conjecture 4.6.** *We have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_{k,\mathcal{A}} = d'_k,$$

where  $d'_k$  is the non-negative integer satisfying the following recursion:

$$d'_k = d'_{k-2} + d'_{k-3} \quad (k \geq 3), \quad d'_0 = 1, \quad d'_1 = 0, \quad d'_2 = 0.$$

The vector space  $\mathcal{Z}_{\mathcal{A}}$  becomes  $\mathbb{Q}$ -subalgebra of  $\mathcal{A}$  since  $\mathcal{Z}_{\mathcal{A}}$  has also the harmonic product as in the classical case. In contrast, “shuffle product” in  $\mathcal{Z}_{\mathcal{A}}$  has not been found.

## 4.3 Conjectures on FMZ(S)Vs

We introduce conjectures on FMZ(S)Vs in this section. To state conjectures, we define the height for  $(k_1, \dots, k_r)$  and the dual index in Hoffman’s sense.

For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \in \mathbb{N}$ , the height of  $(k_1, \dots, k_r)$  is given by the number of components greater than 1.

For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \geq 1$  and with the weight  $k$ , we define  $S((k_1, \dots, k_r))$  to be partial sum sequence of  $(k_1, \dots, k_r)$ :

$$S((k_1, \dots, k_r)) = \{k_1, k_1 + k_2, \dots, k_1 + \dots + k_{r-1}\}$$

as a subset of  $\{1, 2, \dots, k-1\}$ . Obviously, the map  $S$  is a one-to-one correspondence. Then  $(k'_1, \dots, k'_l)$  is said to be the dual index of  $(k_1, \dots, k_r)$  in Hoffman’s sense when

$$(k'_1, \dots, k'_l) = S^{-1}(\{1, 2, \dots, k-1\} \setminus S((k_1, \dots, k_r))),$$

and we denote the dual index of  $(k_1, \dots, k_r)$  by  $(k_1, \dots, k_r)^*$ . It is easy to see that Hoffman's dual operation is an involution. We note that  $k_1 > 1$  if and only if  $k'_1 = 1$  in Hoffman's dual operation.

**Conjecture 4.7** (M. Kaneko [18]). *The space spanned by FMZVs of even weight  $k$  and depth 3 is of dimension  $k/2 - 2 - \dim S_k$ , where  $S_k$  is the space of cusp forms of weight  $k$  on the modular group  $SL_2(\mathbb{Z})$*

**Conjecture 4.8** (M. Kaneko [18]). *Let  $(k_1, \dots, k_r)^* = (k'_1, \dots, k'_r)$  and  $j$  be a non-negative integer. Then we have*

$$\sum_{\substack{e_1 + \dots + e_r = j \\ e_i \geq 0}} \zeta_{\mathcal{A}}(k_1 + e_1, \dots, k_r + e_r) = \sum_{\substack{e'_1 + \dots + e'_r = j \\ e'_i \geq 0}} \zeta_{\mathcal{A}}((k'_1 + e'_1, \dots, k'_r + e'_r)^*).$$

**Conjecture 4.9** (M. Kaneko [18]). *Let  $I(k, s)$  be the set of multi-indices of weight  $k$  and height  $s$  with  $k_1 > 1$ . Then we have*

$$\sum_{\mathbf{k} \in I(k, s)} \zeta_{\mathcal{A}}^*(k_1, \dots, k_r) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \left( \frac{B_{p-k}}{k} \right)_p.$$

Conjecture 4.8 and Conjecture 4.9 are analogues of known relations in MZ(S)V<sub>s</sub>. We end this section by stating the following results proved by S. Saito and N. Wakabayashi which was conjectured by M. Kaneko. These are analogues of sum formulas on MZ(S)V<sub>s</sub>.

**Theorem 4.10** (S. Saito, N. Wakabayashi [26]). *For integers  $k \geq 2$  and  $r \geq 1$ , we have*

$$\begin{aligned} \sum_{\substack{k_1 + \dots + k_r = k \\ k_1 \geq 2}} \zeta_{\mathcal{A}}(k_1, \dots, k_r) &= \left( 1 + (-1)^r \binom{k-1}{r-1} \right) \left( \frac{B_{p-k}}{k} \right)_p, \\ \sum_{\substack{k_1 + \dots + k_r = k \\ k_1 \geq 2}} \zeta_{\mathcal{A}}^*(k_1, \dots, k_r) &= \left( (-1)^r + \binom{k-1}{r-1} \right) \left( \frac{B_{p-k}}{k} \right)_p. \end{aligned}$$

## 5 Multi-Poly-Bernoulli Numbers

In this section, we define two kinds of multi-poly-Bernoulli numbers  $B_n^{(k_1, \dots, k_r)}$ ,  $C_n^{(k_1, \dots, k_r)}$  and multi-poly-Bernoulli-star numbers  $B_{n, \star}^{(k_1, \dots, k_r)}$ ,  $C_{n, \star}^{(k_1, \dots, k_r)}$  which generalize the Bernoulli numbers and poly-Bernoulli numbers. We study fundamental properties for them and discuss a connection to the finite multiple zeta values. As a result, we obtain alternative proof of some relation formulas for the finite multiple zeta values.

### 5.1 Definition of multi-poly-Bernoulli numbers

**Definition 5.1.** For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \in \mathbb{Z}$ , we define two kinds of multi-poly-Bernoulli numbers  $B_n^{(k_1, \dots, k_r)}$ ,  $C_n^{(k_1, \dots, k_r)}$  by the following generating series:

$$\begin{aligned} \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{1 - e^{-t}} &= \sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^n}{n!}, \\ \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} &= \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)} \frac{t^n}{n!}. \end{aligned} \quad (5.1)$$

When  $r = 1$ , these numbers are poly-Bernoulli numbers studied in [2], [16]. Furthermore, when  $r = 1$  and  $k_1 = 1$ , both numbers are classical Bernoulli numbers since  $Li_1(1 - e^{-t}) = t$ . We note that  $B_n^{(1)} = C_n^{(1)}$  ( $n \neq 1$ ) with  $B_1^{(1)} = 1/2$  and  $C_1^{(1)} = -1/2$ . We also use weight and depth in the same manner.

We show an easy evaluation of multi-poly-Bernoulli numbers.

**Example 5.2.**

$$B_n^{\overbrace{(1, \dots, 1)}^r} = \frac{1}{n+1} \binom{n+1}{r} B_{n-r+1}^{(1)}, \quad C_n^{\overbrace{(1, \dots, 1)}^r} = \frac{1}{n+1} \binom{n+1}{r} C_{n-r+1}^{(1)}.$$

*Proof.* Use the identity  $Li_{\underbrace{1, \dots, 1}_r}(z) = (-\log(1-z))^r/r!$  to obtain

$$\sum_{n=0}^{\infty} B_n^{\overbrace{(1, \dots, 1)}^r} \frac{t^n}{n!} = \frac{t^{r-1}}{r!} \frac{t}{1 - e^{-t}} = \frac{t^{r-1}}{r!} \sum_{n=0}^{\infty} B_n^{(1)} \frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty} C_n^{\overbrace{(1, \dots, 1)}^r} \frac{t^n}{n!} = \frac{t^{r-1}}{r!} \frac{t}{e^t - 1} = \frac{t^{r-1}}{r!} \sum_{n=0}^{\infty} C_n^{(1)} \frac{t^n}{n!}.$$

The proposition follows. □

## 5.2 Fundamental properties

In this section, we present fundamental properties for multi-poly-Bernoulli numbers. We first give the recursion formulas for multi-poly-Bernoulli numbers.

