Three term relations of the hypergeometric series and their applications

蛭子, 彰仁

https://doi.org/10.15017/1441039

出版情報：九州大学, 2013, 博士（数理学）, 課程博士
バージョン：

Three term relations of the hypergeometric series and their applications

Akihito Ebisu

To my parents, Mitsuo Ebisu and Mitsuko Ebisu

Preface

The hypergeometric series

$$F(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} x^n,$$

where $(a,n) := \Gamma(a+n)/\Gamma(a)$, has an important property: for a given triple of integers $(k,l,m) \in \mathbf{Z}^3$, there exists a unique pair of rational functions $(Q(x), R(x)) \in (\mathbf{Q}(a,b,c,x))^2$ satisfying

$$F(a+k,b+l;c+m;x) = Q(x)F(a+1,b+1;c+1;x) + R(x)F(a,b;c;x),$$

where $\mathbf{Q}(a,b,c,x)$ is the field generated over $\mathbf{Q}$ by a, b, c and x. This relation is called the **three term relation** of the hypergeometric series. The aim of this article is to get explicit expressions of the coefficients $Q(x)$ and $R(x)$ of the three term relation and to apply this relation to obtaining special values of the hypergeometric series.

In Chapter 1, we show that the coefficients $Q(x)$ and $R(x)$ of the three term relation are given as the sums of the products of the hypergeometric series.

In a series of letters to D.Stanton, R.W.Gosper presented many strange evaluations of hypergeometric series. Recently, we rediscovered one of the strange hypergeometric identities appearing in the series of letters. In Chapter 2, we prove this identity by using some results given in Chapter 1, and find new identities as its generalization.

It is well known that the value at $x = 1$ of the hypergeometric series $F(a,b;c;x)$ can be expressed in terms of gamma functions. The general expression for $F(a,b;c;1)$ is called the Gauss summation formula. In Chapter 3, using three term relations of the hypergeometric series, we show that values of $F(a,b;c;x)$ at some points other than $x = 1$ can also be expressed in terms of gamma functions, together with certain elementary functions. We tabulate the values of $F(a,b;c;x)$ that can be obtained by this method. We find that this catalog includes almost all previously known values (of course, almost all of Gosper's strange evaluations for the hypergeometric series) and many previously unknown values.

The author would like to thank Professors Katsunori Iwasaki, Masanobu Kaneko, Mitsuo Kato, Hiroyuki Ochiai and Raimundas Vidunas, and Dr. Genki Shibukawa and Mr. Hisashi Ando for careful reading and many valuable comments for various chapters. The author also thanks Professor Masaaki Yoshida for his discouragement, which encouraged the author to get some basic ideas.

Akihito Ebisu

Department of Mathematics, Kyushu University
Nishi-ku, Fukuoka 819-0395,Japan
a-ebisu@math.kyushu-u.ac.jp

Contents

Chapter 1

Three term relations of the hypergeometric series

1.1 Introduction

The hypergeometric differential equation $E(a,b,c)$ is defined by $L(a,b,c)y = 0$, where

$$L(a,b,c) := \partial^2 + \frac{c-(a+b+1)x}{x(1-x)}\partial - \frac{ab}{x(1-x)}$$

and $\partial = d/dx$. Throughout this chapter we assume, unless otherwise stated,

- A1: $a,\ b,\ c-a,\ c-b \notin \mathbf{Z}$,
- A2: $c,\ c-a-b,\ a-b \notin \mathbf{Z}$.

A1 is a necessary and sufficient condition for the equation $E(a,b,c)$ to be irreducible (cf. Theorem 4.3.2 of [IKSY] [1]), and A2 is a sufficient condition that every solution is free from logarithmic terms at each singular point.

The equation $E(a,b,c)$ admits the hypergeometric series

$$F(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)}\, x^n$$

as a solution, where $(\alpha, n) := \Gamma(\alpha+n)/\Gamma(\alpha)$. The following is known (cf. Chapter VI §24 of [Poole]): for a given triple of integers $(k,l,m) \in \mathbf{Z}^3$, there exists a unique pair of rational functions $(Q(x), R(x)) \in (\mathbf{Q}(a,b,c,x))^2$ satisfying

$$F(a+k,b+l;c+m;x) = Q(x)F(a+1,b+1;c+1;x) + R(x)F(a,b;c;x), \quad (1.1.1)$$

where $\mathbf{Q}(a,b,c,x)$ is the field generated over $\mathbf{Q}$ by a, b, c and x. This relation is called the **three term relation** of the hypergeometric series. In this chapter, we derive explicit expressions of the coefficients $Q(x)$ and $R(x)$ in (1.1.1).

We use the following series instead of the hypergeometric series:

$$f(a,b;c;x) := \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)}\, x^n = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)} F(a,b;c;x).$$

[1] The references at each chapter are listed at the end of each chapter.

Then, formula (1.1.1) is expressed as

$$f(a+k,b+l;c+m;x) = q(x)f(a+1,b+1;c+1;x) + r(x)f(a,b;c;x), \quad (1.1.2)$$

where

$$q(x) = \frac{c(a,k)(b,l)}{ab(c,m)}Q(x), \; r(x) = \frac{(a,k)(b,l)}{(c,m)}R(x). \quad (1.1.3)$$

Thus, we study $q(x)$ and $r(x)$ instead of $Q(x)$ and $R(x)$ in this chapter.

In Section 1.2, we introduce symbols used in this chapter. In Section 1.3, we show that each of $q(x)$ and $r(x)$ is the product of a power of x, a power of $1-x$ and a polynomial which can be expressible as the sum of products of two hypergeometric series. We also show that these polynomials admit some symmetry.

1.2 Derivation of three term relations

1.2.1 Local solutions of $E(a,b,c)$

Set

$$\begin{aligned}
f_1(a,b;c;x) &:= f(a,b;c;x),\\
f_2(a,b;c;x) &:= f(a,b;a+b+1-c;1-x),\\
f_3(a,b;c;x) &:= x^{-a}f(a,a+1-c;a+1-b;1/x),\\
f_4(a,b;c;x) &:= x^{-b}f(b,b+1-c;b+1-a;1/x),\\
f_5(a,b;c;x) &:= x^{1-c}f(a+1-c,b+1-c;2-c;x),\\
f_6(a,b;c;x) &:= (1-x)^{c-a-b}f(c-a,c-b;c+1-a-b;1-x).
\end{aligned}$$

f_1 and f_5 are solutions around $x=0$ of $E(a,b,c)$. f_2 and f_6 are solutions around $x=1$. f_3 and f_4 are solutions around $x=\infty$. Any two of these functions form a basis of $S(a,b,c)$, which denotes the solution space of $E(a,b,c)$, on a simply connected domain in $\mathbf{C}\setminus\{0,1\}$, say, the upper half-plane.

1.2.2 Contiguity operators

Set

$$\begin{aligned}
&\mathbf{0} := (0,0,0),\; \mathbf{1} := (1,1,1),\; \boldsymbol{k} := (k,l,m),\; \boldsymbol{k}+\mathbf{1} := (k+1,l+1,m+1),\\
&\boldsymbol{e}_1 := (1,0,0),\; \boldsymbol{e}_2 := (0,1,0),\; \boldsymbol{e}_3 := (0,0,1),\; \text{etc},
\end{aligned}$$

where k, l, m, k', l', $m' \in \mathbf{Z}$. For $1 \le i \le 6$, we set

$$\begin{aligned}
&(\mathbf{0})_i := f_i(a,b;c;x),\; (\mathbf{1})_i := f_i(a+1,b+1;c+1;x),\\
&(\boldsymbol{k})_i := f_i(a+k,b+l;c+m;x),\; (\boldsymbol{k}+\mathbf{1})_i := f_i(a+k+1,b+l+1;c+m+1;x),\\
&(\boldsymbol{e}_1)_i := f_i(a+1,b;c;x),\; (\boldsymbol{e}_2)_i := f_i(a,b+1;c;x),\; (\boldsymbol{e}_3)_i := f_i(a,b;c+1;x).
\end{aligned}$$

Lemma 1.2.1. *(Contiguity operators (cf. Theorem 2.1.1 of [IKSY])) Set*

$$H_1(a,b,c) := \vartheta + a, \ \ H_2(a,b,c) := \vartheta + b, \ \ B_3(a,b,c) := \vartheta + c - 1,$$

where $\vartheta := x\,\partial$. *Then we have*

$$\begin{aligned} a(c-a-1) \neq 0 &\Longleftrightarrow H_1(a,b,c) : S(a,b,c) \overset{\simeq}{\to} S(a+1,b,c), \\ b(c-b-1) \neq 0 &\Longleftrightarrow H_2(a,b,c) : S(a,b,c) \overset{\simeq}{\to} S(a,b+1,c), \\ (c-a-1)(c-b-1) \neq 0 &\Longleftrightarrow B_3(a,b,c) : S(a,b,c) \overset{\simeq}{\to} S(a,b,c-1), \end{aligned}$$

where " $\simeq$ *" stands for a linear isomorphism.*

Because these mappings are linearly isomorphic by condition A1, they have the inverse mappings. We define $B_1(a+1,b,c), B_2(a,b+1,c)$ and $H_3(a,b,c-1)$ as the inverse mappings of $H_1(a,b,c), H_2(a,b,c)$ and $B_3(a,b,c)$, respectively. They are also first order linear operators with coefficients in $\mathbf{Q}(a,b,c,x)$. We call H_1, H_2, H_3, B_1, B_2 and B_3 contiguity operators.

By considering the characteristic exponent of each local solution, we find that H_1, H_2 and H_3 send $(\mathbf{0})_i$ to $(\boldsymbol{e}_1)_i$, $(\boldsymbol{e}_2)_i$ and $(\boldsymbol{e}_3)_i$ up to multiplicative factors independent of x, respectively. We evaluate these factors explicitly. To begin with, we remark that

$$\partial f(a,b;c;x) = f(a+1,b+1;c+1,x). \tag{1.2.1}$$

For example, if we operate H_1 to $(\mathbf{0})_1$, from (1.2.1), we have

$$\begin{aligned} H_1(a,b,c)(\mathbf{0})_1 &= (x\partial + a) f(a,b;c;x) \\ &= x f(a+1,b+1;c+1;x) + a f(a,b;c;x) && (1.2.2) \\ &= (const.) f(a+1,b;c;x) && (1.2.3) \\ &= (const.)(\boldsymbol{e}_1)_1. \end{aligned}$$

Substituting zero for x into (1.2.2) and (1.2.3), we get $(const.) = 1$. Similarly, we have

Lemma 1.2.2.

$$\begin{aligned} H_1 :\ & (\mathbf{0})_1 \to (\boldsymbol{e}_1)_1, & H_2 :\ & (\mathbf{0})_1 \to (\boldsymbol{e}_2)_1, \\ & (\mathbf{0})_2 \to (a+1-c)(\boldsymbol{e}_1)_2, & & (\mathbf{0})_2 \to (b+1-c)(\boldsymbol{e}_2)_2, \\ & (\mathbf{0})_3 \to -(\boldsymbol{e}_1)_3, & & (\mathbf{0})_3 \to -(\boldsymbol{e}_2)_3, \\ & (\mathbf{0})_4 \to -(\boldsymbol{e}_1)_4, & & (\mathbf{0})_4 \to -(\boldsymbol{e}_2)_4, \\ & (\mathbf{0})_5 \to (\boldsymbol{e}_1)_5, & & (\mathbf{0})_5 \to (\boldsymbol{e}_2)_5, \\ & (\mathbf{0})_6 \to (a+1-c)(\boldsymbol{e}_1)_6, & & (\mathbf{0})_6 \to (b+1-c)(\boldsymbol{e}_2)_6, \\ H_3 :\ & (\mathbf{0})_1 \to (\boldsymbol{e}_3)_1, & \partial :\ & (\mathbf{0})_1 \to (\mathbf{1})_1, \\ & (\mathbf{0})_2 \to \frac{-1}{(a-c)(b-c)}(\boldsymbol{e}_3)_2, & & (\mathbf{0})_2 \to -(\mathbf{1})_2, \\ & (\mathbf{0})_3 \to -(\boldsymbol{e}_3)_3, & & (\mathbf{0})_3 \to -(\mathbf{1})_3, \\ & (\mathbf{0})_4 \to -(\boldsymbol{e}_3)_4, & & (\mathbf{0})_4 \to -(\mathbf{1})_4, \end{aligned}$$

$$(\mathbf{0})_5 \to (\boldsymbol{e}_3)_5, \qquad (\mathbf{0})_5 \to (\mathbf{1})_5,$$
$$(\mathbf{0})_6 \to \frac{-1}{(a-c)(b-c)}(\boldsymbol{e}_3)_6, \qquad (\mathbf{0})_6 \to -(\mathbf{1})_6,$$

where $H_i = H_i(a,b,c)$.

By composing H_1, H_2, H_3, B_1, B_2 and B_3, we have a linear isomorphism $H(k,l,m) : S(a,b,c) \to S(a+k,b+l,c+m)$. By using $q(x)$ and $r(x)$ in (1.1.2) and Lemma 1.2.2, we are able to express $H(k,l,m)$ as

$$H(k,l,m) = p(\partial)L(a,b,c) + q(x)\partial + r(x), \tag{1.2.4}$$

where $p(\partial) \in \mathbf{Q}(a,b,c,x)[\partial]$ which is the ring of polynomials in ∂ over $\mathbf{Q}(a,b,c,x)$.

In the next section, we study $q(x)$ and $r(x)$.

1.3 The coefficients of three term relations

Lemma 1.2.2 implies that

$$H(k,l,m)((\mathbf{0})_1, (\mathbf{0})_5) = ((\boldsymbol{k})_1, (\boldsymbol{k})_5), \tag{1.3.1}$$
$$H(k,l,m)((\mathbf{0})_2, (\mathbf{0})_6) = A((\boldsymbol{k})_2, (\boldsymbol{k})_6), \tag{1.3.2}$$
$$H(k,l,m)((\mathbf{0})_3, (\mathbf{0})_4) = (-1)^{k+l-m}((\boldsymbol{k})_3, (\boldsymbol{k})_4), \tag{1.3.3}$$

where

$$A = (-1)^m(a+1-c, k-m)(b+1-c, l-m). \tag{1.3.4}$$

If we operate (1.2.4) to $(\mathbf{0})_1$ and $(\mathbf{0})_5$, we get, from (1.3.1),

$$\begin{pmatrix} (\mathbf{1})_1 & (\mathbf{0})_1 \\ (\mathbf{1})_5 & (\mathbf{0})_5 \end{pmatrix} \begin{pmatrix} q(x) \\ r(x) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{k})_1 \\ (\boldsymbol{k})_5 \end{pmatrix},$$

and so,

$$\begin{pmatrix} q(x) \\ r(x) \end{pmatrix} = \frac{1}{(\mathbf{1})_1(\mathbf{0})_5 - (\mathbf{0})_1(\mathbf{1})_5} \begin{pmatrix} (\mathbf{0})_5 & -(\mathbf{0})_1 \\ -(\mathbf{1})_5 & (\mathbf{1})_1 \end{pmatrix} \begin{pmatrix} (\boldsymbol{k})_1 \\ (\boldsymbol{k})_5 \end{pmatrix}. \tag{1.3.5}$$

Since the denominator of the right hand side of (1.3.5) is the Wronskian of two solutions f_1 and f_5, it does not vanish outside of $\{0,1,\infty\}$. So, if $q(x)$ has poles, the poles must be in $\{0,1,\infty\}$. Therefore, $q(x)$ and $r(x)$ can be expressed as

$$q(x) = x^{v_0}(1-x)^{v_1}q_0(x), \quad q_0(x) : \text{a polynomial of degree } g \text{ and } q_0(0)q_0(1) \neq 0,$$
$$r(x) = x^{w_0}(1-x)^{w_1}r_0(x), \quad r_0(x) : \text{a polynomial of degree } h \text{ and } r_0(0)r_0(1) \neq 0.$$

1.3.1 Expressions of $q(x)$

Firstly, we obtain v_0, v_1 and g by counting the order of $q(x)$ at $x = 0$, 1 and ∞, respectively.

Lemma 1.3.1. *We assume* $(k,l,m) \neq (0,0,0)$*. Except the case*

$$E1 : m = 0,\ e_1 := (a,k)(b,l) - (a+1-c,k)(b+1-c,l) = 0,$$

we have

$$(\textit{the order of } q(x) \textit{ at } x=0) = \begin{cases} 1-m & (m \geq 0) \\ 1 & (m \leq 0) \end{cases}.$$

Indeed, by expanding $q(x)$ locally into a series around $x=0$, we have

$$q(x) = \frac{(\boldsymbol{k})_1(\mathbf{0})_5 - (\mathbf{0})_1(\boldsymbol{k})_5}{(\mathbf{1})_1(\mathbf{0})_5 - (\mathbf{0})_1(\mathbf{1})_5} = \frac{\{*\}\,x^{1-c}\{*\} - \{*\}\,x^{1-c-m}\{*\}}{\{*\}\,x^{1-c}\{*\} - \{*\}\,x^{-c}\{*\}} = \frac{x\{*\} + x^{1-m}\{*\}}{\{*\}}, \tag{1.3.6}$$

where each $\{\ *\ \}$ denotes a power series in x of which constant term is not 0.

When $m=0$ in Lemma 1.3.1, the coefficient of x^1 of (1.3.6) is $e_1/(c-1)$. In the case E1, the order of $q(x)$ at $x=0$ is not 1 but greater than 2.

Next, we evaluate the orders of $q(x)$ at $x=1$ and ∞.

Lemma 1.3.2.

$$\begin{aligned}(\boldsymbol{k})_1(\mathbf{0})_5 - (\mathbf{0})_1(\boldsymbol{k})_5 &= \textit{const.}\,\{(\boldsymbol{k})_2(\mathbf{0})_6 - (\mathbf{0})_2(\boldsymbol{k})_6\}, \\ (\boldsymbol{k})_1(\mathbf{0})_5 - (\mathbf{0})_1(\boldsymbol{k})_5 &= \textit{const.}\,\{(\boldsymbol{k})_3(\mathbf{0})_4 - (\mathbf{0})_3(\boldsymbol{k})_4\}.\end{aligned}$$

We give a proof of Lemma 1.3.2 after. From this lemma, we obtain the following in the same way as we got Lemma 1.3.1:

Lemma 1.3.3. *We assume* $(k,l,m) \neq (0,0,0)$*. Except the cases*

$$\begin{aligned}&E2 : m-k-l = 0,\ (a,k)(b,l) - (c-a,l)(c-b,k) = 0, \\ &E3 : k = l,\ (b+1-c,l-m)(b,l) - (a+1-c,k-m)(a,k) = 0,\end{aligned}$$

we have

$$\begin{aligned}(\textit{the order of } q(x) \textit{ at } x=1) &= \begin{cases} m+1-k-l & (m-k-l \leq 0) \\ 1 & (m-k-l \geq 0) \end{cases}, \\ -(\textit{the order of } q(x) \textit{ at } x=\infty) &= \begin{cases} 1-k & (k \leq l) \\ 1-l & (k \geq l) \end{cases}.\end{aligned}$$

Without loss of generality, we asuume, in this section, $k \leq l$. We divide the set

$$\{(k,l,m); k,l,m \in \mathbf{Z}, k \leq l\}$$

into four subsets

$$\begin{aligned}(i) &: \{(k,l,m); m \geq 0,\ m-k-l \leq 0\}, & (ii) &: \{(k,l,m); m \geq 0,\ m-k-l \geq 0\}, \\ (iii) &: \{(k,l,m); m \leq 0,\ m-k-l \leq 0\}, & (iv) &: \{(k,l,m); m \leq 0,\ m-k-l \geq 0\}.\end{aligned}$$

Then, Lemmas 1.3.1 and 1.3.3 imply

Proposition 1.3.4. *Except the cases E1, E2 and E3, we have*

$$(v_0, v_1, g) = \begin{cases} (1-m, m+1-k-l, l-1) & \text{if } (k,l,m) \in (i), \\ (1-m, 1, m-k-1) & \text{if } (k,l,m) \in (ii), \\ (1, m+1-k-l, l-m-1) & \text{if } (k,l,m) \in (iii), \\ (1, 1, -k-1) & \text{if } (k,l,m) \in (iv). \end{cases}$$

Remark 1.3.5. *If we regard a polynomial of degree -1 as 0, (k,l,m) in Proposition 1.3.4 can be extended to $\mathbf{Z}^3$.*

Secondly, we show that $q_0(x)$ can be expressible as the sum of products of two hypergeometric series. To begin with, since $(\mathbf{1})_1(\mathbf{0})_5 - (\mathbf{0})_1(\mathbf{1})_5$ in (1.3.5) is the Wronskian of two solutions f_1 and f_5, we obtain the following:

Lemma 1.3.6.

$$(\mathbf{1})_1(\mathbf{0})_5 - (\mathbf{0})_1(\mathbf{1})_5 = -\frac{\Gamma(a)\Gamma(b)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(c)\Gamma(1-c)} x^{-c}(1-x)^{c-a-b-1}.$$

We consider, for example, the case (i). By Proposition 1.3.4 and Lemma 1.3.6, $(\boldsymbol{k})_1(\mathbf{0})_5 - (\mathbf{0})_1(\boldsymbol{k})_5$ can be expressed as

$$-\frac{\Gamma(a)\Gamma(b)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(c)\Gamma(1-c)} x^{1-c-m}(1-x)^{c-a-b+m-k-l} q_0(x). \tag{1.3.7}$$

On the other hand, it can also be expressed as

$$\begin{aligned} & x^{1-c-m}(1-x)^{c-a-b+m-k-l} \\ & \times \Bigg\{ \frac{\Gamma(a+k)\Gamma(b+l)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(c+m)\Gamma(2-c)} x^m \\ & \times F(c-a+m-k, c-b+m-l; c+m; x) F(a+1-c, b+1-c; 2-c; x) \\ & - \frac{\Gamma(a)\Gamma(b)\Gamma(a+1-c+k-m)\Gamma(b+1-c+l-m)}{\Gamma(c)\Gamma(2-c-m)} \\ & \times F(a,b;c;x) F(1-a-k, 1-b-l; 2-c-m; x) \Bigg\} \end{aligned} \tag{1.3.8}$$

by Kummer's solutions around $x = 0$ (cf. 2.9(1), (2), (17) and (18) of [Erd]). Equating (1.3.7) with (1.3.8), we get an expression of $q_0(x)$. We also get expressions of $q_0(x)$ in other cases similarly:

Theorem 1.3.7. *Except the cases E1, E2 and E3, we have*

$$q_0(x) = \begin{cases} q_1(a,b,c,k,l,m;x) & \text{if } (k,l,m) \in (i), \\ q_2(a,b,c,k,l,m;x) & \text{if } (k,l,m) \in (ii), \\ q_3(a,b,c,k,l,m;x) & \text{if } (k,l,m) \in (iii), \\ q_4(a,b,c,k,l,m;x) & \text{if } (k,l,m) \in (iv). \end{cases}$$

where

$$q_1(a,b,c,k,l,m;x) := C^1 x^m F(c-a+m-k, c-b+m-l; c+m; x)$$

$$
\begin{aligned}
&\times F(a+1-c, b+1-c; 2-c; x) \\
&+ C^2 F(a,b;c;x) F(1-a-k, 1-b-l; 2-c-m; x), \\
q_2(a,b,c,k,l,m;x) &:= C^1 x^m F(a+k, b+l; c+m; x) F(1-a, 1-b; 2-c; x) \\
&+ C^2 F(c-a, c-b; c; x) \\
&\times F(a+1-c+k-m, b+1-c+l-m; 2-c-m; x), \\
q_3(a,b,c,k,l,m;x) &:= C^1 F(c-a+m-k, c-b+m-l; c+m; x) \\
&\times F(a+1-c, b+1-c; 2-c; x) \\
&+ C^2 x^{-m} F(a,b;c;x) F(1-a-k, 1-b-l; 2-c-m; x), \\
q_4(a,b,c,k,l,m;x) &:= C^1 F(a+k, b+l; c+m; x) F(1-a, 1-b; 2-c; x) \\
&+ C^2 x^{-m} F(c-a, c-b; c; x) \\
&\times F(a+1-c+k-m, b+1-c+l-m; 2-c-m; x).
\end{aligned}
$$

and

$$
C^1 := -\frac{(a,k)(b,l)}{(1-c)(c,m)}, \; C^2 := \frac{(a+1-c,k-m)(b+1-c,l-m)}{(1-c)(2-c,-m)}.
$$

Thirdly, we show that each $q_i(a,b,c,k,l,m;x)$ has expressions in six ways by linear transformations of the parameters and linear fractional transformations of the independent variable. For example, in the case (i), we have

$$
\begin{aligned}
q_0(x) = &(1-x)^{l-1} \\
&\times \left\{ C^1 \left(\frac{x}{1-x} \right)^m F(a+k, c-b+m-l; c+m; x/(x-1)) \right. \\
&\times F(1-a, b+1-c; 2-c; x/(x-1)) \\
&+ C^2 F(b, c-a; c; x/(x-1)) \\
&\left. \times F(1-b-l, a+1-c+k-m; 2-c-m; x/(x-1)) \right\}
\end{aligned}
$$

by applying other Kummer's solutions (cf. 2.9(3), (4), (19) and (20) of [Erd]) around $x=0$ to $(\boldsymbol{k})_1(\mathbf{0})_5 - (\mathbf{0})_1(\boldsymbol{k})_5$. This is equal to

$$
\frac{(a+1-c, k-m)}{(1-a,-k)}(1-x)^{l-1} q_1(c-a, b, c, m-k, l, m, x/(x-1)).
$$

So far, we have studied $q(x)$ and $q_0(x)$ by using solutions around $x=0$. We can also use solutions around $x=1$, ∞. For example, if we operate (1.2.4) to $(\mathbf{0})_2$ and $(\mathbf{0})_6$, we obtain

$$
\begin{pmatrix} q(x) \\ r(x) \end{pmatrix} = \frac{-A}{(\mathbf{1})_2(\mathbf{0})_6 - (\mathbf{0})_2(\mathbf{1})_6} \begin{pmatrix} (\mathbf{0})_6 & -(\mathbf{0})_2 \\ (\mathbf{1})_6 & -(\mathbf{1})_2 \end{pmatrix} \begin{pmatrix} (\boldsymbol{k})_2 \\ (\boldsymbol{k})_6 \end{pmatrix}.
$$

Since $(\mathbf{1})_2(\mathbf{0})_6 - (\mathbf{0})_2(\mathbf{1})_6$ is the Wronskian of two solutions f_2 and f_6, this is equal to $(\mathbf{1})_1(\mathbf{0})_5 - (\mathbf{0})_1(\mathbf{1})_5$ up to multiple factor (This gives a proof of Lemma 1.3.2). Hence, we get other expressions of $q_i(a,b,c,k,l,m,x)$ by using Kummer's solutions around $x=1$ (cf. 2.9(5), (6), (21) and (22) of [Erd]):

Theorem 1.3.8. *A is defined by (1.3.4). When* $(k,l,m)\in(i)$*, we have*

$$\begin{aligned}
&q_1(a,b,c,k,l,m;x)\\
&\quad=-Aq_1(a,b,a+b+1-c,k,l,k+l-m;1-x)\\
&\quad=x^{l-1}q_1(b+1-c,b,b+1-a,l-m,l,l-k;1/x)\\
&\quad=\frac{(a+1-c,k-m)}{(1-a,-k)}(1-x)^{l-1}q_1(c-a,b,c,m-k,l,m;x/(x-1))\\
&\quad=\frac{-(c-b,m-l)A}{(1-a,-k)}x^{l-1}\\
&\quad\times q_1(b+1-c,b,a+b+1-c,l-m,l,k+l-m;(x-1)/x)\\
&\quad=\frac{(-1)^{l-1}(a+1-c,k-m)}{(c-b,m-l)}(1-x)^{l-1}\\
&\qquad\times q_1(c-a,b,b+1-a,m-k,l,l-k;1/(1-x)).
\end{aligned}$$

When $(k,l,m)\in(ii)$*, we have*

$$\begin{aligned}
&q_2(a,b,c,k,l,m;x)\\
&\quad=\frac{-(a,k)(b,l)A}{(a+1-c,k-m)(b+1-c,l-m)}\\
&\quad\times q_2(1-b,1-a,c+1-a-b,-l,-k,m-k-l;1-x)\\
&\quad=(-1)^{k+l-m}x^{m-k-1}q_2(b+1-c,b,b+1-a,l-m,l,l-k;1/x)\\
&\quad=\frac{(b+1-c,l-m)}{(1-b,-l)}(1-x)^{m-k-1}q_2(a,c-b,c,k,m-l,m,x/(x-1))\\
&\quad=\frac{-(b,l)A}{(a+1-c,k-m)}x^{m-k-1}\\
&\quad\times q_2(1-b,c-b,c+1-a-b,-l,m-l,m-k-l;(x-1)/x)\\
&\quad=-(a,k)(b,l)(1-x)^{m-k-1}\\
&\quad\times q_2(b+1-c,1-a,b+1-a,l-m,-k,l-k;1/(1-x)).
\end{aligned}$$

When $(k,l,m)\in(iii)$*, we have*

$$\begin{aligned}
&q_3(a,b,c,k,l,m;x)\\
&\quad=\frac{-(a,k)(b,l)A}{(a+1-c,k-m)(b+1-c,l-m)}\\
&\quad\times q_3(1-b,1-a,c+1-a-b,-l,-k,m-k-l;1-x)\\
&\quad=x^{l-m-1}q_3(a,a+1-c,a+1-b,k,k-m,k-l;1/x)\\
&\quad=\frac{(a,k)}{(c-a,m-k)}(1-x)^{l-m-1}q_3(c-a,b,c,m-k,l,m;x/(x-1))\\
&\quad=\frac{-(c-b,m-l)A}{(1-a,-k)}x^{l-m-1}\\
&\quad\times q_3(c-a,1-a,c+1-a-b,m-k,-k,m-k-l;(x-1)/x)\\
&\quad=\frac{(-1)^{l-m-1}(a,k)}{(1-b,-l)}(1-x)^{l-m-1}\\
&\quad\times q_3(1-b,a+1-c,a+1-b,-l,k-m,k-l;1/(1-x)).
\end{aligned}$$

When $(k,l,m)\in(iv)$, *we have*

$$
\begin{aligned}
&q_4(a,b,c,k,l,m;x)\\
&\quad= -Aq_4(a,b,a+b+1-c,k,l,k+l-m;1-x)\\
&\quad= (-1)^{k+l-m}x^{-k-1}q_4(a,a+1-c,a+1-b,k,k-m,k-l;1/x)\\
&\quad= \frac{(b,l)}{(c-b,m-l)}(1-x)^{-k-1}q_4(a,c-b,c,k,m-l,m;x/(x-1))\\
&\quad= \frac{-(b,l)A}{(a+1-c,k-m)}x^{-k-1}\\
&\quad\times q_4(a,a+1-c,a+b+1-c,k,k-m,k+l-m;(x-1)/x)\\
&\quad= -(a+1-c,k-m)(b+1-c,l-m)(1-x)^{-k-1}\\
&\quad\times q_4(a,c-b,a+1-b,k,m-l,k-l,1/(1-x)).
\end{aligned}
$$

1.3.2 Expressions of $r(x)$

We express $r(x)$ as we did for $q(x)$. From (1.3.5), we get

$$r(x) = -\frac{(\boldsymbol{k})_1(\mathbf{1})_5-(\mathbf{1})_1(\boldsymbol{k})_5}{(\mathbf{1})_1(\mathbf{0})_5-(\mathbf{0})_1(\mathbf{1})_5}.$$

Since the numerator $(\boldsymbol{k})_1(\mathbf{1})_5-(\mathbf{1})_1(\boldsymbol{k})_5$ is obtained from $(\boldsymbol{k})_1(\mathbf{0})_5-(\mathbf{0})_1(\boldsymbol{k})_5$ by the change

$$(a,b,c)\to(a+1,b+1,c+1),\ (k,l,m)\to(k-1,l-1,m-1),$$

we can use the discussion in the previous subsection. By the transformation, cases (i), (ii), (iii), (iv), E1, E2 and E3 also change. We call them, respectively, (i'), (ii'), (iii'), (iv'), E1$'$, E2$'$ and E3$'$:

$$
\begin{aligned}
(i'):&\ \{(k,l,m);m\ge 1,\ m-k-l\le -1\},\\
(ii'):&\ \{(k,l,m);m\ge 1,\ m-k-l\ge -1\},\\
(iii'):&\ \{(k,l,m);m\le 1,\ m-k-l\le -1\},\\
(iv'):&\ \{(k,l,m);m\le 1,\ m-k-l\ge -1\},\\
E1':&\ m=1,\ (a+1,k-1)(b+1,l-1)-(a+1-c,k-1)(b+1-c,l-1)=0,\\
E2':&\ m-k-l=-1,\ (a+1,k-1)(b+1,l-1)-(c-a,l-1)(c-b,k-1)=0,\\
E3':&\ k=l,\ (b+1-c,l-m)(b+1,l-1)-(a+1-c,k-m)(a+1,k-1)=0.
\end{aligned}
$$

Recall the expression

$$r(x)=x^{w_0}(1-x)^{w_1}r_0(x),\ r_0(x):\text{a polynomial of degree } h \text{ and } r_0(0)r_0(1)\neq 0.$$

Then, we get the following proposition like Proposition 1.3.4:

Proposition 1.3.9. *Except the cases E1$'$, E2$'$ and E3$'$, we have*

$$
(w_0,w_1,h)=\begin{cases}
(1-m,m+1-k-l,l-2) & \text{if } (k,l,m)\in(i'),\\
(1-m,0,m-k-1) & \text{if } (k,l,m)\in(ii'),\\
(0,m+1-k-l,l-m-1) & \text{if } (k,l,m)\in(iii'),\\
(0,0,-k) & \text{if } (k,l,m)\in(iv'),
\end{cases}
$$

Set

$$
\begin{aligned}
&r_1(a,b,c,k,l,m;x)\\
&\quad := C^3 x^{m-1} F(c-a+m-k, c-b+m-l; c+m; x)\\
&\quad \times F(a+1-c, b+1-c; 1-c; x)\\
&\quad + C^4 F(a+1,b+1;c+1;x) F(1-a-k, 1-b-l; 2-c-m; x),\\
&r_2(a,b,c,k,l,m;x)\\
&\quad := C^3 x^{m-1} F(a+k, b+l; c+m; x) F(-a,-b;1-c;x)\\
&\quad + C^4 F(c-a, c-b; c+1; x)\\
&\quad \times F(a+1-c+k-m, b+1-c+l-m; 2-c-m; x),\\
&r_3(a,b,c,k,l,m;x)\\
&\quad := C^3 F(c-a+m-k, c-b+m-l; c+m; x)\\
&\quad \times F(a+1-c, b+1-c; 1-c; x)\\
&\quad + C^4 x^{1-m} F(a+1,b+1;c+1;x) F(1-a-k, 1-b-l; 2-c-m; x),\\
&r_4(a,b,c,k,l,m;x)\\
&\quad := C^3 F(a+k, b+l; c+m; x) F(-a,-b;1-c;x)\\
&\quad + C^4 x^{1-m} F(c-a, c-b; c+1; x)\\
&\quad \times F(a+1-c+k-m, b+1-c+l-m; 2-c-m; x),
\end{aligned}
$$

where

$$
C^3 := \frac{(a,k)(b,l)}{(c,m)},\ C^4 := -\frac{ab(a+1-c,k-m)(b+1-c,l-m)}{c(1-c)(2-c,-m)}.
$$

Then, we have the following:

Theorem 1.3.10. *We except the cases E1′, E2′ and E3′, and A is defined by (1.3.4). When $(k,l,m) \in (i')$, we have*

$$
\begin{aligned}
r_0(x) &= r_1(a,b,c,k,l,m;x)\\
&= A r_1(a,b,a+b+1-c,k,l,k+l-m;1-x)\\
&= \frac{a}{b-c} x^{l-2} r_1(b-c,b,b-a,l+1-m,l,l+1-k;1/x)\\
&= \frac{(a+1-c,k-m)}{(a+1-c)(1-a,-k)}(1-x)^{l-2}\\
&\quad \times r_1(c-a-1,b,c,m+1-k,l,m;x/(x-1))\\
&= \frac{-(c-b,m-l)A}{(b-c)(1-a,-k)} x^{l-2}\\
&\quad \times r_1(b-c,b,a+b+1-c,l+1-m,l,k+l-m;(x-1)/x)\\
&= \frac{(-1)^{l-2} a(a+1-c,k-m)}{(c-a-1)(c-b,m-l)}(1-x)^{l-2}\\
&\quad \times r_1(c-a-1,b,b-a,m+1-k,l,l+1-k;1/(1-x)).
\end{aligned}
$$

When $(k,l,m) \in (ii')$, we have

$$
r_0(x) = r_2(a,b,c,k,l,m;x)
$$

$$
\begin{aligned}
&= \frac{(a,k)(b,l)A}{(a+1)(b+1)(a+1-c,k-m)(b+1-c;l-m)} \\
&\quad \times r_2(-b-1,-a-1,c-a-b-1, \\
&\qquad\qquad 2-l,2-k,m+2-k-l;1-x) \\
&= \frac{(-1)^{k+l-m-1}\,a}{b-c} x^{m-k-1} r_2(b-c,b,b-a,l+1-m,l,l+1-k;1/x) \\
&= \frac{(b+1-c,l-m)}{(b+1-c)(1-b)^{-l}}(1-x)^{m-k-1} \\
&\quad \times r_2(a,c-b-1,c,k,m+1-l,m,x/(x-1)) \\
&= \frac{a(b,l)A}{(b+1)(b+1-c)(a+1-c,k-m)} x^{m-k-1} \\
&\quad \times r_2(-1-b,c-b-1,c-1-a-b, \\
&\qquad\qquad 2-l,m+1-l,m+2-k-l;(x-1)/x) \\
&= \frac{(a,k)(b,l)}{(a+1)(b-c)}(1-x)^{m-k-1} \\
&\quad \times r_2(b-c,-a-1,b-a,l+1-m,2-k,l+1-k;1/(1-x)).
\end{aligned}
$$

When $(k,l,m) \in (iii')$*, we have*

$$
\begin{aligned}
r_0(x) &= r_3(a,b,c,k,l,m;x) \\
&= \frac{(a,k)(b,l)A}{(a+1)(b+1)(a+1-c,k-m)(b+1-c,l-m)} \\
&\quad \times r_3(-b-1,-a-1,c-a-b-1, \\
&\qquad\qquad 2-l,2-k,m+2-k-l;1-x) \\
&= \frac{b}{a-c} x^{l-m-1} r_3(a,a-c,a-b,k,k+1-m,k+1-l;1/x) \\
&= \frac{-(a,k)}{(a+1-c)(c-a,m-k)}(1-x)^{l-m-1} \\
&\quad \times r_3(c-a-1,b,c,m+1-k,l,m;x/(x-1)) \\
&= \frac{-b(c-b,m-l)A}{(a+1)(a+1-c)(1-a,-k)} x^{l-m-1} \\
&\quad \times r_3(c-a-1,-a-1,c-a-b-1, \\
&\qquad\qquad m+1-k,2-k,m+2-k-l;(x-1)/x) \\
&= \frac{(-1)^{l-m-1}(a,k)}{(a-c)(b+1)(1-b,-l)}(1-x)^{l-m-1} \\
&\quad \times r_3(-b-1,a-c,a-b,2-l,k+1-m,k+1-l;1/(1-x)).
\end{aligned}
$$

When $(k,l,m) \in (iv')$*, we have*

$$
\begin{aligned}
r_0(x) &= r_4(a,b,c,k,l,m;x) \\
&= Ar_4(a,b,a+b+1-c,k,l,k+l-m;1-x) \\
&= \frac{(-1)^{k+l-m-1}\,b}{a-c} x^{-k} r_4(a,a-c,a-b,k,k+1-m,k+1-l;1/x) \\
&= \frac{-(b,l)}{(b+1-c)(c-b,m-l)}(1-x)^{-k}
\end{aligned}
$$

$$\begin{aligned}
&\times r_4(a, c-b-1, c, k, m+1-l, m; x/(x-1)) \\
&= \frac{(b,l)A}{(a-c)(a+1-c,k-m)} x^{-k} \\
&\times r_4(a, a-c, a+b+1-c, k, k+1-m, k+l-m; (x-1)/x) \\
&= \frac{b(a+1-c,k-m)(b+1-c,l-m)}{b+1-c} (1-x)^{-k} \\
&\times r_4(a, c-b-1, a-b, k, m+1-l, k+1-l, 1/(1-x)).
\end{aligned}$$

Remark 1.3.11. *Even if conditions E1, E2, E3, E1′, E2′, E3′ are satisfied, $q(x)$ and $r(x)$ which are derived from Propositions 1.3.4, 1.3.9 and Theorems 1.3.7, 1.3.8, 1.3.10 remain valid. However, $q_0(x)$ and $r_0(x)$ may vanish at $x = 0$ or 1 or may have smaller degree than stated.*

Remark 1.3.12. *Assume $H(k, l, m)$ is an isomorphism, and there exists an expression of q_i (respectively r_i), among the six, which makes sense. Then, even if A1 and A2 are not satisfied, this expression remains valid thanks to Proposition 1.3.4 (respectively Proposition 1.3.9).*

Remark 1.3.13. *If we regard k, l, m not as constants but as variables, from the explicit expressions of $q(x)$ and $r(x)$, we find that $q(x)$, $r(x) \in \boldsymbol{K}$, where $\boldsymbol{K}$ is the field generated over $\mathbf{Q}$ by $a^\alpha, b^\alpha, c^\alpha, k^\alpha, l^\alpha, m^\alpha$ and x^α $(\alpha = 1, k, l, m)$.*

References

[Eb] A.Ebisu, Three term relations for the hypergeometric series(in Japanese), Master thesis at Kyushu University, February 2011.