**Proposition 5.3.** *For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \in \mathbb{Z}$  and  $n \geq 0$ , we have*

$$B_n^{(k_1, \dots, k_r)} = \frac{1}{n+1} \left( B_n^{(k_1-1, k_2, \dots, k_r)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1, \dots, k_r)} \right)$$

and

$$C_n^{(k_1, \dots, k_r)} = \frac{(-1)^n}{n+1} \left( \sum_{m=0}^n (-1)^m \binom{n}{m} C_m^{(k_1-1, k_2, \dots, k_r)} - \sum_{m=1}^{n-1} (-1)^m \binom{n}{m-1} C_m^{(k_1, \dots, k_r)} \right),$$

where an empty sum is understood to be 0.

*Proof.* For  $B_n^{(k_1, \dots, k_r)}$ , we multiply the defining equation (5.1) by  $1 - e^{-t}$  and differentiate with respect to  $t$  to obtain

$$\begin{aligned} & \frac{e^{-t}}{1 - e^{-t}} Li_{k_1-1, k_2, \dots, k_r}(1 - e^{-t}) \\ &= e^{-t} \sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^n}{n!} + (1 - e^{-t}) \sum_{n=1}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^{n-1}}{(n-1)!}, \end{aligned}$$

and from this we have

$$\sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(k_1-1, k_2, \dots, k_r)} \frac{t^n}{n!} - (e^t - 1) \sum_{n=1}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^{n-1}}{(n-1)!}.$$

Comparing the coefficients of  $t^n/n!$  on both sides, we obtain the desired relation for  $B_n^{(k_1, \dots, k_r)}$ . The relation for  $C_n^{(k_1, \dots, k_r)}$  is obtained similarly.  $\square$

Next, we give the explicit formulas for multi-poly-Bernoulli numbers as finite sums involving Stirling numbers of the second kind. Before stating them, we recall the Stirling numbers of the first kind  $\begin{bmatrix} n \\ m \end{bmatrix}$  and the second kind  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ , which are the integers uniquely determined by the following recursions for all integers  $m, n$  respectively (see [9, 20]):

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1, & \begin{bmatrix} n \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ m \end{bmatrix} = \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ m \end{matrix} \right\} = 0 \quad (n, m \neq 0), \\ \begin{bmatrix} n+1 \\ m \end{bmatrix} &= \begin{bmatrix} n \\ m-1 \end{bmatrix} + n \begin{bmatrix} n \\ m \end{bmatrix}, & \left\{ \begin{matrix} n+1 \\ m \end{matrix} \right\} &= \left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\} + m \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \quad (\forall n, m \in \mathbb{Z}). \end{aligned}$$

**Proposition 5.4.** *For any  $r \geq 1$ ,  $k_1, k_2, \dots, k_r \in \mathbb{Z}$ , and  $n \geq 0$ , we have*

$$B_n^{(k_1, \dots, k_r)} = (-1)^n \sum_{n+1 \geq m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1-1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n \\ m_1-1 \end{matrix} \right\}$$

and

$$C_n^{(k_1, \dots, k_r)} = (-1)^n \sum_{n+1 \geq m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1 - 1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n+1 \\ m_1 \end{matrix} \right\}.$$

*Proof.* Using the well-known generating series (cf. [9, §7.4])

$$\frac{(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{t^n}{n!} \quad (m \geq 0), \quad (5.2)$$

we have

$$\begin{aligned} & \sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^n}{n!} \\ &= \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{1 - e^{-t}} \\ &= \sum_{m_1 > m_2 > \dots > m_r > 0} \frac{(1 - e^{-t})^{m_1-1}}{m_1^{k_1} \dots m_r^{k_r}} \\ &= \sum_{m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1 - 1)!}{m_1^{k_1} \dots m_r^{k_r}} \sum_{n=m_1-1}^{\infty} \left\{ \begin{matrix} n \\ m_1 - 1 \end{matrix} \right\} \frac{(-t)^n}{n!} \\ &= \sum_{n=r-1}^{\infty} (-1)^n \sum_{n+1 \geq m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1 - 1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n \\ m_1 - 1 \end{matrix} \right\} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of  $t^n$ , we obtain the formula for  $B_n^{(k_1, \dots, k_r)}$ .

To obtain the formula for  $C_n^{(k_1, \dots, k_r)}$ , we proceed similarly by using the equation

$$\frac{e^t (e^t - 1)^{m-1}}{(m-1)!} = \sum_{n=m-1}^{\infty} \left\{ \begin{matrix} n+1 \\ m \end{matrix} \right\} \frac{t^n}{n!} \quad (m \geq 1),$$

which is derived from (5.2) by differentiation:

$$\begin{aligned} & \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)} \frac{t^n}{n!} \\ &= \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} \\ &= \sum_{m_1 > m_2 > \dots > m_r > 0} \frac{e^{-t} (1 - e^{-t})^{m_1-1}}{m_1^{k_1} \dots m_r^{k_r}} \\ &= \sum_{m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1 - 1)!}{m_1^{k_1} \dots m_r^{k_r}} \sum_{n=m_1-1}^{\infty} \left\{ \begin{matrix} n+1 \\ m_1 \end{matrix} \right\} \frac{(-t)^n}{n!} \\ &= \sum_{n=r-1}^{\infty} (-1)^n \sum_{n+1 \geq m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1 - 1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n+1 \\ m_1 \end{matrix} \right\} \frac{t^n}{n!}. \end{aligned}$$

□

We give some simple relations among the multi-poly-Bernoulli numbers.

**Proposition 5.5.** *For any multi-index  $(k_1, \dots, k_r)$ , we have*

$$C_n^{(k_1, \dots, k_r)} = B_n^{(k_1, \dots, k_r)} - C_{n-1}^{(k_1-1, k_2, \dots, k_r)}.$$

*Proof.* By the explicit formula for  $C_n^{(k_1, \dots, k_r)}$ , we obtain

$$\begin{aligned} C_n^{(k_1, \dots, k_r)} &= \sum_{n+1 > m_1 > \dots > m_r > 0} \frac{(-1)^{m_1+n-1} (m_1-1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n+1 \\ m_1 \end{matrix} \right\} \\ &= \sum_{n+1 > m_1 > \dots > m_r > 0} \frac{(-1)^{m_1+n-1} (m_1-1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n \\ m_1-1 \end{matrix} \right\} \\ &+ \sum_{n+1 > m_1 > \dots > m_r > 0} \frac{(-1)^{m_1+n-1} (m_1-1)!}{m_1^{k_1-1} m_2^{k_2} \dots m_r^{k_r}} \left\{ \begin{matrix} n \\ m_1 \end{matrix} \right\} \\ &= B_n^{(k_1, \dots, k_r)} - C_{n-1}^{(k_1-1, k_2, \dots, k_r)}. \end{aligned}$$

The second equality above is by the recursion for the Stirling numbers of the second kind. □

**Proposition 5.6.** *For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \in \mathbb{Z}$ , we have*

$$B_n^{(k_1, \dots, k_r)} = \sum_{j=0}^n \binom{n}{j} C_j^{(k_1, \dots, k_r)}$$

and

$$C_n^{(k_1, \dots, k_r)} = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_j^{(k_1, \dots, k_r)}.$$

*Proof.* The generating functions of  $B_n^{(k_1, \dots, k_r)}$ ,  $C_n^{(k_1, \dots, k_r)}$  differ by the factor  $e^t$ , and the above identities follow immediately. □

### 5.3 Multi-poly-Bernoulli numbers with negative indices

By the explicit formulas, we immediately find  $B_n^{(-k_1, \dots, -k_r)}$ ,  $C_n^{(-k_1, \dots, -k_r)}$  ( $k_i \geq 0$ ) to be integers. We provide the properties for multi-poly-Bernoulli numbers with negative indices and explicit expressions in special cases.

**Proposition 5.7** (K. Imatomi, M. Kaneko and E. Takeda [14]). *We have the following generating series identities for multi-poly-Bernoulli numbers with non-positive upper indices:*

1)

$$\begin{aligned} & \sum_{k_1, \dots, k_r=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k_1, \dots, -k_r)} \frac{t^n}{n!} \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!} \\ &= \frac{(1 - e^{-t})^{r-1}}{(e^{-x_1} + e^{-t} - 1)(e^{-x_1-x_2} + e^{-t} - 1) \dots (e^{-x_1-\dots-x_r} + e^{-t} - 1)}, \end{aligned}$$

2)

$$\begin{aligned} & \sum_{k_1, \dots, k_r=0}^{\infty} \sum_{n=0}^{\infty} C_n^{(-k_1, \dots, -k_r)} \frac{t^n}{n!} \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!} \\ &= \frac{e^{-t}(1 - e^{-t})^{r-1}}{(e^{-x_1} + e^{-t} - 1)(e^{-x_1-x_2} + e^{-t} - 1) \dots (e^{-x_1-\dots-x_r} + e^{-t} - 1)}. \end{aligned}$$

*Proof.* Use the defining identity

$$\sum_{n=0}^{\infty} B_n^{(-k_1, \dots, -k_r)} \frac{t^n}{n!} = (1 - e^{-t})^{-1} \sum_{m_1 > \dots > m_r > 0} m_1^{k_1} \dots m_r^{k_r} (1 - e^{-t})^{m_1}$$

to obtain

$$\begin{aligned} & \sum_{k_1, \dots, k_r=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k_1, \dots, -k_r)} \frac{t^n}{n!} \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!} \\ &= (1 - e^{-t})^{-1} \sum_{m_1 > \dots > m_r > 0} (1 - e^{-t})^{m_1} e^{m_1 x_1 + \dots + m_r x_r} \\ &= (1 - e^{-t})^{-1} \sum_{n_1, \dots, n_r=1}^{\infty} (1 - e^{-t})^{n_1 + \dots + n_r} e^{(n_1 + \dots + n_r)x_1 + (n_2 + \dots + n_r)x_2 + \dots + n_r x_r} \\ &= (1 - e^{-t})^{-1} \sum_{n_1, \dots, n_r=1}^{\infty} ((1 - e^{-t})e^{x_1})^{n_1} ((1 - e^{-t})e^{x_1+x_2})^{n_2} \dots ((1 - e^{-t})e^{x_1+\dots+x_r})^{n_r} \\ &= (1 - e^{-t})^{r-1} \frac{e^{x_1}}{1 - ((1 - e^{-t})e^{x_1})} \frac{e^{x_1+x_2}}{1 - ((1 - e^{-t})e^{x_1+x_2})} \dots \frac{e^{x_1+\dots+x_r}}{1 - ((1 - e^{-t})e^{x_1+\dots+x_r})} \\ &= \frac{(1 - e^{-t})^{r-1}}{(e^{-x_1} + e^{-t} - 1)(e^{-x_1-x_2} + e^{-t} - 1) \dots (e^{-x_1-\dots-x_r} + e^{-t} - 1)}. \end{aligned}$$

The identity for  $C_n^{(-k_1, \dots, -k_r)}$  readily follows from this because the defining generating series differ only by the factor  $e^{-t}$ .  $\square$

**Corollary 5.8.** For any integers  $k_1, \dots, k_r \geq 0$  and  $n \geq 0$ , the multi-poly-Bernoulli numbers  $B_n^{(-k_1, \dots, -k_r)}$  and  $C_n^{(-k_1, \dots, -k_r)}$  are positive integers.

*Proof.* The right-hand sides of 1) and 2) of the proposition can be rewritten as

$$e^t(e^t - 1)^{r-1} \cdot \frac{e^{x_1}}{1 - (e^t - 1)(e^{x_1} - 1)} \cdot \frac{e^{x_1+x_2}}{1 - (e^t - 1)(e^{x_1+x_2} - 1)} \cdots \frac{e^{x_1+\cdots+x_r}}{1 - (e^t - 1)(e^{x_1+\cdots+x_r} - 1)}$$

and

$$(e^t - 1)^{r-1} \cdot \frac{e^{x_1}}{1 - (e^t - 1)(e^{x_1} - 1)} \cdot \frac{e^{x_1+x_2}}{1 - (e^t - 1)(e^{x_1+x_2} - 1)} \cdots \frac{e^{x_1+\cdots+x_r}}{1 - (e^t - 1)(e^{x_1+\cdots+x_r} - 1)}$$

respectively, from which the positivity of  $B_n^{(-k_1, \dots, -k_r)}$  and  $C_n^{(-k_1, \dots, -k_r)}$  is obvious. That both are integers follows from the explicit formulas in Theorem 5.4.  $\square$

We can give some explicit formulas different from Theorem 5.4 for special indices.