[Erd] A.Erdélyi(editor), Higher transcendental functions vol.1, McGraw-Hill, 1953.

[IKSY] K.Iwasaki, H.Kimura, S.Shimomura and M.Yoshida, *From Gauss to Painlevé —A modern theory of special functions*, Vieweg Verlag, Wiesbaden(1991).

[Poole] E.G.C.Poole, Introduction to the theory of linear differential equations, Oxford University Press, 1936.

Chapter 2

On a strange evaluation of the hypergeometric series by Gosper

2.1 Introduction and main theorem in this chapter

In a series of letters to D.Stanton, R.W.Gosper carried out many strange evaluations of hypergeometric series. Some are presented in [GS]. However, the following identity appears in [Go], but not in [GS]:

$$F\left(1-a, b; b+2; \frac{b}{a+b}\right) = (b+1)\left(\frac{a}{a+b}\right)^a. \tag{2.1.1}$$

Recently, we discovered (2.1.1) independently. Here, we present our proof of this identity and derive a generalization using contiguity operators.

The main theorem in this chapter is the following:

Theorem 2.1.1. *We assume that $\ell \in \mathbf{Z}_{>0}, a \in \mathbf{C}$ and $c \in \mathbf{C} \setminus \mathbf{Z}$. For any root λ of $F(1-a, -\ell; 2-c; x)$, which is a polynomial in the variable x of degree at most ℓ, we have*

$$F(a, 1+\ell; c; \lambda) = -\frac{(1-c)q_0(\lambda)}{(1,\ell)(1-\lambda)^\ell}, \tag{2.1.2}$$

$$F(c-a, c-1-\ell; c; \lambda) = -\frac{(1-c)}{(1,\ell)}(1-\lambda)^{a+1-c}q_0(\lambda), \tag{2.1.3}$$

where $q_0(x)$ is the polynomial in x of degree at most $\ell-1$ given by

$$q_0(x) = -\frac{(1,\ell)}{1-c}(1-x)^{c-a-1}F(c-a, c-1-\ell; c; x) + \frac{(2-c,\ell)}{1-c}F(a,1;c;x)F(1-a,-\ell;2-c;x). \tag{2.1.4}$$

For example, when $\ell = 1$, we have

$$F\left(a, 2; c; \frac{c-2}{a-1}\right) = \frac{(a-1)(c-1)}{a+1-c}, \tag{2.1.5}$$

$$F\left(c-a, c-2; c; \frac{c-2}{a-1}\right) = (c-1)\left(\frac{a+1-c}{a-1}\right)^{a+1-c}. \tag{2.1.6}$$

Remark 2.1.2. *If we assume further the condition $a \notin \mathbf{Z}$, then the degree of $F(1-a, -\ell; 2-c; x)$ is exactly ℓ. In this case, $q_0(x)$ can also be expressed as*

$$q_0(x) = (2-a, \ell-1)(-x)^{\ell-1} \times (\text{the } \ell\text{-th partial sum of the power series in } 1/x \; F(2-c, 1; 2-a; 1/x)F(c-1-\ell, -\ell; a-\ell; 1/x)), \tag{2.1.7}$$

and thus, the degree of $q_0(x)$ is exactly $\ell - 1$.

Note that (2.1.6) is equivalent to (2.1.1).

Many methods for discovering and proving hypergeometric identities are known. In the 19th century, such identities were discovered using algebraic transformations of hypergeometric series (cf. [V] for a treatment of algebraic transformations of hypergeometric series). Moreover, during the last several decades, many new methods that exploit progress in computer technology have been constructed: Gosper's algorithm, the W-Z method, Zeilberger's algorithm, etc. (cf. [Ko] and [PWZ]). These well-known algorithms have been used for discovering and proving hypergeometric identities expressed in closed forms (cf. [WoGo], [WoHy], [WoWZ] and [WoZe]). Indeed, in [Ek] and [AZ], M.Apagodu, S.B.Ekhad and D.Zeilberger report the discovery of such identities using these algorithms. In addition, such identities are proved with the aid of these algorithms in [Ko] and [PWZ]. However, note that although $q_0(x)$ appearing in (2.1.2) and (2.1.3) is expressed explicitly, it is not in a closed form (see (2.1.4)). This is why the identities (2.1.2) and (2.1.3) were not found till now.

Remark 2.1.3. *If we input*

```
> simplify(hypergeom([c − a, c − 2], [c], (c − 2)/(a − 1)))
```

in Maple 16, we obtain (2.1.6) as an output. From this, we can easily obtain (2.1.5) using Euler transformation

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x) \tag{2.1.8}$$

(cf. (2.2.7) in [AAR]). However, when we input

```
> simplify(hypergeom([a, 2], [c], (c − 2)/(a − 1))),
```

Maple 16 does not return (2.1.5). This is mysterious.

2.2 Preliminaries

In this section, modifying some results given in Chapter 1, we obtain Lemmas 2.2.2 and 2.2.3. These are used for proving the main theorem in this chapter.

Recall the hypergeometric (differential) operator in x

$$L(a, b, c) = \partial^2 + \frac{c-(a+b+1)x}{x(1-x)}\partial - \frac{ab}{x(1-x)}, \tag{2.2.1}$$

where $\partial = d/dx$ and a, b and c are complex variables. Further, recall that the contiguity operator $(\vartheta + b)$, where $\vartheta = x\partial$, produces

$$(\vartheta + b)F(a, b; c; x) = bF(a, b+1; c; x) \tag{2.2.2}$$

for $c \notin \mathbf{Z}_{\leq 0}$ (cf. Subsection 1.2.2 in this article or Proposition 2.1.2 in [IKSY]).

Now, we consider the composition of contiguity operators

$$(\vartheta + b + \ell - 1) \cdots (\vartheta + b + 1)(\vartheta + b) =: H(\ell). \tag{2.2.3}$$

Then, $H(\ell)$ can be expressed as

$$H(\ell) = p(\partial)L(a, b, c) + q(x)\partial + r(x), \tag{2.2.4}$$

where $p(\partial) \in \mathbf{Q}(a, b, c, x)[\partial]$ and $q(x), r(x) \in \mathbf{Q}(a, b, c, x)$.

We assume the following:

$$\begin{aligned}
&\text{A1: } a, b, c-a, c-b \notin \mathbf{Z} && \text{(cf. Section 1.1)},\\
&\text{A2: } c, c-a-b, a-b \notin \mathbf{Z} && \text{(cf. Section 1.1)},\\
&\text{E1: } (b, \ell) - (b+1-c, \ell) \neq 0 && \text{(cf. Lemma 1.3.1)},\\
&\text{E2': } \ell \neq 1 \text{ or } \frac{(b+1, \ell-1)}{a} - \frac{(c-a, \ell-1)}{c-b-1} \neq 0 && \text{(cf. Subsection 1.3.2)}.
\end{aligned}$$

Assuming the above, we can directly apply propositions and theorems given in Chapter 1. Specifically, we need only substitute $k = 0$, $l = \ell$ and $m = 0$ into the formulas appearing in the propositions and theorems of Chapter 1. Doing so, $q(x)$ and $r(x)$ can be expressed as

$$\begin{aligned}
&q(x) = x^{v_0}(1-x)^{v_1}q_0(x),\\
&q_0(x) : \text{a polynomial in } x \text{ of degree } g \text{ and } q_0(0)q_0(1) \neq 0,
\end{aligned} \tag{2.2.5}$$

$$\begin{aligned}
&r(x) = x^{w_0}(1-x)^{w_1}r_0(x),\\
&r_0(x) : \text{a polynomial in } x \text{ of degree } h \text{ and } r_0(0)r_0(1) \neq 0
\end{aligned} \tag{2.2.6}$$

(cf. Section 1.3). Moreover, by Propositions 1.3.4 and 1.3.9, we have

$$(v_0, v_1, g) = (1, 1-\ell, \ell-1), \ (w_0, w_1, h) = (0, 1-\ell, \ell-1), \tag{2.2.7}$$

and by Theorems 1.3.7 and 1.3.10, we have

$$\begin{aligned}
q_0(x) = &-\frac{(b, \ell)}{1-c}F(c-a, c-b-\ell; c; x)F(a+1-c, b+1-c; 2-c; x)\\
&+\frac{(b+1-c, \ell)}{1-c}F(a, b; c; x)F(1-a, 1-b-\ell; 2-c; x)
\end{aligned} \tag{2.2.8}$$

$$\begin{aligned}
r_0(x) = &\ (b, \ell)F(c-a, c-b-\ell; c; x)F(a+1-c, b+1-c; 1-c; x)\\
&-\frac{ab(b+1-c, \ell)}{c(1-c)}xF(a+1, b+1; c+1; x)F(1-a, 1-b-\ell; 2-c; x).
\end{aligned} \tag{2.2.9}$$

Remark 2.2.1. *Let us consider the case in which conditions A1, A2, E1 and E2' are not satisfied. Even in this case, because $q(x)$ and $r(x)$ are rational functions in a, b, c and x, we have*

$$q(x) = x(1-x)^{1-\ell} q_0(x), \quad r(x) = (1-x)^{1-\ell} r_0(x), \tag{2.2.10}$$

where $q_0(x)$ and $r_0(x)$ are polynomials in x of degree at most $\ell - 1$. Moreover, if the right-hand sides of (2.2.8) and (2.2.9) are meaningful (that is, $c \notin \mathbf{Z}$), then (2.2.8) and (2.2.9) still hold. Note that $q_0(x)$ and $r_0(x)$ may vanish at $x = 0$ or 1 or may be of smaller degree than that stated in (2.2.7).

Now, we consider the case in which $a \in \mathbf{C}$, $b = 1$ and $c \in \mathbf{C} \setminus \mathbf{Z}$. Then, we obtain the following lemma from Remark 2.2.1:

Lemma 2.2.2. *Let ℓ be a positive integer. We represent $(\vartheta + \ell) \cdots (\vartheta + 2)(\vartheta + 1)$ by $H_1(\ell)$. Then, expressing $H_1(\ell)$ in terms of the hypergeometric operator $L(a, 1, c)$, we have*

$$H_1(\ell) = p(\partial) L(a, 1, c) + q(x)\partial + r(x). \tag{2.2.11}$$

Here, $q(x)$ and $r(x)$ are given by

$$q(x) = x(1-x)^{1-\ell} q_0(x), \quad r(x) = (1-x)^{1-\ell} r_0(x), \tag{2.2.12}$$

where, $q_0(x)$ and $r_0(x)$ are polynomials of degree **at most** *$\ell - 1$ given by*

$$\begin{aligned} q_0(x) = &-\frac{(1,\ell)}{1-c}(1-x)^{c-a-1} F(c-a, c-1-\ell; c; x) \\ &+ \frac{(2-c,\ell)}{1-c} F(a,1;c;x) F(1-a,-\ell;2-c;x), \end{aligned} \tag{2.2.13}$$

$$\begin{aligned} r_0(x) = &\,(1,\ell) F(c-a, c-1-\ell; c; x) F(a+1-c, 2-c; 1-c; x) \\ &- \frac{a(2-c,\ell)}{c(1-c)} x F(a+1, 2; c+1; x) F(1-a, -\ell; 2-c; x). \end{aligned} \tag{2.2.14}$$

We close this section by presenting the following lemma:

Lemma 2.2.3. *Writing*

$$y_1(x) := F(a, 1; c; x), \;\; y_2(x) := x^{1-c}(1-x)^{c-a-1}, \tag{2.2.15}$$

we have

$$y_1(x) = -(1-c) y_2(x) \int_0^x \frac{1}{t(1-t) y_2(t)} dt \;\; (\Re c > 1), \tag{2.2.16}$$

$$H_1(\ell) y_1(x) = (1,\ell) F(a, 1+\ell; c; x), \tag{2.2.17}$$

$$H_1(\ell) y_2(x) = (2-c,\ell) y_2(x)(1-x)^{-\ell} F(1-a, -\ell; 2-c; x). \tag{2.2.18}$$

First, regarding (2.2.16), see (2.1) in [Du]. Note that (2.2.16) implies that $F(a, 1; c; x)$ can be expressed in terms of the incomplete beta function. Next, it is easily shown that (2.2.17) can be obtained from (2.2.2). Finally, we now demonstrate (2.2.18). It is known that

$$(\vartheta+b)x^{1-c}F(a+1-c,b+1-c;2-c;x)$$
$$=(b+1-c)x^{1-c}F(a+1-c,b+2-c;2-c;x) \quad (2.2.19)$$

(cf. Lemma 1.2.2). Then, applying the Euler transformation to both sides of (2.2.19), we obtain

$$(\vartheta+b)x^{1-c}(1-x)^{c-a-b}F(1-a,1-b;2-c;x)$$
$$=(b+1-c)x^{1-c}(1-x)^{c-a-b-1}F(1-a,-b;2-c;x), \quad (2.2.20)$$

and thus,

$$H(\ell)x^{1-c}(1-x)^{c-a-b}F(1-a,1-b;2-c;x)$$
$$=(b+1-c,\ell)x^{1-c}(1-x)^{c-a-b-\ell}F(1-a,1-b-\ell;2-c;x). \quad (2.2.21)$$

Substituting $b=1$ into (2.2.21), we obtain (2.2.18).

2.3 A proof of the theorem

In this section, we prove Theorem 2.1.1 using Lemmas 2.2.2 and 2.2.3.

Operating with $H_1(\ell)$ on $F(a,1;c;x)$, from Lemma 2.2.3, we obtain

$$\begin{aligned}
&F(a,1+\ell;c;x)=\frac{H_1(\ell)}{(1,\ell)}F(a,1;c;x)\\
&=\frac{1}{(1,\ell)}(q(x)\partial+r(x))\left(y_2(x)\int_0^x\frac{-(1-c)}{t(1-t)y_2(t)}dt\right)\\
&=\frac{1}{(1,\ell)}\left[(q(x)\partial+r(x))\,y_2(x)\times\int_0^x\frac{-(1-c)}{t(1-t)y_2(t)}dt-\frac{(1-c)q(x)}{x(1-x)}\right]\\
&=\frac{1}{(1,\ell)}\left[H_1(\ell)y_2(x)\times\int_0^x\frac{-(1-c)}{t(1-t)y_2(t)}dt-\frac{(1-c)q(x)}{x(1-x)}\right]\\
&=\frac{(2-c,\ell)}{(1,\ell)}y_2(x)(1-x)^{-\ell}F(1-a,-\ell;2-c;x)\int_0^x\frac{-(1-c)}{t(1-t)y_2(t)}dt-\frac{(1-c)q(x)}{(1,\ell)x(1-x)}
\end{aligned} \quad (2.3.1)$$

for $\Re c>1$. In particular, for any root λ of the polynomial $F(1-a,-\ell;2-c;x)$, we have

$$F(a,1+\ell;c;\lambda)=-\frac{(1-c)q(\lambda)}{(1,\ell)\lambda(1-\lambda)}, \quad (2.3.2)$$

and thus, from (2.2.12), this implies

$$F(a,1+\ell;c;\lambda)=-\frac{(1-c)q_0(\lambda)}{(1,\ell)(1-\lambda)^{\ell}}. \quad (2.3.3)$$

In addition, substituting λ for x in (2.2.13), we obtain

$$F(c-a, c-1-\ell; c; \lambda) = -\frac{(1-c)}{(1,\ell)}(1-\lambda)^{a+1-c}q_0(\lambda). \tag{2.3.4}$$

Here, we have assumed that $\Re c > 1$. However, because both sides of (2.3.3) are analytic functions of c, (2.3.3) is valid even if this condition is not satisfied, by virtue of analytic continuation. The same holds for (2.3.4). This completes the proof of Theorem 2.1.1.

Finally, using a symmetry of $q_0(x)$, we confirm the statement given in Remark 2.1.2. We begin by assuming that $a \notin \mathbf{Z}$. Then, although we have expressed $q_0(x)$ as in (2.2.13), it can also be expressed as

$$\begin{aligned} q_0(x) &= x^{\ell-1}\times \\ &\left[-\frac{(2-c,\ell)(1,\ell)}{(a-1)(2-a,\ell)}\left(\frac{1}{x}\right)^{\ell}\left(1-\frac{1}{x}\right)^{c-a-1} F\left(c-a, 1-a; 2-a+\ell; \frac{1}{x}\right)\right. \\ &\left.+\frac{(-1)^{\ell}(1-a,\ell)}{a-1}F\left(2-c, 1; 2-a; \frac{1}{x}\right)F\left(c-1-\ell, -\ell; a-\ell; \frac{1}{x}\right)\right] \end{aligned} \tag{2.3.5}$$

(cf. Theorems 1.3.7 and 1.3.8). Note that the right-hand side of (2.3.5) is meaningful by the above assumption. Therefore, we obtain (2.1.7).

References

[AAR] G.E.Andrews, R.Askey and R.Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[AZ] M.Apagodu and D.Zeilberger, *Searching for strange hypergeometric identities by Sheer Brute Force*, Integers, **8** (2008), A36, 6 pp.

[Du] J.Dutka, *The incomplete beta function —a historical profile.* Arch. Hist. Exact Sci. **24**(1981), no. **1**, 11–29.

[Ek] S.B.Ekhad, *Forty "strange" computer-discovered[and computer-proved(of course)] hypergeometric series evaluations*, The Personal Journal of Ekhad and Zeilberger, http://www.math.rutgers.edu/∼zeilberg/mamarim/mamarimhtml/ strange.html, (2004).

[Erd] A.Erdélyi (editor), *Higher transcendental functions Vol.1*, McGraw-Hill, (1953).

[Go] R.W.Gosper, *A letter to D. Stanton*, XEROX Palo Alto Research Center, 21 December 1977.

[GS] I.Gessel and D.Stanton, *Strange evaluations of hypergeometric series*, SIAM J. Math. Anal., **13**(1982), no. **2**, 295–308.

[IKSY] K.Iwasaki, H.Kimura, S.Shimomura and M.Yoshida, *From Gauss to Painlevé —A modern theory of special functions*, Vieweg Verlag, Wiesbaden(1991).

[Ko] W.Koepf, *Hypergeometric summation —An algorithmic approach to summation and special function identities*, Advanced lectures in mathematics, Friedr. Vieweg & Sohn, Braunschweig(1998).

[PWZ] M.Petkovšek, H.Wilf and D.Zeilberger, *A=B*, A.K.Peters, Wellesley(1996).

[V] R.Vidunas, *Algebraic transformations of Gauss hypergeometric functions*, Funkcial. Ekvac., **52**(2009), no. **2**, 139–180.

[WoGo] Wolfram MathWorld, *Gosper's algorithm*, http://mathworld.wolfram.com/GospersAlgorithm.html.

[WoHy] Wolfram MathWorld, *Hypergeometric identity*, http://mathworld.wolfram.com/HypergeometricIdentity.html.

[WoWZ] Wolfram MathWorld, *Wilf-Zeilberger Pair*, http://mathworld.wolfram.com/Wilf-ZeilbergerPair.html.

[WoZe] Wolfram MathWorld, *Zeilberger's algorithm*, http://mathworld.wolfram.com/ZeilbergersAlgorithm.html.

Chapter 3

Special values of the hypergeometric series

3.1 Introduction

There are many known identities for the hypergeometric series $F(a, b; c; x)$. For example,

$$F(a, -n; c; 1) = \frac{(c-a, n)}{(c, n)}, \tag{3.1.1}$$

where $n \in \mathbf{Z}_{\geq 0}$, has been known since the 13th century. Today, this is called the Chu-Vandermonde equality (cf. Corollary 2.2.3 in [AAR]). In 1812, Gauss proved the identity

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{3.1.2}$$

for $\Re(c-a-b) > 0$. This is called the Gauss summation formula (cf. [48] in 24 of [Ga]). The goal of this chapter is to obtain expressions for values of the hypergeometric series at some points other than $x = 1$. These expressions are obtained in terms of gamma functions, together with certain elementary functions, as (3.1.1) and (3.1.2). Such expressions are often called **closed form expressions**.

There are many methods for obtaining closed form expressions for values of the hypergeometric series. In the latter half of the 19th century, such expressions valid for points other than $x = 1$ were derived using algebraic transformations of the hypergeometric series (cf. [V] for a treatment of algebraic transformations of the hypergeometric series). In the last several decades, many new methods that exploit progress in computer technology have been formulated: Gosper's algorithm, the W-Z method, Zeilberger's algorithm, etc. (cf. [Ko] and [PWZ]). These algorithms have been used to obtain and prove hypergeometric identities expressed in closed forms. In this chapter, we employ three term relations of the hypergeometric series as Gauss did (cf. 17 of [Ga]), and thereby find several new identities. Because our method systematically yields almost all known identities of this type, for completeness, we tabulate the entire set.

As stated in Section 1.1, for a given triple of integers $(k, l, m) \in \mathbf{Z}^3$, there exists a unique pair of rational functions $(Q(x), R(x)) \in (\mathbf{Q}(a, b, c, x))^2$ satisfying

$$F(a+k, b+l; c+m; x) = Q(x)F(a+1, b+1; c+1; x) + R(x)F(a, b; c; x). \quad (3.1.3)$$

From this, we obtain

$$\begin{aligned} &F(a+nk, b+nl; c+nm; x) \\ &= Q^{(n)}(x)F(a+(n-1)k+1, b+(n-1)l+1; c+(n-1)m+1; x) \\ &+ R^{(n)}(x)F(a+(n-1)k, b+(n-1)l, c+(n-1)m; x), \end{aligned}$$

where

$$\begin{aligned} Q^{(n)}(x) &:= Q(x)|_{(a,b,c)\to(a+(n-1)k, b+(n-1)l, c+(n-1)m)}, \\ R^{(n)}(x) &:= R(x)|_{(a,b,c)\to(a+(n-1)k, b+(n-1)l, c+(n-1)m)}. \end{aligned}$$

Let (a, b, c) be a triple such that the number of solutions of the system

$$Q^{(n)}(x) = 0 \quad \text{for } n = 1, 2, 3, \cdots \quad (3.1.4)$$

is finite, and let x $(\neq 0, 1)$ be one of its solutions; we call such a quadruple (a, b, c, x) an **admissible quadruple** (cf. Remark 3.2.4). For an admissible quadruple, we have

$$F(a, b; c; x) = \frac{1}{R^{(1)}(x)R^{(2)}(x)\cdots R^{(n)}(x)} \times F(a+nk, b+nl; c+nm; x). \quad (3.1.5)$$

Substituting $a = -nk-k'$, where k' is an integer satisfying $0 \leq k' < |k|$, into (3.1.5), we find

$$\begin{aligned} &F(-nk-k', b; c; x) \\ &= \left(\frac{1}{R^{(1)}(x)R^{(2)}(x)\cdots R^{(n)}(x)} \right)\Bigg|_{a\to -nk-k'} \times F(-k', b+nl; c+nm; x). \end{aligned} \quad (3.1.6)$$

We also obtain

$$\begin{aligned} &F(a, -nl-l'; c; x) \\ &= \left(\frac{1}{R^{(1)}(x)R^{(2)}(x)\cdots R^{(n)}(x)} \right)\Bigg|_{b\to -nl-l'} \times F(a+nk, -l'; c+nm; x) \end{aligned} \quad (3.1.7)$$

by substituting $b = -nl - l'$, where l' is an integer satisfying $0 \leq l' < |l|$, into (3.1.5). In addition, we see that (3.1.6) and (3.1.7) is valid for any integer n from the constitution method of (a, b, c, x) satisfying (3.1.4) (cf. (3.3.1)). In this way, we can find closed form expressions of $F(a, b; c; x)$. In this chapter, we call closed forms obtained from the relation (3.1.5) **special values** of the hypergeometric series.

The hypergeometric equation $E(a, b, c) : L(a, b, c)y = 0$ (cf. Section 1.1) admits 23 hypergeometric solutions in addition to $F(a, b; c; x)$. For each of these solutions, there exist a relation similar to (3.1.5) (cf. Section 1.3). For example, for the solution $x^{-a}F(a, a+1-c; a+1-b; 1/x)$, we have

$$
\begin{aligned}
&x^{-a-k}F(a+k, a+1-c+k-m; a+1-b+k-l; 1/x) \\
&= \frac{(-1)^{m-k-l+1}c(b,l)(a+1-b,k-l)}{b(c,m)(a+1-c,k-m)}Q(x)x^{-a-1}F(a+1, a+1-c; a+1-b; 1/x) \\
&+ \frac{(-1)^{m-k-l}(b,l)(a+1-b,k-l)}{(c,m)(a+1-c,k-m)}R(x)x^{-a}F(a, a+1-c; a+1-b; 1/x).
\end{aligned}
$$

Thus, because

$$
\begin{aligned}
&F(a, a+1-c; a+1-b; 1/x) \\
&= \frac{(-1)^{n(m-k-l)}(c,nm)(a+1-c,n(k-m))}{(b,nl)(a+1-b,n(k-l))} \frac{1}{R^{(1)}(x)R^{(2)}(x)\cdots R^{(n)}(x)} \\
&\times x^{-nk}F(a+nk, a+1-c+n(k-m); a+1-b+n(k-l); 1/x)
\end{aligned}
\tag{3.1.8}
$$

for an admissible quadruple (a, b, c, x), we also get special values of $F(a, a+1-c; a+1-b; 1/x)$ with this quadruple. The same can be done for the other 22 solutions. Thus, for the lattice point (k, l, m), we are able to obtain special values of 24 hypergeometric series with the above quadruple.

The identity (3.1.8) implies that the special values of the 24 hypergeometric series mentioned above for the lattice points $(k, k-m, k-l)$ coincide with those for the lattice point (k, l, m) (cf. Subsection 3.2.2). In other words, $(k, k-m, k-l)$ is equivalent to (k, l, m) with respect to the obtained special values. In fact, this holds generally, as all 24 lattice points represented by triples corresponding to the 24 hypergeomtric solutions are equivalent with respect to these special values. In addition, from the relation $F(a, b; c; x) = F(b, a; c; x)$, which implies that (k, l, m) is equivalent to (l, k, m), it can also be shown that the 48 $(= 24 \cdot 2)$ lattice points are equivalent with respect to these special values. These 48 lattice points form the orbit of (k, l, m) under the action of the group G on $\mathbf{Z}^3$, where G is the group generated by following mappings

$$
\begin{aligned}
&\sigma_1 : (k, l, m) \to (m-k, l, m), && \sigma_2 : (k, l, m) \to (k, l, k+l-m), \\
&\sigma_3 : (k, l, m) \to (l, k, m), && \sigma_4 : (k, l, m) \to (m-k, m-l, m), \\
&\sigma_5 : (k, l, m) \to (-k, -l, -m).
\end{aligned}
$$

We remark that $G = \langle\sigma_1, \sigma_2\rangle \ltimes (\langle\sigma_3\rangle \times \langle\sigma_4\rangle \times \langle\sigma_5\rangle) = S_3 \ltimes (S_2 \times S_2 \times S_2)$, where S_n is the symmetric group of degree n. From this, regarding these 48 lattice points as equivalent, we can take the following set as a complete system of representatives of the quotient $G\backslash\mathbf{Z}^3$ of this action (cf. Subsection 3.2.2):

$$
\{(k, l, m) \in \mathbf{Z}^3 \mid 0 \le k+l-m \le l-k \le m\}. \tag{3.1.9}
$$

Thus, we only need to investigate the lattice points contained in (3.1.9) to obtain the special values of the hypergeometric series. In this chapter, we tabulate the special values for (k, l, m) satisfying $0 \le k+l-m \le l-k \le m \le 6$.

3.2 Preliminaries

As stated in the previous section, for a given (k, l, m), the 24 hypergeometric solutions of $E(a, b, c)$ with an admissible quadruple (a, b, c, x) have two term relations as (3.1.5) and (3.1.8), say degenerate relations. In this section, we list these degenerate relations explicitly. This list will be used subsequently, when we evaluate special values of the corresponding hypergeometric series. After presenting this list, we give a proof that (3.1.9) can be taken as a complete system of representatives of $G\backslash\mathbf{Z}^3$.

3.2.1 Degenerate relations

In this subsection, we list degenerate relations.

Lemma 3.2.1. *$E(a, b, c)$ admits the following 24 hypergeometric solutions (cf. 2.9 in [Erd]):*

$$\begin{aligned}
y_1(a,b,c,x) &:= F(a,b;c;x)\\
&= (1-x)^{c-a-b}F(c-a,c-b;c;x)\\
&= (1-x)^{-a}F(a,c-b,c;x/(x-1))\\
&= (1-x)^{-b}F(c-a,b;c;x/(x-1)),\\
y_2(a,b,c,x) &:= F(a,b;a+b+1-c;1-x)\\
&= x^{1-c}F(a+1-c,b+1-c;a+b+1-c;1-x)\\
&= x^{-a}F(a,a+1-c;a+b+1-c;1-x^{-1})\\
&= x^{-b}F(b+1-c,b;a+b+1-c;1-x^{-1}),\\
y_3(a,b,c,x) &:= x^{-a}F(a,a+1-c;a+1-b;1/x)\\
&= (-1)^a(-x)^{b-c}(1-x)^{c-a-b}F(1-b,c-b;a+1-b;1/x)\\
&= (-1)^a(1-x)^{-a}F(a,c-b;a+1-b;(1-x)^{-1})\\
&= (-1)^a(-x)^{1-c}(1-x)^{c-a-1}F(a+1-c,1-b;a+1-b;(1-x)^{-1}),\\
y_4(a,b,c,x) &:= x^{-b}F(b+1-c,b;b+1-a;1/x)\\
&= (-1)^b(-x)^{a-c}(1-x)^{c-a-b}F(1-a,c-a;b+1-a;1/x)\\
&= (-1)^b(1-x)^{-b}F(b,c-a;b+1-a;(1-x)^{-1})\\
&= (-1)^b(-x)^{1-c}(1-x)^{c-b-1}F(b+1-c,1-a;b+1-a;(1-x)^{-1}),\\
y_5(a,b,c,x) &:= x^{1-c}F(a+1-c,b+1-c;2-c;x)\\
&= x^{1-c}(1-x)^{c-a-b}F(1-a,1-b;2-c;x)\\
&= x^{1-c}(1-x)^{c-a-1}F(a+1-c,1-b;2-c;x/(x-1))\\
&= x^{1-c}(1-x)^{c-b-1}F(b+1-c,1-a,2-c;x/(x-1)),\\
y_6(a,b,c,x) &:= (1-x)^{c-a-b}F(c-a,c-b;c+1-a-b;1-x)\\
&= x^{1-c}(1-x)^{c-a-b}F(1-a,1-b;c+1-a-b;1-x)\\
&= x^{a-c}(1-x)^{c-a-b}F(c-a,1-a;c+1-a-b;1-x^{-1})\\
&= x^{b-c}(1-x)^{c-a-b}F(c-b,1-b;c+1-a-b;1-x^{-1}),
\end{aligned}$$

In the above, we must take the appropriate branches of $(-1)^a$ and $(-1)^b$.

We now apply both sides of (1.2.4) to y_i $(i=1,2,\cdots,6)$. First, we have

$$\begin{aligned}y_1(a+k,b+l,c+m,x)=&\frac{ab(c,m)}{c(a,k)(b,l)}q(x)y_1(a+1,b+1,c+1,x)\\&+\frac{(c,m)}{(a,k)(b,l)}r(x)y_1(a,b,c,x)\end{aligned}$$

from (1.1.2). Recall that

$$Q(x)=\frac{ab(c,m)}{c(a,k)(b,l)}q(x),\ R(x)=\frac{(c,m)}{(a,k)(b,l)}r(x) \tag{3.2.1}$$

(cf. (1.1.3)). Next, we apply both sides of (1.2.4) to y_2. This yields

$$\begin{aligned}&\frac{(a,k)(b,l)(c-a-b,m-k-l)}{(c-a,m-k)(c-b,m-l)}y_2(a+k,b+l,c+m,x)\\&=-\frac{ab}{a+b+1-c}q(x)y_2(a+1,b+1,c+1,x)+r(x)y_2(a,b,c,x)\end{aligned}$$

(cf. (1.3.2) and Lemma 1.2.2). Combining this and (3.2.1), we obtain

$$\begin{aligned}&y_2(a+k,b+l,c+m,x)\\&=\frac{c(c-a,m-k)(c-b,m-l)}{(c-a-b-1)(c-a-b,m-k-l)(c,m)}Q(x)y_2(a+1,b+1,c+1,x)\\&+\frac{(c-a,m-k)(c-b,m-l)}{(c-a-b,m-k-l)(c,m)}R(x)y_2(a,b,c,x)\end{aligned}$$

Finally, applying both sides of (1.2.4) to y_i $(i=3,4,5,6)$, we have the following:

Lemma 3.2.2. *We define $Q(x)$ and $R(x)$ as (3.1.3). Then, we have*

$$\begin{aligned}&y_1(a+k,b+l,c+m,x)=Q(x)y_1(a+1,b+1,c+1,x)+R(x)y_1(a,b,c,x),\\&y_2(a+k,b+l,c+m,x)\\&=\frac{c(c-a,m-k)(c-b,m-l)}{(c-a-b-1)(c,m)(c-a-b,m-k-l)}Q(x)y_2(a+1,b+1,c+1,x)\\&+\frac{(c-a,m-k)(c-b,m-l)}{(c,m)(c-a-b,m-k-l)}R(x)y_2(a,b,c,x),\\&y_3(a+k,b+l,c+m,x)\\&=\frac{(-1)^{m+1-k-l}c(b,l)(a+1-b,k-l)}{b(c,m)(a+1-c,k-m)}Q(x)y_3(a+1,b+1,c+1,x)\\&+\frac{(-1)^{m-k-l}(b,l)(a+1-b,k-l)}{(c,m)(a+1-c,k-m)}R(x)y_3(a,b,c,x),\\&y_4(a+k,b+l,c+m,x)\\&=\frac{(-1)^{m+1-k-l}c(a,k)(b+1-a,l-k)}{a(c,m)(b+1-c,l-m)}Q(x)y_4(a+1,b+1,c+1,x)\\&+\frac{(-1)^{m-k-l}(a,k)(b+1-a,l-k)}{(c,m)(b+1-c,l-m)}R(x)y_4(a,b,c,x),\end{aligned}$$

$$\begin{aligned}&y_5(a+k,b+l,c+m,x)\\&=\frac{c(1-c)(a,k)(b,l)(2-c,-m)}{ab(c,m)(a+1-c,k-m)(b+1-c,l-m)}Q(x)y_5(a+1,b+1,c+1,x)\\&+\frac{(a,k)(b,l)(2-c,-m)}{(c,m)(a+1-c,k-m)(b+1-c,l-m)}R(x)y_5(a,b,c,x),\\&y_6(a+k,b+l,c+m,x)\\&=\frac{c(a+b-c)(a,k)(b,l)}{ab(c,m)(a+b-c,k+l-m)}Q(x)y_6(a+1,b+1,c+1,x)\\&+\frac{(a,k)(b,l)}{(c,m)(a+b-c,k+l-m)}R(x)y_6(a,b,c,x).\end{aligned}$$

Using Lemmas 3.2.1 and 3.2.2, we are able to obtain 24 degenerate relations. For example, substituting $y_1(a,b,c,x)=(1-x)^{c-a-b}F(c-a,c-b;c;x)$ into the top formula in Lemma 3.2.2, we obtain

$$\begin{aligned}&(1-x)^{m-k-l}F(c-a+m-k,c-b+m-l;c+m;x)\\&\quad=Q(x)(1-x)^{-1}F(c-a,c-b;c+1;x)+R(x)F(c-a,c-b;c;x).\end{aligned}\tag{3.2.2}$$

Therefore, defining

$$S^{(n)}:=\frac{1}{R^{(1)}(x)R^{(2)}(x)\cdots R^{(n)}(x)},$$

where $R^{(n)}(x):=R(x)|_{(a,b,c)\mapsto(a+(n-1)k,b+(n-1)l,c+(n-1)m)}$, we find that

$$\begin{aligned}&F(c-a,c-b;c;x)\\&=S^{(n)}(1-x)^{n(m-k-l)}F(c-a+n(m-k),c-b+n(m-l);c+nm;x)\end{aligned}$$

for an admissible quadruple (a,b,c,x). The following is obtained similarly from Lemmas 3.2.1 and 3.2.2.