**Proposition 5.9.** *For  $r \geq 1$ ,  $k \geq 0$  and  $n \geq 0$ , we have*

1)

$$B_n^{(-k, \overbrace{0, \dots, 0}^{r-1})} = (-1)^n \sum_{j=0}^n (-1)^j (j+1)^k j! \binom{j}{r-1} \left\{ \begin{matrix} n \\ j \end{matrix} \right\},$$

$$C_n^{(-k, \overbrace{0, \dots, 0}^{r-1})} = (-1)^n \sum_{j=0}^n (-1)^j (j+1)^k j! \binom{j}{r-1} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\}.$$

2)

$$B_n^{\overbrace{(0, \dots, 0, -k)}^{r-1}} = \sum_{j=0}^{\min\{n+1-r, k\}} (r+j-1)! j! \left\{ \begin{matrix} n+1 \\ r+j \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\},$$

$$C_n^{\overbrace{(0, \dots, 0, -k)}^{r-1}} = \sum_{j=0}^{\min\{n+1-r, k\}} (r+j-1)! j! \left\{ \begin{matrix} n \\ r+j-1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\}.$$

*Proof.* Set  $x_2 = \cdots = x_r = 0$  in 1) of Proposition 5.7 to obtain

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k, \overbrace{0, \dots, 0}^{r-1})} \frac{t^n x^k}{n! k!} = \frac{(1 - e^{-t})^{r-1}}{(e^{-x} + e^{-t} - 1)^r}.$$



From this we have

$$\begin{aligned}
\sum_{r=1}^{\infty} \sum_{k,n=0}^{\infty} B_n^{(-k, \overbrace{0, \dots, 0}^{r-1})} \frac{t^n x^k}{n! k!} y^{r-1} &= \frac{1}{(e^{-x} + e^{-t} - 1)} \sum_{r=1}^{\infty} \left( \frac{1 - e^{-t}}{e^{-x} + e^{-t} - 1} \right)^{r-1} y^{r-1} \\
&= \frac{1}{(e^{-x} + e^{-t} - 1)} \frac{1}{1 - \frac{1 - e^{-t}}{e^{-x} + e^{-t} - 1} y} \\
&= \frac{1}{e^{-x} + e^{-t} - 1 - (1 - e^{-t})y} \\
&= \frac{e^x}{1 - e^x(1 + y)(1 - e^{-t})} \\
&= \sum_{j=0}^{\infty} e^{(j+1)x} (1 + y)^j (1 - e^{-t})^j \\
&= \sum_{j=0}^{\infty} e^{(j+1)x} (1 + y)^j (-1)^j j! \sum_{n=j}^{\infty} (-1)^n \begin{Bmatrix} n \\ j \end{Bmatrix} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^n e^{(j+1)x} (1 + y)^j (-1)^{n+j} j! \begin{Bmatrix} n \\ j \end{Bmatrix} \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of  $\frac{t^n x^k}{n! k!} y^{r-1}$  on both sides, we obtain 1) for  $B_n^{(-k, 0, \dots, 0)}$ .

Similarly, we compute

$$\begin{aligned}
\sum_{r=1}^{\infty} \sum_{k,n=0}^{\infty} C_n^{(-k, \overbrace{0, \dots, 0}^{r-1})} \frac{t^n x^k}{n! k!} y^{r-1} &= e^{-t} \cdot \frac{e^x}{1 - e^x(1 + y)(1 - e^{-t})} \\
&= \sum_{j=0}^{\infty} e^{(j+1)x} (1 + y)^j e^{-t} (1 - e^{-t})^j \\
&= \sum_{j=0}^{\infty} e^{(j+1)x} (1 + y)^j (-1)^j j! \sum_{n=j}^{\infty} (-1)^n \begin{Bmatrix} n + 1 \\ j + 1 \end{Bmatrix} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^n e^{(j+1)x} (1 + y)^j (-1)^{n+j} j! \begin{Bmatrix} n + 1 \\ j + 1 \end{Bmatrix} \frac{t^n}{n!}
\end{aligned}$$

and obtain the formula for  $C_n^{(-k, 0, \dots, 0)}$ .

For 2), we set  $x_1 = \dots = x_{r-1} = 0$  in the formulas in Proposition 5.7 and have

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{\overbrace{(0, \dots, 0, -k)}^{r-1}} \frac{t^n x^k}{n! k!} &= \frac{(e^t - 1)^{r-1}}{e^{-x} + e^{-t} - 1} \\
&= (e^t - 1)^{r-1} \cdot \frac{e^{t+x}}{1 - (e^t - 1)(e^x - 1)} \\
&= \sum_{j=0}^{\infty} e^t (e^t - 1)^{r+j-1} e^x (e^x - 1)^j \\
&= \sum_{j=0}^{\infty} (r+j-1)! \sum_{n=r+j-1}^{\infty} \begin{Bmatrix} n+1 \\ r+j \end{Bmatrix} \frac{t^n}{n!} \cdot j! \sum_{k=j}^{\infty} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix} \frac{x^k}{k!}
\end{aligned}$$

and

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} C_n^{\overbrace{(0, \dots, 0, -k)}^{r-1}} \frac{t^n x^k}{n! k!} = \sum_{j=0}^{\infty} (r+j-1)! \sum_{n=r+j-1}^{\infty} \begin{Bmatrix} n \\ r+j-1 \end{Bmatrix} \frac{t^n}{n!} \cdots j! \sum_{k=j}^{\infty} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix} \frac{x^k}{k!}.$$

Comparing the coefficients of  $\frac{t^n x^k}{n! k!}$  on both sides and noting that  $\begin{Bmatrix} n \\ m \end{Bmatrix} = 0$  if  $n < m$ , we obtain 2).  $\square$

## 5.4 Multi-poly-Bernoulli-star numbers

We describe multi-poly-Bernoulli-star numbers in this section. Multi-poly-Bernoulli-star numbers are defined in a similar way as multi-poly-Bernoulli numbers:

**Definition 5.10.**

$$\begin{aligned}
\frac{Li_{k_1, \dots, k_r}^*(1 - e^{-t})}{1 - e^{-t}} &= \sum_{n=0}^{\infty} B_{n, \star}^{(k_1, \dots, k_r)} \frac{t^n}{n!}, \\
\frac{Li_{k_1, \dots, k_r}^*(1 - e^{-t})}{e^t - 1} &= \sum_{n=0}^{\infty} C_{n, \star}^{(k_1, \dots, k_r)} \frac{t^n}{n!},
\end{aligned}$$

where  $Li_{k_1, \dots, k_r}^*(z)$  is called non-strict multiple polylogarithm defined by the following series:

$$Li_{k_1, \dots, k_r}^*(z) = \sum_{m_1 \geq \dots \geq m_r \geq 1} \frac{z^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

We omit our proofs of fundamental properties for multi-poly-Bernoulli-star numbers since their proofs are almost identical to the case of multi-poly-Bernoulli numbers.