Proposition 3.2.3. *Fix $(k,l,m)\in\mathbf{Z}^3$. For an admissible quadruple (a,b,c,x), we obtain the following 24 degenerate relations:*

(i) $F(a,b;c;x)=S^{(n)}F(a+nk,b+nl;c+nm;x)$,

(ii) $F(c-a,c-b;c;x)$
$=S^{(n)}(1-x)^{n(m-k-l)}F(c-a+n(m-k),c-b+n(m-l);c+nm;x)$,

(iii) $F(a,c-b;c;x/(x-1))$
$=S^{(n)}(1-x)^{-nk}F(a+nk,c-b+n(m-l);c+nm;x/(x-1))$,

(iv) $F(c-a,b;c;x/(x-1))$
$=S^{(n)}(1-x)^{-nl}F(c-a+n(m-k),b+nl;c+nm;x/(x-1))$,

(v) $F(a,b;a+b+1-c;1-x)$
$=\dfrac{(c,nm)(c-a-b,n(m-k-l))}{(c-a,n(m-k))(c-b,n(m-l))}S^{(n)}$
$\times F(a+nk,b+nl;a+b+1-c+n(k+l-m);1-x)$,

(vi) $F(a+1-c,b+1-c;a+b+1-c;1-x)$

$$= \frac{(c,nm)(c-a-b,n(m-k-l))}{(c-a,n(m-k))(c-b,n(m-l))} S^{(n)} x^{-nm}$$

$$\times F(a+1-c+n(k-m),b+1-c+n(l-m)$$

$$;a+b+1-c+n(k+l-m);1-x),$$

(vii) $F(a,a+1-c;a+b+1-c;1-x^{-1})$

$$= \frac{(c,nm)(c-a-b,n(m-k-l))}{(c-a,n(m-k))(c-b,n(m-l))} S^{(n)} x^{-nk}$$

$$\times F(a+nk,a+1-c+n(k-m);a+b+1-c+n(k+l-m);1-x^{-1}),$$

(viii) $F(b+1-c,b;a+b+1-c;1-x^{-1})$

$$= \frac{(c,nm)(c-a-b,n(m-k-l))}{(c-a,n(m-k))(c-b,n(m-l))} S^{(n)} x^{-nl}$$

$$\times F(b+1-c+n(l-m),b+nl;a+b+1-c+n(k+l-m);1-x^{-1}),$$

(ix) $F(a,a+1-c;a+1-b;1/x)$

$$= \frac{(-1)^{n(m-k-l)}(c,nm)(a+1-c,n(k-m))}{(b,nl)(a+1-b,n(k-l))} S^{(n)} x^{-nk}$$

$$\times F(a+nk,a+1-c+n(k-m);a+1-b+n(k-l);1/x),$$

(x) $F(1-b,c-b;a+1-b;1/x)$

$$= \frac{(-1)^{n(m-l)}(c,nm)(a+1-c,n(k-m))}{(b,nl)(a+1-b,n(k-l))} S^{(n)} (-x)^{n(l-m)}(1-x)^{n(m-k-l)}$$

$$\times F(1-b-nl,c-b+n(m-l);a+1-b+n(k-l);1/x),$$

(xi) $F(a,c-b;a+1-b;(1-x)^{-1})$

$$= \frac{(-1)^{n(m-l)}(c,nm)(a+1-c,n(k-m))}{(b,nl)(a+1-b,n(k-l))} S^{(n)} (1-x)^{-nk}$$

$$\times F(a+nk,c-b+n(m-l);a+1-b+n(k-l);(1-x)^{-1}),$$

(xii) $F(a+1-c,1-b;a+1-b;(1-x)^{-1})$

$$= \frac{(-1)^{n(m-l)}(c,nm)(a+1-c,n(k-m))}{(b,nl)(a+1-b,n(k-l))} S^{(n)} (-x)^{-nm}(1-x)^{n(m-k)}$$

$$\times F(a+1-c+n(k-m),1-b-nl;a+1-b+n(k-l);(1-x)^{-1}),$$

(xiii) $F(b+1-c,b;b+1-a;1/x)$

$$= \frac{(-1)^{n(m-k-l)}(c,nm)(b+1-c,n(l-m))}{(a,nk)(b+1-a,n(l-k))} S^{(n)} x^{-nl}$$

$$\times F(b+1-c+n(l-m),b+nl;b+1-a+n(l-k);1/x),$$

(xiv) $F(1-a,c-a;b+1-a;1/x)$

$$= \frac{(-1)^{n(m-k)}(c,nm)(b+1-c,n(l-m))}{(a,nk)(b+1-a,n(l-k))} S^{(n)} (-x)^{n(k-m)}(1-x)^{n(m-k-l)}$$

$$\times F(1-a-nk,c-a+n(m-k);b+1-a+n(l-k);1/x),$$

(xv) $F(b,c-a;b+1-a;(1-x)^{-1})$

$$= \frac{(-1)^{n(m-k)}(c,nm)(b+1-c,n(l-m))}{(a,nk)(b+1-a,n(l-k))} S^{(n)}(1-x)^{-nl}$$
$$\times F(b+nl, c-a+n(m-k); b+1-a+n(l-k); (1-x)^{-1}),$$
(xvi) $F(b+1-c, 1-a; b+1-a; (1-x)^{-1})$
$$= \frac{(-1)^{n(m-k)}(c,nm)(b+1-c,n(l-m))}{(a,nk)(b+1-a,n(l-k))} S^{(n)}(-x)^{-nm}(1-x)^{n(m-l)}$$
$$\times F(b+1-c+n(l-m), 1-a-nk; b+1-a+n(l-k); (1-x)^{-1}),$$
(xvii) $F(a+1-c, b+1-c; 2-c; x)$
$$= \frac{(c,nm)(a+1-c,n(k-m))(b+1-c,n(l-m))}{(a,nk)(b,nl)(2-c,-nm)} S^{(n)}x^{-nm}$$
$$\times F(a+1-c+n(k-m), b+1-c+n(l-m); 2-c-nm; x),$$
(xviii) $F(1-a, 1-b; 2-c; x)$
$$= \frac{(c,nm)(a+1-c,n(k-m))(b+1-c,n(l-m))}{(a,nk)(b,nl)(2-c,-nm)} S^{(n)}x^{-nm}(1-x)^{n(m-k-l)}$$
$$\times F(1-a-nk, 1-b-nl; 2-c-nm; x),$$
(xix) $F(a+1-c, 1-b; 2-c; x/(x-1))$
$$= \frac{(c,nm)(a+1-c,n(k-m))(b+1-c,n(l-m))}{(a,nk)(b,nl)(2-c,-nm)} S^{(n)}x^{-nm}(1-x)^{n(m-k)}$$
$$\times F(a+1-c+n(k-m), 1-b-nl; 2-c-nm; x/(x-1)),$$
(xx) $F(b+1-c, 1-a; 2-c; x/(x-1))$
$$= \frac{(c,nm)(a+1-c,n(k-m))(b+1-c,n(l-m))}{(a,nk)(b,nl)(2-c,-nm)} S^{(n)}x^{-nm}(1-x)^{n(m-l)}$$
$$\times F(b+1-c+n(l-m), 1-a-nk; 2-c-nm; x/(x-1)),$$
(xxi) $F(c-a, c-b; c+1-a-b; 1-x)$
$$= \frac{(c,nm)(a+b-c,n(k+l-m))}{(a,nk)(b,nl)} S^{(n)}(1-x)^{n(m-k-l)}$$
$$\times F(c-a+n(m-k), c-b+n(m-l); c+1-a-b+n(m-k-l); 1-x),$$
(xxii) $F(1-a, 1-b; c+1-a-b; 1-x)$
$$= \frac{(c,nm)(a+b-c,n(k+l-m))}{(a,nk)(b,nl)} S^{(n)}x^{-nm}(1-x)^{n(m-k-l)}$$
$$\times F(1-a-nk, 1-b-nl; c+1-a-b+n(m-k-l); 1-x),$$
(xxiii) $F(c-a, 1-a; c+1-a-b; 1-x^{-1})$
$$= \frac{(c,nm)(a+b-c,n(k+l-m))}{(a,nk)(b,nl)} S^{(n)}x^{n(k-m)}(1-x)^{n(m-k-l)}$$
$$\times F(c-a+n(m-k), 1-a-nk; c+1-a-b+n(m-k-l); 1-x^{-1}),$$
(xxiv) $F(c-b, 1-b; c+1-a-b; 1-x^{-1})$
$$= \frac{(c,nm)(a+b-c,n(k+l-m))}{(a,nk)(b,nl)} S^{(n)}x^{n(l-m)}(1-x)^{n(m-k-l)}$$
$$\times F(c-b+n(m-l), 1-b-nl; c+1-a-b+n(m-k-l); 1-x^{-1}).$$

Remark 3.2.4. *From the treatment presented in Section 3.1, it is understood that*

x in an admissible quadruple is not a free parameter. The reason for this is the following. If x were a free parameter, then some of the 24 degenerate relations may be incorrect, while the special values obtained from the remaining correct relations would be trivial. (Here 'trivial' means obvious from the definition of $F(a,b;c;x)$.) For example, in the case $(k,l,m) = (0,1,1)$, we have

$$F\left(a,b+1;c+1;x\right) = \frac{a\left(1-x\right)F\left(a+1,b+1;c+1;x\right)}{a-c} - \frac{cF\left(a,b;c;x\right)}{a-c}. \quad (3.2.3)$$

This implies that

$$Q^{(n)}(x) = \frac{a(1-x)}{a-c-n+1},$$

and, therefore, the (a,b,c,x) satisfying (3.1.4) are $(a,b,c,x) = (0,b,c,x)$ and $(a,b,c,1)$. In the former case, we find that the 12 degenerate relations (xiii)-(xxiv) *are incorrect because the denominator of the coefficient of $y_i(a+1,b+1,c+1,x)$ ($i = 4,5,6$) in Lemma 3.2.2 contains a as a factor. In addition, the special values obtained from the 12 correct degenerate relations are trivial. For this reason, we exclude cases in which x is a free parameter from consideration.*

3.2.2 A complete system of representatives of $G\backslash\mathbf{Z}^3$

In this subsection, we prove that we only need to investigate the lattice points contained in (3.1.9) to obtain the special values of the hypergeometric series.

First, we show that the 48 lattice points that form the orbit of (k,l,m) under the action of G on $\mathbf{Z}^3$ (cf. Section 3.1) are equivalent with respect to the obtained special values. We do this by considering two particular examples, $(m-k,m-l,m)$ and $(k,m-l,m)$, and demonstrating that they are equivalent to (k,l,m) with respect to the special values. The general result follows by analogy.

To begin with, we consider the case of $(m-k,m-l,m)$. We start by replacing (a,b,c,x) with $(c-a,c-b,c,x)$. Then, the three term relation for $(m-k,m-l,m)$ is expressed as

$$\begin{aligned}&F(c-a+m-k,c-b+m-l;c+m;x)\\&=Q'(x)F(c-a+1;c-b+1;c+1;x)+R'(x)F(c-a,c-b;c;x),\end{aligned}$$

and that for $(0,0,1)$ is

$$\begin{aligned}F(c-a,c-b;c+1;x) = {}&\frac{(c-a)(c-b)(1-x)}{ab}F(c-a+1;c-b+1;c+1;x)\\&+\frac{c(a+b-c)}{ab}F(c-a,c-b;c;x).\end{aligned}$$

These two three term relations lead to

$$\begin{aligned}&\frac{(c-a)(c-b)(1-x)}{ab}F(c-a+m-k,c-b+m-l;c+m;x)\\&=Q'(x)F(c-a,c-b,c+1;x)\end{aligned}$$

$$+\left\{\frac{(c-a)(c-b)(1-x)}{ab}R'(x)-\frac{(c-a)(a+b-c)}{ab}Q'(x)\right\}F(c-a,c-b;c;x). \tag{3.2.4}$$

Equating (3.2.2) with (3.2.4), we have

$$Q(x)=\frac{ab}{(c-a)(c-b)}(1-x)^{m-k-l}Q'(x). \tag{3.2.5}$$

Hence, the admissible quadruples for (k,l,m) coincide with those for $(m-k,m-l,m)$. This implies that the special values for (k,l,m) coincide with those for $(m-k,m-l,m)$; that is, $(m-k,m-l,m)$ is equivalent to (k,l,m) with respect to the obtained special values.

We next show that $(k,m-l,m)$ is equivalent to (k,l,m) with respect to the obtained special values. From the top formula in Lemma 3.2.2, we have

$$\begin{aligned}&(1-x)^{-k}F(a+k,c-b+m-l;c+m;x/(x-1))\\&=Q(x)(1-x)^{-1}F(a+1,c-b;c+1;x/(x-1))+R(x)F(a,c-b;c;x/(x-1)).\end{aligned} \tag{3.2.6}$$

Also, making the replacement $(a,b,c,x)\to(a,c-b,c,x/(x-1))$, we find that the three term relation for $(k,m-l,m)$ is expressible as

$$\begin{aligned}&F(a+k,c-b+m-l;c+m;x/(x-1))\\&=Q''(x)F(a+1,c-b+1;c+1;x/(x-1))+R''(x)F(a,c-b;c;x/(x-1)).\end{aligned} \tag{3.2.7}$$

Then, using

$$\begin{aligned}F(a+1,c-b,c+1;x/(x-1))=&\frac{(b-c)F(a+1,c-b+1;c+1;x/(x-1))}{b(1-x)}\\&+\frac{cF(a,c-b;c;x/(x-1))}{b}\end{aligned}$$

and (3.2.7), we get

$$\begin{aligned}&\frac{b-c}{b(1-x)}F(a+k,c-b+m-l;c+m;x/(x-1))\\&=Q''(x)F(a+1,c-b;c+1;x/(x-1))\\&+\left\{\frac{(b-c)}{b(1-x)}R''(x)-\frac{c}{b}Q''(x)\right\}F(a,c-b;c;x/(x-1)).\end{aligned} \tag{3.2.8}$$

Finally, equating (3.2.6) and (3.2.8), we obtain

$$Q(x)=\frac{b(1-x)^{2-k}}{b-c}Q''(x). \tag{3.2.9}$$

Therefore, the admissible quadruples for (k,l,m) coincide with those for $(k,m-l,m)$. This means that $(k,m-l,m)$ is equivalent to (k,l,m) with respect to the obtained special values.

We can show that the other 46 lattice points are equivalent to (k,l,m) with respect to the obtained special values analogously.

Remark 3.2.5. *The reason that we assumed $x \neq 0, 1$ in the definition of an admissible quadruple (a, b, c, x) is that if we do not make this assumption, there are cases in which the admissible quadruples for (k, l, m) do not coincide with those for the other 47 points (cf. (3.2.5), (3.2.9) and Subsection 3.3.1).*

Next, we determine a complete system of representatives of the quotient of the action of G on $\mathbf{Z}^3$. Recall that $G = \langle \sigma_1, \sigma_2 \rangle \ltimes (\langle \sigma_3 \rangle \times \langle \sigma_4 \rangle \times \langle \sigma_5 \rangle) = S_3 \ltimes (S_2 \times S_2 \times S_2)$. To begin with, we find a complete system of representatives of the quotient of the action of $(\langle \sigma_3 \rangle \times \langle \sigma_4 \rangle \times \langle \sigma_5 \rangle)$ on $\mathbf{Z}^3$. From σ_5, we see that $(-k, -l, -m)$ is identical to (k, l, m). Hence, we can assume $0 \leq m$. Further, because $(m-k, m-l, m)$ is identical to (k, l, m) by σ_4, we can assume $0 \leq k + l - m$. In addition, because (l, k, m) is identical to (k, l, m) by σ_3, we can assume $0 \leq l - k$. Thus, we can take $\{(k, l, m) \in \mathbf{Z}^3 \mid 0 \leq k+l-m, l-k, m\}$, which we call D, as a complete system of representatives of the quotient of the action of $(\langle \sigma_3 \rangle \times \langle \sigma_4 \rangle \times \langle \sigma_5 \rangle)$ on $\mathbf{Z}^3$. We now proceed to obtain a complete system of representatives of the quotient of the action of G on $\mathbf{Z}^3$. If $(k, l, m) \in D$, then D also contains $(m-k, l, m)$, $(k, l, k+l-m)$, $(l-m, l, k+l-m)$, $(l-m, l, l-k)$ and $(m-k, l, l-k)$. Then, with

$$\alpha = k + l - m, \ \beta = l - k, \ \gamma = m$$
$$\mathbf{e_1} = (1/2, 1/2, 0), \ \mathbf{e_2} = (-1/2, 1/2, 0), \ \mathbf{e_3} = (1/2, 1/2, 1),$$

we have the following:

$$(k, l, m) = \alpha \mathbf{e_1} + \beta \mathbf{e_2} + \gamma \mathbf{e_3}, \ (m-k, l, m) = \alpha \mathbf{e_2} + \beta \mathbf{e_1} + \gamma \mathbf{e_3},$$
$$(k, l, k+l-m) = \alpha \mathbf{e_3} + \beta \mathbf{e_2} + \gamma \mathbf{e_1}, \ (l-m, l, k+l-m) = \alpha \mathbf{e_3} + \beta \mathbf{e_1} + \gamma \mathbf{e_2},$$
$$(l-m, l, l-k) = \alpha \mathbf{e_1} + \beta \mathbf{e_3} + \gamma \mathbf{e_2}, \ (m-k, l, l-k) = \alpha \mathbf{e_2} + \beta \mathbf{e_3} + \gamma \mathbf{e_1}.$$

Together, these imply that we are able to adopt (3.1.9) as a complete system of the quotient of the action of G on $\mathbf{Z}^3$. Thus, we only have to investigate the lattice points contained in (3.1.9) to obtain the special values of the hypergeometric series.

3.3 Derivation of special values

In this section, we derive the special values for some lattice points as examples.

For a given lattice point, the numerator of $Q^{(n)}(x)$ is a polynomial in n over $\mathbf{Z}[a, b, c, x]$. Therefore, in order to obtain the admissible quadruples, we seek the (a, b, c, x) that eliminate all of the coefficients of this polynomial.

Although we implicitly assumed that the parameter c of $F(a, b; c; x)$ is an element of $\mathbf{C} \setminus \{0, -1, -2, \cdots\}$, for the rest of chapter, we expand the definition of $F(a, b; c; x)$. From this point, even if the parameter c is a non-positive integer, we define $F(a, b; c; x)$ as follows if the parameter a is a non-positive integer satisfying $c < a$:

$$F(a, b; c; x) := \sum_{n=0}^{|a|} \frac{(a, n)(b, n)}{(c, n)(1, n)} x^n.$$

3.3.1 Example 1: $(k, l, m) = (0, 1, 1)$

We first consider the case $m = 1$. Then, the only (k, l, m) satisfying (3.1.9) is $(0, 1, 1)$. For this point, the corresponding three term relation is (3.2.3), and this leads to

$$Q^{(n)}(x) = \frac{a(1-x)}{a-c-n+1}.$$

Hence, $(a, b, c, x) = (a, b, c, 1)$ satisfies (3.1.4) (and so does $(0, b, c, x)$, but for the reason explained in Remark 3.2.4, we do not consider this case).

In the case $(a, b, c, x) = (a, b, c, 1)$, because

$$S^{(n)} = \frac{(c-a, n)}{(c, n)},$$

we have

$$F(a, b; c; 1) = \frac{(c-a, n)}{(c, n)} F(a, b+n; c+n; 1).$$

From this, substituting $b = -n$, where $n \in \mathbf{Z}_{\geq 0}$, we get the Chu-Vandermonde equality, (3.1.1). Note that this equality holds even if c is a non-positive integer with $c < b = -n$.

Because $(a, b, c, 1)$ is not an admissible quadruple, we should investigate the other 47 points. For example, considering the case $(k, l, m) = (0, 0, 1)$, we get the Gauss summation formula, (3.1.2). Similarly, the Chu-Vandermonde equality and the Gauss summation formula are provided by the other 46 points.

3.3.2 Example 2: $(k, l, m) = (1, 2, 2)$

In the case $(k, l, m) = (1, 2, 2)$, because

$$\begin{aligned} &F(a+1, b+2, c+2, x) \\ &= \frac{(c+1)(-c+xa) F(a+1, b+1, c+1, x)}{x(a-c)(b+1)} + \frac{c(c+1) F(a, b, c, x)}{x(a-c)(b+1)}, \end{aligned}$$

the numerator of $Q^{(n)}(x)$ is

$$(-4+2x)n^2 + ((c+2a-3)x - 4c+6)n + (1-c+ca-a)x - c^2 + 3c - 2. \tag{3.3.1}$$

Therefore, the admissible quadruple (a, b, c, x) is $(a, b, 2a, 2)$, and

$$S^{(n)} = \frac{(-1)^n (1/2\,b + 1/2, n)}{(a+1/2, n)},$$

where we denote $\frac{1}{2}b$ by $1/2\,b$, and we also denote thus for the rest of this chapter. Hence, (i) in Proposition 3.2.3 leads to

$$F(a, b;\, 2a;\, 2) = \frac{(-1)^n (1/2\,b + 1/2, n)}{(a+1/2, n)} F(a+n, b+2n;\, 2a+2n;\, 2) \tag{3.3.2}$$

Although (3.3.2) is valid by virtue of analytic continuation, this equality regarded as an infinite series expression does not make sense. For this reason, we carry out the degeneration of this into a finite series expression that does make sense. This is done separately in the following cases: $a = -n-1$, $b = -2\,n$ and $b = -2\,n-1$, where $n \in \mathbf{Z}_{\geq 0}$. If $a = -n-1$, then the right hand side of (3.3.2) becomes

$$\frac{(-1)^n\,(1/2\,b+1/2,n)\,F(-1,b+2\,n;-2;2)}{(-1/2-n,n)} = \frac{(2\,n+b+1)\,(1/2\,b+1/2,n)}{(3/2,n)}.$$

If $b = -2\,n$, then the right hand side of (3.3.2) is

$$\frac{(1/2,n)}{(a+1/2,n)}.$$

If $b = -2\,n-1$, then the right hand side of (3.3.2) is equal to

$$\frac{(1,n)\,F\,(a+n,-1;2\,a+2\,n;2)}{(a+1/2,n)} = 0.$$

The special values obtained from (ii), (iii) and (iv) are identical to the above.

Next, using (v) in Proposition 3.2.3, we have

$$F(a,b;\,b+1-a;\,-1) = \frac{2^{2\,n}\,(1/2\,b+1/2,n)}{(b+1-a,n)}F(a+n,b+2\,n;\,b+1-a+n;\,-1). \tag{3.3.3}$$

Substituting $a=-n$ into (3.3.3), this becomes

$$F(-n,b;\,b+1+n;\,-1) = \frac{2^{2\,n}\Gamma\,(1/2\,b+1/2+n)\,\Gamma\,(n+b+1)}{\Gamma\,(1/2\,b+1/2)\,\Gamma\,(2\,n+b+1)}. \tag{3.3.4}$$

Note that the validity of (3.3.4) for any integer n follows from the constitution method of (a,b,c,x) satisfying (3.1.4) (cf. (3.3.1)). The following lemma is known (cf. 5.3 in [Ba]):

Lemma 3.3.1. *(Carlson's theorem) We assume that $f(z)$ and $g(z)$ are regular and of the form $O(e^{k|z|})$, where $k<\pi$, for $\Re z \geq 0$, and $f(z) = g(z)$ for $z = 0,1,2\cdots$. Then, $f(z) = g(z)$ on $\{z \mid \Re z \geq 0\}$.*

It is easily confirmed that both sides of (3.3.4) satisfy the assumption of the above lemma. Hence, (3.3.4) holds for any complex number n for which the left hand side is meaningful. Resultingly, substituting $n=-a$ into (3.3.4), we find

$$F(a,b;\,b+1-a;\,-1) = \frac{2^{-2a}\Gamma\,(1/2\,b+1/2-a)\,\Gamma\,(b+1-a)}{\Gamma\,(1/2\,b+1/2)\,\Gamma\,(b+1-2\,a)}.$$

Now, we consider the case $a,\ b+1-a \in \mathbf{Z}_{\leq 0}$ with $b+1-a<a$. Specifically, we consider the case $a=-n, b=-2\,n-n_1-2$, where $n,\ n_1 \in \mathbf{Z}_{\geq 0}$. Then, (3.3.3) leads to

$$F(a,b;b+1-a;-1) = \frac{2^{2\,n}\,(1/2\,n_1+3/2,n)}{(n_1+2,n)}.$$

Next, we consider the case b, $b+1-a \in \mathbf{Z}_{\leq 0}$ with $b+1-a < b$. Then, if $a = n_1 + 2$, $b = -2\,n$, where n_1, $n \in \mathbf{Z}_{\geq 0}$, (3.3.3) implies

$$F(a, b;\, b+1-a;\, -1) = \frac{(1/2, n)(n_1+2, n)}{(1/2\,n_1+1, n)(1/2\,n_1+3/2, n)}.$$

If $a = n_1 + 2$, $b = -2\,n - 1$, where n_1, $n \in \mathbf{Z}_{\geq 0}$, we find

$$F(a, b;\, b+1-a;\, -1) = 0$$

from (3.3.3).

We are able to obtain special values from the other 19 degenerate relations similarly. Those values are tabulated in the following section.

3.3.3 Example 3: $(k, l, m) = (1, 2, 3)$

For an admissible quadruple $(a, 2a-1/3, 3a, 9)$, we have

$$S^{(n)} = \frac{(-4)^n\,(a+1/2, n)}{(a+2/3, n)}.$$

Now, we consider the special values obtained from (vii) in Proposition 3.2.3. Because (vii) in this case is

$$F(a, 1-2\,a;\, 2/3;\, 8/9) = (-3)^n\, F(a+n, -2\,a+1-2\,n;\, 2/3;\, 8/9),$$

we find

$$F(a, 1-2\,a;\, 2/3;\, 8/9) = \begin{cases} (-3)^n & \text{if } a = -n, \\ (-3)^{-n} & \text{if } a = n+1/2, \\ (-3)^{-n-1} & \text{if } a = n+1. \end{cases}$$

However, we can not directly apply Lemma 3.3.1 to the above identity, because its left hand side does not satisfy the assumption of that lemma. For this reason, we get the special value of $F(a, 1-2\,a;\, 2/3;\, 8/9)$ using the following algebraic transformation of the hypergeometric series:

$$F\left(a, 1-2\,a;\, 2/3;\, x\right) = (1-x)^{1/2\,a-1/4}\, F\left(1/2\,a-1/12, 1/4-1/2\,a;\, 2/3;\, u(x)\right), \tag{3.3.5}$$

with

$$u(x) = \frac{1}{64}\,\frac{x\,(-8+9\,x)^3}{x-1}.$$

We remark that this holds near $x = 0$. Now, we carry out an analytic continuation of each side of (3.3.5) along a curve starting at $x = 0$ and ending at $x = 8/9$, as depicted in Figure 1. Noting that $u(x)$ encircles $u = 1$ once counterclockwise, as shown in Figure 2, we obtain

$$F\left(a, 1-2\,a;\, 2/3;\, 8/9\right) = 2\cdot 3^{-a}\sin\left((5/6-a)\,\pi\right)$$

using the following lemma (cf. Theorem 4.7.2 in [IKSY]):

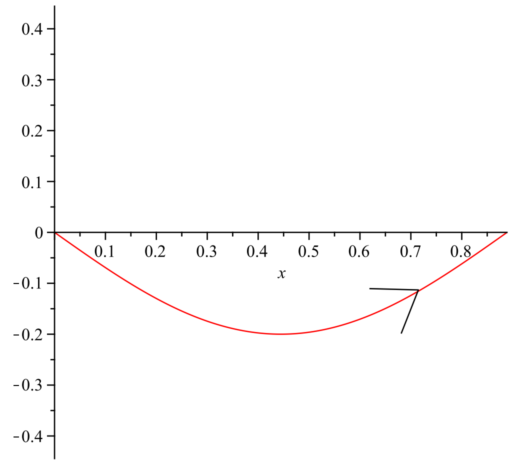

Figure 3.1: x-plane

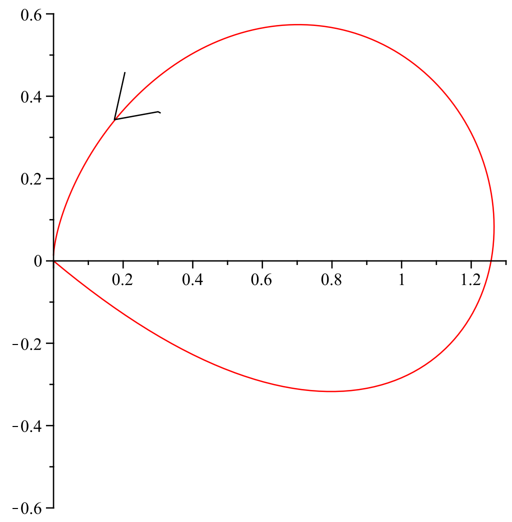

Figure 3.2: u-plane

Lemma 3.3.2. *Let γ_1 be a loop starting and ending at $x = a$, where $0 < a < 1$, and encircling $x = 1$ once in the counterclockwise direction. Then, analytic continuation of $(y_1(a, b, c, x), y_5(a, b, c, x))$ along γ_1 is given by*

$$(y_1(a, b, c, x), y_5(a, b, c, x))P^{-1}AP,$$

where

$$P = \begin{bmatrix} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(2-c)(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} & \frac{\Gamma(a+b-c)\Gamma(2-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\,i\pi\,(c-a-b)} \end{bmatrix}.$$

3.4 Tables of special values

In this section, using Proposition 3.2.3, we tabulate the special values for (k, l, m) satisfying $0 \leq k+l-m \leq l-k \leq m \leq 6$. However, we exclude the Chu-Vandermonde equality, (3.1.1) and the Gauss summation formula, (3.1.2).

Let n, n_1, $n_2 \in \mathbf{Z}_{\geq 0}$. Then, recall that

$$F(a, b; c; x) := \sum_{n=0}^{|a|} \frac{(a, n)(b, n)}{(c, n)(1, n)} x^n$$

when a and c are non-positive integers satisfying $c < a$. Further, recall that we denote $\frac{1}{2}a$ by $1/2\,a$, and so on.

For a given (k, l, m), we use the expression (**)$\leq$(*) to indicate that the values obtained from (**) in Proposition 3.2.3 coincide with or are contained in those obtained from (*).

3.4.1 $m = 1$

$(k, l, m) = (0, 1, 1)$

In this case, there is no admissible quadruple (cf. Subsection 3.1).

3.4.2 $m = 2$

$(k, l, m) = (1, 2, 2)$

In the case that $(k, l, m) = (1, 2, 2)$, we obtain

$$(a, b, c, x) = (a, b, 2\,a, 2),\ S^{(n)} = \frac{(-1)^n (1/2\,b + 1/2, n)}{(a + 1/2, n)}. \tag{1,2,2-1}$$

(1,2,2-1)

$$\text{(i) } F(a, b; 2\,a; 2) = \begin{cases} \dfrac{(b + 1 + 2\,n)\,(1/2\,b + 1/2, n)}{(3/2, n)} & \text{if } a = -n - 1, \\ \dfrac{(1/2, n)}{(a + 1/2, n)} \quad \text{if } b = -2\,n, \\ 0 \quad \text{if } b = -2\,n - 1 \end{cases}$$

(The first case is identical to (4.11) in [Ge]). We find that (ii), (iii), (iv) $\leq$ (i).

(v) $F(a, b; b + 1 - a; -1)$

$$= \begin{cases} \dfrac{2^{-2\,a}\Gamma\,(1/2\,b + 1/2 - a)\,\Gamma\,(b + 1 - a)}{\Gamma\,(1/2\,b + 1/2)\,\Gamma\,(b + 1 - 2\,a)}, \\ \dfrac{(1/2, n_2)(n_1 + 2, n_2)}{(1/2\,n_1 + 1, n_2)(1/2\,n_1 + 3/2, n_2)} \quad \text{if } a = n_1 + 2,\ b = -2\,n_2, \\ 0 \quad \text{if } a = n_1 + 2,\ b = -2\,n_2 - 1, \\ \dfrac{2^{2\,n_1}\,(1/2\,n_2 + 3/2, n_1)}{(n_2 + 2, n_1)} \quad \text{if } a = -n_1, b = -2\,n_1 - n_2 - 2 \end{cases}$$

(The first case is identical to 2.8(47) in [Erd]). We find that (vi)$\leq$(v).

(vii) $F(a, 1 - a;\ b + 1 - a;\ 1/2)$

$$= \begin{cases} \dfrac{2^{-a}\Gamma\,(1/2\,b + 1/2 - a)\,\Gamma\,(b + 1 - a)}{\Gamma\,(1/2\,b + 1/2)\,\Gamma\,(b + 1 - 2\,a)}, \\ \dfrac{2^{n_1}\,(1/2\,n_2 + 3/2, n_1)}{(n_2 + 2, n_1)} \quad \text{if } a = -n_1,\ b = -2\,n_1 - n_2 - 2\ , \\ \dfrac{(n_1 + n_2 + 2, n_1)}{2^{n_1}\,(1/2\,n_2 + 1, n_1)} \text{ if } a = n_1 + 1,\ b = -n_2 - 1 \end{cases}$$

(The first case is identical to 2.8(51) in [Erd]).

(viii) $F(b, b + 1 - 2\,a;\ b + 1 - a;\ 1/2)$

$$= \begin{cases} \dfrac{2^{b-2\,a}\Gamma\,(1/2\,b + 1/2 - a)\,\Gamma\,(b + 1 - a)}{\Gamma\,(1/2\,b + 1/2)\,\Gamma\,(b + 1 - 2\,a)}, \\ \dfrac{(1/2, n_2)\,(n_1 + 2, n_2)}{2^{2\,n_2}\,(1/2\,n_1 + 3/2, n_2)\,(1/2\,n_1 + 1, n_2)} \quad \text{if } a = n_1 + 2,\ b = -2\,n_2\ , \\ 0 \quad \text{if } a = n_1 + 2,\ b = -2\,n_2 - 1 \end{cases}$$

(The first case is identical to 2.8(50) in [Erd]). The special values obtained from (ix)-(xxiv) are contained in the above.

$(k, l, m) = (0, 2, 2)$

When $(k, l, m) = (0, 2, 2)$, we obtain

$$\begin{cases} (a, b, c, x) = (a, b, b + 1 - a, -1), \\ S^{(n)} = \dfrac{(1/2\, b + 1/2, n)\,(1/2\, b + 1 - a, n)}{(1/2\, b + 1/2 - 1/2\, a, n)\,(1/2\, b + 1 - 1/2\, a, n)}. \end{cases} \tag{0,2,2-1}$$

(0,2,2-1) The special values obtained from (0,2,2-1) are contained in those obtained from (1,2,2-1) except the following:

(iv) $F(b, b + 1 - 2\, a;\ b + 1 - a;\ 1/2)$

$$= \begin{cases} \dfrac{(3 + 2\, n_1, n_2)\,(1/2, n_2)}{2^{2\, n_2}\,(n_1 + 2, n_2)\,(n_1 + 3/2, n_2)} & \text{if } a = -2\, n_1 - 2,\ b = -4\, n_1 - 2\, n_2 - 5\ , \\ 0 \quad \text{if } a = -2\, n_1 - 2,\ b = -4\, n_1 - 2\, n_2 - 6. \end{cases}$$

$(k, l, m) = (1, 1, 2)$

In this case, there is no admissible quadruple.

$(k, l, m) = (1, 3, 2)$

In this case, we have

$$(a, b, c, x) = (a, 3\, a - 1, 2\, a, 1/2 + 1/2\, i\sqrt{3}),\ S^{(n)} = \frac{\left(-3/4\, i\sqrt{3}\right)^n (a + 1/3, n)}{(a + 1/2, n)}, \tag{1,3,2-1}$$

$$(a, b, c, x) = (a, 3\, a - 1, 2\, a, 1/2 - 1/2\, i\sqrt{3}),\ S^{(n)} = \frac{\left(3/4\, i\sqrt{3}\right)^n (a + 1/3, n)}{(a + 1/2, n)}. \tag{1,3,2-2}$$

(1,3,2-1)

(i) $F(a, 3\, a - 1;\ 2\, a;\ 1/2 + 1/2\, i\sqrt{3})$

$$= \begin{cases} \dfrac{2^{2\, a - 2/3} e^{1/2\, i(a - 1/3)\pi}\,\Gamma(2/3)\,\Gamma(a + 1/2)}{3^{3/2\, a - 1/2}\,\Gamma(5/6)\,\Gamma(a + 1/3)}, \\ \dfrac{(-i)^{n+1}\, 3^{3/2\, n + 1/2}\,(5/3, n)}{2^{2\, n}\,(3/2, n)} \quad \text{if } a = -1 - n \end{cases}$$

(The first case is identical to 2.8(55) in [Erd] and the second case is identical to Theorem 11 in [Ek]).

(ii) $F(a, 1 - a;\ 2\, a;\ 1/2 + 1/2\, i\sqrt{3})$

$$= \begin{cases} \dfrac{2^{2a-2/3} e^{1/6\, i(1-a)\pi} \Gamma(2/3)\, \Gamma(a+1/2)}{3^{3/2\, a-1/2} \Gamma(5/6)\, \Gamma(a+1/3)}, \\ \dfrac{-(-i)^{n+1}\, 3^{3/2\, n+1/2} \left(1+i\sqrt{3}\right)^{2n-1} (5/3,n)}{2^{4n-1}\, (3/2,n)} & \text{if } a=-1-n. \end{cases}$$

(iii) $F(a, 1-a;\, 2a;\, 1/2 - 1/2\, i\sqrt{3})$

$$= \begin{cases} \dfrac{2^{2a-2/3} e^{1/6\, i(a-1)\pi} \Gamma(2/3)\, \Gamma(a+1/2)}{3^{3/2\, a-1/2} \Gamma(5/6)\, \Gamma(a+1/3)}, \\ \dfrac{-i^{n+1} 3^{3/2\, n+1/2} \left(1-i\sqrt{3}\right)^{2n-1} (5/3,n)}{2^{4n-1}\, (3/2,n)} & \text{if } a=-1-n. \end{cases}$$

(iv) $F(a, 3a-1;\, 2a;\, 1/2 - 1/2\, i\sqrt{3})$

$$= \begin{cases} \dfrac{2^{2a-2/3} e^{1/6\, i(1-3a)\pi} \Gamma(2/3)\, \Gamma(a+1/2)}{3^{3/2\, a-1/2} \Gamma(5/6)\, \Gamma(a+1/3)}, \\ \dfrac{i^{n+1} 3^{3/2\, n+1/2}\, (5/3,n)}{2^{2n}\, (3/2,n)} & \text{if } a=-1-n \end{cases}$$

(The first case is identical to 2.8(56) in [Erd] and the second case is identical to Theorem 11 in [Ek]). The special values obtained from (v)-(xxiv) are contained in the above.

(1,3,2-2) The special values obtained from (1,3,2-2) coincide with those obtained from (1,3,2-1).

3.4.3 $m = 3$

$(k, l, m) = (0, 3, 3)$

In this case, we get

$$(a,b,c,x) = (1, b, b, -1/2 + 1/2\, i\sqrt{3}),\ S^{(n)} = 1, \quad (0,3,3\text{-}1)$$

$$(a,b,c,x) = (1, b, b, -1/2 - 1/2\, i\sqrt{3}),\ S^{(n)} = 1, \quad (0,3,3\text{-}2)$$

$$(a,b,c,x) = (0, b, b+1, -1/2 + 1/2\, i\sqrt{3}),\ S^{(n)} = 1, \quad (0,3,3\text{-}3)$$

$$(a,b,c,x) = (0, b, b+1, -1/2 - 1/2\, i\sqrt{3}),\ S^{(n)} = 1. \quad (0,3,3\text{-}4)$$

(0,3,3-1) The special values obtained from (0,3,3-1) except trivial values are special cases of

$$F(1, b; 2; x) = \frac{(1-x)^{1-b} - 1}{(b-1)x}. \quad (3.4.1)$$

(0,3,3-2), (0,3,3-3), (0,3,3-4) The special values obtained from (0,3,3-2), (0,3,3-3) and (0,3,3-4) coincide with those obtained from (0,3,3-1).

$(k, l, m) = (1, 2, 3)$

In this case, we have

$$(a, b, c, x) = (a, 2\,a - 1/3, 3\,a, 9),\ S^{(n)} = \frac{(-4)^n\,(a+1/2,n)}{(a+2/3,n)}, \tag{1,2,3-1}$$

$$(a, b, c, x) = (a, 2\,a - 2/3, 3\,a - 1, 9),\ S^{(n)} = \frac{(-4)^n\,(a+1/2,n)}{(a+2/3,n)}. \tag{1,2,3-2}$$

(1,2,3-1)

$$\text{(i) } F(a, 2\,a-1/3;\ 3\,a;\ 9) = \begin{cases} \dfrac{(-6)\,(-4)^n\,(3/2,n)}{(4/3,n)} & \text{if } a = -1-n, \\ \dfrac{(-4)^n\,(1/3,n)}{(1/6,n)} & \text{if } a = 1/6-n, \\ \dfrac{10\,(-4)^n\,(11/6,n)}{(5/3,n)} & \text{if } a = -4/3-n \end{cases}$$

(The first case is identical to Theorem 17 in [Ek]).

$$\text{(ii) } F(2\,a, a+1/3;\ 3\,a;\ 9) = \begin{cases} \dfrac{(-6)\,(-4)^n\,(3/2,n)}{(4/3,n)} & \text{if } a = -1-n, \\ 0 \quad \text{if } a = -1/2-n, \\ \dfrac{(-4)^n\,(5/6,n)}{(2/3,n)} & \text{if } a = -1/3-n \end{cases}$$

(The third case is identical to Theorem 18 in [Ek]).

$$\text{(iii) } F(a, a+1/3;\ 3\,a;\ 9/8) = \begin{cases} \dfrac{3\,(3/2,n)}{2^{n+2}\,(4/3,n)} & \text{if } a = -1-n, \\ \dfrac{(5/6,n)}{2^n\,(2/3,n)} & \text{if } a = -1/3-n. \end{cases}$$

$$\text{(iv) } F(2\,a, 2\,a-1/3;\ 3\,a;\ 9/8) = \begin{cases} \dfrac{3}{2}\,\dfrac{(3/2,n)}{(-16)^{n+1}\,(4/3,n)} & \text{if } a = -1-n, \\ 0 \quad \text{if } a = -1/2-n, \\ \dfrac{(1/3,n)}{(-16)^n\,(1/6,n)} & \text{if } a = 1/6-n, \\ \dfrac{-5\,(11/6,n)}{(-16)^{n+2}\,(5/3,n)} & \text{if } a = -4/3-n. \end{cases}$$

$$\text{(v) } F(a, 2\,a-1/3;\ 2/3;\ -8) = \begin{cases} (-3)^{3\,n} & \text{if } a = -n, \\ (-3)^{3\,n} & \text{if } a = 1/6-n, \\ (-3)^{3\,n+1} & \text{if } a = -1/3-n \end{cases}$$

(The second case is identical to (3.12) in [GS] and the first case is identical to (5.23) in [GS]). We find that (vi)$\leq$(v).

(vii) $F\,(a, 1-2\,a;\ 2/3;\ 8/9) = 2\cdot 3^{-a}\sin\left((5/6-a)\,\pi\right)$

(The above is identical to (8/9.1) in [Go] and (3.2) in [Ka]). We find that (viii)$\leq$(vii).

(ix) $F(a, 1-2\,a;\ 4/3-a;\ 1/9) = \dfrac{3^{-a}\sqrt{\pi}\Gamma\,(4/3-a)}{2^{1/3-2\,a}\Gamma\,(2/3)\,\Gamma\,(7/6-a)}$

(The above is identical to (1/9.4) in [Go] and (1.2) in [Ka]).

(x) $F(4/3-2\,a, a+1/3;\ 4/3-a;\ 1/9) = \dfrac{3^{2/3-a}\sqrt{\pi}\Gamma\,(4/3-a)}{2^{4/3-2\,a}\Gamma\,(2/3)\,\Gamma\,(7/6-a)}$

(The above is identical to (1/9.5) in [Go] and (1.3) in [Ka]).

(xi) $F(a, a+1/3;\ 4/3-a;\ -1/8) = \dfrac{3^{-3\,a}\sqrt{\pi}\Gamma\,(4/3-a)}{2^{1/3-5\,a}\Gamma\,(2/3)\,\Gamma\,(7/6-a)}$.

(xii) $F(1-2\,a, 4/3-2\,a;\ 4/3-a;\ -1/8) = \dfrac{3^{3\,a-2}\sqrt{\pi}\Gamma\,(4/3-a)}{2^{4\,a-8/3}\Gamma\,(2/3)\,\Gamma\,(7/6-a)}$

(The above is a generalization of Theorem 32 in [Ek]). We find (xiii)$\leq$(ix), (xiv)$\leq$(x), (xv)$\leq$(xi), (xvi)$\leq$(xii), (xvii)$\leq$(i), (xviii)$\leq$(ii), (xix)$\leq$(iii), (xx)$\leq$(iv).

(xxi) $F(2\,a, a+1/3;\ 4/3;\ -8) = \begin{cases} \dfrac{(-3)^{3\,n}\,(1/2,n)}{(7/6,n)} & \text{if } a=-n, \\ 0 \quad \text{if } a=-1/2-n, \\ \dfrac{(-3)^{3\,n}\,(5/6,n)}{(3/2,n)} & \text{if } a=-1/3-n \end{cases}$

(The third case is identical to (3.7) in [GS]). We find (xxii)$\leq$(xxi).

(xxiii) $F(2\,a, 1-a;\ 4/3;\ 8/9) = \dfrac{3^{a-1}\sqrt{\pi}\Gamma\,(1/6)}{2\Gamma\,(7/6-a)\,\Gamma\,(a+1/2)}$

(The above is identical to (8/9.2) in [Go], (5.24) in [DS] and (3.3) in [Ka]).

(1,2,3-2) The special values obtained from (1,2,3-2) coincide with those obtained from (1,2,3-1).

$(k, l, m) = (1, 3, 3)$

In this case, we get

$$(a, b, c, x) = (a, 3\,a-1/2, 3\,a, -3),\ S^{(n)} = \frac{2^{2\,n}\,(a+1/2,n)^2}{(a+1/3,n)\,(a+2/3,n)}, \qquad (1,3,3\text{-}1)$$

$$(a, b, c, x) = (a, 3\,a-3/2, 3\,a-1, -3),\ S^{(n)} = \frac{2^{2\,n}\,(a-1/6,n)\,(a+1/6,n)}{(a-1/3,n)\,(a+1/3,n)}, \qquad (1,3,3\text{-}2)$$

$$(a, b, c, x) = (a, 3\,a+1, 3\,a, 3/2),\ S^{(n)} = \frac{(-3)^{3\,n}\,(a+1,n)\,(2\,a,2\,n)}{2^{3\,n}\,(3\,a,3\,n)}, \qquad (1,3,3\text{-}3)$$

$$(a, b, c, x) = (a, 3\,a-3, 3\,a-1, 3/2),\ S^{(n)} = \frac{(3\,a-2)}{(-2)^n\,(3\,a-2+3\,n)}. \qquad (1,3,3\text{-}4)$$

(1,3,3-1)

$$\text{(i) } F(a, 3\,a-1/2;\; 3\,a;\; -3) = \begin{cases} \dfrac{9\cdot 2^{2\,n-1}\,(3/2,n)^2}{(5/3,n)\,(4/3,n)} & \text{if } a=-1-n,\\ \dfrac{2^{2\,n}\,(1/3,n)^2}{(1/2,n)\,(1/6,n)} & \text{if } a=1/6-n,\\ \dfrac{2^{2\,n+1}\,(2/3,n)^2}{(5/6,n)\,(1/2,n)} & \text{if } a=-1/6-n,\\ 0 \quad \text{if } a=-1/2-n. \end{cases}$$

$$\text{(ii) } F(1/2, 2\,a;\; 3\,a;\; -3) = \begin{cases} \dfrac{9}{8}\,\dfrac{(3/2,n)^2}{(5/3,n)\,(4/3,n)} & \text{if } a=-1-n,\\ 0 \quad \text{if } a=-1/2-n. \end{cases}$$

$$\text{(iii) } F(1/2, a;\; 3\,a;\; 3/4) = \begin{cases} \dfrac{2}{\sqrt{3}}\,\dfrac{\Gamma\,(a+1/3)\,\Gamma\,(a+2/3)}{(\Gamma\,(a+1/2))^2},\\ \dfrac{9}{8}\,\dfrac{(3/2,n)^2}{(5/3,n)\,(4/3,n)} & \text{if } a=-1-n \end{cases}$$

(The first case is identical to (3/4.1) in [Go]).

$$\text{(iv) } F(2\,a, 3\,a-1/2;\; 3\,a;\; 3/4) = \begin{cases} \dfrac{2^{4\,a}}{\sqrt{3}}\,\dfrac{\Gamma\,(a+1/3)\,\Gamma\,(a+2/3)}{(\Gamma\,(a+1/2))^2},\\ \dfrac{9\,(3/2,n)^2}{2^{4\,n+7}\,(5/3,n)\,(4/3,n)} & \text{if } a=-1-n. \end{cases}$$

$$\text{(v) } F(a, 3\,a-1/2;\; a+1/2;\; 4) = \begin{cases} (-3)^{3\,n} & \text{if } a=-n,\\ (-3)^{3\,n} & \text{if } a=1/6-n,\\ -\,(-3)^{3\,n+1} & \text{if } a=-1/6-n \end{cases}$$

(The first case is identical to Theorem 12 in [Ek]).

$$\text{(vi) } F(1/2, 1-2\,a;\; a+1/2;\; 4) = \begin{cases} 1 & \text{if } a=1/2+n,\\ -1/3 & \text{if } a=1+n. \end{cases}$$

$$\text{(vii) } F(a, 1-2\,a;\; a+1/2;\; 4/3) = \begin{cases} 3^{2\,n} & \text{if } a=-n,\\ 3^{-2\,n} & \text{if } a=1/2+n,\\ 3^{-2\,n-2} & \text{if } a=1+n. \end{cases}$$

$$\text{(viii) } F(1/2, 3\,a-1/2;\; a+1/2;\; 4/3) = \begin{cases} 1 & \text{if } a=1/6-n,\\ -1 & \text{if } a=-1/6-n. \end{cases}$$

(ix) $F(a, 1-2\,a;\, 3/2-2\,a;\, -1/3) = \dfrac{2^{3/2-2\,a}\Gamma\,(5/4-a)\,\Gamma\,(3/4-a)}{3^{1-a}\Gamma\,(7/6-a)\,\Gamma\,(5/6-a)}$.

(x) $F(1/2, 3/2-3\,a;\, 3/2-2\,a;\, -1/3) = \sqrt{\dfrac{2}{3}}\,\dfrac{\Gamma\,(5/4-a)\,\Gamma\,(3/4-a)}{\Gamma\,(7/6-a)\,\Gamma\,(5/6-a)}$

(The above is a generalization of (28.1) in [Ge]).

(xi) $F(1/2, a;\, 3/2-2\,a;\, 1/4) = \dfrac{2^{3/2}\,\Gamma\,(5/4-a)\,\Gamma\,(3/4-a)}{3\,\Gamma\,(7/6-a)\,\Gamma\,(5/6-a)}$

(The above is identical to (1/4.1) in [Go] and (5.22) in [GS]).

(xii) $F(1-2\,a, 3/2-3\,a;\, 3/2-2\,a;\, 1/4) = \dfrac{2^{7/2-6\,a}\Gamma\,(5/4-a)\,\Gamma\,(3/4-a)}{3^{2-3\,a}\Gamma\,(7/6-a)\,\Gamma\,(5/6-a)}$

(The above is a generalization of Theorem 30 in [Ek]).

(xiii) $F(1/2, 3\,a-1/2;\, 2\,a+1/2;\, -1/3) = \dfrac{2^{1/3-2\,a}\,(\Gamma\,(2/3))^2\,\Gamma\,(2\,a+1/2)}{\Gamma\,(5/6)\,(\Gamma\,(a+1/2))^2}$.

(xiv) $F(2\,a, 1-a;\, 2\,a+1/2;\, -1/3) = \dfrac{2^{-2/3}}{3^{a-1/2}}\,\dfrac{(\Gamma\,(2/3))^2\,\Gamma\,(2\,a+1/2)}{\Gamma\,(5/6)\,(\Gamma\,(a+1/2))^2}$.

(xv) $F(2\,a, 3\,a-1/2;\, 2\,a+1/2;\, 1/4) = \dfrac{2^{4\,a-2/3}}{3^{3\,a-1/2}}\,\dfrac{(\Gamma\,(2/3))^2\,\Gamma\,(2\,a+1/2)}{\Gamma\,(5/6)\,(\Gamma\,(a+1/2))^2}$.

(xvi) $F(1/2, 1-a;\, 2\,a+1/2;\, 1/4) = \dfrac{2^{4/3-2\,a}}{\sqrt{3}}\,\dfrac{(\Gamma\,(2/3))^2\,\Gamma\,(2\,a+1/2)}{\Gamma\,(5/6)\,(\Gamma\,(a+1/2))^2}$

(The above is identical to (1/4.2) in [Go]).

(xvii) $F(1/2, 1-2\,a;\, 2-3\,a;\, -3) = \begin{cases} \dfrac{(1/3,n)\,(2/3,n)}{(1/6,n)\,(5/6,n)} & \text{if } a = 1/2+n, \\ -\dfrac{35}{64}\,\dfrac{(11/6,n)\,(13/6,n)}{(5/3,n)\,(7/3,n)} & \text{if } a = 2+n. \end{cases}$

(xviii) $F(1-a, 3/2-3\,a;\, 2-3\,a;\, -3)$

$$= \begin{cases} \dfrac{2^{2\,n}\,(5/6,n)\,(7/6,n)}{(2/3,n)\,(4/3,n)} & \text{if } a = 1+n, \\ \dfrac{2^{2\,n}\,(1/3,n)\,(2/3,n)}{(1/6,n)\,(5/6,n)} & \text{if } a = 1/2+n, \\ 0 \quad \text{if } a = 5/6+n, \\ 0 \quad \text{if } a = 7/6+n. \end{cases}$$

(xix) $F(1-2\,a, 3/2-3\,a;\ 2-3\,a;\ 3/4)$

$$= \begin{cases} \dfrac{2^{2-4\,a}\Gamma\,(4/3-a)\,\Gamma\,(2/3-a)}{\sqrt{3}\,\Gamma\,(7/6-a)\,\Gamma\,(5/6-a)}, \\ \dfrac{-35\,(11/6,n)\,(13/6,n)}{2^{4\,n+12}\,(5/3,n)\,(7/3,n)} \quad \text{if } a = 2+n. \end{cases}$$

(xx) $F(1/2, 1-a;\ 2-3\,a;\ 3/4)$

$$= \begin{cases} \dfrac{2\,\Gamma\,(4/3-a)\,\Gamma\,(2/3-a)}{\sqrt{3}\,\Gamma\,(7/6-a)\,\Gamma\,(5/6-a)}, \\ \dfrac{(5/6,n)\,(7/6,n)}{(2/3,n)\,(4/3,n)} \quad \text{if } a = 1+n \end{cases}$$

(The first case is identical to (3/4.2) in [Go]).

(xxi) $F(1/2, 2\,a;\ 3/2-a;\ 4) = \begin{cases} \dfrac{(3/2,n)\,(1/2,n)}{(5/6,n)\,(7/6,n)} \quad \text{if } a = -n, \\ 0 \quad \text{if } a = -1/2-n \end{cases}$

(The above are identical to (5.25) in [GS]).

(xxii) $F(1-a, 3/2-3\,a;\ 3/2-a;\ 4)$

$$= \begin{cases} \dfrac{(-3)^{3\,n}\,(5/6,n)\,(7/6,n)}{(3/2,n)\,(1/2,n)} \quad \text{if } a = 1+n, \\ 0 \quad \text{if } a = 5/6+n, \\ 0 \quad \text{if } a = 7/6+n \end{cases}$$

(The first case is identical to Theorem 13 in [Ek]).

(xxiii) $F(2\,a, 1-a;\ 3/2-a;\ 4/3) = \begin{cases} \dfrac{(3/2,n)\,(1/2,n)}{3^{2\,n}\,(5/6,n)\,(7/6,n)} \quad \text{if } a = -n, \\ 0 \quad \text{if } a = -1/2-n, \\ \dfrac{3^{2\,n}\,(5/6,n)\,(7/6,n)}{(3/2,n)\,(1/2,n)} \quad \text{if } a = 1+n. \end{cases}$

(xxiv) $F(1/2, 3/2-3\,a;\ 3/2-a;\ 4/3) = \begin{cases} 0 \quad \text{if } a = 5/6+n, \\ 0 \quad \text{if } a = 7/6+n. \end{cases}$

(1,3,3-2) The special values obtained from (1,3,3-2) coincide with those obtained from (1,3,3-1).

(1,3,3-3)

(i) $F(a, 3\,a+1;\ 3\,a;\ 3/2) = \begin{cases} 0 \quad \text{if } a = -1-n, \\ \dfrac{(-3)^{3\,n}\,(1/3,n)\,(5/3,2\,n)}{2^{3\,n}\,(2,3\,n)} \quad \text{if } a = -1/3-n, \\ \dfrac{(-3)^{3\,n}\,(2/3,n)\,(7/3,2\,n)}{2^{3\,n+1}\,(3,3\,n)} \quad \text{if } a = -2/3-n. \end{cases}$

The special values obtained from (ii) and (iii) are trivial.

$$\text{(iv) } F(2\,a, 3\,a+1;\ 3\,a;\ 3) = \begin{cases} 0 \quad \text{if } a = -1/2 - 1/2\,n, \\ \dfrac{3^{3\,n}\,(1/3, n)\,(5/3, 2\,n)}{(2, 3\,n)} \quad \text{if } a = -1/3 - n, \\ \dfrac{-3^{3\,n}\,(2/3, n)\,(7/3, 2\,n)}{(3, 3\,n)} \quad \text{if } a = -2/3 - n. \end{cases}$$

(v) $F(a, 3\,a+1;\ a+2;\ -1/2) = 2^{3\,a}3^{-3\,a}\,(a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(vi) $F(2, 1-2\,a;\ a+2;\ -1/2) = 2/3\,a + 2/3$

(The above is a special case of (1.5) in [Eb2]).

(vii) $F(a, 1-2\,a;\ a+2;\ 1/3) = 2^{2\,a}3^{-2\,a}\,(a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(viii) $F(2, 3\,a+1;\ a+2;\ 1/3) = 3/2\,a + 3/2$

(The above is a special case of (1.5) in [Eb2]).

$$\text{(ix) } F(a, 1-2\,a;\ -2\,a;\ 2/3) = \begin{cases} 0, \\ \dfrac{2^{2\,n}\,(3/2, 3\,n)}{3^{2\,n}\,(3/2, n)\,(1, 2\,n)} \quad \text{if } a = 1/2 + n, \\ \dfrac{2^{2\,n+2}\,(3, 3\,n)}{3^{2\,n+1}\,(2, n)\,(2, 2\,n)} \quad \text{if } a = 1 + n. \end{cases}$$

The special values obtained from (x) and (xi) are trivial.

$$\text{(xii) } F(-3\,a, 1-2\,a;\ -2\,a;\ -2) = \begin{cases} 0 \quad \text{if } a = 1/3 + n, \\ 0 \quad \text{if } a = 2/3 + n, \\ \dfrac{2^{2\,n}\,(3/2, 3\,n)}{(3/2, n)\,(1, 2\,n)} \quad \text{if } a = 1/2 + n, \\ \dfrac{2^{2\,n+2}\,(3, 3\,n)}{(2, n)\,(2, 2\,n)} \quad \text{if } a = 1 + n. \end{cases}$$

(xiii) $F(2, 3\,a+1;\ 2\,a+2;\ 2/3) = 6\,a + 3$

(The above is a special case of (1.5) in [Eb2]).

(xiv) $F(2\,a, 1-a;\ 2\,a+2;\ 2/3) = 3^{-a}\,(2\,a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

$$\text{(xv) } F(2\,a, 3\,a+1;\ 2\,a+2;\ -2) = \begin{cases} 3^{3\,n}\,(1-6\,n) \quad \text{if } a = -1/3 - n, \\ 3^{3\,n+1}\,(-1-6\,n) \quad \text{if } a = -2/3 - n \end{cases}$$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xvi) $F(2, 1-a;\ 2\,a+2;\ -2) = 2/3\,n + 1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]).

(xvii) $F(2, 1-2\,a;\ 2-3\,a;\ 3/2) = \begin{cases} 6\,n+1 & \text{if } a = 1/2+n, \\ 6\,n+10 & \text{if } a = 2+n \end{cases}$

(The above are special cases of (1.5) in [Eb2]).

(xviii) $F(-3\,a, 1-a;\ 2-3\,a;\ 3/2) = (-2)^{-1-n}\,(-3\,n-2) \quad \text{if } a = 1+n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xix) $F(-3\,a, 1-2\,a;\ 2-3\,a;\ 3) = \begin{cases} 2^{2\,n}\,(6\,n+1) & \text{if } a = 1/2+n, \\ -2^{2\,n+3}\,(6\,n+10) & \text{if } a = 2+n \end{cases}$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xx) $F(2, 1-a;\ 2-3\,a;\ 3) = 3/2\,n + 1 \quad \text{if } a = 1+n$

The above is a special case of (1.5) in [Eb2]). The special values obtained from (xxi) are trivial.

(xxii) $F(-3\,a, 1-a;\ -a;\ -1/2) = \begin{cases} 0, \\ \dfrac{(3, 3\,n)}{2^n\,(2, n)\,(2, 2\,n)} & \text{if } a = 1+n. \end{cases}$

(xxiii) $F(2\,a, 1-a;\ -a;\ 1/3) = \begin{cases} 0, \\ \dfrac{(3, 3\,n)}{3^n\,(2, n)\,(2, 2\,n)} & \text{if } a = 1+n. \end{cases}$

The special values obtained from (xxiv) are trivial.

(1,3,3-4) The special values obtained from (1,3,3-4) coincide with those obtained from (1,3,3-3).

$(k, l, m) = (1, 4, 3)$

In this case, we have

$$(a, b, c, x) = (a, b, b+1-a, -1), \tag{1,4,3-1}$$

$$(a, b, c, x) = (a, 4\,a-1/2, 3\,a, -1),\ S^{(n)} = \frac{2^{6\,n}\,(a+3/8, n)\,(a+5/8, n)}{3^{3\,n}\,(a+1/3, n)\,(a+2/3, n)}, \tag{1,4,3-2}$$

$$(a, b, c, x) = (a, 4\,a-5/2, 3\,a-1, -1), S^{(n)} = \frac{2^{6\,n}\,(a-1/8, n)\,(a+1/8, n)}{3^{3\,n}\,(a-1/3, n)\,(a+1/3, n)}. \tag{1,4,3-3}$$

(1,4,3-1) The special values obtained from (1,4,3-1) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1),

(1,4,3-2)

(i) $F(a, 4\,a-1/2;\ 3\,a;\ -1)$

$$= \begin{cases} \dfrac{2^{3/4-6\,a}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/3)\,\Gamma\,(a+2/3)}{3^{3/8-3\,a}\Gamma\,(11/24)\,\Gamma\,(19/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{5\cdot 2^{6\,n-1}\,(13/8,n)\,(11/8,n)}{3^{3\,n}\,(5/3,n)\,(4/3,n)} \quad \text{if } a=-1-n \end{cases}$$

(The first case is a generalization of Theorem 1 in [Ek]).

(ii) $F(2\,a, 1/2-a;\ 3\,a;\ -1)$

$$= \begin{cases} \dfrac{2^{1/4-4\,a}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/3)\,\Gamma\,(a+2/3)}{3^{3/8-3\,a}\Gamma\,(11/24)\,\Gamma\,(19/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{5\cdot 2^{4\,n-3}\,(13/8,n)\,(11/8,n)}{3^{3\,n}\,(5/3,n)\,(4/3,n)} \quad \text{if } a=-1-n. \end{cases}$$

(iii) $F(a, 1/2-a;\ 3\,a;\ 1/2)$

$$= \begin{cases} \dfrac{2^{3/4-5\,a}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/3)\,\Gamma\,(a+2/3)}{3^{3/8-3\,a}\Gamma\,(11/24)\,\Gamma\,(19/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{5\cdot 2^{5\,n-2}\,(13/8,n)\,(11/8,n)}{3^{3\,n}\,(5/3,n)\,(4/3,n)} \quad \text{if } a=-1-n. \end{cases}$$

(iv) $F(2\,a, 4\,a-1/2;\ 3\,a;\ 1/2)$

$$= \begin{cases} \dfrac{2^{1/4-2\,a}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/3)\,\Gamma\,(a+2/3)}{3^{3/8-3\,a}\Gamma\,(11/24)\,\Gamma\,(19/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{5\cdot 2^{2\,n-5}\,(13/8,n)\,(11/8,n)}{3^{3\,n}\,(5/3,n)\,(4/3,n)} \quad \text{if } a=-1-n. \end{cases}$$

(v) $F(a, 4\,a-1/2;\ 2\,a+1/2;\ 2)$

$$= \begin{cases} \dfrac{(-4)^n\,(5/8,n)\,(3/8,n)}{(3/4,\,n)\,(1/4,\,n)} \quad \text{if } a=-n, \\ \dfrac{(-4)^n\,(1/2,n)\,(1/4,n)}{(5/8,\,n)\,(1/8,\,n)} \quad \text{if } a=1/8-n, \\ \dfrac{2\,(-4)^n\,(3/4,n)\,(1/2,n)}{(7/8,\,n)\,(3/8,\,n)} \quad \text{if } a=-1/8-n, \\ 0 \quad \text{if } a=-3/8-n, \\ 0 \quad \text{if } a=-5/8-n. \end{cases}$$

(vi) $F(1-2\,a, a+1/2;\ 2\,a+1/2;\ 2)$

$$= \begin{cases} \dfrac{(3/4,n)\,(5/4,n)}{2^{2\,n}\,(7/8,n)\,(9/8,n)} & \text{if } a = 1/2+n, \\ \dfrac{-\,(5/4,n)\,(7/4,n)}{5\cdot 2^{2\,n}\,(11/8,n)\,(13/8,n)} & \text{if } a = 1+n, \\ \dfrac{2^{2\,n}\,(9/8,n)\,(7/8,n)}{(5/4,n)\,(3/4,n)} & \text{if } a = -1/2-n. \end{cases}$$

$$\text{(vii) } F(a, 1-2\,a;\ 2\,a+1/2;\ 2) = \begin{cases} \dfrac{2^{2\,n}\,(5/8,n)\,(3/8,n)}{(3/4,n)\,(1/4,n)} & \text{if } a = -n, \\ \dfrac{(3/4,n)\,(5/4,n)}{2^{2\,n}\,(7/8,n)\,(9/8,n)} & \text{if } a = 1/2+n, \\ \dfrac{(5/4,n)\,(7/4,n)}{5\cdot 2^{2\,n}\,(11/8,n)\,(13/8,n)} & \text{if } a = 1+n. \end{cases}$$

(viii) $F(4\,a-1/2, a+1/2;\ 2\,a+1/2;\ 2)$

$$= \begin{cases} \dfrac{(-4)^n\,(1/2,n)\,(1/4,n)}{(5/8,n)\,(1/8,n)} & \text{if } a = 1/8-n, \\ \dfrac{(-4)^{n+1}\,(3/4,n)\,(1/2,n)}{2\,(7/8,n)\,(3/8,n)} & \text{if } a = -1/8-n, \\ 0 \quad \text{if } a = -3/8-n, \\ 0 \quad \text{if } a = -5/8-n, \\ \dfrac{(-4)^n\,(9/8,n)\,(7/8,n)}{(5/4,n)\,(3/4,n)} & \text{if } a = -1/2-n. \end{cases}$$

We find that (ix)≤(ii), (x)≤(i), (xi)≤(iii), (xii)≤(iv).

(xiii) $F(4\,a-1/2, a+1/2;\ 3\,a+1/2;\ -1)$

$$= \begin{cases} \dfrac{3^{3\,a-3/8}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/6)\,\Gamma\,(a+5/6)}{2^{6\,a-3/4}\Gamma\,(7/24)\,\Gamma\,(23/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{2^{6\,n}\,(9/8,n)\,(7/8,n)}{3^{3\,n}\,(2/3,n)\,(4/3,n)} & \text{if } a = -1/2-n \end{cases}$$

(The second case is identical to Theorem 2 in [Ek]).

(xiv) $F(2\,a, 1-a;\ 3\,a+1/2;\ -1)$

$$= \begin{cases} \dfrac{3^{3\,a-3/8}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/6)\,\Gamma\,(a+5/6)}{2^{4\,a-1/4}\Gamma\,(7/24)\,\Gamma\,(23/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{-7\cdot 2^{4\,n-6}\,(17/8,n)\,(15/8,n)}{3^{3\,n-1}\,(5/3,n)\,(7/3,n)} & \text{if } a = -3/2-n. \end{cases}$$

(xv) $F(2\,a, 4\,a-1/2;\ 3\,a+1/2;\ 1/2)$

$$= \begin{cases} \dfrac{3^{3\,a-3/8}\sqrt{\pi}\Gamma\,(3/4)\,\Gamma\,(a+1/6)\,\Gamma\,(a+5/6)}{2^{2\,a-1/4}\Gamma\,(7/24)\,\Gamma\,(23/24)\,\Gamma\,(a+3/8)\,\Gamma\,(a+5/8)}, \\ \dfrac{-7\cdot 2^{2\,n-9}\,(17/8,n)\,(15/8,n)}{3^{3\,n-1}\,(5/3,n)\,(7/3,n)} & \text{if } a = -3/2-n. \end{cases}$$

(xvi) $F(1-a, a+1/2;\ 3a+1/2;\ 1/2)$

$$= \begin{cases} \dfrac{3^{3a-3/8}\sqrt{\pi}\Gamma(3/4)\Gamma(a+1/6)\Gamma(a+5/6)}{2^{5a-5/4}\Gamma(7/24)\Gamma(23/24)\Gamma(a+3/8)\Gamma(a+5/8)}, \\ \dfrac{2^{5n}(9/8,n)(7/8,n)}{3^{3n}(2/3,n)(4/3,n)} \quad \text{if } a=-1/2-n. \end{cases}$$

The special values obtained from (xvii)-(xxiv) are contained in the above.

(1,4,3-3) The special values obtained from (1,4,3-3) coincide with those obtained from (1,4,3-2).

3.4.4 $m = 4$

$(k, l, m) = (0, 4, 4)$

In this case, we have

$$(a, b, c, x) = (a, b, b+1-a, -1), \qquad \text{(0,4,4-1)}$$
$$(a, b, c, x) = (1, b, b, \lambda),\ S^{(n)} = 1, \qquad \text{(0,4,4-2)}$$
$$(a, b, c, x) = (0, b, b+1, \lambda,\ S^{(n)} = 1 \qquad \text{(0,4,4-3)}$$

where, λ is a solution of $x^2+1=0$.

(0,4,4-1) The special values obtained from (0,4,4-1) are evaluated in the case (0,2,2-1).

(0,4,4-2) The special values obtained from (0,4,4-2) except trivial values are the special cases of (3.4.1).

(0,4,4-3) The special values obtained from (0,4,4-3) coincide with those obtained from (0,4,4-2)

$(k, l, m) = (1, 3, 4)$

In this case, we have

$$(a, b, c, x) = (a, 3a-1/2, 4a, 4),\ S^{(n)} = \frac{(-3)^{3n}(a+2/3,n)(a+1/6,n)}{2^{4n}(a+1/4,n)(a+3/4,n)}, \qquad \text{(1,3,4-1)}$$

$$(a, b, c, x) = (a, 3a-3/2, 4a-2, 4)\ S^{(n)} = \frac{(-3)^{3n}(a-1/3,n)(a+1/6,n)}{2^{4n}(a-1/4,n)(a+1/4,n)}, \qquad \text{(1,3,4-2)}$$

$$(a, b, c, x) = (a, 3a-1/2, 4a, -8),\ S^{(n)} = \frac{3^{3n}(a+1/3,n)(a+2/3,n)}{2^{2n}(a+1/4,n)(a+3/4,n)}, \qquad \text{(1,3,4-3)}$$

$$(a, b, c, x) = (a, 3a-1/4, 4a, -8),\ S^{(n)} = \frac{3^{3n}(a+2/3,n)(a+7/12,n)}{2^{2n}(a+1/2,n)(a+3/4,n)}, \qquad \text{(1,3,4-4)}$$

$$(a,b,c,x)=(a,3\,a-3/4,4\,a-1,-8),\ S^{(n)}=\frac{3^{3\,n}\,(a+1/3,n)\,(a+5/12,n)}{2^{2\,n}\,(a+1/2,n)\,(a+1/4,n)},\tag{1,3,4-5}$$

$$(a,b,c,x)=(a,3\,a-5/4,4\,a-1,-8),\ S^{(n)}=\frac{3^{3\,n}\,(a+1/3,n)\,(a-1/12,n)}{2^{2\,n}\,(a+1/2,n)\,(a-1/4,n)},\tag{1,3,4-6}$$

$$(a,b,c,x)=(a,3\,a-3/2,4\,a-2,-8),\ S^{(n)}=\frac{3^{3\,n}\,(a-1/6,n)\,(a+1/6,n)}{2^{2\,n}\,(a-1/4,n)\,(a+1/4,n)},\tag{1,3,4-7}$$

$$(a,b,c,x)=(a,3\,a-7/4,4\,a-2,-8),\ S^{(n)}=\frac{3^{3\,n}\,(a+1/3,n)\,(a+5/12,n)}{2^{2\,n}\,(a+1/2,n)\,(a+1/4,n)}.\tag{1,3,4-8}$$

(1,3,4-1)

$$\text{(i) } F(a,3\,a-1/2;\,4\,a;\,4)=\begin{cases}\dfrac{-5\cdot(-3)^{3\,n}\,(4/3,n)\,(11/6,n)}{2^{4\,n+1}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{(-3)^{3\,n}\,(1/6,n)\,(2/3,n)}{2^{4\,n}\,(7/12,n)\,(1/12,n)} & \text{if } a=1/6-n,\\ 0 \quad \text{if } a=-1/6-n, & \\ \dfrac{-(-3)^{3\,n+1}\,(11/6,n)\,(7/3,n)}{2^{4\,n-1}\,(9/4,n)\,(7/4,n)} & \text{if } a=-3/2-n\end{cases}$$

(The first case is identical to Theorem 3 in [Ek]).

$$\text{(ii) } F(3\,a,a+1/2;\,4\,a;\,4)=\begin{cases}\dfrac{-5\,(-3)^{3\,n}\,(4/3,n)\,(11/6,n)}{2^{4\,n+1}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{-(-3)^{3\,n+1}\,(2/3,n)\,(7/6,n)}{2^{4\,n+1}\,(13/12,n)\,(7/12,n)} & \text{if } a=-1/3-n,\\ 0 \quad \text{if } a=-2/3-n, & \\ \dfrac{(-3)^{3\,n}\,(5/6,n)\,(4/3,n)}{2^{4\,n}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1/2-n\end{cases}$$

(The fourth case is identical to Theorem 9 in [Ek]).

$$\text{(iii) } F(a,a+1/2;\,4\,a;\,4/3)=\begin{cases}\dfrac{5\cdot 3^{2\,n-1}\,(4/3,n)\,(11/6,n)}{2^{4\,n+1}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{3^{2\,n}\,(5/6,n)\,(4/3,n)}{2^{4\,n}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1/2-n.\end{cases}$$

(iv) $F(3\,a,3\,a-1/2;\,4\,a;\,4/3)$

$$= \begin{cases} \dfrac{5}{27}\,\dfrac{(4/3,n)\,(11/6,n)}{2^{4\,n+1}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n, \\ \dfrac{-\,(2/3,n)\,(7/6,n)}{2^{4\,n+1}\,(13/12,n)\,(7/12,n)} & \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{(1/6,n)\,(2/3,n)}{2^{4\,n}\,(7/12,n)\,(1/12,n)} & \text{if } a=1/6-n, \\ 0 \quad \text{if } a=-1/6-n, \\ -\dfrac{1}{81}\,\dfrac{(11/6,n)\,(7/3,n)}{2^{4\,n-1}\,(9/4,n)\,(7/4,n)} & \text{if } a=-3/2-n, \end{cases}$$

$$\text{(v) } F(a,3\,a-1/2;\;1/2;\;-3) = \begin{cases} \dfrac{(-16)^n\,(5/6,n)}{(2/3,n)} & \text{if } a=-n, \\ \dfrac{(-16)^n\,(2/3,n)}{(1/2,n)} & \text{if } a=1/6-n, \\ 0 \quad \text{if } a=-1/6-n, \\ \dfrac{(-16)^{n+1}\,(4/3,n)}{2\,(7/6,n)} & \text{if } a=-1/2-n. \end{cases}$$

We find (vi)≤(v).

$$\text{(vii) } F(a,1-3\,a;\;1/2;\;3/4) = \frac{2^{2/3-2\,a}\sqrt{\pi}\Gamma(1/3)}{\Gamma(a+1/6)\Gamma(2/3-a)}$$

(The above is identical to (3/4.4) in [Go]). We find (viii)≤(vii).

$$\text{(ix) } F(a,1-3\,a;\;3/2-2\,a;\;1/4) = \frac{2^{2/3-2\,a}\Gamma(1/3)\Gamma\,(3/2-2\,a)}{3^{1-3\,a}\Gamma\,(7/6-a)\,\Gamma\,(2/3-a)}.$$

$$\text{(x) } F(a+1/2,3/2-3\,a;\;3/2-2\,a;\;1/4) = \frac{2^{5/3-2\,a}\Gamma(1/3)\Gamma\,(3/2-2\,a)}{3^{3/2-3\,a}\Gamma\,(7/6-a)\,\Gamma\,(2/3-a)}.$$

$$\text{(xi) } F(a,a+1/2;\;3/2-2\,a;\;-1/3) = \frac{2^{2/3-4\,a}\Gamma(1/3)\Gamma\,(3/2-2\,a)}{3^{1-4\,a}\Gamma\,(7/6-a)\,\Gamma\,(2/3-a)}.$$

$$\text{(xii) } F(1-3\,a,3/2-3\,a;\;3/2-2\,a;\;-1/3) = \frac{2^{4\,a-4/3}\Gamma(1/3)\Gamma\,(3/2-2\,a)}{\Gamma\,(7/6-a)\,\Gamma\,(2/3-a)}.$$

The special values obtained from (xiii)-(xx) are contained in the above those.

$$\text{(xxi) } F(3\,a,a+1/2;\;3/2;\;-3) = \begin{cases} \dfrac{(-16)^n\,(1/3,n)}{(7/6,n)} & \text{if } a=-n, \\ \dfrac{4\,(-16)^n\,(2/3,n)}{3\,(3/2,n)} & \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{(-16)^n\,(5/6,n)}{(5/3,n)} & \text{if } a=-1/2-n. \end{cases}$$

We find (xxii)≤(xxi).

(xxiii) $F(3\,a, 1-a;\ 3/2;\ 3/4) = \dfrac{2^{2\,a-1/3}\sqrt{\pi}\Gamma\,(4/3)}{\Gamma\,(7/6-a)\,\Gamma\,(a+2/3)}.$

(The above is identical to (3/4.3) in [Go]). We find (xxiv)≤(xxiii).

(1,3,4-2) The special values obtained from (1,3,4-2) coincide with those obtained from (1,3,4-1).

(1,3,4-3)

$$\text{(i) } F(a, 3\,a-1/2;\ 4\,a;\ -8) = \begin{cases} \dfrac{3^{3\,n}\,(4/3,n)\,(5/3,n)}{2^{2\,n-3}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n, \\ \dfrac{3^{3\,n}\,(1/6,n)\,(1/2,n)}{2^{2\,n}\,(7/12,n)\,(1/12,n)} & \text{if } a=1/6-n, \\ \dfrac{3^{3\,n+1}\,(1/2,n)\,(5/6,n)}{2^{2\,n}\,(11/12,n)\,(5/12,n)} & \text{if } a=-1/6-n, \\ \dfrac{-7\cdot 3^{3\,n+1}\,(11/6,n)\,(13/6,n)}{2^{2\,n}\,(9/4,n)\,(7/4,n)} & \text{if } a=-3/2-n \end{cases}$$

(The first case is identical to Theorem 7 in [Ek]).

$$\text{(ii) } F(3\,a, a+1/2;\ 4\,a;\ -8) = \begin{cases} \dfrac{3^{3\,n}\,(4/3,n)\,(5/3,n)}{2^{2\,n-3}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n, \\ 0 \quad \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{3^{3\,n}\,(5/6,n)\,(7/6,n)}{2^{2\,n}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1/2-n \end{cases}$$

(The fourth case is identical to Theorem 5 in [Ek]).

$$\text{(iii) } F(a, a+1/2;\ 4\,a;\ 8/9) = \begin{cases} \dfrac{2^{2\,a-1/2}\Gamma\,(a+3/4)\,\Gamma\,(a+1/4)}{3^{a-1/2}\Gamma\,(a+2/3)\,\Gamma\,(a+1/3)}, \\ \dfrac{3^{n-2}\,(4/3,n)\,(5/3,n)}{2^{2\,n-3}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n, \\ \dfrac{3^{n}\,(5/6,n)\,(7/6,n)}{2^{2\,n}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1/2-n. \end{cases}$$

$$\text{(iv) } F(3\,a, 3\,a-1/2;\ 4\,a;\ 8/9) = \begin{cases} \dfrac{2^{2\,a-1/2}\Gamma\,(a+3/4)\,\Gamma\,(a+1/4)}{3^{1/2-3\,a}\Gamma\,(a+2/3)\,\Gamma\,(a+1/3)}, \\ \dfrac{(4/3,n)\,(5/3,n)}{2^{2\,n-3}3^{3\,n+6}\,(7/4,n)\,(5/4,n)} & \text{if } a=-1-n, \\ \dfrac{-7\,(11/6,n)\,(13/6,n)}{2^{2\,n}3^{3\,n+9}\,(9/4,n)\,(7/4,n)} & \text{if } a=-3/2-n \end{cases}$$

(The first case is identical to (3.1) in [Ka]).

$$\text{(v) } F(a, 3\,a-1/2;\,1/2;\,9) = \begin{cases} 2^{6\,n} \quad \text{if } a=-n, \\ 2^{6\,n} \quad \text{if } a=1/6-n, \\ 2^{6\,n+2} \quad \text{if } a=-1/6-n, \\ -2^{6\,n+3} \quad \text{if } a=-1/2-n. \end{cases}$$

We find (vi)≤(v).

$$\text{(vii) } F(a, 1-3\,a;\,1/2;\,9/8) = \begin{cases} (-2)^{3\,n} \quad \text{if } a=-n, \\ (-2)^{-3\,n} \quad \text{if } a=1/3+n, \\ (-2)^{-3\,n-1} \quad \text{if } a=2/3+n, \\ (-2)^{-3\,n-3} \quad \text{if } a=1+n. \end{cases}$$

We find (viii)≤(vii).

$$\text{(ix) } F(a, 1-3\,a;\,3/2-2\,a;\,-1/8) = \left(\frac{2}{3}\right)^{1-3\,a} \frac{\sqrt{\pi}\Gamma\left(3/2-2\,a\right)}{\Gamma\left(7/6-a\right)\Gamma\left(5/6-a\right)}.$$

$$\text{(x) } F(a+1/2, 3/2-3\,a;\,3/2-2\,a;\,-1/8) = \frac{2^{5/2-3\,a}\sqrt{\pi}\Gamma\left(3/2-2\,a\right)}{3^{2-3\,a}\Gamma\left(7/6-a\right)\Gamma\left(5/6-a\right)}.$$

$$\text{(xi) } F(a, a+1/2;\,3/2-2\,a;\,1/9) = \frac{2^{1-6\,a}\sqrt{\pi}\Gamma\left(3/2-2\,a\right)}{3^{1-5\,a}\Gamma\left(7/6-a\right)\Gamma\left(5/6-a\right)}$$

(The above is identical to (1/9.1) in [Go]).

$$\text{(xii) } F(1-3\,a, 3/2-3\,a;\,3/2-2\,a;\,1/9) = \left(\frac{4}{3}\right)^{3\,a-1} \frac{\sqrt{\pi}\Gamma\left(3/2-2\,a\right)}{\Gamma\left(7/6-a\right)\Gamma\left(5/6-a\right)}$$

(The above is identical to (1.1) in [Ka]). The special values obtained from (xiii)-(xx) are contained in the above those.

$$\text{(xxi) } F(3\,a, a+1/2;\,3/2;\,9) = \begin{cases} \dfrac{2^{6\,n}\,(1/3,n)\,(2/3,n)}{(5/6,n)\,(7/6,n)} \quad \text{if } a=-n, \\ 0 \quad \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{2^{6\,n}\,(5/6,n)\,(7/6,n)}{(4/3,n)\,(5/3,n)} \quad \text{if } a=-1/2-n, \end{cases}$$

We find (xxii)≤(xxi).

$$\text{(xxiii) } F(3\,a, 1-a;\,3/2;\,9/8) = \begin{cases} \dfrac{(1/3,n)\,(2/3,n)}{(-2)^{3\,n}\,(5/6,n)\,(7/6,n)} \quad \text{if } a=-n, \\ 0 \quad \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{(-2)^{3\,n}\,(5/6,n)\,(7/6,n)}{(5/3,n)\,(4/3,n)} \quad \text{if } a=1+n. \end{cases}$$

We find (xxiv)≤(xxiii).