**Proposition 5.11.** For any multi-index  $(k_1, \dots, k_r)$ , we have the following recursions:

$$B_{n,\star}^{(k_1, \dots, k_r)} = \frac{1}{n+1} \left( B_{n,\star}^{(k_1-1, k_2, \dots, k_r)} - \sum_{j=1}^{n-1} \binom{n}{j-1} B_{j,\star}^{(k_1, \dots, k_r)} \right),$$

$$C_{n,\star}^{(k_1, \dots, k_r)} = \frac{1}{n+1} \left( C_{n,\star}^{(k_1-1, k_2, \dots, k_r)} - \sum_{j=0}^{n-1} \binom{n+1}{j} C_{j,\star}^{(k_1, \dots, k_r)} \right).$$

**Proposition 5.12.** For any multi-index  $(k_1, \dots, k_r)$ ,  $k_i \in \mathbb{Z}$ , we have

$$B_{n,\star}^{(k_1, \dots, k_r)} = \sum_{n+1 \geq m_1 \geq \dots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1} (m_1-1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n \\ m_1-1 \end{matrix} \right\}$$

and

$$C_{n,\star}^{(k_1, \dots, k_r)} = \sum_{n+1 \geq m_1 \geq \dots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1} (m_1-1)!}{m_1^{k_1} \dots m_r^{k_r}} \left\{ \begin{matrix} n+1 \\ m_1 \end{matrix} \right\}.$$

**Proposition 5.13.** For any multi-index  $(k_1, \dots, k_r)$ , we have

$$C_{n,\star}^{(k_1, \dots, k_r)} = B_{n,\star}^{(k_1, \dots, k_r)} - C_{n-1,\star}^{(k_1-1, k_2, \dots, k_r)}.$$

**Proposition 5.14.** For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \in \mathbb{Z}$ , we have

$$B_{n,\star}^{(k_1, \dots, k_r)} = \sum_{j=0}^n \binom{n}{j} C_{j,\star}^{(k_1, \dots, k_r)}$$

and

$$C_{n,\star}^{(k_1, \dots, k_r)} = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_{j,\star}^{(k_1, \dots, k_r)}.$$

We present the sum formulas and duality relation for multi-poly-Bernoulli-star numbers. These results hold not for multi-poly-Bernoulli numbers but for multi-poly-Bernoulli-star numbers.

The former one is the following:

**Theorem 5.15.** We have

$$\sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^r B_{n,\star}^{(k_1, \dots, k_r)} = \frac{(-1)^k}{k} \binom{n}{k-1} B_{n-k+1,\star}^{(1)}$$

and

$$\sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^r C_{n,\star}^{(k_1, \dots, k_r)} = \frac{(-1)^k}{k} \binom{n}{k-1} C_{n-k+1,\star}^{(1)}. \quad (5.3)$$

*Proof.* We consider the generating functions of both sides by multiplying  $t^n/n!$  and taking the summation on  $n$ :

$$\begin{aligned}
(\text{LHS}) &= \sum_{n=0}^{\infty} \sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^r B_{n,\star}^{(k_1, \dots, k_r)} \frac{t^n}{n!} \\
&= \sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^r \frac{Li_{k_1, \dots, k_r}^{\star}(1 - e^{-t})}{1 - e^{-t}}, \\
(\text{RHS}) &= \frac{(-1)^k}{k} \sum_{n=0}^{\infty} \binom{n}{k-1} B_{n-k+1,\star}^{(1)} \frac{t^n}{n!} \\
&= \frac{(-1)^k}{k!} \sum_{n=0}^{\infty} B_{n-k+1,\star}^{(1)} \frac{t^n}{(n-k+1)!} \\
&= \frac{(-1)^k}{k!} \sum_{n=0}^{\infty} B_{n,\star}^{(1)} \frac{t^{n+k-1}}{n!} \\
&= \frac{(-1)^k}{k!} \frac{t^k}{1 - e^{-t}}.
\end{aligned}$$

Since both sides have the same denominator, it suffices to prove the following identity:

$$\sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^r Li_{k_1, \dots, k_r}^{\star}(1 - e^{-t}) = \frac{(-1)^k}{k!} t^k. \quad (5.4)$$

This equality is proved by the induction on the weight. When  $k = 1$ , the left-hand side is  $-Li_1^{\star}(1 - e^{-t}) = -t$  and is equal to the right-hand side. Next we assume the identity holds when the weight is  $k$ . Then by differentiating the left-hand side of the identity of weight  $k + 1$ , we obtain

$$\begin{aligned}
&\frac{d}{dt} \left( \sum_{\substack{k_1+\dots+k_r=k+1 \\ 1 \leq r \leq k+1, k_i \geq 1}} (-1)^r Li_{k_1, \dots, k_r}^{\star}(1 - e^{-t}) \right) \\
&= \sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^r \left( \frac{e^{-t}}{1 - e^{-t}} - \frac{1}{1 - e^{-t}} \right) Li_{k_1, \dots, k_r}^{\star}(1 - e^{-t}) \\
&= \sum_{\substack{k_1+\dots+k_r=k \\ 1 \leq r \leq k, k_i \geq 1}} (-1)^{r+1} Li_{k_1, \dots, k_r}^{\star}(1 - e^{-t}) \\
&= \frac{(-1)^{k+1}}{k!} t^k.
\end{aligned}$$

We used the induction hypothesis in the last equality. Therefore we have

$$\sum_{\substack{k_1+\dots+k_r=k+1 \\ 1\leq r\leq k+1, k_i\geq 1}} (-1)^r Li_{k_1, \dots, k_r}^*(1 - e^{-t}) = \frac{(-1)^{k+1}}{(k+1)!} t^{k+1} + C$$

with some constant  $C$ , which we find is 0 by putting  $t = 0$ . The sum formula for  $C_{n, \star}$  can also be obtained from (5.4) since the generating function of  $C_{n, \star}$  differs only by the factor  $e^t$  to the one of  $B_{n, \star}$ .  $\square$

Next we present the duality relation for the multi-poly-Bernoulli-star numbers.

**Theorem 5.16** (K. Imatomi [13]). *For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \geq 1$  ( $1 \leq i \leq r$ ), we have*

$$C_{n, \star}^{(k_1, \dots, k_r)} = (-1)^n B_{n, \star}^{(k'_1, \dots, k'_i)},$$

where  $(k'_1, \dots, k'_i)$  is the dual index of  $(k_1, \dots, k_r)$  in Hoffman's sense.

*Proof.* We also consider the generating functions of both sides:

$$\begin{aligned} \text{(LHS)} &= \sum_{n=0}^{\infty} C_{n, \star}^{(k_1, \dots, k_r)} \frac{t^n}{n!} = \frac{Li_{k_1, \dots, k_r}^*(1 - e^{-t})}{e^t - 1}, \\ \text{(RHS)} &= \sum_{n=0}^{\infty} (-1)^n B_{n, \star}^{(k'_1, \dots, k'_i)} \frac{t^n}{n!} = \frac{Li_{k'_1, \dots, k'_i}^*(1 - e^t)}{1 - e^t}. \end{aligned}$$

Hence we have to show the following identity:

$$Li_{k_1, \dots, k_r}^*(1 - e^{-t}) + Li_{k'_1, \dots, k'_i}^*(1 - e^t) = 0. \quad (5.5)$$