(1,3,4-4)

$$\text{(i) } F(a,3\,a-1/4;\;4\,a;\;-8)=\begin{cases}\dfrac{5\cdot 3^{3\,n+1}\,(4/3,n)\,(17/12,n)}{2^{2\,n+1}\,(3/2,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{3^{3\,n}\,(1/4,n)\,(1/3,n)}{2^{2\,n}\,(5/12,n)\,(1/6,n)} & \text{if } a=1/12-n,\\ \dfrac{7\cdot 3^{3\,n+1}\,(19/12,n)\,(5/3,n)}{2^{2\,n}\,(7/4,n)\,(3/2,n)} & \text{if } a=-5/4-n,\\ 0 \quad \text{if } a=-7/12-n\end{cases}$$

(The first case is identical to Theorem 6 in [Ek]).

$$\text{(ii) } F(3\,a,a+1/4;\;4\,a;\;-8)=\begin{cases}\dfrac{5\cdot 3^{3\,n+1}\,(4/3,n)\,(17/12,n)}{2^{2\,n+1}\,(3/2,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{3^{3\,n+1}\,(2/3,n)\,(3/4,n)}{2^{2\,n+1}\,(5/6,n)\,(7/12,n)} & \text{if } a=-1/3-n,\\ 0 \quad \text{if } a=-2/3-n,\\ \dfrac{3^{3\,n}\,(7/12,n)\,(2/3,n)}{2^{2\,n}\,(3/4,n)\,(1/2,n)} & \text{if } a=-1/4-n\end{cases}$$

(The fourth case is identical to Theorem 4 in [Ek]).

$$\text{(iii) } F(a,a+1/4;\;4\,a;\;8/9)=\begin{cases}\dfrac{2^{2\,a-1/6}\Gamma\,(2/3)\,\Gamma\,(3/4)\,\Gamma\,(a+3/4)\,\Gamma\,(a+1/2)}{3^{a-1/4}\Gamma\,(5/6)\,\Gamma\,(7/12)\,\Gamma\,(a+2/3)\,\Gamma\,(a+7/12)},\\ \dfrac{5\cdot 3^{n-1}\,(4/3,n)\,(17/12,n)}{2^{2\,n+1}\,(3/2,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{3^{n}\,(7/12,n)\,(2/3,n)}{2^{2\,n}\,(3/4,n)\,(1/2,n)} & \text{if } a=-1/4-n.\end{cases}$$

(iv) $F(3\,a,3\,a-1/4;\;4\,a;\;8/9)$

$$=\begin{cases}\dfrac{2^{2\,a-1/6}\Gamma\,(2/3)\,\Gamma\,(3/4)\,\Gamma\,(a+3/4)\,\Gamma\,(a+1/2)}{3^{1/4-3\,a}\Gamma\,(5/6)\,\Gamma\,(7/12)\,\Gamma\,(a+2/3)\,\Gamma\,(a+7/12)},\\ \dfrac{5\,(4/3,n)\,(17/12,n)}{2^{2\,n+1}3^{3\,n+5}\,(3/2,n)\,(5/4,n)} & \text{if } a=-1-n,\\ \dfrac{7\,(19/12,n)\,(5/3,n)}{2^{2\,n}3^{3\,n+7}\,(7/4,n)\,(3/2,n)} & \text{if } a=-5/4-n\end{cases}$$

(The first case is identical to (3.4) in [Ka]).

$$\text{(v) } F(a,3\,a-1/4;\;3/4;\;9)=\begin{cases}\dfrac{2^{6\,n}\,(5/12,n)}{(2/3,n)} & \text{if } a=-n,\\ \dfrac{2^{6\,n}\,(1/3,n)}{(7/12,n)} & \text{if } a=1/12-n,\\ \dfrac{2^{6\,n+2}\,(2/3,n)}{(11/12,n)} & \text{if } a=-1/4-n,\\ 0 \quad \text{if } a=-7/12-n.\end{cases}$$

$$\text{(vi) } F(1-3\,a, 3/4-a;\ 3/4;\ 9) = \begin{cases} \dfrac{2^{6\,n}\,(2/3,n)}{(11/12,n)} & \text{if } a = 1/3+n, \\ 0 \quad \text{if } a = 2/3+n, \\ \dfrac{-2^{6\,n+5}\,(4/3,n)}{7\,(19/12,n)} & \text{if } a = 1+n, \\ \dfrac{2^{6\,n}\,(13/12,n)}{(4/3,n)} & \text{if } a = 3/4+n. \end{cases}$$

$$\text{(vii) } F(a, 1-3\,a;\ 3/4;\ 9/8) = \begin{cases} \dfrac{(-2)^{3\,n}\,(5/12,n)}{(2/3,n)} & \text{if } a = -n, \\ \dfrac{(2/3,n)}{(-2)^{3\,n}\,(11/12,n)} & \text{if } a = 1/3+n, \\ 0 \quad \text{if } a = 2/3+n, \\ \dfrac{(4/3,n)}{7(-2)^{3\,n+1}\,(19/12,n)} & \text{if } a = 1+n. \end{cases}$$

(viii) $F(3\,a-1/4, 3/4-a;\ 3/4;\ 9/8)$

$$= \begin{cases} \dfrac{(1/3,n)}{(-2)^{3\,n}\,(7/12,n)} & \text{if } a = 1/12-n, \\ \dfrac{(2/3,n)}{(-2)^{3\,n+1}\,(11/12,n)} & \text{if } a = -1/4-n, \\ 0 \quad \text{if } a = -7/12-n, \\ \dfrac{(-2)^{3\,n}\,(13/12,n)}{(4/3,n)} & \text{if } a = 3/4+n. \end{cases}$$

(ix) $F(a, 1-3\,a;\ 5/4-2\,a;\ -1/8)$

$$= \frac{3^{3\,a+5/4}\Gamma\,(4/3)\,\Gamma\,(11/12)\,\Gamma\,(5/8-a)\,\Gamma\,(9/8-a)}{2^{5\,a+2}\Gamma\,(7/8)\,\Gamma\,(11/8)\,\Gamma\,(13/12-a)\,\Gamma\,(2/3-a)}.$$

(x) $F(5/4-3\,a, a+1/4;\ 5/4-2\,a;\ -1/8)$

$$= \frac{3^{3\,a+3/4}\Gamma\,(4/3)\,\Gamma\,(11/12)\,\Gamma\,(5/8-a)\,\Gamma\,(9/8-a)}{2^{5\,a+5/4}\Gamma\,(7/8)\,\Gamma\,(11/8)\,\Gamma\,(13/12-a)\,\Gamma\,(2/3-a)}.$$

(xi) $F(a, a+1/4;\ 5/4-2\,a;\ 1/9)$

$$= \frac{3^{5\,a+5/4}\Gamma\,(4/3)\,\Gamma\,(11/12)\,\Gamma\,(5/8-a)\,\Gamma\,(9/8-a)}{2^{8\,a+2}\Gamma\,(7/8)\,\Gamma\,(11/8)\,\Gamma\,(13/12-a)\,\Gamma\,(2/3-a)}$$

(The above is identical to (1/9.2) in [Go], (6.5) in [GS] and (1.4) in [Ka]).

(xii) $F(1-3\,a, 5/4-3\,a;\ 5/4-2\,a;\ 1/9)$

$$= \frac{3^{13/4-3\,a}\Gamma\,(4/3)\,\Gamma\,(11/12)\,\Gamma\,(5/8-a)\,\Gamma\,(9/8-a)}{2^{5-4\,a}\Gamma\,(7/8)\,\Gamma\,(11/8)\,\Gamma\,(13/12-a)\,\Gamma\,(2/3-a)}.$$

(xiii) $F(3/4 - a, 3\,a - 1/4;\ 2\,a + 3/4;\ -1/8)$

$$= \frac{3^{1/2-3\,a}\Gamma\,(2/3)\,\Gamma\,(7/12)\,\Gamma\,(a+7/8)\,\Gamma\,(a+3/8)}{2^{3/4-5\,a}\Gamma\,(7/8)\,\Gamma\,(3/8)\,\Gamma\,(a+2/3)\,\Gamma\,(a+7/12)}.$$

(xiv) $F(3\,a, 1 - a;\ 2\,a + 3/4;\ -1/8)$

$$= \frac{3^{-3\,a}\Gamma\,(2/3)\,\Gamma\,(7/12)\,\Gamma\,(a+7/8)\,\Gamma\,(a+3/8)}{2^{-5\,a}\Gamma\,(7/8)\,\Gamma\,(3/8)\,\Gamma\,(a+2/3)\,\Gamma\,(a+7/12)}.$$

(xv) $F(3\,a, 3\,a - 1/4;\ 2\,a + 3/4;\ 1/9)$

$$= \frac{3^{3\,a}\Gamma\,(2/3)\,\Gamma\,(7/12)\,\Gamma\,(a+7/8)\,\Gamma\,(a+3/8)}{2^{4\,a}\Gamma\,(7/8)\,\Gamma\,(3/8)\,\Gamma\,(a+2/3)\,\Gamma\,(a+7/12)}.$$

(xvi) $F(1 - a, 3/4 - a;\ 2\,a + 3/4;\ 1/9)$

$$= \frac{3^{2-5\,a}\Gamma\,(2/3)\,\Gamma\,(7/12)\,\Gamma\,(a+7/8)\,\Gamma\,(a+3/8)}{2^{3-8\,a}\Gamma\,(7/8)\,\Gamma\,(3/8)\,\Gamma\,(a+2/3)\,\Gamma\,(a+7/12)}$$

(The above is identical to (1/9.3) in [Go], (6.6) in [GS] and (1.5) in [Ka]).

(xvii) $F(1 - 3\,a, 3/4 - a;\ 2 - 4\,a;\ -8)$

$$= \begin{cases} \dfrac{3^{3\,n}\,(1/4,n)\,(2/3,n)}{2^{2\,n}\,(1/12,n)\,(5/6,n)} \quad \text{if } a = 1/3+n, \\ 0 \quad \text{if } a = 2/3+n, \\ \dfrac{-11\cdot 3^{3\,n}\,(23/12,n)\,(7/3,n)}{2^{2\,n-1}\,(7/4,n)\,(5/2,n)} \quad \text{if } a = 2+n, \\ \dfrac{3^{3\,n}\,(2/3,n)\,(13/12,n)}{2^{2\,n}\,(1/2,n)\,(5/4,n)} \quad \text{if } a = 3/4+n \end{cases}$$

(The fourth case is identical to Theorem 8 in [Ek]).

(xviii) $F(1 - a, 5/4 - 3\,a;\ 2 - 4\,a;\ -8)$

$$= \begin{cases} \dfrac{3^{3\,n}\,(11/12,n)\,(4/3,n)}{2^{2\,n}\,(3/2,n)\,(3/4,n)} \quad \text{if } a = 1+n, \\ \dfrac{3^{3\,n}\,(3/4,n)\,(1/3,n)}{2^{2\,n}\,(11/12,n)\,(1/6,n)} \quad \text{if } a = 5/12+n, \\ -\dfrac{13}{5}\,\dfrac{3^{3\,n+1}\,(25/12,n)\,(5/3,n)}{2^{2\,n}\,(9/4,n)\,(3/2,n)} \quad \text{if } a = 7/4+n, \\ 0 \quad \text{if } a = 13/12+n \end{cases}$$

(The first case is identical to Theorem 10 in [Ek]).

(xix) $F(1 - 3\,a, 5/4 - 3\,a;\ 2 - 4\,a;\ 8/9)$

$$= \begin{cases} \dfrac{2^{5/6-2\,a}\Gamma\,(1/4)\,\Gamma\,(2/3)\,\Gamma\,(1/2-a)\,\Gamma\,(5/4-a)}{3^{3\,a-5/4}\Gamma\,(1/12)\,\Gamma\,(5/6)\,\Gamma\,(2/3-a)\,\Gamma\,(13/12-a)}, \\ \dfrac{-11\ (7/3,n)\,(23/12,n)}{2^{2\,n-1}3^{3\,n+10}\,(5/2,n)\,(7/4,n)} \quad \text{if } a = 2+n, \\ -\dfrac{13}{5}\,\dfrac{(5/3,n)\,(25/12,n)}{2^{2\,n}3^{3\,n+7}\,(9/4,n)\,(3/2,n)} \quad \text{if } a = 7/4+n \end{cases}$$

(The first case is identical to (3.5) in [Ka]).

(xx) $F(1-a, 3/4-a;\, 2-4\,a;\, 8/9)$

$$= \begin{cases} \dfrac{2^{5/6-2\,a}\Gamma\left(1/4\right)\Gamma\left(2/3\right)\Gamma\left(1/2-a\right)\Gamma\left(5/4-a\right)}{3^{-a-3/4}\Gamma\left(1/12\right)\Gamma\left(5/6\right)\Gamma\left(2/3-a\right)\Gamma\left(13/12-a\right)}, \\ \dfrac{3^n\left(4/3,n\right)\left(11/12,n\right)}{2^{2\,n}\left(3/2,n\right)\left(3/4,n\right)} \quad \text{if } a=1+n, \\ \dfrac{3^n\left(13/12,n\right)\left(2/3,n\right)}{2^{2\,n}\left(5/4,n\right)\left(1/2,n\right)} \quad \text{if } a=3/4+n. \end{cases}$$

$$\text{(xxi) } F(3\,a, a+1/4;\, 5/4;\, 9) = \begin{cases} \dfrac{2^{6\,n}\left(1/3,n\right)}{\left(13/12,n\right)} \quad \text{if } a=-n, \\ \dfrac{2^{6\,n+3}\left(2/3,n\right)}{5\left(17/12,n\right)} \quad \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{2^{6\,n}\left(7/12,n\right)}{\left(4/3,n\right)} \quad \text{if } a=-1/4-n. \end{cases}$$

$$\text{(xxii) } F(1-a, 5/4-3\,a;\, 5/4;\, 9) = \begin{cases} \dfrac{2^{6\,n}\left(11/12,n\right)}{\left(5/3,n\right)} \quad \text{if } a=1+n, \\ \dfrac{2^{6\,n}\left(1/3,n\right)}{\left(13/12,n\right)} \quad \text{if } a=5/12+n, \\ \dfrac{-2^{6\,n+2}\left(2/3,n\right)}{5\left(17/12,n\right)} \quad \text{if } a=3/4+n, \\ 0 \quad \text{if } a=13/12+n. \end{cases}$$

$$\text{(xxiii) } F(3\,a, 1-a;\, 5/4;\, 9/8) = \begin{cases} \dfrac{\left(1/3,n\right)}{\left(-2\right)^{3\,n}\left(13/12,n\right)} \quad \text{if } a=-n, \\ \dfrac{-\left(2/3,n\right)}{5\left(-2\right)^{3\,n}\left(17/12,n\right)} \quad \text{if } a=-1/3-n, \\ 0 \quad \text{if } a=-2/3-n, \\ \dfrac{\left(-2\right)^{3\,n}\left(11/12,n\right)}{\left(5/3,n\right)} \quad \text{if } a=1+n. \end{cases}$$

(xxiv) $F(5/4-3\,a, a+1/4;\, 5/4;\, 9/8)$

$$= \begin{cases} \dfrac{\left(1/3,n\right)}{\left(-2\right)^{3\,n}\left(13/12,n\right)} \quad \text{if } a=5/12+n, \\ \dfrac{-\left(2/3,n\right)}{5\left(-2\right)^{3\,n+1}\left(17/12,n\right)} \quad \text{if } a=3/4+n, \\ 0 \quad \text{if } a=13/12+n, \\ \dfrac{\left(-2\right)^{3\,n}\left(7/12,n\right)}{\left(4/3,n\right)} \quad \text{if } a=-1/4-n. \end{cases}$$

(1,3,4-5), (1,3,4-6), (1,3,4-8) The special values obtained from (1,3,4-5), (1,3,4-6) and (1,3,4-8) coincide with those obtained from (1,3,4-4).

(1,3,4-7) The special values obtained from (1,3,4-7) coincide with those obtained from (1,3,4-3).

$(k, l, m) = (1, 4, 4)$

In this case, we have

$$(a, b, c, x) = (a, 4\,a + 1, 4\,a, 4/3),\ S^{(n)} = \frac{(-1)^n\, 2^{8\,n}\,(a+1, n)\,(3\,a, 3\,n)}{3^{4\,n}\,(4\,a, 4\,n)}, \qquad (1,4,4\text{-}1)$$

$$(a, b, c, x) = (a, 4\,a - 4, 4\,a - 2, 4/3),\ S^{(n)} = \frac{4\,a-3}{(-3)^n\,(4\,a - 3 + 4\,n)}. \qquad (1,4,4\text{-}2)$$

(1,4,4-1)

(i) $F(a, 4\,a + 1;\ 4\,a;\ 4/3)$

$$= \begin{cases} 0 \quad \text{if } a = -1 - n, \\ \dfrac{(-1)^n\, 2^{8\,n}\,(1/4, n)\,(7/4, 3\,n)}{3^{4\,n}\,(2, 4\,n)} \quad \text{if } a = -1/4 - n, \\ \dfrac{(-1)^n\, 2^{8\,n+1}\,(1/2, n)\,(5/2, 3\,n)}{3^{4\,n+1}\,(3, 4\,n)} \quad \text{if } a = -1/2 - n, \\ \dfrac{5\,(-1)^n\, 2^{8\,n-1}\,(3/4, n)\,(13/4, 3\,n)}{3^{4\,n+2}\,(4, 4\,n)} \quad \text{if } a = -3/4 - n. \end{cases}$$

The special values obtained from (ii) and (iii) are trivial.

(iv) $F(3\,a, 4\,a + 1;\ 4\,a;\ 4)$

$$= \begin{cases} 0 \quad \text{if } a = -1/3 - 1/3\,n, \\ \dfrac{(-1)^n\, 2^{8\,n}\,(1/4, n)\,(7/4, 3\,n)}{(2, 4\,n)} \quad \text{if } a = -1/4 - n, \\ \dfrac{(-1)^{n+1}\, 2^{8\,n+1}\,(1/2, n)\,(5/2, 3\,n)}{(3, 4\,n)} \quad \text{if } a = -1/2 - n, \\ \dfrac{5\,(-1)^n\, 2^{8\,n-1}\,(3/4, n)\,(13/4, 3\,n)}{(4, 4\,n)} \quad \text{if } a = -3/4 - n. \end{cases}$$

(v) $F(a, 4\,a + 1;\ a + 2;\ -1/3) = 2^{-8\,a} 3^{4\,a}\,(a + 1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(vi) $F(2, 1 - 3\,a;\ a + 2;\ -1/3) = 3/4\,a + 3/4$

(The above is a special case of (1.5) in [Eb2]).

(vii) $F(a, 1 - 3\,a;\ a + 2;\ 1/4) = 2^{-6\,a} 3^{3\,a}\,(a + 1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(viii) $F(2, 4\,a + 1;\ a + 2;\ 1/4) = 4/3\,a + 4/3$

(The above is a special case of (1.5) in [Eb2]).

$$\text{(ix) } F(a, 1-3\,a;\; -3\,a;\; 3/4) = \begin{cases} 0, \\ \dfrac{3^{3\,n}\,(4/3, 4\,n)}{2^{6\,n}\,(4/3, n)\,(1, 3\,n)} & \text{if } a = 1/3+n, \\ \dfrac{5\cdot 3^{3\,n}\,(8/3, 4\,n)}{2^{6\,n+2}\,(5/3, n)\,(2, 3\,n)} & \text{if } a = 2/3+n, \\ \dfrac{3^{3\,n+3}\,(4, 4\,n)}{2^{6\,n+4}\,(2, n)\,(3, 3\,n)} & \text{if } a = 1+n. \end{cases}$$

The special values obtained from (x) and (xi) are trivial.

$$\text{(xii) } F(-4\,a, 1-3\,a;\; -3\,a;\; -3) = \begin{cases} 0 & \text{if } a = 1/4+n, \\ 0 & \text{if } a = 1/2+n, \\ 0 & \text{if } a = 3/4+n, \\ \dfrac{3^{3\,n}\,(4/3, 4\,n)}{(4/3, n)\,(1, 3\,n)} & \text{if } a = 1/3+n, \\ \dfrac{5\cdot 3^{3\,n}\,(8/3, 4\,n)}{(5/3, n)\,(2, 3\,n)} & \text{if } a = 2/3+n, \\ \dfrac{3^{3\,n+3}\,(4, 4\,n)}{(2, n)\,(3, 3\,n)} & \text{if } a = 1+n. \end{cases}$$

(xiii) $F(2, 4\,a+1;\; 3\,a+2;\; 3/4) = 12\,a+4$

(The above is a special case of (1.5) in [Eb2]).

(xiv) $F(3\,a, 1-a;\; 3\,a+2;\; 3/4) = 2^{-2\,a}\,(3\,a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

$$\text{(xv) } F(3\,a, 4\,a+1;\; 3\,a+2;\; -3) = \begin{cases} 2^{8\,n}\,(1-12\,n) & \text{if } a = -1/4-n, \\ 2^{8\,n+2}\,(-2-12\,n) & \text{if } a = -1/2-n, \\ 2^{8\,n+4}\,(-5-12\,n) & \text{if } a = -3/4-n \end{cases}$$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xvi) $F(2, 1-a;\; 3\,a+2;\; -3) = 3/4\,n+1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]).

(xvii) $F(2, 1-3\,a;\; 2-4\,a;\; 4/3) = 4\,n+1 \quad \text{if } a = 1/3+1/3\,n$

(The above is a special case of (1.5) in [Eb2]).

(xviii) $F(-4\,a, 1-a;\; 2-4\,a;\; 4/3) = -\,(-3)^{-n-1}\,(4\,n+3) \quad \text{if } a = 1+n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xix) $F(-4\,a, 1-3\,a;\; 2-4\,a;\; 4) = (-3)^{n}\,(4\,n+1) \quad \text{if } a = 1/3+1/3\,n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xx) $F(2, 1-a;\ 2-4\,a;\ 4) = 4/3\,n+1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]). The special values obtained from (xxi) are trivial.

(xxii) $F(-4\,a, 1-a;\ -a;\ -1/3) = \begin{cases} 0, \\ \dfrac{(4, 4\,n)}{3^n\,(2, n)\,(3, 3\,n)} & \text{if } a = 1+n. \end{cases}$

(xxiii) $F(3\,a, 1-a;\ -a;\ 1/4) = \begin{cases} 0, \\ \dfrac{(4, 4\,n)}{2^{2\,n}\,(2, n)\,(3, 3\,n)} & \text{if } a = 1+n \end{cases}$

(The first case is identical to (29.3) in [Ge]). The special values obtained from (xxiv) are trivial.

(1,4,4-2) The special values obtained from (1,4,4-2) coincide with those obtained from (1,4,4-1).

$(k, l, m) = (1, 5, 4)$

In this case, there is no admissible quadruple.

$(k, l, m) = (2, 2, 4)$

In this case, we have

$$(a, b, c, x) = (a, b, 2\,a, 2), \qquad \text{(2,2,4-1)}$$
$$(a, b, c, x) = (a, b, 2\,b, 2). \qquad \text{(2,2,4-2)}$$

(2,2,4-1), (2,2,4-2) The special values obtained from (2,2,4-1) and (2,2,4-2) coincide with those obtained from (1,2,2-1).

$(k, l, m) = (2, 3, 4)$

In this case, we have

$$\begin{cases} (a, b, c, x) = (a, 3/2\,a - 1/4, 2\,a, 8 + 4\sqrt{3}), \\ S^{(n)} = \dfrac{3^{3/2\,n}\left(\sqrt{3}+2\right)^{3\,n}(1/2\,a + 7/12, n)}{2^{2\,n}\,(1/2\,a + 3/4, n)}, \end{cases} \qquad \text{(2,3,4-1)}$$

$$\begin{cases} (a, b, c, x) = (a, 3/2\,a - 1/4, 2\,a, 8 - 4\sqrt{3}), \\ S^{(n)} = \dfrac{3^{3/2\,n}\left(\sqrt{3}-2\right)^{3\,n}(1/2\,a + 7/12, n)}{2^{2\,n}\,(1/2\,a + 3/4, n)}. \end{cases} \qquad \text{(2,3,4-2)}$$

(2,3,4-1)

(i) $F(a, 3/2\,a - 1/4;\ 2\,a;\ 8 + 4\sqrt{3})$

$$= \begin{cases} \dfrac{5 \cdot 3^{3/2\,n+1/2} \left(2+\sqrt{3}\right)^{3\,n+3} (17/12, n)}{2^{2\,n+2} (5/4, n)} & \text{if } a = -2 - 2\,n, \\ \dfrac{-3^{3/2\,n+1/2} \left(2+\sqrt{3}\right)^{3\,n+2} (11/12, n)}{2^{2\,n+1} (3/4, n)} & \text{if } a = -1 - 2\,n, \\ \dfrac{3^{3/2\,n} \left(2+\sqrt{3}\right)^{3\,n} (1/3, n)}{2^{2\,n} (1/6, n)} & \text{if } a = 1/6 - 2\,n, \\ \dfrac{-3^{3/2\,n+1} \left(2+\sqrt{3}\right)^{3\,n+4} (5/3, n)}{2^{2\,n} (3/2, n)} & \text{if } a = -5/2 - 2\,n, \\ 0 \quad \text{if } a = -7/6 - 2\,n \end{cases}$$

(The first case is identical to Theorem 28 in [Ek]).

(ii) $F(a, 1/2\,a + 1/4;\ 2\,a;\ 8 + 4\sqrt{3})$

$$= \begin{cases} \dfrac{5 \cdot 3^{3/2\,n+1/2} \left(-2-\sqrt{3}\right)^{n+1} (17/12, n)}{2^{2\,n+2} (5/4, n)} & \text{if } a = -2 - 2\,n, \\ \dfrac{-3^{3/2\,n+1/2} \left(-2-\sqrt{3}\right)^{n} (11/12, n)}{2^{2\,n+1} (3/4, n)} & \text{if } a = -1 - 2\,n, \\ \dfrac{3^{3/2\,n} \left(-2-\sqrt{3}\right)^{n} (2/3, n)}{2^{2\,n} (1/2, n)} & \text{if } a = -1/2 - 2\,n \end{cases}$$

(The third case is identical to Theorem 16 in [Ek]).

(iii) $F(a, 1/2\,a + 1/4;\ 2\,a;\ 8 - 4\sqrt{3})$

$$= \begin{cases} \dfrac{5 \cdot 3^{3/2\,n+1/2} \left(2-\sqrt{3}\right)^{n+1} (17/12, n)}{2^{2\,n+2} (5/4, n)} & \text{if } a = -2 - 2\,n, \\ \dfrac{3^{3/2\,n+1/2} \left(2-\sqrt{3}\right)^{n} (11/12, n)}{2^{2\,n+1} (3/4, n)} & \text{if } a = -1 - 2\,n, \\ \dfrac{3^{3/2\,n} \left(2-\sqrt{3}\right)^{n} (2/3, n)}{2^{2\,n} (1/2, n)} & \text{if } a = -1/2 - 2\,n \end{cases}$$

(The third case is identical to Theorem 16 in [Ek]).

(iv) $F(a, 3/2\,a - 1/4;\ 2\,a;\ 8 - 4\sqrt{3})$

$$= \begin{cases} \dfrac{5 \cdot 3^{3/2\,n+1/2} \left(\sqrt{3}-2\right)^{3\,n+3} (17/12, n)}{2^{2\,n+2} (5/4, n)} & \text{if } a = -2 - 2\,n, \\ \dfrac{3^{3/2\,n+1/2} \left(\sqrt{3}-2\right)^{3\,n+2} (11/12, n)}{2^{2\,n+1} (3/4, n)} & \text{if } a = -1 - 2\,n, \\ \dfrac{3^{3/2\,n} \left(\sqrt{3}-2\right)^{3\,n} (1/3, n)}{2^{2\,n} (1/6, n)} & \text{if } a = 1/6 - 2\,n, \\ \dfrac{-3^{3/2\,n+1} \left(\sqrt{3}-2\right)^{3\,n+4} (5/3, n)}{2^{2\,n} (3/2, n)} & \text{if } a = -5/2 - 2\,n, \\ 0 \quad \text{if } a = -7/6 - 2\,n \end{cases}$$

(The first case is identical to Theorem 28 in [Ek]).

(v) $F(a, 3/2\,a - 1/4;\ 1/2\,a + 3/4;\ -7 - 4\sqrt{3})$

$$= \begin{cases} \dfrac{2^{4\,n} 3^{3/2\,n} \left(-2-\sqrt{3}\right)^{3\,n} (5/12, n)}{(1/4, n)} & \text{if } a = -2\,n, \\ \dfrac{-2^{4\,n+2} 3^{3/2\,n+1/2} \left(-2-\sqrt{3}\right)^{3\,n+2} (11/12, n)}{(3/4, n)} & \text{if } a = -1 - 2\,n, \\ \dfrac{2^{4\,n} 3^{3/2\,n} \left(-2-\sqrt{3}\right)^{3\,n} (1/3, n)}{(1/6, n)} & \text{if } a = 1/6 - 2\,n, \\ \dfrac{2^{4\,n+1} 3^{3/2\,n+1/2} \left(-2-\sqrt{3}\right)^{3\,n+1} (2/3, n)}{(1/2, n)} & \text{if } a = -1/2 - 2\,n, \\ 0 \quad \text{if } a = -7/6 - 2\,n \end{cases}$$

(The first case is identical to Theorem 31 in [Ek]).

(vi) $F(1 - a, 3/4 - 1/2\,a;\ 1/2\,a + 3/4;\ -7 - 4\sqrt{3})$

$$= \begin{cases} \dfrac{2^{4\,n} \left(-2-\sqrt{3}\right)^{n} (5/4, n)}{3^{3/2\,n} (13/12, n)} & \text{if } a = 1 + 2\,n, \\ \dfrac{-2^{4\,n+2} \left(-2-\sqrt{3}\right)^{n} (7/4, n)}{7 \cdot 3^{3/2\,n-1/2} (19/12, n)} & \text{if } a = 2 + 2\,n, \\ \dfrac{2^{4\,n} \left(-2-\sqrt{3}\right)^{n} (3/2, n)}{3^{3/2\,n} (4/3, n)} & \text{if } a = 3/2 + 2\,n. \end{cases}$$

(vii) $F(a, 1 - a;\ 1/2\,a + 3/4;\ 1/2 + 1/4\sqrt{3})$

$$= \frac{4\left(2-\sqrt{3}\right)^{1/2\,a-1/4} \sqrt{\pi} \sin\left(\pi\left(1/2\,a + 1/12\right)\right) \Gamma\left(1/2\,a + 3/4\right)}{3^{3/4\,a+3/8} \Gamma\left(2/3\right) \Gamma\left(1/2\,a + 7/12\right)}$$

(The above is a generalization of Theorem 37 in [Ek]).

(viii) $F(3/4 - 1/2\,a, 3/2\,a - 1/4;\ 1/2\,a + 3/4;\ 1/2 + 1/4\sqrt{3})$

$$= \frac{2^{a+3/2} \sqrt{\pi} \sin\left(\pi\left(1/2\,a + 1/12\right)\right) \Gamma\left(1/2\,a + 3/4\right)}{3^{3/4\,a+3/8} \Gamma\left(2/3\right) \Gamma\left(1/2\,a + 7/12\right)}.$$

(ix) $F(a, 1 - a;\ 5/4 - 1/2\,a;\ 1/2 - 1/4\sqrt{3})$

$$= \frac{\left(2-\sqrt{3}\right)^{1/2\,a-1/4} \Gamma(1/3) \Gamma\left(5/4 - 1/2\,a\right)}{3^{5/8-3/4\,a} \sqrt{\pi} \Gamma\left(13/12 - 1/2\,a\right)}$$

(The above is a generalization of Theorem 37 in [Ek]).

(x) $F(5/4 - 3/2\,a, 1/2\,a + 1/4;\ 5/4 - 1/2\,a;\ 1/2 - 1/4\sqrt{3})$

$$= \frac{2^{1/2-a} \Gamma(1/3) \Gamma\left(5/4 - 1/2\,a\right)}{3^{5/8-3/4\,a} \sqrt{\pi} \Gamma\left(13/12 - 1/2\,a\right)}.$$

(xi) $F(a,1/2\,a+1/4;\;5/4-1/2\,a;\;-7+4\sqrt{3})$

$$= \frac{2^{-2a}\,\left(2+\sqrt{3}\right)^{1/2\,a+1/4}\Gamma(1/3)\Gamma\left(5/4-1/2\,a\right)}{3^{5/8-3/4\,a}\sqrt{\pi}\Gamma\left(13/12-1/2\,a\right)}.$$

(xii) $F(1-a,5/4-3/2\,a;\;5/4-1/2\,a;\;-7+4\sqrt{3})$

$$= \frac{2^{2a-2}\,\left(2-\sqrt{3}\right)^{3/2\,a-5/4}\Gamma(1/3)\Gamma\left(5/4-1/2\,a\right)}{3^{5/8-3/4\,a}\sqrt{\pi}\Gamma\left(13/12-1/2\,a\right)}$$

(The above is a generalization of Theorem 31 in [Ek]). The special values obtained from (xiii)-(xxiv) are contained in the above those.

(2,3,4-2) The special values obtained from (2,3,4-2) coincide with those obtained from (2,3,4-1).

$(k,l,m)=(2,4,4)$

In this case, we have

$$(a,b,c,x)=(a,b,2\,a,2), \tag{2,4,4-1}$$

$$\begin{cases}(a,b,c,x)=(a,2\,a-1/2,2\,a,-2+2\sqrt{2}),\\ S^{(n)}=\dfrac{2^{2n}\left(\sqrt{2}-1\right)^{4n}(1/2\,a+3/8,n)\,(1/2\,a+5/8,n)}{(a+1/2,2\,n)},\end{cases} \tag{2,4,4-2}$$

$$\begin{cases}(a,b,c,x)=(a,2\,a-1/2,2\,a,-2-2\sqrt{2}),\\ S^{(n)}=\dfrac{2^{2n}\left(\sqrt{2}+1\right)^{4n}(1/2\,a+3/8,n)\,(1/2\,a+5/8,n)}{(a+1/2,2\,n)}.\end{cases} \tag{2,4,4-3}$$

(2,4,4-1) The special values obtained from (2,4,4-1) are evaluated in the case (1,2,2-1).

(2,4,4-2)

(i) $F(a,2\,a-1/2;\;2\,a;\;-2+2\sqrt{2})$

$$=\begin{cases}\dfrac{2^{1/4-a}\left(\sqrt{2}-1\right)^{1/2-2a}\sqrt{\pi}\Gamma\left(a+1/2\right)}{\Gamma\left(1/2\,a+3/8\right)\Gamma\left(1/2\,a+5/8\right)},\\ \dfrac{5\left(\sqrt{2}-1\right)^{4n+4}(13/8,n)\,(11/8,n)}{4\,(7/4,n)\,(5/4,n)} & \text{if } a=-2-2\,n,\\ \dfrac{-\left(\sqrt{2}-1\right)^{4n+3}(9/8,n)\,(7/8,n)}{2\,(5/4,n)\,(3/4,n)} & \text{if } a=-1-2\,n.\end{cases}$$

(ii) $F(1/2,a;\;2\,a;\;-2+2\sqrt{2})$

$$= \begin{cases} \dfrac{2^{1/4-a}\left(1+\sqrt{2}\right)^{1/2}\sqrt{\pi}\Gamma\left(a+1/2\right)}{\Gamma\left(1/2\,a+3/8\right)\Gamma\left(1/2\,a+5/8\right)}, \\ \dfrac{5\,(13/8,n)\,(11/8,n)}{4\,(7/4,n)\,(5/4,n)} \quad \text{if } a=-2-2\,n, \\ \dfrac{\left(1+\sqrt{2}\right)(9/8,n)\,(7/8,n)}{2\,(5/4,n)\,(3/4,n)} \quad \text{if } a=-1-2\,n \end{cases}$$

(The second case is identical to Theorem 33 in [Ek]).

(iii) $F(1/2, a;\ 2\,a;\ -2-2\,\sqrt{2})$

$$= \begin{cases} \dfrac{5\,(13/8,n)\,(11/8,n)}{4\,(7/4,n)\,(5/4,n)} \quad \text{if } a=-2-2\,n, \\ \dfrac{\left(1-\sqrt{2}\right)(9/8,n)\,(7/8,n)}{2\,(5/4,n)\,(3/4,n)} \quad \text{if } a=-1-2\,n \end{cases}$$

(The first case is identical to Theorem 33 in [Ek]).

(iv) $F(a, 2\,a-1/2;\ 2\,a;\ -2-2\,\sqrt{2})$

$$= \begin{cases} \dfrac{5\left(1+\sqrt{2}\right)^{4\,n+4}(13/8,n)\,(11/8,n)}{4\,(7/4,n)\,(5/4,n)} \quad \text{if } a=-2-2\,n, \\ \dfrac{\left(1+\sqrt{2}\right)^{4\,n+3}(9/8,n)\,(7/8,n)}{2\,(5/4,n)\,(3/4,n)} \quad \text{if } a=-1-2\,n, \\ \dfrac{\left(1+\sqrt{2}\right)^{4\,n}(1/2,n)\,(1/4,n)}{(5/8,n)\,(1/8,n)} \quad \text{if } a=1/4-2\,n, \\ \dfrac{\sqrt{2}\left(1+\sqrt{2}\right)^{4\,n+1}(3/4,n)\,(1/2,n)}{(7/8,n)\,(3/8,n)} \quad \text{if } a=-1/4-2\,n, \\ 0 \quad \text{if } a=-3/4-2\,n, \\ 0 \quad \text{if } a=-5/4-2\,n. \end{cases}$$

(v) $F(a, 2\,a-1/2;\ a+1/2;\ 3-2\,\sqrt{2}) = \dfrac{2^{3/4-3\,a}\left(\sqrt{2}-1\right)^{1/2-2\,a}\sqrt{\pi}\Gamma\left(a+1/2\right)}{\Gamma\left(1/2\,a+3/8\right)\Gamma\left(1/2\,a+5/8\right)}$

(The above is a generalization of Theorem 25 in [Ek]).

(vi) $F(1/2, 1-a;\ a+1/2;\ 3-2\,\sqrt{2}) = \dfrac{2^{-a-1/4}(1+\sqrt{2})^{1/2}\sqrt{\pi}\Gamma\left(a+1/2\right)}{\Gamma\left(1/2\,a+3/8\right)\Gamma\left(1/2\,a+5/8\right)}.$

(vii) $F(a, 1-a;\ a+1/2;\ 1/2-1/2\,\sqrt{2}) = \dfrac{2^{3/4-2\,a}\left(\sqrt{2}-1\right)^{1/2-a}\sqrt{\pi}\Gamma\left(a+1/2\right)}{\Gamma\left(1/2\,a+3/8\right)\Gamma\left(1/2\,a+5/8\right)}$

(The above is a generalization of Theorem 36 in [Ek]).