This identity also follows from the induction on the weight. First, it is trivial in the case  $k = 1$ . We assume the above identity holds when the weight is  $k$ . Since  $k_1 = 1$  is equivalent to  $k'_1 \neq 1$ , we may assume  $k_1 = 1$  by the symmetry of the identity. Then when the weight is  $k + 1$ , the derivative of the left-hand side yields

$$\begin{aligned} & \frac{d}{dt} \left( Li_{1, k_2, \dots, k_r}^*(1 - e^{-t}) + Li_{k'_1, \dots, k'_i}^*(1 - e^t) \right) \\ &= \frac{1}{1 - e^{-t}} Li_{k_2, \dots, k_r}^*(1 - e^{-t}) + \frac{-e^t}{1 - e^t} Li_{k'_1 - 1, k'_2, \dots, k'_i}^*(1 - e^t) \\ &= \frac{1}{1 - e^{-t}} \left( Li_{k_2, \dots, k_r}^*(1 - e^{-t}) + Li_{k'_1 - 1, k'_2, \dots, k'_i}^*(1 - e^t) \right) \\ &= 0. \end{aligned}$$

Therefore we obtain

$$Li_{1, k_2, \dots, k_r}^*(1 - e^{-t}) + Li_{k'_1, \dots, k'_i}^*(1 - e^t) = C$$

with some constant  $C$ , and by putting  $t = 0$ , we conclude  $C = 0$ .  $\square$

## 5.5 Connection to finite multiple zeta(-star) values

First, we introduce the following congruence formula between multi-poly-Bernoulli(-star) numbers and finite multiple zeta(-star) values.

**Proposition 5.17.** *We have*

$$\begin{aligned} H_p(k_1, \dots, k_r) &\equiv -C_{p-2}^{(k_1-1, k_2, \dots, k_r)} \pmod{p}, \\ H_p^*(k_1, \dots, k_r) &\equiv -C_{p-2, \star}^{(k_1-1, k_2, \dots, k_r)} \pmod{p}. \end{aligned}$$

More generally, for (strict) multiple harmonic sums, we have for any  $r \geq 1$ ,  $j \geq 0$ , and  $k_i \in \mathbb{Z}$

$$H_p(\underbrace{1, \dots, 1}_j, k_1, \dots, k_r) \equiv -C_{p-j-2}^{(k_1-1, k_2, \dots, k_r)} \pmod{p}.$$

*Proof.* By the explicit formula in Theorem 5.4, we have (we may assume  $p$  is odd)

$$\begin{aligned} C_{p-2}^{(k_1-1, k_2, \dots, k_r)} &= - \sum_{p-1 \geq m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1-1} (m_1-1)! \left\{ \begin{matrix} p-1 \\ m_1 \end{matrix} \right\}}{m_1^{k_1-1} m_2^{k_2} \dots m_r^{k_r}} \\ &= \sum_{p-1 \geq m_1 > m_2 > \dots > m_r > 0} \frac{(-1)^{m_1} m_1! \left\{ \begin{matrix} p-1 \\ m_1 \end{matrix} \right\}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}}. \end{aligned}$$

By the well-known formula (cf. [9, §6.1])

$$(-1)^{m_1} m_1! \left\{ \begin{matrix} p-1 \\ m_1 \end{matrix} \right\} = \sum_{l=0}^{m_1} (-1)^l \binom{m_1}{l} l^{p-1},$$

we see

$$(-1)^{m_1} m_1! \left\{ \begin{matrix} p-1 \\ m_1 \end{matrix} \right\} \equiv \sum_{l=1}^{m_1} (-1)^l \binom{m_1}{l} \pmod{p},$$

the sum on the right being equal to  $(1-1)^{m_1} - 1 = -1$ . This proves that

$$C_{p-2}^{(k_1-1, k_2, \dots, k_r)} \equiv -H_p(k_1, \dots, k_r) \pmod{p}.$$

Congruence formula for star version is proved in the same manner.

For the second identity, we proceed as follows. First note

$$\begin{aligned} &\sum_{p-1 \geq i_1 > \dots > i_j > m_1 > \dots > m_r \geq 1} \frac{1}{i_1 \dots i_j m_1^{k_1} \dots m_r^{k_r}} \\ &= \sum_{p-j > m_1 > \dots > m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \sum_{p-1 \geq i_1 > \dots > i_j > m_1} \frac{1}{i_1 \dots i_j}. \end{aligned}$$

By changing  $i_l \rightarrow p - i_l$ , we see that

$$\sum_{p-1 \geq i_1 > \dots > i_j > m_1} \frac{1}{i_1 \dots i_j} \equiv (-1)^j \sum_{p-m_1 > i_j > \dots > i_1 \geq 1} \frac{1}{i_1 \dots i_j} \pmod{p}.$$

Using the formula

$$\sum_{p-m_1 > i_j > \dots > i_1 \geq 1} \frac{1}{i_1 \cdots i_j} = \frac{1}{(p-m_1-1)!} \left[ \begin{matrix} p-m_1 \\ j+1 \end{matrix} \right]$$

and the congruences  $\left[ \begin{matrix} n \\ m \end{matrix} \right] \equiv \left\{ \begin{matrix} p-m \\ p-n \end{matrix} \right\} \pmod{p}$  ( $1 \leq m \leq n \leq p-1$ ) (see [11, §5]) and  $\frac{1}{(p-m_1-1)!} \equiv (-1)^{m_1+1} m_1! \pmod{p}$ , we obtain (for odd  $p$ )

$$\begin{aligned} \zeta_{\mathcal{A}}(\underbrace{1, \dots, 1}_j, k_1, \dots, k_r)_{(p)} &= \sum_{p-j > m_1 > \dots > m_r \geq 1} \frac{(-1)^{j+m_1+1} m_1! \left\{ \begin{matrix} p-j-1 \\ m_1 \end{matrix} \right\}}{m_1^{k_1} \cdots m_r^{k_r}} \pmod{p} \\ &= -C_{p-j-2}^{(k_1-1, k_2, \dots, k_r)} \pmod{p}. \end{aligned}$$

This concludes the proof of the theorem.  $\square$

Combining the proposition with Example 5.2, we see that

$$\begin{aligned} \zeta_{\mathcal{A}}(2, \overbrace{1, \dots, 1}^{k-2}) &= (-C_{p-2}^{\overbrace{(1, \dots, 1)}^{k-1}} \pmod{p})_p \\ &= \left( -\frac{1}{p-1} \binom{p-1}{k-1} B_{p-k} \pmod{p} \right)_p \\ &= (B_{p-k} \pmod{p})_p. \end{aligned}$$

(We used  $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod{p}$  and  $(-1)^{p-k} B_{p-k} = B_{p-k}$  for large enough  $p$ .) As is discussed in [18], the element  $(B_{p-k} \pmod{p})_p$  on the right is regarded as an analogue of  $k\zeta(k)$ .

**Corollary 5.18** (M. Hoffman [11]). *For any multi-index  $(k_1, \dots, k_r)$  with  $k_i \geq 1$  ( $1 \leq i \leq r$ ), let  $(k'_1, \dots, k'_l)$  be the dual index for  $(k_1, \dots, k_r)$  in Hoffman's sense. Then we have*

$$\zeta_{\mathcal{A}}^*(k_1, \dots, k_r) = -\zeta_{\mathcal{A}}^*(k'_1, \dots, k'_l).$$

*Proof.* It is sufficient to prove the case  $k_1 = 1$ . By (5.6) and the duality relation for the multi-poly-Bernoulli-star numbers, we obtain

$$\begin{aligned} (\text{LHS}) &= \left( -C_{p-2, \star}^{(0, k_2, \dots, k_r)} \pmod{p} \right)_p, \\ (\text{RHS}) &= \left( C_{p-2, \star}^{(k'_1-1, k'_2, \dots, k'_l)} \pmod{p} \right)_p \\ &= \left( (-1)^p B_{p-2, \star}^{(k_2, \dots, k_r)} \pmod{p} \right)_p. \end{aligned}$$

Hence we complete the proof if we prove  $C_{n, \star}^{(0, k_2, \dots, k_r)} = B_{n, \star}^{(k_2, \dots, k_r)}$  for all  $n$ . We consider

the generating functions of these numbers:

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,\star}^{(0,k_2,\dots,k_r)} \frac{t^n}{n!} &= \frac{Li_{0,k_2,\dots,k_r}^{\star}(1-e^{-t})}{e^t-1} \\
&= \frac{1}{e^t-1} \sum_{m_2 \geq \dots \geq m_r \geq 1} \frac{1}{m_2^{k_2} \dots m_r^{k_r}} \sum_{m_1=m_2}^{\infty} (1-e^{-t})^{m_1} \\
&= \frac{1}{e^{-t}(e^t-1)} Li_{k_2,\dots,k_r}^{\star}(1-e^{-t}) \\
&= \sum_{n=0}^{\infty} B_{n,\star}^{(k_2,\dots,k_r)} \frac{t^n}{n!}.
\end{aligned}$$

From this we have

$$-C_{p-2,\star}^{(0,k_2,\dots,k_r)} = (-1)^p B_{p-2,\star}^{(k_2,\dots,k_r)}$$

for any odd prime  $p$ . □

The following corollary is a weaker version of Theorem 4.10.

**Corollary 5.19.** *We have*

$$\sum_{\substack{k_1+\dots+k_r=k \\ r \geq 1, k_1 \geq 2, k_i \geq 1}} (-1)^r \zeta_{\mathcal{A}}^{\star}(k_1, \dots, k_r) = (B_{p-k} \bmod p)_p,$$

where  $B_n$  is the classical Bernoulli number.

*Proof.* Equations (5.3) and (5.6) yield

$$\begin{aligned}
\sum_{\substack{k_1+\dots+k_r=k+1 \\ r \geq 1, k_1 \geq 2, k_i \geq 1}} (-1)^{r+1} \zeta_{\mathcal{A}}^{\star}(k_1, \dots, k_r) &= \left( \frac{(-1)^k}{k} \binom{p-2}{k-1} C_{p-k-1,\star}^{(1)} \bmod p \right)_p \\
&= \left( -C_{p-k-1,\star}^{(1)} \bmod p \right)_p.
\end{aligned}$$

So replacing  $k$  by  $k-1$ , we obtain the desired identity. □



## 6 Multi-Poly-Bernoulli Polynomials

In this section, we give multi-poly-Bernoulli polynomials which generalize the classical Bernoulli polynomials, and generalized Arakawa-Kaneko zeta function whose values at non-positive integers are multi-poly-Bernoulli polynomials .

### 6.1 Definition of multi-poly-Bernoulli polynomials

**Definition 6.1.** For any multi-index  $(k_1, \dots, k_r)$ , multi-poly-Bernoulli(-star) polynomials  $C_n^{(k_1, \dots, k_r)}(x)$ ,  $C_{n, \star}^{(k_1, \dots, k_r)}(x)$  are defined by the following series:

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} &= \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{xt}, \\ \sum_{n=0}^{\infty} C_{n, \star}^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} &= \frac{Li_{k_1, \dots, k_r}^{\star}(1 - e^{-t})}{e^t - 1} e^{xt}. \end{aligned} \quad (6.1)$$

When  $r = 1$  and  $k_1 = 1$ ,  $C_n^{(1)}(x)$  and  $C_{n, \star}^{(1)}(x)$  are the classical Bernoulli polynomials whose generating function is  $te^{xt}/(e^t - 1)$ , and when  $x = 0$ ,  $C_n^{(k_1, \dots, k_r)}(0)$  and  $C_{n, \star}^{(k_1, \dots, k_r)}(0)$  are the multi-poly-Bernoulli(-star) numbers. We need not consider ‘‘B-version’’ of multi-poly-Bernoulli(-star) polynomials since we obtain ‘‘B-version’’ by replacing  $x$  by  $x + 1$  in Definition 6.1.

We present basic results for multi-poly-Bernoulli(-star) polynomials. We first give the explicit expression in terms of multi-poly-Bernoulli(-star) numbers.

**Proposition 6.2.** For any multi-index  $(k_1, \dots, k_r)$  and  $n \geq 0$ , we have

$$\begin{aligned} C_n^{(k_1, \dots, k_r)}(x) &= \sum_{j=0}^n \binom{n}{j} C_{n-j}^{(k_1, \dots, k_r)} x^j, \\ C_{n, \star}^{(k_1, \dots, k_r)}(x) &= \sum_{j=0}^n \binom{n}{j} C_{n-j, \star}^{(k_1, \dots, k_r)} x^j. \end{aligned}$$

*Proof.* By using Definition 6.1, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} &= \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{xt} \\ &= \left( \sum_{i=0}^{\infty} C_i^{(k_1, \dots, k_r)} \frac{t^i}{i!} \right) \left( \sum_{j=0}^{\infty} \frac{(xt)^j}{j!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} C_{n-j}^{(k_1, \dots, k_r)} x^j \frac{t^n}{n!}. \end{aligned}$$

The proof for multi-poly-Bernoulli-star polynomials is identical. □

**Proposition 6.3.** For any multi-index  $(k_1, \dots, k_r)$  and  $n \geq 1$ , we have

$$\begin{aligned}\frac{d}{dx} C_n^{(k_1, \dots, k_r)}(x) &= n C_{n-1}^{(k_1, \dots, k_r)}(x), \\ \frac{d}{dx} C_{n, \star}^{(k_1, \dots, k_r)}(x) &= n C_{n-1, \star}^{(k_1, \dots, k_r)}(x).\end{aligned}$$

*Proof.* We differentiate both hands of (6.1) with respect to  $x$ :

$$\begin{aligned}\frac{d}{dx} \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} &= \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} t e^{xt} \\ &= \frac{d}{dx} \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)}(x) \frac{t^{n+1}}{n!}.\end{aligned}$$

We can prove the identity for multi-poly-Bernoulli-star polynomials in the same manner.  $\square$

Next, we give the symmetric relation for multi-poly-Bernoulli-star polynomials. We note that the similar relation does not hold for multi-poly-Bernoulli polynomials.