(viii) $F(1/2, 2\,a-1/2;\ a+1/2;\ 1/2-1/2\,\sqrt{2}) = \dfrac{2^{1/4-a}\sqrt{\pi}\Gamma\left(a+1/2\right)}{\Gamma\left(1/2\,a+3/8\right)\Gamma\left(1/2\,a+5/8\right)}.$

(ix) $F(a, 1-a;\, 3/2-a;\, 1/2+1/2\sqrt{2})$

$$= \begin{cases} \dfrac{(4n+1)\left(\sqrt{2}-1\right)^{2n}(1/4,n)\,(3/4,n)}{2^{2n}\,(7/8,n)\,(9/8,n)} & \text{if } a=-2n, \\ \dfrac{1}{15}\,\dfrac{(4n+3)\left(\sqrt{2}-1\right)^{2n+2}(3/4,n)\,(5/4,n)}{2^{2n}\,(11/8,n)\,(13/8,n)} & \text{if } a=-1-2n, \\ \dfrac{2^{2n}\left(1+\sqrt{2}\right)^{2n}(3/8,n)\,(5/8,n)}{(1/4,n)\,(3/4,n)} & \text{if } a=1+2n, \\ \dfrac{2^{2n}\left(1+\sqrt{2}\right)^{2n+2}(7/8,n)\,(9/8,n)}{(3/4,n)\,(5/4,n)} & \text{if } a=2+2n \end{cases}$$

(The third case is identical to Theorem 36 in [Ek]).

(x) $F(1/2, 3/2-2a;\, 3/2-a;\, 1/2+1/2\sqrt{2})$

$$= \begin{cases} \dfrac{(1/4,n)\,(1/2,n)}{(1/8,n)\,(5/8,n)} & \text{if } a=3/4+2n, \\ \dfrac{-\sqrt{2}\,(1/2,n)\,(3/4,n)}{(3/8,n)\,(7/8,n)} & \text{if } a=5/4+2n, \\ 0 \quad \text{if } a=7/4+2n, \\ 0 \quad \text{if } a=9/4+2n. \end{cases}$$

(xi) $F(1/2, a;\, 3/2-a;\, 3+2\sqrt{2})$

$$= \begin{cases} \dfrac{(3/4,n)\,(5/4,n)}{(7/8,n)\,(9/8,n)} & \text{if } a=-2n, \\ \dfrac{2\left(1-\sqrt{2}\right)(5/4,n)\,(7/4,n)}{5\,(11/8,n)\,(13/8,n)} & \text{if } a=-1-2n. \end{cases}$$

(xii) $F(1-a, 3/2-2a;\, 3/2-a;\, 3+2\sqrt{2})$

$$= \begin{cases} \dfrac{(4n+1)\,2^{4n}\left(1+\sqrt{2}\right)^{4n}(3/8,n)\,(5/8,n)}{(3/4,n)\,(5/4,n)} & \text{if } a=1+2n, \\ \dfrac{-2^{4n+1}\left(1+\sqrt{2}\right)^{4n+3}(7/8,n)\,(9/8,n)}{(3/4,n)\,(5/4,n)} & \text{if } a=2+2n, \\ \dfrac{2^{4n}\left(1+\sqrt{2}\right)^{4n}(1/4,n)\,(1/2,n)}{(1/8,n)\,(5/8,n)} & \text{if } a=3/4+2n, \\ \dfrac{2^{4n+3/2}\left(1+\sqrt{2}\right)^{4n+1}(1/2,n)\,(3/4,n)}{(3/8,n)\,(7/8,n)} & \text{if } a=5/4+2n, \\ 0 \quad \text{if } a=7/4+2n, \\ 0 \quad \text{if } a=9/4+2n \end{cases}$$

(The first case is identical to Theorem 25 in [Ek]). The special values obtained from (xiii)-(xxiv) coincide with the above.

(2,4,4-3) The special values obtained from (2,4,4-3) coincide with those obtained from (2,4,4-2).

$(k,l,m)=(2,5,4)$

In this case, we have

$$\begin{cases}(a,b,c,x)=(a,5/2\,a-1,2\,a,-1/2+1/2\,\sqrt{5}),\\ S^{(n)}=\dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{5\,n}(1/2\,a+2/5,n)\,(1/2\,a+3/5,n)}{2^{11\,n}\,(1/2\,a+1/4,n)\,(1/2\,a+3/4,n)},\end{cases} \tag{2,5,4-1}$$

$$\begin{cases}(a,b,c,x)=(a,5/2\,a-1,2\,a,-1/2-1/2\,\sqrt{5}),\\ S^{(n)}=\dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5\,n}(1/2\,a+2/5,n)\,(1/2\,a+3/5,n)}{2^{11\,n}\,(1/2\,a+1/4,n)\,(1/2\,a+3/4,n)},\end{cases} \tag{2,5,4-2}$$

$$\begin{cases}(a,b,c,x)=(a,5/2\,a-1/2,2\,a,-1/2+1/2\,\sqrt{5}),\\ S^{(n)}=\dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{5\,n}(1/2\,a+3/10,n)\,(1/2\,a+7/10,n)}{2^{11\,n}\,(1/2\,a+1/4,n)\,(1/2\,a+3/4,n)},\end{cases} \tag{2,5,4-3}$$

$$\begin{cases}(a,b,c,x)=(a,5/2\,a-1/2,2\,a,-1/2-1/2\,\sqrt{5}),\\ S^{(n)}=\dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5\,n}(1/2\,a+3/10,n)\,(1/2\,a+7/10,n)}{2^{11\,n}\,(1/2\,a+1/4,n)\,(1/2\,a+3/4,n)}.\end{cases} \tag{2,5,4-4}$$

(2,5,4-1)

(i) $F(a,5/2\,a-1;\;2\,a;\;-1/2+1/2\,\sqrt{5})$

$$=\begin{cases}\dfrac{5^{1/2-5/4\,a}\left(\sqrt{5}-1\right)^{1-5/2\,a}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(a+1/2)}{2^{9/5-9/2\,a}\Gamma(9/10)\,\Gamma(1/2\,a+2/5)\,\Gamma(1/2\,a+3/5)},\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{5\,n+5}(8/5,n)\,(7/5,n)}{2^{11\,n+6}\,(7/4,n)\,(5/4,n)} & \text{if } a=-2-2\,n,\\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{5\,n+4}(11/10,n)\,(9/10,n)}{2^{11\,n+6}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1-2\,n.\end{cases}$$

(ii) $F(a,1-1/2\,a;\;2\,a;\;-1/2+1/2\,\sqrt{5})$

$$=\begin{cases}\dfrac{5^{1/2-5/4\,a}\left(\sqrt{5}-1\right)^{1/2\,a-1}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(a+1/2)}{2^{-3/2\,a-1/5}\Gamma(9/10)\,\Gamma(1/2\,a+2/5)\,\Gamma(1/2\,a+3/5)},\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{n+1}(8/5,n)\,(7/5,n)}{2^{7\,n+2}\,(7/4,n)\,(5/4,n)} & \text{if } a=-2-2\,n,\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{n+2}(11/10,n)\,(9/10,n)}{2^{7\,n+4}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1-2\,n\end{cases}$$

(The first case is a generalization of Theorem 24 in [Ek]).

(iii) $F(a,1-1/2\,a;\;2\,a;\;-1/2-1/2\,\sqrt{5})$

$$=\begin{cases}\dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{n+1}(8/5,n)\,(7/5,n)}{2^{7\,n+2}\,(7/4,n)\,(5/4,n)} & \text{if } a=-2-2\,n,\\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{n+2}(11/10,n)\,(9/10,n)}{2^{7\,n+4}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1-2\,n,\\ \dfrac{2^{5\,n}\left(\sqrt{5}+1\right)^{n}(5/4,n)\,(7/4,n)}{5^{5/2\,n}\,(7/5,n)\,(8/5,n)} & \text{if } a=2+2\,n.\end{cases}$$

(iv) $F(a, 5/2\,a-1;\ 2\,a;\ -1/2-1/2\,\sqrt{5})$

$$= \begin{cases} \frac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5\,n+5}(8/5,n)\,(7/5,n)}{2^{11\,n+6}\,(7/4,n)\,(5/4,n)} & \text{if } a=-2-2\,n, \\ \frac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5\,n+4}(11/10,n)\,(9/10,n)}{2^{11\,n+6}\,(5/4,n)\,(3/4,n)} & \text{if } a=-1-2\,n, \\ \frac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5\,n}(1/5,n)\,(2/5,n)}{2^{11\,n}\,(11/20,n)\,(1/20,n)} & \text{if } a=2/5-2\,n, \\ \frac{5^{5/2\,n+1}\left(\sqrt{5}+1\right)^{5\,n+2}(4/5,n)\,(3/5,n)}{2^{11\,n+3}\,(19/20,n)\,(9/20,n)} & \text{if } a=-2/5-2\,n, \\ 0 & \text{if } a=-4/5-2\,n, \\ 0 & \text{if } a=-6/5-2\,n. \end{cases}$$

(v) $F(a, 5/2\,a-1;\ 3/2\,a;\ 3/2-1/2\,\sqrt{5})$

$$= \begin{cases} \frac{3^{3/2\,a-3/5}\left(\sqrt{5}+1\right)^{5/2\,a-1}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(1/2\,a+1/3)\,\Gamma(1/2\,a+2/3)}{2^{5/2\,a-1}5^{5/4\,a-1/2}\Gamma(8/15)\,\Gamma(13/15)\,\Gamma(1/2\,a+2/5)\,\Gamma(1/2\,a+3/5)}, \\ \frac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{5\,n+5}(8/5,n)\,(7/5,n)}{2^{5\,n+5}3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n. \end{cases}$$

(vi) $F(1-a, 1/2\,a;\ 3/2\,a;\ 3/2-1/2\,\sqrt{5})$

$$= \begin{cases} \frac{3^{3/2\,a-3/5}\left(\sqrt{5}+1\right)^{1/2\,a}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(1/2\,a+1/3)\,\Gamma(1/2\,a+2/3)}{2^{1/2\,a}5^{5/4\,a-1/2}\Gamma(8/15)\,\Gamma(13/15)\,\Gamma(1/2\,a+2/5)\,\Gamma(1/2\,a+3/5)}, \\ \frac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{n+1}(8/5,n)\,(7/5,n)}{2^{n+1}3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n \end{cases}$$

(The second case is identical to Theorem 21 in [Ek]).

(vii) $F(a, 1-a;\ 3/2\,a;\ 1/2-1/2\,\sqrt{5})$

$$= \begin{cases} \frac{3^{3/2\,a-3/5}\left(\sqrt{5}+1\right)^{3/2\,a-1}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(1/2\,a+1/3)\,\Gamma(1/2\,a+2/3)}{2^{3/2\,a-1}5^{5/4\,a-1/2}\Gamma(8/15)\,\Gamma(13/15)\,\Gamma(1/2\,a+2/5)\,\Gamma(1/2\,a+3/5)}, \\ \frac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{3\,n+3}(8/5,n)\,(7/5,n)}{2^{3\,n+3}3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n. \end{cases}$$

(viii) $F(1/2\,a, 5/2\,a-1;\ 3/2\,a;\ 1/2-1/2\,\sqrt{5})$

$$= \begin{cases} \frac{3^{3/2\,a-3/5}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(1/2\,a+1/3)\,\Gamma(1/2\,a+2/3)}{5^{5/4\,a-1/2}\Gamma(8/15)\,\Gamma(13/15)\,\Gamma(1/2\,a+2/5)\,\Gamma(1/2\,a+3/5)}, \\ \frac{5^{5/2\,n+1/2}(8/5,n)\,(7/5,n)}{3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n. \end{cases}$$

(ix) $F(a, 1-a;\ 2-3/2\,a;\ 1/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{3^{3n}\left(\sqrt{5}-1\right)^{3n}(4/3,n)\,(2/3,n)}{2^{3n}5^{5/2\,n}\,(6/5,n)\,(4/5,n)} & \text{if } a=-2\,n,\\ \dfrac{3^{3n}\left(\sqrt{5}-1\right)^{3n+3}(11/6,n)\,(7/6,n)}{7\cdot 2^{3n+3}5^{5/2\,n-1/2}\,(17/10,n)\,(13/10,n)} & \text{if } a=-1-2\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{3n}(3/10,n)\,(7/10,n)}{2^{3n}3^{3n}\,(1/6,n)\,(5/6,n)} & \text{if } a=1+2\,n,\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3n+6}(9/5,n)\,(11/5,n)}{2^{3n+6}3^{3n}\,(5/3,n)\,(7/3,n)} & \text{if } a=4+2\,n \end{cases}$$

(The third case is identical to Theorem 34 in [Ek]).

(x) $F(2-5/2\,a, 1-1/2\,a;\ 2-3/2\,a;\ 1/2+1/2\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n}\,(1/5,n)\,(3/5,n)}{(-3)^{3n}\,(1/15,n)\,(11/15,n)} & \text{if } a=4/5+2\,n,\\ \dfrac{-5^{5/2\,n+1/2}\,(2/5,n)\,(4/5,n)}{(-3)^{3n}\,(4/15,n)\,(14/15,n)} & \text{if } a=6/5+2\,n,\\ 0 \quad \text{if } a=8/5+2\,n, & \\ \dfrac{5^{5/2\,n}\,(4/5,n)\,(6/5,n)}{(-3)^{3n}\,(2/3,n)\,(4/3,n)} & \text{if } a=2+2\,n,\\ 0 \quad \text{if } a=12/5+2\,n. & \end{cases}$$

(xi) $F(a, 1-1/2\,a;\ 2-3/2\,a;\ 3/2+1/2\sqrt{5})$

$$= \begin{cases} \dfrac{3^{3n}\left(\sqrt{5}-1\right)^{n}(4/3,n)\,(2/3,n)}{2^{n}5^{5/2\,n}\,(6/5,n)\,(4/5,n)} & \text{if } a=-2\,n,\\ \dfrac{-3^{3n}\left(\sqrt{5}-1\right)^{n+2}(11/6,n)\,(7/6,n)}{7\cdot 2^{n+2}5^{5/2\,n-1/2}\,(17/10,n)\,(13/10,n)} & \text{if } a=-1-2\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{n}(4/5,n)\,(6/5,n)}{2^{n}3^{3n}\,(2/3,n)\,(4/3,n)} & \text{if } a=2+2\,n \end{cases}$$

(The third case is identical to Theorem 22 in [Ek]).

(xii) $F(1-a, 2-5/2\,a;\ 2-3/2\,a;\ 3/2+1/2\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5n}(3/10,n)\,(7/10,n)}{2^{5n}3^{3n}\,(1/6,n)\,(5/6,n)} & \text{if } a=1+2\,n,\\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5n+9}(9/5,n)\,(11/5,n)}{2^{5n+9}3^{3n}\,(5/3,n)\,(7/3,n)} & \text{if } a=4+2\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5n}(1/5,n)\,(3/5,n)}{2^{5n}3^{3n}\,(1/15,n)\,(11/15,n)} & \text{if } a=4/5+2\,n,\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5n+1}(2/5,n)\,(4/5,n)}{2^{5n+1}3^{3n}\,(4/15,n)\,(14/15,n)} & \text{if } a=6/5+2\,n,\\ 0 \quad \text{if } a=8/5+2\,n, & \\ 0 \quad \text{if } a=12/5+2\,n. & \end{cases}$$

(xiii) $F(1/2\,a, 5/2\,a-1;\ 3/2\,a;\ 1/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{-5^{5/2\,n+1/2}\,(8/5,n)\,(7/5,n)}{(-3)^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n, \\ \dfrac{5^{5/2\,n}\,(2/5,n)\,(1/5,n)}{(-3)^{3\,n}\,(7/15,n)\,(2/15,n)} & \text{if } a=2/5-2\,n, \\ \dfrac{-5^{5/2\,n+1}\,(4/5,n)\,(3/5,n)}{(-3)^{3\,n+1}\,(13/15,n)\,(8/15,n)} & \text{if } a=-2/5-2\,n, \\ 0 & \text{if } a=-4/5-2\,n, \\ 0 & \text{if } a=-6/5-2\,n. \end{cases}$$

(xiv) $F(a, 1-a;\ 3/2\,a;\ 1/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3\,n+3}(8/5,n)\,(7/5,n)}{2^{3\,n+3}3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3\,n+3}(11/10,n)\,(9/10,n)}{2^{3\,n+3}3^{3\,n+1}\,(7/6,n)\,(5/6,n)} & \text{if } a=-1-2\,n, \\ \dfrac{3^{3\,n}\left(\sqrt{5}-1\right)^{3\,n}(5/6,n)\,(7/6,n)}{2^{3\,n}5^{5/2\,n}\,(9/10,n)\,(11/10,n)} & \text{if } a=1+2\,n, \\ \dfrac{-3^{3\,n-1}\left(\sqrt{5}-1\right)^{3\,n+3}(4/3,n)\,(5/3,n)}{2^{3\,n+3}5^{5/2\,n}\,(7/5,n)\,(8/5,n)} & \text{if } a=2+2\,n \end{cases}$$

(The first case is identical to Theorem 35 in [Ek]).

(xv) $F(a, 5/2\,a-1;\ 3/2\,a;\ 3/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5\,n+5}(8/5,n)\,(7/5,n)}{2^{5\,n+5}3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n, \\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5\,n+4}(11/10,n)\,(9/10,n)}{2^{5\,n+4}3^{3\,n+1}\,(7/6,n)\,(5/6,n)} & \text{if } a=-1-2\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5\,n}(2/5,n)\,(1/5,n)}{2^{5\,n}3^{3\,n}\,(7/15,n)\,(2/15,n)} & \text{if } a=2/5-2\,n, \\ \dfrac{5^{5/2\,n+1}\left(\sqrt{5}+1\right)^{5\,n+2}(4/5,n)\,(3/5,n)}{2^{5\,n+2}3^{3\,n+1}\,(13/15,n)\,(8/15,n)} & \text{if } a=-2/5-2\,n, \\ 0 & \text{if } a=-4/5-2\,n, \\ 0 & \text{if } a=-6/5-2\,n. \end{cases}$$

(xvi) $F(1/2\,a, 1-a;\ 3/2\,a;\ 3/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{n+1}(8/5,n)\,(7/5,n)}{2^{n+1}3^{3\,n}\,(5/3,n)\,(4/3,n)} & \text{if } a=-2-2\,n, \\ \dfrac{3^{3\,n}\left(\sqrt{5}-1\right)^{n}(5/6,n)\,(7/6,n)}{2^{n}5^{5/2\,n}\,(9/10,n)\,(11/10,n)} & \text{if } a=1+2\,n, \\ \dfrac{3^{3\,n-1}\left(\sqrt{5}-1\right)^{n+2}(4/3,n)\,(5/3,n)}{2^{n+2}5^{5/2\,n}\,(7/5,n)\,(8/5,n)} & \text{if } a=2+2\,n \end{cases}$$

(The first case is identical to Theorem 21 in [Ek]).

(xvii) $F(1/2\,a, 1-a;\; 2-2\,a;\; -1/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{2^{-3/2\,a}\left(\sqrt{5}-1\right)^{-1/2\,a}\Gamma(6/5)\,\Gamma(4/5)\,\Gamma(3/2-a)}{5^{-5/4\,a}\Gamma(3/2)\,\Gamma(6/5-1/2\,a)\,\Gamma(4/5-1/2\,a)}, \\ \dfrac{7\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{n+1}(13/10,n)\,(17/10,n)}{2^{7\,n+5}\,(5/4,n)\,(7/4,n)} \quad \text{if } a=3+2\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{n+2}(4/5,n)\,(6/5,n)}{2^{7\,n+3}\,(3/4,n)\,(5/4,n)} \quad \text{if } a=2+2\,n \end{cases}$$

(The first case is a generalization of Theorem 23 in [Ek]).

(xviii) $F(1-a, 2-5/2\,a;\; 2-2\,a;\; -1/2+1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{2^{2-9/2\,a}\left(\sqrt{5}-1\right)^{5/2\,a-2}\Gamma(6/5)\,\Gamma(4/5)\,\Gamma(3/2-a)}{5^{-5/4\,a}\Gamma(3/2)\Gamma(6/5-1/2\,a)\,\Gamma(4/5-1/2\,a)}, \\ \dfrac{7\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{5\,n+5}(13/10,n)\,(17/10,n)}{2^{11\,n+9}\,(5/4,n)\,(7/4,n)} \quad \text{if } a=3+2\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{5\,n+4}(4/5,n)\,(6/5,n)}{2^{11\,n+5}\,(3/4,n)\,(5/4,n)} \quad \text{if } a=2+2\,n, \end{cases}$$

(xix) $F(1-a, 2-5/2\,a;\; 2-2\,a;\; -1/2-1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{7\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5\,n+5}(13/10,n)\,(17/10,n)}{2^{11\,n+9}\,(5/4,n)\,(7/4,n)} \quad \text{if } a=3+2\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5\,n+4}(4/5,n)\,(6/5,n)}{2^{11\,n+5}\,(3/4,n)\,(5/4,n)} \quad \text{if } a=2+2\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{5\,n}(1/5,n)\,(3/5,n)}{2^{11\,n}\,(3/20,n)\,(13/20,n)} \quad \text{if } a=4/5+2\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{5\,n+1}(2/5,n)\,(4/5,n)}{2^{11\,n+2}\,(7/20,n)\,(17/20,n)} \quad \text{if } a=6/5+2\,n, \\ 0 \quad \text{if } a=8/5+2\,n, \\ 0 \quad \text{if } a=12/5+2\,n. \end{cases}$$

(xx) $F(1/2\,a, 1-a;\; 2-2\,a;\; -1/2-1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{2^{5\,n}\left(\sqrt{5}+1\right)^{n}(5/4,n)\,(3/4,n)}{5^{5/2\,n}\,(6/5,n)\,(4/5,n)} \quad \text{if } a=-2\,n, \\ \dfrac{7\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{n+1}(13/10,n)\,(17/10,n)}{2^{7\,n+5}\,(5/4,n)\,(7/4,n)} \quad \text{if } a=3+2\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{n+2}(4/5,n)\,(6/5,n)}{2^{7\,n+3}\,(3/4,n)\,(5/4,n)} \quad \text{if } a=2+2\,n. \end{cases}$$

(xxi) $F(a, 1-1/2\,a;\; 2-3/2\,a;\; 3/2-1/2\,\sqrt{5})$

$$= \begin{cases} \dfrac{3^{-3/2\,a}\left(\sqrt{5}+1\right)^{-1/2\,a}\Gamma\left(6/5\right)\Gamma\left(4/5\right)\Gamma\left(4/3-1/2\,a\right)\Gamma\left(2/3-1/2\,a\right)}{2^{-1/2\,a}5^{-5/4\,a}\Gamma\left(4/3\right)\Gamma\left(2/3\right)\Gamma\left(6/5-1/2\,a\right)\Gamma\left(4/5-1/2\,a\right)}, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{n}\left(4/5,n\right)\left(6/5,n\right)}{2^{n}3^{3\,n}\left(2/3,n\right)\left(4/3,n\right)} \quad \text{if } a=2+2\,n \end{cases}$$

(The second case is identical to Theorem 22 in [Ek]).

(xxii) $F(1-a, 2-5/2\,a;\ 2-3/2\,a;\ 3/2-1/2\sqrt{5})$

$$= \begin{cases} \dfrac{3^{-3/2\,a}\left(\sqrt{5}+1\right)^{1-5/2\,a}\Gamma\left(6/5\right)\Gamma\left(4/5\right)\Gamma\left(4/3-1/2\,a\right)\Gamma\left(2/3-1/2\,a\right)}{2^{1-5/2\,a}5^{-5/4\,a}\Gamma\left(4/3\right)\Gamma\left(2/3\right)\Gamma\left(6/5-1/2\,a\right)\Gamma\left(4/5-1/2\,a\right)}, \\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{5\,n+9}\left(9/5,n\right)\left(11/5,n\right)}{2^{5\,n+9}3^{3\,n}\left(5/3,n\right)\left(7/3,n\right)} \quad \text{if } a=4+2\,n. \end{cases}$$

(xxiii) $F(a, 1-a;\ 2-3/2\,a;\ 1/2-1/2\sqrt{5})$

$$= \begin{cases} \dfrac{3^{-3/2\,a}\left(\sqrt{5}+1\right)^{-3/2\,a}\Gamma\left(6/5\right)\Gamma\left(4/5\right)\Gamma\left(4/3-1/2\,a\right)\Gamma\left(2/3-1/2\,a\right)}{2^{-3/2\,a}5^{-5/4\,a}\Gamma\left(4/3\right)\Gamma\left(2/3\right)\Gamma\left(6/5-1/2\,a\right)\Gamma\left(4/5-1/2\,a\right)}, \\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{3\,n+6}\left(9/5,n\right)\left(11/5,n\right)}{2^{3\,n+6}3^{3\,n}\left(5/3,n\right)\left(7/3,n\right)} \quad \text{if } a=4+2\,n. \end{cases}$$

(xxiv) $F(2-5/2\,a, 1-1/2\,a;\ 2-3/2\,a;\ 1/2-1/2\sqrt{5})$

$$= \begin{cases} \dfrac{3^{-3/2\,a}\left(\sqrt{5}-1\right)\Gamma\left(6/5\right)\Gamma\left(4/5\right)\Gamma\left(4/3-1/2\,a\right)\Gamma\left(2/3-1/2\,a\right)}{2\cdot 5^{-5/4\,a}\Gamma\left(4/3\right)\Gamma\left(2/3\right)\Gamma\left(6/5-1/2\,a\right)\Gamma\left(4/5-1/2\,a\right)}, \\ \dfrac{5^{5/2\,n}\left(4/5,n\right)\left(6/5,n\right)}{3^{3\,n}\left(2/3,n\right)\left(4/3,n\right)} \quad \text{if } a=2+2\,n. \end{cases}$$

(2,5,4-2), (2,5,4-3), (2,5,4-4) The special values obtained from (2,5,4-2), (2,5,4-3) and (2,5,4-4) coincide with those obtained from (2,5,4-1).

$(k, l, m) = (2, 6, 4)$

In this case, we have

$$(a,b,c,x) = (a,b,2\,a,2), \tag{2,6,4-1}$$
$$(a,b,c,x) = (a,b,b+1-a,-1), \tag{2,6,4-2}$$
$$(a,b,c,x) = (a,b,1/2\,a+1/2\,b+1/2,1/2), \tag{2,6,4-3}$$
$$(a,b,c,x) = (a,3\,a-1,2\,a,1/2+1/2\,i\sqrt{3}), \tag{2,6,4-4}$$
$$(a,b,c,x) = (a,3\,a-1,2\,a,1/2-1/2\,i\sqrt{3}). \tag{2,6,4-5}$$

(2,6,4-1), (2,6,4-2), (2,6,4-3) The special values obtained from (2,6,4-1), (2,6,4-2) and (2,6,4-3) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1).

(2,6,4-4), (2,6,4-5) The special values obtained from (2,6,4-4) and (2,6,4-5) coincide with those obtained from (1,3,2-1).

3.4.5 $m = 5$

$(k, l, m) = (0, 5, 5)$

In this case, we have

$$(a, b, c, x) = (1, b, b, \lambda),\ S^{(n)} = 1, \tag{0,5,5-1}$$

$$(a, b, c, x) = (0, b, b+1, \lambda),\ S^{(n)} = 1, \tag{0,5,5-2}$$

where λ is a solution of $x^4 + x^3 + x^2 + x + 1 = 0$.

(0,5,5-1) The special values obtained from (0,5,5-1) except trivial values are the special cases of (3.4.1).

(0,5,5-2) The special values obtained from (0,5,5-2) coincide with those obtained from (0,5,5-1).

$(k, l, m) = (1, 4, 5)$

In this case, there is no admissible quadruple.

$(k, l, m) = (1, 5, 5)$

In this case, we have

$$(a, b, c, x) = (a, 5\,a+1, 5\,a, 5/4),\ S^{(n)} = \frac{(-5)^{5\,n}\,(a+1, n)\,(4\,a, 4\,n)}{2^{10\,n}\,(5\,a, 5\,n)}, \tag{1,5,5-1}$$

$$(a, b, c, x) = (a, 5\,a-5, 5\,a-3, 5/4),\ S^{(n)} = \frac{5\,a-4}{(-4)^n\,(5\,a-4+5\,n)}. \tag{1,5,5-2}$$

(1,5,5-1)

$$\text{(i) } F(a, 5\,a+1;\ 5\,a;\ 5/4) = \begin{cases} 0 \quad \text{if } a = -1-n, \\ \dfrac{(-5)^{5\,n}\,(1/5, n)\,(9/5, 4\,n)}{2^{10\,n}\,(2, 5\,n)} \quad \text{if } a = -1/5-n, \\ \dfrac{3\,(-5)^{5\,n}\,(2/5, n)\,(13/5, 4\,n)}{2^{10\,n+2}\,(3, 5\,n)} \quad \text{if } a = -2/5-n, \\ \dfrac{7\,(-5)^{5\,n}\,(3/5, n)\,(17/5, 4\,n)}{2^{10\,n+4}\,(4, 5\,n)} \quad \text{if } a = -3/5-n, \\ \dfrac{11\,(-5)^{5\,n}\,(4/5, n)\,(21/5, 4\,n)}{2^{10\,n+6}\,(5, 5\,n)} \quad \text{if } a = -4/5-n. \end{cases}$$

The special values obtained from (ii) and (iii) are trivial.

$$\text{(iv) } F(4\,a, 5\,a+1;\, 5\,a;\, 5) = \begin{cases} 0 \quad \text{if } a = -1/4 - 1/4\,n, \\ \dfrac{5^{5\,n}\,(1/5,n)\,(9/5,4\,n)}{(2,5\,n)} \quad \text{if } a = -1/5 - n, \\ \dfrac{-3\cdot 5^{5\,n}\,(2/5,n)\,(13/5,4\,n)}{(3,5\,n)} \quad \text{if } a = -2/5 - n, \\ \dfrac{7\cdot 5^{5\,n}\,(3/5,n)\,(17/5,4\,n)}{(4,5\,n)} \quad \text{if } a = -3/5 - n, \\ \dfrac{-11\cdot 5^{5\,n}\,(4/5,n)\,(21/5,4\,n)}{(5,5\,n)} \quad \text{if } a = -4/5 - n. \end{cases}$$

(v) $F(a, 5\,a+1;\, a+2;\, -1/4) = 2^{10\,a}5^{-5\,a}\,(a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(vi) $F(2, 1-4\,a;\, a+2;\, -1/4) = 4/5\,a + 4/5$

(The above is a special case of (1.5) in [Eb2]).

(vii) $F(a, 1-4\,a;\, a+2;\, 1/5) = 2^{8\,a}5^{-4\,a}\,(a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(viii) $F(2, 5\,a+1;\, a+2;\, 1/5) = 5/4\,a + 5/4$

(The above is a special case of (1.5) in [Eb2]).

$$\text{(ix) } F(a, 1-4\,a;\, -4\,a;\, 4/5) = \begin{cases} 0, \\ \dfrac{2^{8\,n}\,(9/4,5\,n)}{5^{4\,n}\,(5/4,n)\,(2,4\,n)} \quad \text{if } a = 1/4 + n, \\ \dfrac{3\cdot 2^{8\,n+1}\,(7/2,5\,n)}{5^{4\,n+1}\,(3/2,n)\,(3,4\,n)} \quad \text{if } a = 1/2 + n, \\ \dfrac{77\cdot 2^{8\,n-1}\,(19/4,5\,n)}{5^{4\,n+2}\,(7/4,n)\,(4,4\,n)} \quad \text{if } a = 3/4 + n, \\ \dfrac{2^{8\,n+8}\,(6,5\,n)}{5^{4\,n+3}\,(2,n)\,(5,4\,n)} \quad \text{if } a = 1 + n. \end{cases}$$

The special values obtained from (x) and (xi) are trivial.

$$\text{(xii) } F(-5\,a, 1-4\,a;\, -4\,a;\, -4) = \begin{cases} 0 \quad \text{if } a = 1/5 + n, \\ 0 \quad \text{if } a = 2/5 + n, \\ 0 \quad \text{if } a = 3/5 + n, \\ 0 \quad \text{if } a = 4/5 + n, \\ \dfrac{2^{8\,n}\,(9/4,5\,n)}{(5/4,n)\,(2,4\,n)} \quad \text{if } a = 1/4 + n, \\ \dfrac{3\cdot 2^{8\,n+1}\,(7/2,5\,n)}{(3/2,n)\,(3,4\,n)} \quad \text{if } a = 1/2 + n, \\ \dfrac{77\cdot 2^{8\,n-1}\,(19/4,5\,n)}{(7/4,n)\,(4,4\,n)} \quad \text{if } a = 3/4 + n, \\ \dfrac{2^{8\,n+8}\,(6,5\,n)}{(2,n)\,(5,4\,n)} \quad \text{if } a = 1 + n. \end{cases}$$

(xiii) $F(2, 5\,a+1;\; 4\,a+2;\; 4/5) = 5\,(4\,a+1)$

(The above is a special case of (1.5) in [Eb2]).

(xiv) $F(4\,a, 1-a;\; 4\,a+2;\; 4/5) = 5^{-a}\,(4\,a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

$$\text{(xv) } F(4\,a, 5\,a+1;\; 4\,a+2;\; -4) = \begin{cases} -5^{5\,n}\,(20\,n-1) & \text{if } a = -1/5-n, \\ -5^{5\,n+1}\,(20\,n+3) & \text{if } a = -2/5-n, \\ -5^{5\,n+2}\,(20\,n+7) & \text{if } a = -3/5-n, \\ -5^{5\,n+3}\,(20\,n+11) & \text{if } a = -4/5-n \end{cases}$$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xvi) $F(2, 1-a;\; 4\,a+2;\; -4) = 4/5\,n+1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]).

(xvii) $F(2, 1-4\,a;\; 2-5\,a;\; 5/4) = 5\,n+1 \quad \text{if } a = 1/4+1/4\,n$

(The above is a special case of (1.5) in [Eb2]).

(xviii) $F(-5\,a, 1-a;\; 2-5\,a;\; 5/4) = -\,(-4)^{-n-1}\,(5\,n+4) \quad \text{if } a = 1+n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

$$\text{(xix) } F(-5\,a, 1-4\,a;\; 2-5\,a;\; 5) = \begin{cases} 2^{8\,n}\,(20\,n+1) & \text{if } a = 1/4+n, \\ -2^{8\,n+2}\,(20\,n+6) & \text{if } a = 1/2+n, \\ 2^{8\,n+4}\,(20\,n+11) & \text{if } a = 3/4+n, \\ -2^{8\,n+14}\,(20\,n+36) & \text{if } a = 2+n \end{cases}$$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xx) $F(2, 1-a;\; 2-5\,a;\; 5) = 5/4\,n+1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]). The special values obtained from (xxi) are trivial.

$$\text{(xxii) } F(-5\,a, 1-a;\; -a;\; -1/4) = \begin{cases} 0, & \\ \dfrac{(5, 5\,n)}{2^{2\,n}\,(2, n)\,(4, 4\,n)} & \text{if } a = 1+n \end{cases}$$

(The first case is identical to (29.6) in [Ge]).

$$\text{(xxiii) } F(4\,a, 1-a;\; -a;\; 1/5) = \begin{cases} 0, & \\ \dfrac{(5, 5\,n)}{5^{n}\,(2, n)\,(4, 4\,n)} & \text{if } a = 1+n. \end{cases}$$

The special values obtained from (xxiv) are trivial.

(1,5,5-2) The special values obtained from (1,5,5-2) coincide with those obtained from (1,5,5-1).

$(k, l, m) = (1, 6, 5)$

In this case, we have

$$(a, b, c, x) = (a, b, b + 1 - a, -1). \qquad \text{(1,6,5-1)}$$

(1,6,5-1) The special values obtained from (1,6,5-1) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1).

$(k, l, m) = (2, 3, 5)$

In this case, there is no admissible quadruple.

$(k, l, m) = (2, 4, 5)$

In this case, there is no admissible quadruple.

$(k, l, m) = (2, 5, 5)$

In this case, we have

$$(a, b, c, x) = (a, 5/2\,a + 1, 5/2\,a, 5/3),\ S^{(n)} = \frac{5^{5\,n}\,(a+1, 2\,n)\,(3/2\,a, 3\,n)}{3^{5\,n}\,(5/2\,a, 5\,n)}, \qquad \text{(2,5,5-1)}$$

$$(a, b, c, x) = (a, 5/2\,a - 5/2, 5/2\,a - 1/2, 5/3),\ S^{(n)} = \frac{2^{2\,n}(5\,a - 3)}{5\,a - 3 + 10\,n}. \qquad \text{(2,5,5-2)}$$

(2,5,5-1)

(i) $F(a, 5/2\,a + 1;\ 5/2\,a;\ 5/3)$

$$= \begin{cases} 0 \quad \text{if } a = -1 - n, \\ \dfrac{5^{5\,n}\,(2/5, 2\,n)\,(8/5, 3\,n)}{3^{5\,n}\,(2, 5\,n)} & \text{if } a = -2/5 - 2\,n, \\ \dfrac{5^{5\,n}\,(4/5, 2\,n)\,(11/5, 3\,n)}{3^{5\,n+1}\,(3, 5\,n)} & \text{if } a = -4/5 - 2\,n, \\ \dfrac{-2 \cdot 5^{5\,n}\,(6/5, 2\,n)\,(14/5, 3\,n)}{3^{5\,n+2}\,(4, 5\,n)} & \text{if } a = -6/5 - 2\,n, \\ \dfrac{-7 \cdot 5^{5\,n}\,(8/5, 2\,n)\,(17/5, 3\,n)}{3^{5\,n+3}\,(5, 5\,n)} & \text{if } a = -8/5 - 2\,n. \end{cases}$$

The special values obtained from (ii) and (iii) are trivial.

(iv) $F(3/2\,a, 5/2\,a + 1;\ 5/2\,a;\ 5/2)$

$$= \begin{cases} 0 \quad \text{if } a = -2/3 - 2/3\,n, \\ \dfrac{(-5)^{5\,n}\,(2/5, 2\,n)\,(8/5, 3\,n)}{2^{5\,n}\,(2, 5\,n)} \quad \text{if } a = -2/5 - 2\,n, \\ \dfrac{-(-5)^{5\,n}\,(4/5, 2\,n)\,(11/5, 3\,n)}{2^{5\,n+1}\,(3, 5\,n)} \quad \text{if } a = -4/5 - 2\,n, \\ \dfrac{-(-5)^{5\,n}\,(6/5, 2\,n)\,(14/5, 3\,n)}{2^{5\,n+1}\,(4, 5\,n)} \quad \text{if } a = -6/5 - 2\,n, \\ \dfrac{7\,(-5)^{5\,n}\,(8/5, 2\,n)\,(17/5, 3\,n)}{2^{5\,n+3}\,(5, 5\,n)} \quad \text{if } a = -8/5 - 2\,n. \end{cases}$$

(v) $F(a, 5/2\,a + 1;\; a + 2;\; -2/3) = 3^{5/2\,a} 5^{-5/2\,a}\,(a + 1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(vi) $F(2, 1 - 3/2\,a;\; a + 2;\; -2/3) = 3/5\,a + 3/5$

(The above is a special case of (1.5) in [Eb2]).

(vii) $F(a, 1 - 3/2\,a;\; a + 2;\; 2/5) = 3^{3/2\,a} 5^{-3/2\,a}\,(a + 1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2] and is a generalization of Theorem 38 in [Ek]).

(viii) $F(2, 5/2\,a + 1;\; a + 2;\; 2/5) = 5/3\,a + 5/3$

(The above is a special case of (1.5) in [Eb2]).

(ix) $F(a, 1 - 3/2\,a;\; -3/2\,a;\; 3/5)$

$$= \begin{cases} 0, \\ \dfrac{3^{3\,n}\,(8/3, 5\,n)}{5^{3\,n}\,(5/3, 2\,n)\,(2, 3\,n)} \quad \text{if } a = 2/3 + 2\,n, \\ \dfrac{7 \cdot 3^{3\,n}\,(13/3, 5\,n)}{5^{3\,n+1}\,(7/3, 2\,n)\,(3, 3\,n)} \quad \text{if } a = 4/3 + 2\,n, \\ \dfrac{18 \cdot 3^{3\,n+1}\,(6, 5\,n)}{5^{3\,n+2}\,(3, 2\,n)\,(4, 3\,n)} \quad \text{if } a = 2 + 2\,n. \end{cases}$$

The special values obtained from (x) and (xi) are trivial.

(xii) $F(-5/2\,a, 1 - 3/2\,a;\; -3/2\,a;\; -3/2)$

$$= \begin{cases} 0 \quad \text{if } a = 2/5 + 2\,n, \\ 0 \quad \text{if } a = 4/5 + 2\,n, \\ 0 \quad \text{if } a = 6/5 + 2\,n, \\ 0 \quad \text{if } a = 8/5 + 2\,n, \\ \dfrac{3^{3\,n}\,(8/3, 5\,n)}{2^{3\,n}\,(5/3, 2\,n)\,(2, 3\,n)} \quad \text{if } a = 2/3 + 2\,n, \\ \dfrac{7 \cdot 3^{3\,n}\,(13/3, 5\,n)}{2^{3\,n+1}\,(7/3, 2\,n)\,(3, 3\,n)} \quad \text{if } a = 4/3 + 2\,n, \\ \dfrac{3^{3\,n+3}\,(6, 5\,n)}{2^{3\,n+1}\,(3, 2\,n)\,(4, 3\,n)} \quad \text{if } a = 2 + 2\,n. \end{cases}$$

(xiii) $F(2, 5/2\,a+1;\ 3/2\,a+2;\ 3/5) = 15/4\,a + 5/2$

(The above is a special case of (1.5) in [Eb2]).

(xiv) $F(3/2\,a, 1-a;\ 3/2\,a+2;\ 3/5) = 2^{a-1} 5^{-a}\,(3\,a+2)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xv) $F(3/2\,a, 5/2\,a+1;\ 3/2\,a+2;\ -3/2)$

$$= \begin{cases} 2^{-5\,n-1} 5^{5\,n}\,(2-15\,n) & \text{if } a = -2/5 - 2\,n, \\ 2^{-5\,n-2} 5^{5\,n+1}\,(-1-15\,n) & \text{if } a = -4/5 - 2\,n, \\ 2^{-5\,n-3} 5^{5\,n+2}\,(-4-15\,n) & \text{if } a = -6/5 - 2\,n, \\ 2^{-5\,n-4} 5^{5\,n+3}\,(-7-15\,n) & \text{if } a = -8/5 - 2\,n \end{cases}$$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xvi) $F(2, 1-a;\ 3/2\,a+2;\ -3/2) = 3/5\,n + 1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]).

(xvii) $F(2, 1-3/2\,a;\ 2-5/2\,a;\ 5/3) = 5/2\,n + 1 \quad \text{if } a = 2/3 + 2/3\,n$

(The above is a special case of (1.5) in [Eb2]).

(xviii) $F(-5/2\,a, 1-a;\ 2-5/2\,a;\ 5/3) = (-2)^n\, 3^{-n-1}\,(5\,n+3) \quad \text{if } a = 1+n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xix) $F(-5/2\,a, 1-3/2\,a;\ 2-5/2\,a;\ 5/2)$
$= 2^{-n-1}\,(-3)^n\,(5\,n+2) \quad \text{if } a = 2/3 + 2/3\,n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xx) $F(2, 1-a;\ 2-5/2\,a;\ 5/2) = 5/3\,n + 1 \quad \text{if } a = 1+n$

(The above is a special case of (1.5) in [Eb2]). The special values obtained from (xxi) are trivial.

$$\text{(xxii) } F(-5/2\,a, 1-a;\ -a;\ -2/3) = \begin{cases} 0, & \\ \dfrac{2^{2\,n}\,(5/2, 5\,n)}{3^{2\,n}\,(2, 2\,n)\,(3/2, 3\,n)} & \text{if } a = 1+2\,n, \\ \dfrac{2^{2\,n+3}\,(5, 5\,n)}{3^{2\,n+1}\,(3, 2\,n)\,(3, 3\,n)} & \text{if } a = 2+2\,n. \end{cases}$$

$$\text{(xxiii) } F(3/2\,a, 1-a;\ -a;\ 2/5) = \begin{cases} 0, & \\ \dfrac{2^{2\,n}\,(5/2, 5\,n)}{5^{2\,n}\,(2, 2\,n)\,(3/2, 3\,n)} & \text{if } a = 1+2\,n, \\ \dfrac{2^{2\,n+3}\,(5, 5\,n)}{5^{2\,n+1}\,(3, 2\,n)\,(3, 3\,n)} & \text{if } a = 2+2\,n. \end{cases}$$

The special values obtained from (xxiv) are trivial.

(2,5,5-2) The special values obtained from (2,5,5-2) coincide with those obtained from (2,5,5-1).

$(k, l, m) = (2, 6, 5)$

In this case, there is no admissible quadruple.

$(k, l, m) = (2, 7, 5)$

In this case, there is no admissible quadruple.

3.4.6 $m = 6$

$(k, l, m) = (0, 6, 6)$

In this case, we have

$$(a, b, c, x) = (a, b, b + 1 - a, -1), \tag{0,6,6-1}$$
$$(a, b, c, x) = (1, b, b, \lambda), \tag{0,6,6-2}$$
$$(a, b, c, x) = (0, b, b + 1, \lambda), \tag{0,6,6-3}$$
$$(a, b, c, x) = (1, b, b, \mu), \tag{0,6,6-4}$$
$$(a, b, c, x) = (0, b, b + 1, \mu)., \tag{0,6,6-5}$$

where λ and μ are solutions of $x^2 + x + 1 = 0$ and $x^2 - x + 1 = 0$, respectively.

(0,6,6-1) The special values obtained from (0,6,6-1) are contained in those from (1,2,2-1) and (0,2,2-1).

(0,6,6-2) The special values obtained from (0,6,6-2) except trivial values are special cases of (3.4.1).

(0,6,6-3) The special values obtained from (0,6,6-3) coincide with those obtained from (0,6,6-2).

(0,6,6-4) The special values obtained from (0,6,6-4) except trivial values are special cases of (3.4.1).

(0,6,6-5) The special values obtained from (0,6,6-5) coincide with those obtained from (0,6,6-4).

$(k, l, m) = (1, 5, 6)$

In this case, we have

$$\begin{cases} (a, b, c, x) = (a, 5\,a - 1/2, 6\,a, -4), \\ S(n) = \dfrac{5^{5\,n}\,(a + 1/5, n)\,(a + 4/5, n)\,(a + 3/10, n)\,(a + 7/10, n)}{3^{6\,n}\,(a + 1/3, n)\,(a + 2/3, n)\,(a + 1/6, n)\,(a + 5/6, n)}, \end{cases} \tag{1,5,6-1}$$

$$\begin{cases}(a,b,c,x)=(a,5\,a-3/2,6\,a-1,-4),\\ S^(n)=\dfrac{5^{5\,n}\,(a+2/5,n)\,(a+3/5,n)\,(a-1/10,n)\,(a+1/10,n)}{3^{6\,n}\,(a+1/3,n)\,(a+2/3,n)\,(a-1/6,n)\,(a+1/6,n)},\end{cases}\tag{1,5,6-2}$$

$$\begin{cases}(a,b,c,x)=(a,5\,a-5/2,6\,a-3,-4),\\ S^(n)=\dfrac{5^{5\,n}\,(a-1/5,n)\,(a+1/5,n)\,(a-3/10,n)\,(a+3/10,n)}{3^{6\,n}\,(a-1/3,n)\,(a+1/3,n)\,(a-1/6,n)\,(a+1/6,n)},\end{cases}\tag{1,5,6-3}$$

$$\begin{cases}(a,b,c,x)=(a,5\,a-7/2,6\,a-4,-4),\\ S^(n)=\dfrac{5^{5\,n}\,(a-3/5,n)\,(a-2/5,n)\,(a-1/10,n)\,(a+1/10,n)}{3^{6\,n}\,(a-2/3,n)\,(a-1/3,n)\,(a-1/6,n)\,(a+1/6,n)}.\end{cases}\tag{1,5,6-4}$$

(1,5,6-1)

(i) $F(a,5\,a-1/2;\;6\,a;\;-4)$

$$=\begin{cases}\dfrac{14\cdot 5^{5\,n}\,(9/5,n)\,(6/5,n)\,(17/10,n)\,(13/10,n)}{3^{6\,n+1}\,(5/3,n)\,(4/3,n)\,(11/6,n)\,(7/6,n)} & \text{if } a=-1-n,\\ \dfrac{5^{5\,n}\,(7/10,n)\,(1/10,n)\,(3/5,n)\,(1/5,n)}{3^{6\,n}\,(17/30,n)\,(7/30,n)\,(11/15,n)\,(1/15,n)} & \text{if } a=1/10-n,\\ \dfrac{5^{5\,n+1}\,(9/10,n)\,(3/10,n)\,(4/5,n)\,(2/5,n)}{3^{6\,n+1}\,(23/30,n)\,(13/30,n)\,(14/15,n)\,(4/15,n)} & \text{if } a=-1/10-n,\\ 0 \quad \text{if } a=-3/10-n,\\ \dfrac{13\cdot 5^{5\,n+1}\,(23/10,n)\,(17/10,n)\,(11/5,n)\,(9/5,n)}{3^{6\,n+1}\,(13/6,n)\,(11/6,n)\,(7/3,n)\,(5/3,n)} & \text{if } a=-3/2-n,\\ 0 \quad \text{if } a=-7/10-n.\end{cases}$$

(ii) $F(5\,a,a+1/2;\;6\,a;\;-4)$

$$=\begin{cases}\dfrac{14\cdot 5^{5\,n}\,(9/5,n)\,(6/5,n)\,(17/10,n)\,(13/10,n)}{3^{6\,n+1}\,(5/3,n)\,(4/3,n)\,(11/6,n)\,(7/6,n)} & \text{if } a=-1-n,\\ 0 \quad \text{if } a=-1/5-n,\\ \dfrac{5^{5\,n+2}\,(6/5,n)\,(3/5,n)\,(11/10,n)\,(7/10,n)}{7\cdot 3^{6\,n+1}\,(16/15,n)\,(11/15,n)\,(37/30,n)\,(17/30,n)} & \text{if } a=-2/5-n,\\ \dfrac{5^{5\,n+3}\,(7/5,n)\,(4/5,n)\,(13/10,n)\,(9/10,n)}{26\cdot 3^{6\,n+1}\,(19/15,n)\,(14/15,n)\,(43/30,n)\,(23/30,n)} & \text{if } a=-3/5-n,\\ 0 \quad \text{if } a=-4/5-n,\\ \dfrac{5^{5\,n}\,(13/10,n)\,(7/10,n)\,(6/5,n)\,(4/5,n)}{3^{6\,n}\,(7/6,n)\,(5/6,n)\,(4/3,n)\,(2/3,n)} & \text{if } a=-1/2-n.\end{cases}$$

(iii) $F(a,a+1/2;\;6\,a;\;4/5)$

$$\begin{cases}\dfrac{3^{6\,a-3/5}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(2\,a+1/3)\,\Gamma(2\,a+2/3)}{5^{4\,a-1/2}\Gamma(8/15)\,\Gamma(13/15)\,\Gamma(2\,a+2/5)\,\Gamma(2\,a+3/5)},\\ \dfrac{14\cdot 5^{4\,n-1}\,(9/5,n)\,(6/5,n)\,(17/10,n)\,(13/10,n)}{3^{6\,n+1}\,(5/3,n)\,(4/3,n)\,(11/6,n)\,(7/6,n)} & \text{if } a=-1-n,\\ \dfrac{5^{4\,n}\,(13/10,n)\,(7/10,n)\,(6/5,n)\,(4/5,n)}{3^{6\,n}\,(7/6,n)\,(5/6,n)\,(4/3,n)\,(2/3,n)} & \text{if } a=-1/2-n.\end{cases}$$

(iv) $F(5\,a, 5\,a-1/2;\; 6\,a;\; 4/5)$

$$\begin{cases} \dfrac{3^{6\,a-3/5}\Gamma(3/5)\,\Gamma(4/5)\,\Gamma(2\,a+1/3)\,\Gamma(2\,a+2/3)}{\Gamma(8/15)\,\Gamma(13/15)\,\Gamma(2\,a+2/5)\,\Gamma(2\,a+3/5)}, \\ \dfrac{14}{5^5}\,\dfrac{(9/5,n)\,(6/5,n)\,(17/10,n)\,(13/10,n)}{3^{6\,n+1}\,(5/3,n)\,(4/3,n)\,(11/6,n)\,(7/6,n)} \quad \text{if } a=-1-n, \\ \dfrac{13}{5^7}\,\dfrac{(11/5,n)\,(9/5,n)\,(23/10,n)\,(17/10,n)}{3^{6\,n+1}\,(7/3,n)\,(5/3,n)\,(13/6,n)\,(11/6,n)} \quad \text{if } a=-3/2-n. \end{cases}$$

$$\text{(v) } F(a, 5\,a-1/2;\; 1/2;\; 5) = \begin{cases} \dfrac{2^{6\,n}\,(7/10,n)\,(3/10,n)}{(3/5,n)\,(2/5,n)} \quad \text{if } a=-n, \\ \dfrac{2^{6\,n}\,(3/5,n)\,(1/5,n)}{(1/2,n)\,(3/10,n)} \quad \text{if } a=1/10-n, \\ \dfrac{2^{6\,n+1}\,(4/5,n)\,(2/5,n)}{(7/10,n)\,(1/2,n)} \quad \text{if } a=-1/10-n, \\ 0 \quad \text{if } a=-3/10-n, \\ \dfrac{2^{6\,n+4}\,(6/5,n)\,(4/5,n)}{(11/10,n)\,(9/10,n)} \quad \text{if } a=-1/2-n, \\ 0 \quad \text{if } a=-7/10-n. \end{cases}$$

$$\text{(vi) } F(1-5\,a, 1/2-a;\; 1/2;\; 5) = \begin{cases} \dfrac{2^{6\,n}\,(3/5,n)\,(4/5,n)}{(1/2,n)\,(9/10,n)} \quad \text{if } a=1/5+n, \\ 0 \quad \text{if } a=2/5+n, \\ 0 \quad \text{if } a=3/5+n, \\ \dfrac{2^{6\,n+6}\,(6/5,n)\,(7/5,n)}{5\,(11/10,n)\,(3/2,n)} \quad \text{if } a=4/5+n, \\ \dfrac{-2^{6\,n+7}\,(7/5,n)\,(8/5,n)}{7\,(13/10,n)\,(17/10,n)} \quad \text{if } a=1+n, \\ \dfrac{2^{6\,n}\,(9/10,n)\,(11/10,n)}{(4/5,n)\,(6/5,n)} \quad \text{if } a=1/2+n. \end{cases}$$

(vii) $F(a, 1-5\,a;\; 1/2;\; 5/4)$

$$= \begin{cases} \dfrac{(-16)^n\,(7/10,n)\,(3/10,n)}{(3/5,n)\,(2/5,n)} \quad \text{if } a=-n, \\ \dfrac{(3/5,n)\,(4/5,n)}{(-16)^n\,(1/2,n)\,(9/10,n)} \quad \text{if } a=1/5+n, \\ 0 \quad \text{if } a=2/5+n, \\ 0 \quad \text{if } a=3/5+n, \\ \dfrac{-\,(6/5,n)\,(7/5,n)}{5\,(-16)^n\,(11/10,n)\,(3/2,n)} \quad \text{if } a=4/5+n, \\ \dfrac{-\,(7/5,n)\,(8/5,n)}{14\,(-16)^n\,(13/10,n)\,(17/10,n)} \quad \text{if } a=1+n. \end{cases}$$

(viii) $F(5\,a-1/2, 1/2-a;\ 1/2;\ 5/4)$

$$= \begin{cases} \dfrac{(3/5,n)\,(1/5,n)}{(-16)^n\,(1/2,n)\,(3/10,n)} & \text{if } a = 1/10-n, \\ \dfrac{-\,(4/5,n)\,(2/5,n)}{2\,(-16)^n\,(7/10,n)\,(1/2,n)} & \text{if } a = -1/10-n, \\ 0 \quad \text{if } a = -3/10-n, \\ \dfrac{-\,(6/5,n)\,(4/5,n)}{4\,(-16)^n\,(11/10,n)\,(9/10,n)} & \text{if } a = -1/2-n, \\ 0 \quad \text{if } a = -7/10-n, \\ \dfrac{(-16)^n\,(9/10,n)\,(11/10,n)}{(4/5,n)\,(6/5,n)} & \text{if } a = 1/2+n. \end{cases}$$

(ix) $F(a, 1-5\,a;\ 3/2-4\,a;\ -1/4)$

$$= \frac{5^{5\,a}\Gamma\,(4/5)\,\Gamma\,(6/5)\,\Gamma\,(3/2-4\,a)}{2^{8\,a}\Gamma\,(3/2)\,\Gamma\,(4/5-2\,a)\,\Gamma\,(6/5-2\,a)}.$$

(x) $F(3/2-5\,a, a+1/2;\ 3/2-4\,a;\ -1/4)$

$$= \frac{5^{5\,a-1/2}\Gamma\,(4/5)\,\Gamma\,(6/5)\,\Gamma\,(3/2-4\,a)}{2^{8\,a-1}\Gamma\,(3/2)\,\Gamma\,(4/5-2\,a)\,\Gamma\,(6/5-2\,a)}.$$

(xi) $F(a, a+1/2;\ 3/2-4\,a;\ 1/5) = \dfrac{5^{6\,a}\Gamma\,(4/5)\,\Gamma\,(6/5)\,\Gamma\,(3/2-4\,a)}{2^{10\,a}\Gamma\,(3/2)\,\Gamma\,(4/5-2\,a)\,\Gamma\,(6/5-2\,a)}$

(The above is a generalization of Theorem 20 in [Ek]).

(xii) $F(1-5\,a, 3/2-5\,a;\ 3/2-4\,a;\ 1/5)$

$$= \frac{5\,\Gamma\,(4/5)\,\Gamma\,(6/5)\,\Gamma\,(3/2-4\,a)}{2^{2-2\,a}\Gamma\,(3/2)\,\Gamma\,(4/5-2\,a)\,\Gamma\,(6/5-2\,a)}.$$

(xiii) $F(5\,a-1/2, 1/2-a;\ 4\,a+1/2;\ -1/4)$

$$= \frac{2^{8\,a-4}\Gamma\,(7/5)\,\Gamma\,(8/5)\,\Gamma\,(4\,a+1/2)}{5^{5\,a-5/2}\Gamma\,(5/2)\,\Gamma\,(2\,a+2/5)\,\Gamma\,(2\,a+3/5)}.$$

(xiv) $F(5\,a, 1-a;\ 4\,a+1/2;\ -1/4) = \dfrac{2^{8\,a-3}\Gamma\,(7/5)\,\Gamma\,(8/5)\,\Gamma\,(4\,a+1/2)}{5^{5\,a-2}\Gamma\,(5/2)\,\Gamma\,(2\,a+2/5)\,\Gamma\,(2\,a+3/5)}.$

(xv) $F(5\,a, 5\,a-1/2;\ 4\,a+1/2;\ 1/5) = \dfrac{25\cdot 2^{-2\,a-3}\Gamma\,(7/5)\,\Gamma\,(8/5)\,\Gamma\,(4\,a+1/2)}{\Gamma\,(5/2)\,\Gamma\,(2\,a+2/5)\,\Gamma\,(2\,a+3/5)}.$

(xvi) $F(1-a, 1/2-a;\ 4\,a+1/2;\ 1/5) = \dfrac{2^{10\,a-5}\Gamma\,(7/5)\,\Gamma\,(8/5)\,\Gamma\,(4\,a+1/2)}{5^{6\,a-3}\Gamma\,(5/2)\,\Gamma\,(2\,a+2/5)\,\Gamma\,(2\,a+3/5)}$

(The above is a generalization of Theorem 19 in [Ek]).

(xvii) $F(1-5\,a, 1/2-a;\ 2-6\,a;\ -4)$

$$=\begin{cases} \dfrac{5^{5\,n}\,(1/10,n)\,(3/10,n)\,(3/5,n)\,(4/5,n)}{3^{6\,n}\,(1/30,n)\,(11/30,n)\,(8/15,n)\,(13/15,n)} & \text{if } a=1/5+n, \\ 0 \quad \text{if } a=2/5+n, \\ 0 \quad \text{if } a=3/5+n, \\ \dfrac{5^{5\,n+3}\,(7/10,n)\,(9/10,n)\,(6/5,n)\,(7/5,n)}{7\cdot 3^{6\,n+1}\,(19/30,n)\,(29/30,n)\,(17/15,n)\,(22/15,n)} & \text{if } a=4/5+n, \\ \dfrac{-22\cdot 5^{5\,n}\,(19/10,n)\,(21/10,n)\,(12/5,n)\,(13/5,n)}{3^{6\,n}\,(11/6,n)\,(13/6,n)\,(7/3,n)\,(8/3,n)} & \text{if } a=2+n, \\ \dfrac{5^{5\,n}\,(2/5,n)\,(3/5,n)\,(9/10,n)\,(11/10)}{3^{6\,n}\,(1/3,n)\,(2/3,n)\,(5/6,n)\,(7/6,n)} & \text{if } a=1/2+n. \end{cases}$$

(xviii) $F(1-a, 3/2-5\,a;\ 2-6\,a;\ -4)$

$$=\begin{cases} \dfrac{5^{5\,n}\,(9/10,n)\,(11/10,n)\,(7/5,n)\,(8/5,n)}{3^{6\,n}\,(5/6,n)\,(7/6,n)\,(4/3,n)\,(5/3,n)} & \text{if } a=1+n, \\ \dfrac{5^{5\,n}\,(1/5,n)\,(2/5,n)\,(7/10,n)\,(9/10,n)}{3^{6\,n}\,(2/15,n)\,(7/15,n)\,(19/30,n)\,(29/30,n)} & \text{if } a=3/10+n, \\ \dfrac{-11\cdot 5^{5\,n}\,(7/5,n)\,(8/5,n)\,(19/10,n)\,(21/10,n)}{7\cdot 3^{6\,n-1}\,(4/3,n)\,(5/3,n)\,(11/6,n)\,(13/6,n)} & \text{if } a=3/2+n, \\ \dfrac{5^{5\,n+2}\,(3/5,n)\,(4/5,n)\,(11/10,n)\,(13/10,n)}{11\cdot 3^{6\,n}\,(8/15,n)\,(13/15,n)\,(31/30,n)\,(41/30,n)} & \text{if } a=7/10+n, \\ 0 \quad \text{if } a=9/10+n, \\ 0 \quad \text{if } a=11/10+n. \end{cases}$$

(xix) $F(1-5\,a, 3/2-5\,a;\ 2-6\,a;\ 4/5)$

$$=\begin{cases} \dfrac{3^{6/5-6\,a}\Gamma\,(2/5)\,\Gamma\,(4/5)\,\Gamma\,(2/3-2\,a)\,\Gamma\,(4/3-2\,a)}{\Gamma\,(4/15)\,\Gamma\,(14/15)\,\Gamma\,(4/5-2\,a)\,\Gamma\,(6/5-2\,a)}, \\ \dfrac{-22\,(19/10,n)\,(21/10,n)\,(12/5,n)\,(13/5,n)}{5^9 3^{6\,n}\,(11/6,n)\,(13/6,n)\,(7/3,n)\,(8/3,n)} & \text{if } a=2+n, \\ \dfrac{-11\,(7/5,n)\,(8/5,n)\,(19/10,n)\,(21/10,n)}{5^6\cdot 7\cdot 3^{6\,n-1}\,(4/3,n)\,(5/3,n)\,(11/6,n)\,(13/6,n)} & \text{if } a=3/2+n. \end{cases}$$

(xx) $F(1-a, 1/2-a;\ 2-6\,a;\ 4/5)$

$$=\begin{cases} \dfrac{3^{6/5-6\,a}\Gamma\,(2/5)\,\Gamma\,(4/5)\,\Gamma\,(2/3-2\,a)\,\Gamma\,(4/3-2\,a)}{5^{1/2-4\,a}\Gamma\,(4/15)\,\Gamma\,(14/15)\,\Gamma\,(4/5-2\,a)\,\Gamma\,(6/5-2\,a)}, \\ \dfrac{5^{4\,n}\,(9/10,n)\,(11/10,n)\,(7/5,n)\,(8/5,n)}{3^{6\,n}\,(5/6,n)\,(7/6,n)\,(4/3,n)\,(5/3,n)} & \text{if } a=1+n, \\ \dfrac{5^{4\,n}\,(2/5,n)\,(3/5,n)\,(9/10,n)\,(11/10,n)}{3^{6\,n}\,(1/3,n)\,(2/3,n)\,(5/6,n)\,(7/6,n)} & \text{if } a=1/2+n. \end{cases}$$

$$\text{(xxi) } F(5\,a, a+1/2;\ 3/2;\ 5) = \begin{cases} \dfrac{2^{6\,n}\,(4/5,n)\,(1/5,n)}{(11/10,n)\,(9/10,n)} & \text{if } a=-n, \\ 0 \quad \text{if } a=-1/5-n, \\ \dfrac{2^{6\,n+4}\,(6/5,n)\,(3/5,n)}{15\,(3/2,n)\,(13/10,n)} & \text{if } a=-2/5-n, \\ \dfrac{2^{6\,n+6}\,(7/5,n)\,(4/5,n)}{35\,(17/10,n)\,(3/2,n)} & \text{if } a=-3/5-n, \\ 0 \quad \text{if } a=-4/5-n, \\ \dfrac{2^{6\,n}\,(13/10,n)\,(7/10,n)}{(8/5,n)\,(7/5,n)} & \text{if } a=-1/2-n. \end{cases}$$

$$\text{(xxii) } F(1-a, 3/2-5\,a;\ 3/2;\ 5) = \begin{cases} \dfrac{2^{6\,n}\,(9/10,n)\,(11/10,n)}{(6/5,n)\,(9/5,n)} & \text{if } a=1+n, \\ \dfrac{2^{6\,n}\,(1/5,n)\,(2/5,n)}{(1/2,n)\,(11/10,n)} & \text{if } a=3/10+n, \\ \dfrac{-2^{6\,n+1}\,(2/5,n)\,(3/5,n)}{3\,(7/10,n)\,(13/10,n)} & \text{if } a=1/2+n, \\ \dfrac{2^{6\,n+3}\,(3/5,n)\,(4/5,n)}{5\,(9/10,n)\,(3/2,n)} & \text{if } a=7/10+n, \\ 0 \quad \text{if } a=9/10+n, \\ 0 \quad \text{if } a=11/10+n. \end{cases}$$

(xxiii) $F(5\,a, 1-a;\ 3/2;\ 5/4)$

$$= \begin{cases} \dfrac{(4/5,n)\,(1/5,n)}{(-16)^n\,(11/10,n)\,(9/10,n)} & \text{if } a=-n, \\ 0 \quad \text{if } a=-1/5-n, \\ \dfrac{(6/5,n)\,(3/5,n)}{15\,(-16)^n\,(3/2,n)\,(13/10,n)} & \text{if } a=-2/5-n, \\ \dfrac{-\,(7/5,n)\,(4/5,n)}{35\,(-16)^n\,(17/10,n)\,(3/2,n)} & \text{if } a=-3/5-n, \\ 0 \quad \text{if } a=-4/5-n, \\ \dfrac{(-16)^n\,(9/10,n)\,(11/10,n)}{(6/5,n)\,(9/5,n)} & \text{if } a=1+n. \end{cases}$$

(xxiv) $F(a+1/2, 3/2-5\,a;\ 3/2;\ 5/4)$

$$= \begin{cases} \dfrac{(-16)^n\,(13/10,n)\,(7/10,n)}{(8/5,n)\,(7/5,n)} & \text{if } a=-1/2-n, \\ \dfrac{(1/5,n)\,(2/5,n)}{(-16)^n\,(1/2,n)\,(11/10,n)} & \text{if } a=3/10+n, \\ \dfrac{(2/5,n)\,(3/5,n)}{6\,(-16)^n\,(7/10,n)\,(13/10,n)} & \text{if } a=1/2+n, \\ \dfrac{(3/5,n)\,(4/5,n)}{10\,(-16)^n\,(9/10,n)\,(3/2,n)} & \text{if } a=7/10+n, \\ 0 \quad \text{if } a=9/10+n, \\ 0 \quad \text{if } a=11/10+n. \end{cases}$$

(1,5,6-2), (1,5,6-3), (1,5,6-4) The special values obtained from (1,5,6-2), (1,5,6-3) and (1,5,6-4) coincide with those obtained from (1,5,6-1).

$(k,l,m) = (1,6,6)$

In this case, we have

$$(a,b,c,x) = (a, 6\,a+1, 6\,a, 6/5),\ S^{(n)} = \frac{(-1)^n\, 2^{6\,n} 3^{6\,n}\,(a+1,n)\,(5\,a,5\,n)}{5^{6\,n}\,(6\,a,6\,n)}, \quad (1,6,6\text{-}1)$$

$$(a,b,c,x) = (a, 6\,a-6, 6\,a-4, 6/5),\ S^{(n)} = \frac{6\,a-5}{(-5)^n\,(6\,a-5+6\,n)}. \quad (1,6,6\text{-}2)$$

(1,6,6-1)

(i) $F(a, 6\,a+1;\ 6\,a;\ 6/5)$

$$= \begin{cases} 0 \quad \text{if } a = -1-n, \\ \dfrac{(-1)^n\, 2^{6\,n} 3^{6\,n}\,(1/6,n)\,(11/6,5\,n)}{5^{6\,n}\,(2,6\,n)} & \text{if } a = -1/6-n, \\ \dfrac{(-1)^n\, 2^{6\,n+2} 3^{6\,n}\,(1/3,n)\,(8/3,5\,n)}{5^{6\,n+1}\,(3,6\,n)} & \text{if } a = -1/3-n, \\ \dfrac{(-1)^n\, 2^{6\,n-1} 3^{6\,n+3}\,(1/2,n)\,(7/2,5\,n)}{5^{6\,n+2}\,(4,6\,n)} & \text{if } a = -1/2-n, \\ \dfrac{7\,(-1)^n\, 2^{6\,n+4} 3^{6\,n-1}\,(2/3,n)\,(13/3,5\,n)}{5^{6\,n+3}\,(5,6\,n)} & \text{if } a = -2/3-n, \\ \dfrac{1729\,(-1)^n\, 2^{6\,n-3} 3^{6\,n-1}\,(5/6,n)\,(31/6,5\,n)}{5^{6\,n+4}\,(6,6\,n)} & \text{if } a = -5/6-n. \end{cases}$$

The special values obtained from (ii) and (iii) are trivial.

(iv) $F(5\,a, 6\,a+1;\ 6\,a;\ 6)$

$$= \begin{cases} 0 \quad \text{if } a = -1/5-1/5\,n, \\ \dfrac{(-1)^n\, 2^{6\,n} 3^{6\,n}\,(1/6,n)\,(11/6,5\,n)}{(2,6\,n)} & \text{if } a = -1/6-n, \\ \dfrac{(-1)^{n+1}\, 2^{6\,n+2} 3^{6\,n}\,(1/3,n)\,(8/3,5\,n)}{(3,6\,n)} & \text{if } a = -1/3-n, \\ \dfrac{(-1)^n\, 2^{6\,n-1} 3^{6\,n+3}\,(1/2,n)\,(7/2,5\,n)}{(4,6\,n)} & \text{if } a = -1/2-n, \\ \dfrac{7\,(-1)^{n+1}\, 2^{6\,n+4} 3^{6\,n-1}\,(2/3,n)\,(13/3,5\,n)}{(5,6\,n)} & \text{if } a = -2/3-n, \\ \dfrac{1729\,(-1)^n\, 2^{6\,n-3} 3^{6\,n-1}\,(5/6,n)\,(31/6,5\,n)}{(6,6\,n)} & \text{if } a = -5/6-n. \end{cases}$$

(v) $F(a, 6\,a+1;\ a+2;\ -1/5) = 5^{6\,a} 6^{-6\,a}\,(a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(vi) $F(2, 1-5\,a;\ a+2;\ -1/5) = 5/6\,a + 5/6$

(The above is a special case of (1.5) in [Eb2]).

(vii) $F(a, 1-5\,a;\; a+2;\; 1/6) = 5^{5\,a}6^{-5\,a}\,(a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(viii) $F(2, 6\,a+1;\; a+2;\; 1/6) = 6/5\,a + 6/5$

(The above is a special case of (1.5) in [Eb2]).

(ix) $F(a, 1-5\,a;\; -5\,a;\; 5/6)$

$$= \begin{cases} 0, \\ \dfrac{5^{5\,n}\,(6/5, 6\,n)}{2^{5\,n}3^{5\,n}\,(6/5, n)\,(1, 5\,n)} & \text{if } a = 1/5+n, \\ \dfrac{7\cdot 5^{5\,n}\,(12/5, 6\,n)}{2^{5\,n+1}3^{5\,n+1}\,(7/5, n)\,(2, 5\,n)} & \text{if } a = 2/5+n, \\ \dfrac{13\cdot 5^{5\,n}\,(18/5, 6\,n)}{2^{5\,n}3^{5\,n+2}\,(8/5, n)\,(3, 5\,n)} & \text{if } a = 3/5+n, \\ \dfrac{133\cdot 5^{5\,n}\,(24/5, 6\,n)}{2^{5\,n+3}3^{5\,n+2}\,(9/5, n)\,(4, 5\,n)} & \text{if } a = 4/5+n, \\ \dfrac{5^{5\,n+5}\,(6, 6\,n)}{2^{5\,n+4}3^{5\,n+4}\,(2, n)\,(5, 5\,n)} & \text{if } a = 1+n. \end{cases}$$

The special values obtained from (x) and (xi) are trivial.

$$\text{(xii) } F(-6\,a, 1-5\,a;\; -5\,a;\; -5) = \begin{cases} 0 & \text{if } a = 1/6+n, \\ 0 & \text{if } a = 1/3+n, \\ 0 & \text{if } a = 1/2+n, \\ 0 & \text{if } a = 2/3+n, \\ 0 & \text{if } a = 5/6+n, \\ \dfrac{5^{5\,n}\,(6/5, 6\,n)}{(6/5, n)\,(1, 5\,n)} & \text{if } a = 1/5+n, \\ \dfrac{7\cdot 5^{5\,n}\,(12/5, 6\,n)}{(7/5, n)\,(2, 5\,n)} & \text{if } a = 2/5+n, \\ \dfrac{52\cdot 5^{5\,n}\,(18/5, 6\,n)}{(8/5, n)\,(3, 5\,n)} & \text{if } a = 3/5+n, \\ \dfrac{399\cdot 5^{5\,n}\,(24/5, 6\,n)}{(9/5, n)\,(4, 5\,n)} & \text{if } a = 4/5+n, \\ \dfrac{5^{5\,n+5}\,(6, 6\,n)}{(2, n)\,(5, 5\,n)} & \text{if } a = 1+n. \end{cases}$$

(xiii) $F(2, 6\,a+1;\; 5\,a+2;\; 5/6) = 6\,(5\,a+1)$

(The above is a special case of (1.5) in [Eb2]).

(xiv) $F(5\,a, 1-a;\; 5\,a+2;\; 5/6) = 6^{-a}\,(5\,a+1)$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

$$\text{(xv) } F(5\,a, 6\,a+1;\; 5\,a+2;\; -5) = \begin{cases} 6^{6\,n}\,(1-30\,n) & \text{if } a=-1/6-n, \\ 6^{6\,n+1}\,(-4-30\,n) & \text{if } a=-1/3-n, \\ 6^{6\,n+2}\,(-9-30\,n) & \text{if } a=-1/2-n, \\ 6^{6\,n+3}\,(-14-30\,n) & \text{if } a=-2/3-n, \\ 6^{6\,n+4}\,(-19-30\,n) & \text{if } a=-5/6-n \end{cases}$$

(The above are special cases of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xvi) $F(2, 1-a;\; 5\,a+2;\; -5) = 5/6\,n+1 \quad \text{if } a=1+n$

(The above is a special case of (1.5) in [Eb2]).

(xvii) $F(2, 1-5\,a;\; 2-6\,a;\; 6/5) = 6\,n+1 \quad \text{if } a=1/5+1/5\,n$

(The above is a special case of (1.5) in [Eb2]).

(xviii) $F(-6\,a, 1-a;\; 2-6\,a;\; 6/5) = -5^{-n-1}\,(6\,n+5) \quad \text{if } a=1+n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xix) $F(-6\,a, 1-5\,a;\; 2-6\,a;\; 6) = (-5)^n\,(6\,n+1) \quad \text{if } a=1/5+1/5\,n$

(The above is a special case of (a/b.1) in [Go] and (1.6) in [Eb2]).

(xx) $F(2, 1-a;\; 2-6\,a;\; 6) = 6/5\,n+1 \quad \text{if } a=1+n$

(The above is a special case of (1.5) in [Eb2]). The special values obtained from (xxi) are trivial.

$$\text{(xxii) } F(-6\,a, 1-a;\; -a;\; -1/5) = \begin{cases} 0, & \\ \dfrac{(6, 6\,n)}{5^n\,(2,n)\,(5,5\,n)} & \text{if } a=1+n. \end{cases}$$

$$\text{(xxiii) } F(5\,a, 1-a;\; -a;\; 1/6) = \begin{cases} 0, & \\ \dfrac{(6, 6\,n)}{2^n 3^n\,(2,n)\,(5,5\,n)} & \text{if } a=1+n. \end{cases}$$

The special values obtained from (xxiv) are trivial.

(1,6,6-2) The special values obtained from (1,6,6-2) coincide with those obtained from (1,6,6-1).

$(k, l, m) = (1, 7, 6)$

In this case, there is no admissible quadruple.

$(k, l, m) = (2, 4, 6)$

In this case, we have

$$(a, b, c, x) = (a, 2a - 1/3, 3a, 9), \qquad (2,4,6\text{-}1)$$
$$(a, b, c, x) = (a, 2a - 2/3, 3a - 1, 9), \qquad (2,4,6\text{-}2)$$
$$\begin{cases} (a, b, c, x) = (a, 2a - 1/3, 3a, -9 + 6\sqrt{3}), \\ S^{(n)} = \dfrac{(-2)^n \left(\sqrt{3} - 1\right)^{6n} (1/2\,a + 3/4, n)\,(1/2\,a + 5/12, n)}{3^{3/2\,n}\,(1/2\,a + 1/3, n)\,(1/2\,a + 5/6, n)}, \end{cases} \qquad (2,4,6\text{-}3)$$
$$\begin{cases} (a, b, c, x) = (a, 2a - 1/3, 3a, -9 - 6\sqrt{3}), \\ S^{(n)} = \dfrac{2^n \left(\sqrt{3} + 1\right)^{6n} (1/2\,a + 3/4, n)\,(1/2\,a + 5/12, n)}{3^{3/2\,n}\,(1/2\,a + 1/3, n)\,(1/2\,a + 5/6, n)}, \end{cases} \qquad (2,4,6\text{-}4)$$
$$\begin{cases} (a, b, c, x) = (a, 2a - 2/3, 3a - 1, -9 + 6\sqrt{3}), \\ S^{(n)} = \dfrac{(-2)^n \left(\sqrt{3} - 1\right)^{6n} (1/2\,a + 1/4, n)\,(1/2\,a + 7/12, n)}{3^{3/2\,n}\,(1/2\,a + 2/3, n)\,(1/2\,a + 1/6, n)}, \end{cases} \qquad (2,4,6\text{-}5)$$
$$\begin{cases} (a, b, c, x) = (a, 2a - 2/3, 3a - 1, -9 - 6\sqrt{3}), \\ S^{(n)} = \dfrac{2^n \left(\sqrt{3} + 1\right)^{6n} (1/2\,a + 1/4, n)\,(1/2\,a + 7/12, n)}{3^{3/2\,n}\,(1/2\,a + 2/3, n)\,(1/2\,a + 1/6, n)}. \end{cases} \qquad (2,4,6\text{-}6)$$

(2,4,6-1), (2,4,6-2) The special values obtained from (2,4,6-1) and (2,4,6-2) coincide with those obtained from (1,2,3-1).