**Proposition 6.4.** Let  $(k_1, \dots, k_r)^* = (k'_1, \dots, k'_r)$ . Then the relation

$$C_{n, \star}^{(k_1, \dots, k_r)}(x) = (-1)^n C_{n, \star}^{(k'_1, \dots, k'_r)}(1 - x)$$

holds.

*Proof.* Using the identity (5.5), we obtain

$$\begin{aligned}0 &= \frac{Li_{k_1, \dots, k_r}^*(1 - e^{-t})}{e^t - 1} e^{xt} + \frac{Li_{k'_1, \dots, k'_r}^*(1 - e^t)}{e^t - 1} e^{xt} \\ &= \frac{Li_{k_1, \dots, k_r}^*(1 - e^{-t})}{1 - e^{-t}} e^{xt} - \frac{Li_{k'_1, \dots, k'_r}^*(1 - e^t)}{e^{-t} - 1} e^{(1-x)(-t)}.\end{aligned}$$

Hence we have the required identity by comparing the coefficients of  $t^n/n!$ .  $\square$

## 6.2 Generalized Arakawa-Kaneko zeta function

In this section, we introduce generalized Arakawa-Kaneko zeta function whose values at non-positive integers are multi-poly-Bernoulli(-star) polynomials.

**Definition 6.5.** For complex numbers  $s$  and  $x$  with  $\Re(s) > 1 - r$  and  $\Re(x) > 0$ , we define  $Z_{k_1, \dots, k_r}(s, x)$  by the following integral:

$$Z_{k_1, \dots, k_r}(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{-xt} t^{s-1} dt.$$

**Proposition 6.6.** *The function  $Z_{k_1, \dots, k_r}(s, x)$  can be analytically continued to whole  $s$ -plane as an entire function. Further, we have*

$$Z_{k_1, \dots, k_r}(-n, x) = (-1)^n C_n^{(k_1, \dots, k_r)}(-x).$$

*Proof.* We decompose the interval of integration of defining integral:

$$\begin{aligned} Z_{k_1, \dots, k_r}(s, x) &= \frac{1}{\Gamma(s)} \int_0^1 \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{-xt} t^{s-1} dt \\ &+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{-xt} t^{s-1} dt. \end{aligned}$$

The latter integral converges for  $s \in \mathbb{C}$  since

$$\begin{aligned} &\left| \int_1^\infty \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{-xt} t^{s-1} dt \right| \\ &\leq \int_1^\infty \frac{Li_{1, \dots, 1}(1 - e^{-t})}{e^t - 1} |e^{-xt} t^{s-1}| dt \\ &\leq \exists M \int_1^\infty e^{-\Re(x)t} t^{\Re(s)+r-1} dt. \end{aligned}$$

Also we find that the latter one vanishes at non-positive integers. The former one can be rewritten as the following form:

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^1 \frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} e^{-xt} t^{s-1} dt &= \frac{1}{\Gamma(s)} \int_0^1 \sum_{n=0}^\infty C_n^{(k_1, \dots, k_r)}(-x) \frac{t^{n+s-1}}{n!} dt \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty C_n^{(k_1, \dots, k_r)}(-x) \frac{1}{n!} \frac{1}{n+s}. \end{aligned}$$

Second equality holds from Lebesgue's convergence theorem and the estimate

$$\begin{aligned} \sum_{n=r}^\infty \int_0^1 \left| C_n^{(k_1, \dots, k_r)}(-x) \frac{t^{n+s-1}}{n!} \right| dt &= \sum_{n=r}^\infty |C_n^{(k_1, \dots, k_r)}(-x)| \frac{1}{n!} \frac{1}{n + \Re(s)} \\ &\leq \sum_{n=r}^\infty |C_n^{(k_1, \dots, k_r)}(-x)| \frac{1}{n!} < \infty \end{aligned}$$

for  $\Re(s) > 1 - r$ . Therefore, we can continue  $Z_{k_1, \dots, k_r}(s, x)$  to  $s$ -plane and moreover, when  $s = -n$  for  $n \geq 0$ , we have

$$\begin{aligned} Z_{k_1, \dots, k_r}(-n, x) &= \lim_{s \rightarrow -n} \frac{1}{\Gamma(s)(s+n)} C_n^{(k_1, \dots, k_r)}(-x) \frac{1}{n!} \\ &= (-1)^n C_n^{(k_1, \dots, k_r)}(-x). \end{aligned}$$

□

**Proposition 6.7.** For any multi-index  $(k_1, \dots, k_r)$ , we have

$$\begin{aligned} Z_{k_1, \dots, k_r}(s, x) &= \sum_{m_1 > \dots > m_r > 0} \frac{(-\Delta)^{m_1}}{m_1^{k_1} \dots m_r^{k_r}} \zeta(s, x+1), \\ &=: Li_{k_1, \dots, k_r}(-\Delta) \zeta(s, x+1). \end{aligned}$$

where  $\zeta(s, x)$  is Hurwitz zeta function and  $\Delta$  is the forward difference operator defined by

$$\Delta f(x) = f(x+1) - f(x).$$

*Proof.* We rewrite multiple polylogarithm as series in Definition 6.5:

$$\begin{aligned} Z_{k_1, \dots, k_r}(s, x) &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{m_1 > \dots > m_r > 0} \frac{(1-e^{-t})^{m_1} e^{-xt} t^{s-1}}{m_1^{k_1} \dots m_r^{k_r} e^t - 1} dt \\ &= \frac{1}{\Gamma(s)} \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \sum_{j=0}^{m_1} (-1)^j \binom{m_1}{j} \int_0^\infty \frac{e^{-(j+x)t} t^{s-1}}{e^t - 1} dt \\ &= \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \sum_{j=0}^{m_1} (-1)^j \binom{m_1}{j} \zeta(s, x+j+1) \\ &= \sum_{m_1 > \dots > m_r > 0} \frac{(-\Delta)^{m_1}}{m_1^{k_1} \dots m_r^{k_r}} \zeta(s, x+1). \end{aligned}$$

□

We derive a formula for  $Z_{1, \dots, 1}(s, x)$  using Proposition 6.7.

**Example 6.8.** For  $m \geq 0$ , we have

$$Z_{\{1\}^m}(s, x) = \frac{(-1)^m}{m!} s(s+1) \cdots (s+m-1) \zeta(s+m, x+1).$$

*Proof.* By using Proposition 6.7, we obtain

$$\begin{aligned} Z_{\{1\}^m}(s, x) &= Li_{\{1\}^m}(-\Delta) \zeta(s, x+1) \\ &= \frac{1}{m!} (Li_1(-\Delta))^m \zeta(s, x+1) \\ &= \frac{1}{m!} (-D_x)^m \zeta(s, x+1) \\ &= \frac{1}{m!} s(s+1) \cdots (s+m-1) \zeta(s+m, x+1), \end{aligned}$$

where  $D_x$  is the differential operator on  $x$ . Notice that  $1 + \Delta = \exp(D_x)$  holds since

$$e^{D_x} f(x) = \sum_{n=0}^{\infty} \frac{D_x^n}{n!} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} 1^n = f(x+1).$$

□

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