(2,4,6-3)

(i) $F(a, 2a - 1/3;\ 3a;\ -9 + 6\sqrt{3})$

$$= \begin{cases} \dfrac{7\,(-2)^{n-3} \left(\sqrt{3} - 1\right)^{6n+6} (5/4, n)\,(19/12, n)}{3^{3/2\,n+1/2}\,(5/3, n)\,(7/6, n)} & \text{if } a = -2 - 2n, \\ \dfrac{(-2)^{n-1} \left(\sqrt{3} - 1\right)^{6n+4} (3/4, n)\,(13/12, n)}{3^{3/2\,n+1/2}\,(7/6, n)\,(2/3, n)} & \text{if } a = -1 - 2n, \\ \dfrac{(-2)^{n} \left(\sqrt{3} - 1\right)^{6n} (1/6, n)\,(1/2, n)}{3^{3/2\,n}\,(7/12, n)\,(1/12, n)} & \text{if } a = 1/6 - 2n, \\ \dfrac{-(-2)^{n-2} \left(\sqrt{3} - 1\right)^{6n+8} (17/12, n)\,(7/4, n)}{3^{3/2\,n-1/2}\,(11/6, n)\,(4/3, n)} & \text{if } a = -7/3 - 2n \\ 0 \quad \text{if } a = -5/6 - 2n, \\ \dfrac{(-2)^{n-1} \left(\sqrt{3} - 1\right)^{6n+4} (11/12, n)\,(5/4, n)}{3^{3/2\,n}\,(4/3, n)\,(5/6, n)} & \text{if } a = -4/3 - 2n. \end{cases}$$

(ii) $F(2a, a + 1/3;\ 3a;\ -9 + 6\sqrt{3})$

$$= \begin{cases} \dfrac{7\,(-2)^{n-3}\left(\sqrt{3}-1\right)^{6\,n+6}(5/4,n)\,(19/12,n)}{3^{3/2\,n+1/2}\,(5/3,n)\,(7/6,n)} & \text{if } a=-2-2\,n,\\ \dfrac{-(-2)^{n+1}\left(\sqrt{3}-1\right)^{6\,n+1}(1/2,n)\,(5/6,n)}{3^{3/2\,n+1/2}\,(11/12,n)\,(5/12,n)} & \text{if } a=-1/2-2\,n,\\ \dfrac{(-2)^{n}\left(\sqrt{3}-1\right)^{6\,n+2}(3/4,n)\,(13/12,n)}{3^{3/2\,n+1/2}\,(7/6,n)\,(2/3,n)} & \text{if } a=-1-2\,n,\\ 0 \quad \text{if } a=-3/2-2\,n,\\ \dfrac{(-2)^{n}\left(\sqrt{3}-1\right)^{6\,n}(5/12,n)\,(3/4,n)}{3^{3/2\,n}\,(5/6,n)\,(1/3,n)} & \text{if } a=-1/3-2\,n,\\ \dfrac{(-2)^{n-2}\left(\sqrt{3}-1\right)^{6\,n+4}(11/12,n)\,(5/4,n)}{3^{3/2\,n}\,(4/3,n)\,(5/6,n)} & \text{if } a=-4/3-2\,n. \end{cases}$$

(iii) $F(a, a+1/3;\ 3\,a;\ 9/4+3/4\,\sqrt{3})$

$$= \begin{cases} \dfrac{7\,(-2)^{n-3}(5/4,n)\,(19/12,n)}{3^{3/2\,n+1/2}\,(5/3,n)\,(7/6,n)} & \text{if } a=-2-2\,n,\\ \dfrac{-\,(-2)^{n-1}\left(\sqrt{3}-1\right)(3/4,n)\,(13/12,n)}{3^{3/2\,n+1/2}\,(7/6,n)\,(2/3,n)} & \text{if } a=-1-2\,n,\\ \dfrac{(-2)^{n}(5/12,n)\,(3/4,n)}{3^{3/2\,n}\,(5/6,n)\,(1/3,n)} & \text{if } a=-1/3-2\,n,\\ \dfrac{-\,(-2)^{n-2}\left(\sqrt{3}-1\right)(11/12,n)\,(5/4,n)}{3^{3/2\,n}\,(4/3,n)\,(5/6,n)} & \text{if } a=-4/3-2\,n. \end{cases}$$

(iv) $F(2\,a, 2\,a-1/3;\ 3\,a;\ 9/4+3/4\,\sqrt{3})$

$$= \begin{cases} \dfrac{7\left(\sqrt{3}+1\right)^{6\,n+6}(5/4,n)\,(19/12,n)}{(-2)^{5\,n+9}\,3^{3/2\,n+1/2}\,(5/3,n)\,(7/6,n)} & \text{if } a=-2-2\,n,\\ \dfrac{\left(\sqrt{3}+1\right)^{6\,n+2}(1/2,n)\,(5/6,n)}{(-2)^{5\,n+1}\,3^{3/2\,n+1/2}\,(11/12,n)\,(5/12,n)} & \text{if } a=-1/2-2\,n,\\ \dfrac{\left(\sqrt{3}+1\right)^{6\,n+4}(3/4,n)\,(13/12,n)}{(-2)^{5\,n+4}\,3^{3/2\,n+1/2}\,(7/6,n)\,(2/3,n)} & \text{if } a=-1-2\,n,\\ 0 \quad \text{if } a=-3/2-2\,n,\\ \dfrac{\left(\sqrt{3}+1\right)^{6\,n}(1/6,n)\,(1/2,n)}{(-2)^{5\,n}\,3^{3/2\,n}\,(7/12,n)\,(1/12,n)} & \text{if } a=1/6-2\,n,\\ \dfrac{-\left(\sqrt{3}+1\right)^{6\,n+7}(17/12,n)\,(7/4,n)}{(-2)^{5\,n+9}\,3^{3/2\,n-1/2}\,(11/6,n)\,(4/3,n)} & \text{if } a=-7/3-2\,n,\\ 0 \quad \text{if } a=-5/6-2\,n,\\ \dfrac{\left(\sqrt{3}+1\right)^{6\,n+5}(11/12,n)\,(5/4,n)}{(-2)^{5\,n+6}\,3^{3/2\,n}\,(4/3,n)\,(5/6,n)} & \text{if } a=-4/3-2\,n. \end{cases}$$

(v) $F(a, 2\,a-1/3;\ 2/3;\ 10-6\,\sqrt{3}) = \dfrac{3^{3/8-9/4\,a}\left(\sqrt{3}-1\right)^{1/2-3\,a}\sqrt{\pi}\,\Gamma\,(2/3)}{2^{1/4-3/2\,a}\,\Gamma\,(3/4-1/2\,a)\,\Gamma\,(1/2\,a+5/12)}$

(The above is a generalization of Theorem 26 in [Ek]). We find (vi)≤(v).

(vii) $F(a, 1-2\,a;\ 2/3;\ 2/3-2/9\,\sqrt{3}) = \dfrac{3^{3/8-3/4\,a}\left(\sqrt{3}-1\right)^{1/2-a}\sqrt{\pi}\Gamma\left(2/3\right)}{2^{1/4-1/2\,a}\Gamma\left(3/4-1/2\,a\right)\Gamma\left(1/2\,a+5/12\right)}$

(The above is a generalization of Theorem 40 in [Ek]). We find (viii)≤(vii).

(ix) $F(a, 1-2\,a;\ 4/3-a;\ 1/3+2/9\,\sqrt{3})$

$$= \frac{3^{-3/4\,a}\left(\sqrt{3}-1\right)^{-a}\sin\left(\pi\left(5/12-1/2\,a\right)\right)\Gamma\left(3/4\right)\Gamma\left(5/12\right)\Gamma\left(4/3-a\right)}{2^{1/2-7/2\,a}\Gamma\left(2/3\right)\Gamma\left(13/12-1/2\,a\right)\Gamma\left(3/4-1/2\,a\right)}.$$

(x) $F(4/3-2\,a, a+1/3;\ 4/3-a;\ 1/3+2/9\,\sqrt{3})$

$$= \frac{3^{1/2-3/4\,a}\left(\sqrt{3}-1\right)^{-a-1/3}\sin\left(\pi\left(5/12-1/2\,a\right)\right)\Gamma\left(3/4\right)\Gamma\left(5/12\right)\Gamma\left(4/3-a\right)}{2^{5/6-7/2\,a}\Gamma\left(2/3\right)\Gamma\left(13/12-1/2\,a\right)\Gamma\left(3/4-1/2\,a\right)}.$$

(xi) $F(a, a+1/3;\ 4/3-a;\ -5/4-3/4\,\sqrt{3})$

$$= \begin{cases} \dfrac{3^{9/2\,n}\left(7/6,n\right)\left(2/3,n\right)}{\left(-2\right)^{7\,n}\left(13/12,n\right)\left(3/4,n\right)} & \text{if } a=-2\,n,\\ \dfrac{-3^{9/2\,n+3/2}\left(\sqrt{3}-1\right)\left(5/3,n\right)\left(7/6,n\right)}{7\left(-2\right)^{7\,n+1}\left(19/12,n\right)\left(5/4,n\right)} & \text{if } a=-1-2\,n,\\ \dfrac{3^{9/2\,n}\left(4/3,n\right)\left(5/6,n\right)}{\left(-2\right)^{7\,n}\left(5/4,n\right)\left(11/12,n\right)} & \text{if } a=-1/3-2\,n,\\ \dfrac{3^{9/2\,n+1}\left(\sqrt{3}-1\right)\left(11/6,n\right)\left(4/3,n\right)}{\left(-2\right)^{7\,n+3}\left(7/4,n\right)\left(17/12,n\right)} & \text{if } a=-4/3-2\,n. \end{cases}$$

(xii) $F(1-2\,a, 4/3-2\,a;\ 4/3-a;\ -5/4-3/4\,\sqrt{3})$

$$= \begin{cases} \dfrac{3^{9/2\,n}\left(\sqrt{3}+1\right)^{6\,n}\left(1/6,n\right)\left(1/2,n\right)}{\left(-2\right)^{5\,n}\left(1/12,n\right)\left(7/12,n\right)} & \text{if } a=1/2+2\,n,\\ \dfrac{3^{9/2\,n+1}\left(\sqrt{3}+1\right)^{6\,n+1}\left(5/12,n\right)\left(3/4,n\right)}{\left(-2\right)^{5\,n+1}\left(1/3,n\right)\left(5/6,n\right)} & \text{if } a=1+2\,n,\\ 0 \quad \text{if } a=3/2+2\,n, & \\ \dfrac{-3^{9/2\,n+4}\left(\sqrt{3}+1\right)^{6\,n+5}\left(11/12,n\right)\left(5/4,n\right)}{\left(-2\right)^{5\,n+6}\left(5/6,n\right)\left(4/3,n\right)} & \text{if } a=2+2\,n,\\ \dfrac{3^{9/2\,n}\left(\sqrt{3}+1\right)^{6\,n}\left(1/4,n\right)\left(7/12,n\right)}{\left(-2\right)^{5\,n}\left(1/6,n\right)\left(2/3,n\right)} & \text{if } a=2/3+2\,n,\\ \dfrac{3^{9/2\,n+3/2}\left(\sqrt{3}+1\right)^{6\,n+2}\left(1/2,n\right)\left(5/6,n\right)}{\left(-2\right)^{5\,n+1}\left(5/12,n\right)\left(11/12,n\right)} & \text{if } a=7/6+2\,n,\\ \dfrac{-3^{9/2\,n+5/2}\left(\sqrt{3}+1\right)^{6\,n+4}\left(3/4,n\right)\left(13/12,n\right)}{\left(-2\right)^{5\,n+4}\left(2/3,n\right)\left(7/6,n\right)} & \text{if } a=5/3+2\,n,\\ 0 \quad \text{if } a=13/6+2\,n. & \end{cases}$$

The special values obtained from (xiii)-(xx) are contained in the above.

(xxi) $F(2\,a, a+1/3;\ 4/3;\ 10-6\sqrt{3})$

$$= \frac{3^{-9/4\,a-5/8}\left(\sqrt{3}-1\right)^{-3\,a-1/2}\sqrt{\pi}\Gamma\left(1/3\right)}{2^{5/4-3/2\,a}\,\Gamma\left(13/12-1/2\,a\right)\Gamma\left(1/2\,a+3/4\right)}.$$

(The above is a generalization of Theorem 27 in [Ek]). We find (xxii)≤ (xxi).

(xxiii) $F(2\,a, 1-a;\ 4/3;\ 2/3-2/9\sqrt{3})$

$$= \frac{3^{3/4\,a-5/8}\left(\sqrt{3}-1\right)^{a-1/2}\sqrt{\pi}\Gamma\left(1/3\right)}{2^{1/2\,a+5/4}\,\Gamma\left(13/12-1/2\,a\right)\Gamma\left(1/2\,a+3/4\right)}$$

(The above is a generalization of Theorem 39 in [Ek]). We find (xxiv)≤ (xxiii).

(2,4,6-4)

(i) $F(a, 2\,a-1/3;\ 3\,a;\ -9-6\sqrt{3})$

$$= \begin{cases} \dfrac{7\cdot 2^{n-3}\left(\sqrt{3}+1\right)^{6\,n+6}(5/4,n)\,(19/12,n)}{3^{3/2\,n+1/2}\,(5/3,n)\,(7/6,n)} & \text{if } a=-2-2\,n,\\ \dfrac{2^{n-1}\left(\sqrt{3}+1\right)^{6\,n+4}(3/4,n)\,(13/12,n)}{3^{3/2\,n+1/2}\,(7/6,n)\,(2/3,n)} & \text{if } a=-1-2\,n,\\ \dfrac{2^{n}\left(\sqrt{3}+1\right)^{6\,n}(1/6,n)\,(1/2,n)}{3^{3/2\,n}\,(7/12,n)\,(1/12,n)} & \text{if } a=1/6-2\,n,\\ \dfrac{2^{n-2}\left(\sqrt{3}+1\right)^{6\,n+8}(17/12,n)\,(7/4,n)}{3^{3/2\,n-1/2}\,(11/6,n)\,(4/3,n)} & \text{if } a=-7/3-2\,n\\ 0 \quad \text{if } a=-5/6-2\,n, & \\ \dfrac{-2^{n-1}\left(\sqrt{3}+1\right)^{6\,n+4}(11/12,n)\,(5/4,n)}{3^{3/2\,n}\,(4/3,n)\,(5/6,n)} & \text{if } a=-4/3-2\,n. \end{cases}$$

(ii) $F(2\,a, a+1/3;\ 3\,a;\ -9-6\sqrt{3})$

$$= \begin{cases} \dfrac{7\cdot 2^{n-3}\left(\sqrt{3}+1\right)^{6\,n+6}(5/4,n)\,(19/12,n)}{3^{3/2\,n+1/2}\,(5/3,n)\,(7/6,n)} & \text{if } a=-2-2\,n,\\ \dfrac{2^{n+1}\left(\sqrt{3}+1\right)^{6\,n+1}(1/2,n)\,(5/6,n)}{3^{3/2\,n+1/2}\,(11/12,n)\,(5/12,n)} & \text{if } a=-1/2-2\,n,\\ \dfrac{-2^{n}\left(\sqrt{3}+1\right)^{6\,n+2}(3/4,n)\,(13/12,n)}{3^{3/2\,n+1/2}\,(7/6,n)\,(2/3,n)} & \text{if } a=-1-2\,n,\\ 0 \quad \text{if } a=-3/2-2\,n, & \\ \dfrac{2^{n}\left(\sqrt{3}+1\right)^{6\,n}(5/12,n)\,(3/4,n)}{3^{3/2\,n}\,(5/6,n)\,(1/3,n)} & \text{if } a=-1/3-2\,n,\\ \dfrac{2^{n-2}\left(\sqrt{3}+1\right)^{6\,n+4}(11/12,n)\,(5/4,n)}{3^{3/2\,n}\,(4/3,n)\,(5/6,n)} & \text{if } a=-4/3-2\,n. \end{cases}$$

(iii) $F(a, a+1/3;\ 3\,a;\ 9/4-3/4\sqrt{3})$

$$= \begin{cases} \dfrac{3^{3/4\,a-1/8}\left(\sqrt{3}+1\right)^{1/2}\sqrt{\pi}\Gamma\left(a+2/3\right)}{2^{3/2\,a-1/4}\Gamma\left(1/2\,a+5/12\right)\Gamma\left(1/2\,a+3/4\right)}, \\ \dfrac{7\cdot 2^{n-3}\left(5/4,n\right)\left(19/12,n\right)}{3^{3/2\,n+1/2}\left(5/3,n\right)\left(7/6,n\right)} \quad \text{if } a=-2-2\,n, \\ \dfrac{2^{n-1}\left(\sqrt{3}+1\right)\left(3/4,n\right)\left(13/12,n\right)}{3^{3/2\,n+1/2}\left(7/6,n\right)\left(2/3,n\right)} \quad \text{if } a=-1-2\,n, \\ \dfrac{2^{n}\left(5/12,n\right)\left(3/4,n\right)}{3^{3/2\,n}\left(5/6,n\right)\left(1/3,n\right)} \quad \text{if } a=-1/3-2\,n, \\ \dfrac{2^{n-2}\left(\sqrt{3}+1\right)\left(11/12,n\right)\left(5/4,n\right)}{3^{3/2\,n}\left(4/3,n\right)\left(5/6,n\right)} \quad \text{if } a=-4/3-2\,n. \end{cases}$$

(iv) $F(2\,a, 2\,a-1/3;\ 3\,a;\ 9/4-3/4\,\sqrt{3})$

$$= \begin{cases} \dfrac{3^{3/4\,a-1/8}\left(\sqrt{3}+1\right)^{3\,a-1/2}\sqrt{\pi}\Gamma\left(a+2/3\right)}{2^{3/2\,a-1/4}\Gamma\left(1/2\,a+5/12\right)\Gamma\left(1/2\,a+3/4\right)}, \\ \dfrac{7\left(\sqrt{3}-1\right)^{6\,n+6}\left(5/4,n\right)\left(19/12,n\right)}{2^{5\,n+9}3^{3/2\,n+1/2}\left(5/3,n\right)\left(7/6,n\right)} \quad \text{if } a=-2-2\,n, \\ \dfrac{-\left(\sqrt{3}-1\right)^{6\,n+4}\left(3/4,n\right)\left(13/12,n\right)}{2^{5\,n+4}3^{3/2\,n+1/2}\left(7/6,n\right)\left(2/3,n\right)} \quad \text{if } a=-1-2\,n, \\ \dfrac{\left(\sqrt{3}-1\right)^{6\,n+7}\left(17/12,n\right)\left(7/4,n\right)}{2^{5\,n+9}3^{3/2\,n-1/2}\left(11/6,n\right)\left(4/3,n\right)} \quad \text{if } a=-7/3-2\,n, \\ \dfrac{-\left(\sqrt{3}-1\right)^{6\,n+5}\left(11/12,n\right)\left(5/4,n\right)}{2^{5\,n+6}3^{3/2\,n}\left(4/3,n\right)\left(5/6,n\right)} \quad \text{if } a=-4/3-2\,n. \end{cases}$$

(v) $F(a, 2\,a-1/3;\ 2/3;\ 10+6\,\sqrt{3})$

$$= \begin{cases} \dfrac{3^{9/2\,n}\left(\sqrt{3}+1\right)^{6\,n}\left(7/12,n\right)}{2^{3\,n}\left(3/4,n\right)} \quad \text{if } a=-2\,n, \\ \dfrac{3^{9/2\,n+3/2}\left(\sqrt{3}+1\right)^{6\,n+4}\left(13/12,n\right)}{2^{3\,n+2}\left(5/4,n\right)} \quad \text{if } a=-1-2\,n, \\ \dfrac{3^{9/2\,n}\left(\sqrt{3}+1\right)^{6\,n}\left(1/2,n\right)}{2^{3\,n}\left(2/3,n\right)} \quad \text{if } a=1/6-2\,n, \\ \dfrac{3^{9/2\,n+1}\left(\sqrt{3}+1\right)^{6\,n+2}\left(3/4,n\right)}{2^{3\,n+1}\left(11/12,n\right)} \quad \text{if } a=-1/3-2\,n, \\ 0 \quad \text{if } a=-5/6-2\,n, \\ \dfrac{-3^{9/2\,n+4}\left(\sqrt{3}+1\right)^{6\,n+4}\left(5/4,n\right)}{5\cdot 2^{3\,n+2}\left(17/12,n\right)} \quad \text{if } a=-4/3-2\,n. \end{cases}$$

(The first case is identical to Theorem 26 in [Ek]). We find (vi)$\le$(v).

(vii) $F(a, 1-2\,a;\ 2/3;\ 2/3+2/9\,\sqrt{3})$

$$= \begin{cases} \dfrac{3^{3/2\,n}\left(\sqrt{3}+1\right)^{2\,n}(7/12,n)}{2^{n}\,(3/4,n)} & \text{if } a=-2\,n, \\ \dfrac{-3^{3/2\,n}\left(\sqrt{3}+1\right)^{2\,n+2}(13/12,n)}{2^{n+1}\,(5/4,n)} & \text{if } a=-1-2\,n, \\ \dfrac{\left(\sqrt{3}-1\right)^{2\,n}(1/2,n)}{2^{n}3^{3/2\,n}\,(2/3,n)} & \text{if } a=1/2+2\,n, \\ \dfrac{-\left(\sqrt{3}-1\right)^{2\,n}(3/4,n)}{2^{n}3^{3/2\,n+1/2}\,(11/12,n)} & \text{if } a=1+2\,n, \\ 0 \quad \text{if } a=3/2+2\,n, \\ \dfrac{\left(\sqrt{3}-1\right)^{2\,n+2}(5/4,n)}{5\cdot 2^{n+1}3^{3/2\,n+1/2}\,(17/12,n)} & \text{if } a=2+2\,n \end{cases}$$

(The first case is identical to Theorem 40 in [Ek]). We find (viii)≤(vii).

(ix) $F(a,1-2\,a;\;4/3-a;\;1/3-2/9\,\sqrt{3})$

$$= \frac{3^{-3/4\,a}\left(\sqrt{3}+1\right)^{-a}\Gamma(3/4)\,\Gamma(13/12)\,\Gamma(2/3-1/2\,a)\,\Gamma(7/6-1/2\,a)}{2^{-5/2\,a}\Gamma(2/3)\,\Gamma(7/6)\,\Gamma(3/4-1/2\,a)\,\Gamma(13/12-1/2\,a)}.$$

(x) $F(a+1/3,4/3-2\,a;\;4/3-a;\;1/3-2/9\,\sqrt{3})$

$$= \frac{3^{1/2-3/4\,a}\left(\sqrt{3}+1\right)^{-a-1/3}\Gamma(3/4)\,\Gamma(13/12)\,\Gamma(2/3-1/2\,a)\,\Gamma(7/6-1/2\,a)}{2^{1/3-5/2\,a}\Gamma(2/3)\,\Gamma(7/6)\,\Gamma(3/4-1/2\,a)\,\Gamma(13/12-1/2\,a)}.$$

(xi) $F(a,a+1/3;\;4/3-a;\;-5/4+3/4\,\sqrt{3})$

$$= \frac{3^{-9/4\,a}\Gamma(3/4)\,\Gamma(13/12)\,\Gamma(2/3-1/2\,a)\,\Gamma(7/6-1/2\,a)}{2^{-7/2\,a}\Gamma(2/3)\,\Gamma(7/6)\,\Gamma(3/4-1/2\,a)\,\Gamma(13/12-1/2\,a)}.$$

(xii) $F(1-2\,a,4/3-2\,a;\;4/3-a;\;-5/4+3/4\,\sqrt{3})$

$$= \frac{3^{9/4\,a-3/2}\left(\sqrt{3}+1\right)^{1-3\,a}\Gamma(3/4)\,\Gamma(13/12)\,\Gamma(2/3-1/2\,a)\,\Gamma(7/6-1/2\,a)}{2^{-1/2\,a-1}\Gamma(2/3)\,\Gamma(7/6)\,\Gamma(3/4-1/2\,a)\,\Gamma(13/12-1/2\,a)}.$$

The special values obtained from (xiii)-(xx) are contained in the above.

(xxi) $F(2\,a,a+1/3;\;4/3;\;10+6\,\sqrt{3})$

$$= \begin{cases} \dfrac{3^{9/2\,n}\left(\sqrt{3}+1\right)^{6\,n}(1/4,n)}{2^{3\,n}\,(13/12,n)} & \text{if } a=-2\,n, \\ \dfrac{3^{9/2\,n+3/2}\left(\sqrt{3}+1\right)^{6\,n+1}(1/2,n)}{2^{3\,n+2}\,(4/3,n)} & \text{if } a=-1/2-2\,n, \\ \dfrac{-3^{9/2\,n+5/2}\left(\sqrt{3}+1\right)^{6\,n+2}(3/4,n)}{7\cdot 2^{3\,n+1}\,(19/12,n)} & \text{if } a=-1-2\,n, \\ 0 \quad \text{if } a=-3/2-2\,n, \\ \dfrac{3^{9/2\,n}\left(\sqrt{3}+1\right)^{6\,n}(5/12,n)}{2^{3\,n}\,(5/4,n)} & \text{if } a=-1/3-2\,n, \\ \dfrac{3^{9/2\,n+1}\left(\sqrt{3}+1\right)^{6\,n+4}(11/12,n)}{2^{3\,n+2}\,(7/4,n)} & \text{if } a=-4/3-2\,n \end{cases}$$

(The fifth case is identical to Theorem 27 in [Ek]). We find (xxii)≤(xxi).

(xxiii) $F(2\,a, 1-a;\ 4/3;\ 2/3+2/9\,\sqrt{3})$

$$= \begin{cases} \dfrac{\left(\sqrt{3}-1\right)^{2\,n}(1/4,n)}{2^n 3^{3/2\,n}\,(13/12,n)} \quad \text{if } a=-2\,n, \\ \dfrac{-\left(\sqrt{3}-1\right)^{2\,n+1}(1/2,n)}{2^{n+2} 3^{3/2\,n}\,(4/3,n)} \quad \text{if } a=-1/2-2\,n, \\ \dfrac{-\left(\sqrt{3}-1\right)^{2\,n+2}(3/4,n)}{7\cdot 2^{n+1} 3^{3/2\,n+1/2}\,(19/12,n)} \quad \text{if } a=-1-2\,n, \\ 0 \quad \text{if } a=-3/2-2\,n, \\ \dfrac{3^{3/2\,n}\left(\sqrt{3}+1\right)^{2\,n}(5/12,n)}{2^n\,(5/4,n)} \quad \text{if } a=1+2\,n, \\ \dfrac{-3^{3/2\,n-1/2}\left(\sqrt{3}+1\right)^{2\,n+2}(11/12,n)}{2^{n+1}\,(7/4,n)} \quad \text{if } a=2+2\,n \end{cases}$$

(The fifth case is identical to Theorem 39 in [Ek]). We find (xxiv)≤(xxiii).

(2,4,6-5) The special values obtained from (2,4,6-5) coincide with those (2,4,6-3).

(2,4,6-6) The special values obtained from (2,4,6-6) coincide with those (2,4,6-4).

$(k,l,m)=(2,5,6)$

In this case, there is no admissible quadruple.

$(k,l,m)=(2,6,6)$

In this case, we get

$$(a,b,c,x)=(a,3\,a-1/2,3\,a,-3), \qquad (2,6,6\text{-}1)$$
$$(a,b,c,x)=(a,3\,a-3/2,3\,a-1,-3), \qquad (2,6,6\text{-}2)$$
$$(a,b,c,x)=(a,3\,a+1,3\,a,3/2), \qquad (2,6,6\text{-}3)$$
$$(a,b,c,x)=(a,3\,a-3,3\,a-1,3/2). \qquad (2,6,6\text{-}4)$$

(2,6,6-1), (2,6,6-2) The special values obtained from (2,6,6-1) and (2,6,6-2) coincide those obtained from (1,3,3-1).

(2,6,6-3), (2,6,6-4) The special values obtained from (2,6,6-3) and (2,6,6-4) coincide those obtained from (1,3,3-3).

$(k,l,m)=(2,7,6)$

In this case, there is no admissible quadruple.

$(k, l, m) = (2, 8, 6)$

In this case, we have

$$(a, b, c, x) = (a, b, b + 1 - a, -1), \quad (2,8,6\text{-}1)$$
$$(a, b, c, x) = (a, 4\,a - 1/2, 3\,a, -1), \quad (2,8,6\text{-}2)$$
$$(a, b, c, x) = (a, 4\,a - 5/2, 3\,a - 1, -1). \quad (2,8,6\text{-}3)$$

(2,8,6-1) The special values obtained from (2,8,6-1) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1).

(2,8,6-2), (2,8,6-3) The special values obtained from (2,8,6-2) and (2,8,6-3) coincide with those obtained from (1,4,3-2).

$(k, l, m) = (3, 3, 6)$

In this case, we have

$$(a, b, c, x) = (a, a - 1/2, 2\,a, 4),\ S^{(n)} = 1, \quad (3,3,6\text{-}1)$$
$$(a, b, c, x) = (a, a + 1/2, 2\,a, 4),\ S^{(n)} = 1, \quad (3,3,6\text{-}2)$$
$$(a, b, c, x) = (a, a - 1/2, 2\,a - 1, 4),\ S^{(n)} = 1, \quad (3,3,6\text{-}3)$$
$$(a, b, c, x) = (a, a + 1/2, 2\,a + 1, 4),\ S^{(n)} = 1, \quad (3,3,6\text{-}4)$$
$$(a, b, c, x) = (a, a + 3/2, 2\,a - 1, 4),\ S^{(n)} = \frac{(2\,a - 3)\,(2\,a + 1 + 6\,n)}{(2\,a + 1)\,(2\,a - 3 + 6\,n)}, \quad (3,3,6\text{-}5)$$
$$(a, b, c, x) = (a, a - 3/2, 2\,a + 1, 4),\ S^{(n)} = \frac{(2\,a - 1)\,(2\,a + 3 + 6\,n)}{(2\,a + 3)\,(2\,a - 1 + 6\,n)}, \quad (3,3,6\text{-}6)$$
$$(a, b, c, x) = (a, a - 3/2, 2\,a - 4, 4),\ S^{(n)} = \frac{(a - 3)\,(a - 1 + 3\,n)}{(a - 1)\,(a - 3 + 3\,n)}, \quad (3,3,6\text{-}7)$$
$$(a, b, c, x) = (a, a + 3/2, 2\,a + 4, 4),\ S^{(n)} = \frac{(a + 1)\,(a + 3 + 3\,n)}{(a + 3)\,(a + 1 + 3\,n)}, \quad (3,3,6\text{-}8)$$
$$(a, b, c, x) = (a, a - 1/2, 2\,a, 4/3),\ S^{(n)} = (-3)^{-3\,n}, \quad (3,3,6\text{-}9)$$
$$(a, b, c, x) = (a, a + 1/2, 2\,a, 4/3),\ S^{(n)} = (-3)^{-3\,n}, \quad (3,3,6\text{-}10)$$
$$(a, b, c, x) = (a, a - 1/2, 2\,a - 1, 4/3),\ S^{(n)} = (-3)^{-3\,n}, \quad (3,3,6\text{-}11)$$
$$(a, b, c, x) = (a, a + 1/2, 2\,a + 1, 4/3),\ S^{(n)} = (-3)^{-3\,n}, \quad (3,3,6\text{-}12)$$
$$(a, b, c, x) = (a, a - 5/2, 2\,a - 1, 4/3),\ S^{(n)} = \frac{(2\,a - 3)\,(2\,a + 1 + 6\,n)}{(-3)^{3\,n}\,(2\,a + 1)\,(2\,a - 3 + 6\,n)}, \quad (3,3,6\text{-}13)$$
$$(a, b, c, x) = (a, a + 5/2, 2\,a + 1, 4/3),\ S^{(n)} = \frac{(2\,a - 1)\,(2\,a + 3 + 6\,n)}{(-3)^{3\,n}\,(2\,a + 3)\,(2\,a - 1 + 6\,n)}, \quad (3,3,6\text{-}14)$$
$$(a, b, c, x) = (a, a - 5/2, 2\,a - 4, 4/3),\ S^{(n)} = \frac{(a - 3)\,(a - 1 + 3\,n)}{(-3)^{3\,n}\,(a - 1)\,(a - 3 + 3\,n)}, \quad (3,3,6\text{-}15)$$

$$(a,b,c,x)=(a,a+5/2,2\,a+4,4/3),\ S^{(n)}=\frac{(a+1)\,(a+3+3\,n)}{(-3)^{3\,n}\,(a+3)\,(a+1+3\,n)}. \tag{3,3,6-16}$$

(3,3,6-1)

$$\text{(i) } F(a,a-1/2;\,2\,a;\,4)=\begin{cases}1 & \text{if } a=-3-3\,n,\\ -2 & \text{if } a=-1-3\,n,\\ 1 & \text{if } a=-2-3\,n,\\ 2 & \text{if } a=-5/2-3\,n,\\ -1 & \text{if } a=-7/2-3\,n,\\ -1 & \text{if } a=-3/2-3\,n.\end{cases}$$

$$\text{(ii) } F(a,a+1/2;\,2\,a;\,4)=\begin{cases}1 & \text{if } a=-3-3\,n,\\ 0 & \text{if } a=-1-3\,n,\\ -1 & \text{if } a=-2-3\,n,\\ 1 & \text{if } a=-1/2-3\,n,\\ -1 & \text{if } a=-3/2-3\,n,\\ 0 & \text{if } a=-5/2-3\,n.\end{cases}$$

$$\text{(iii) } F(a,a+1/2;\,2\,a;\,4/3)=\begin{cases}(-3)^{-3\,n-3} & \text{if } a=-3-3\,n,\\ -2\,(-3)^{-3\,n-1} & \text{if } a=-1-3\,n,\\ (-3)^{-3\,n-2} & \text{if } a=-2-3\,n,\\ (-3)^{-3\,n} & \text{if } a=-1/2-3\,n,\\ -(-3)^{-3\,n-1} & \text{if } a=-3/2-3\,n,\\ 0 & \text{if } a=-5/2-3\,n.\end{cases}$$

$$\text{(iv) } F(a,a-1/2;\,2\,a;\,4/3)=\begin{cases}(-3)^{-3\,n-3} & \text{if } a=-3-3\,n,\\ 0 & \text{if } a=-1-3\,n,\\ -(-3)^{-3\,n-2} & \text{if } a=-2-3\,n,\\ 2\,(-3)^{-3\,n-3} & \text{if } a=-5/2-3\,n,\\ -(-3)^{-3\,n-4} & \text{if } a=-7/2-3\,n,\\ -(-3)^{-3\,n-2} & \text{if } a=-3/2-3\,n.\end{cases}$$

$$\text{(v) } F(a,a-1/2;\,1/2;\,-3)=\begin{cases}2^{6\,n} & \text{if } a=-3\,n,\\ -2^{6\,n+3} & \text{if } a=-1-3\,n,\\ 2^{6\,n+4} & \text{if } a=-2-3\,n,\\ 2^{6\,n} & \text{if } a=1/2-3\,n,\\ -2^{6\,n+1} & \text{if } a=-1/2-3\,n,\\ -2^{6\,n+3} & \text{if } a=-3/2-3\,n.\end{cases}$$

We find (vi)≤(v).

(vii) $F(a, 1-a;\ 1/2;\ 3/4) = 2\ \sin\left(1/6\left(4\,a+1\right)\pi\right).$

(viii) $F(1/2-a, a-1/2;\ 1/2;\ 3/4) = \sin\left(1/6\left(4\,a+1\right)\pi\right).$

(ix) $F(a, 1-a;\ 3/2;\ 1/4) = \dfrac{2\ \sin\left(1/6\left(2\,a-1\right)\pi\right)}{2\,a-1}.$

(x) $F(3/2-a, a+1/2;\ 3/2;\ 1/4) = \dfrac{4\ \sin\left(1/6\left(2\,a-1\right)\pi\right)}{\sqrt{3}\left(2\,a-1\right)}.$

(xi) $F(a, a+1/2;\ 3/2;\ -1/3) = \dfrac{3^{a}\sin\left(1/6\left(2\,a-1\right)\pi\right)}{2^{2\,a-1}\left(2\,a-1\right)}.$

We find (xii)≤(xi).

(xiii) $F(1/2-a, a-1/2;\ 1/2;\ 1/4) = \sin\left(1/3\left(2-a\right)\pi\right).$

(xiv) $F(a, 1-a;\ 1/2;\ 1/4) = \dfrac{2}{\sqrt{3}}\sin\left(1/3\left(2-a\right)\pi\right).$

(xv) $F(a, a-1/2;\ 1/2;\ -1/3) = \dfrac{3^{a-1/2}\sin\left(1/3\left(2-a\right)\pi\right)}{2^{2\,a-1}}.$

We find (xvi)≤(xv), (xvii)≤(i), (xviii)≤(ii), (xix)≤(iii) and (xx)≤(iv).

$$\text{(xxi) } F(a, a+1/2;\ 3/2;\ -3) = \begin{cases} \dfrac{2^{6\,n}}{6\,n+1} & \text{if } a=-3\,n, \\ 0 \quad \text{if } a=-1-3\,n, \\ \dfrac{-2^{6\,n+4}}{6\,n+5} & \text{if } a=-2-3\,n, \\ \dfrac{2^{6\,n}}{3\,n+1} & \text{if } a=-1/2-3\,n, \\ \dfrac{-2^{6\,n+2}}{3\,n+2} & \text{if } a=-3/2-3\,n, \\ 0 \quad \text{if } a=-5/2-3\,n. \end{cases}$$

We find (xxii)≤(xxi).

(xxiii) $F(a, 1-a;\ 3/2;\ 3/4) = \dfrac{2\ \sin\left(1/3\left(2\,a-1\right)\pi\right)}{\sqrt{3}\left(2\,a-1\right)}.$

(xxiv) $F(3/2-a, a+1/2;\ 3/2;\ 3/4) = \dfrac{4\ \sin\left(1/3\left(2\,a-1\right)\pi\right)}{\sqrt{3}\left(2\,a-1\right)}.$

(3,3,6-2), (3,3,6-3), (3,3,6-4), (3,3,6-9), (3,3,6-10), (3,3,6-11), (3,3,6-12)
The special values obtained from (3,3,6-2), (3,3,6-3), (3,3,6-4), (3,3,6-9), (3,3,6-10), (3,3,6-11) and (3,3,6-12) coincide with those obtained from (3,3,6-1).

(3,3,6-5)

$$\text{(i) } F(a, a+3/2;\, 2a-1;\, 4) = \begin{cases} -\dfrac{2n+1}{6n-1} & \text{if } a = -3n, \\ \dfrac{6n+5}{18n+3} & \text{if } a = -1-3n, \\ 0 & \text{if } a = -2-3n, \\ \dfrac{n+1}{3n+1} & \text{if } a = -3/2-3n, \\ -\dfrac{3n+4}{9n+6} & \text{if } a = -5/2-3n, \\ 0 & \text{if } a = -7/2-3n. \end{cases}$$

$$\text{(ii) } F(a-1, a-5/2;\, 2a-1;\, 4) = \begin{cases} -\dfrac{12n+14}{2n+1} & \text{if } a = -2-3n, \\ \dfrac{18n+27}{6n+5} & \text{if } a = -3-3n, \\ \dfrac{18n+15}{6n+1} & \text{if } a = -1-3n, \\ \dfrac{6n+10}{n+1} & \text{if } a = -7/2-3n, \\ -\dfrac{9n+18}{3n+4} & \text{if } a = -9/2-3n, \\ -\dfrac{9n+12}{3n+2} & \text{if } a = -5/2-3n. \end{cases}$$

$$\text{(iii) } F(a, a-5/2;\, 2a-1;\, 4/3) = \begin{cases} -\dfrac{2n+1}{(-3)^{3n}(6n-1)} & \text{if } a = -3n, \\ -\dfrac{6n+5}{(-3)^{3n+2}(6n+1)} & \text{if } a = -1-3n, \\ 0 & \text{if } a = -2-3n, \\ \dfrac{6n+10}{(-3)^{3n+6}(n+1)} & \text{if } a = -7/2-3n, \\ \dfrac{3n+6}{(-3)^{3n+6}(3n+4)} & \text{if } a = -9/2-3n, \\ \dfrac{3n+4}{(-3)^{3n+4}(3n+2)} & \text{if } a = -5/2-3n. \end{cases}$$

$$(\mathrm{iv})\; F(a-1,a+3/2;\;2\,a-1;\;4/3)=\begin{cases}-\dfrac{12\,n+14}{(-3)^{3\,n+3}\,(2\,n+1)} & \text{if } a=-2-3\,n,\\ \dfrac{2\,n+3}{(-3)^{3\,n+2}\,(6\,n+5)} & \text{if } a=-3-3\,n,\\ -\dfrac{6\,n+5}{(-3)^{3\,n+1}\,(6\,n+1)} & \text{if } a=-1-3\,n,\\ \dfrac{n+1}{(-3)^{3\,n}\,(3\,n+1)} & \text{if } a=-3/2-3\,n,\\ \dfrac{3\,n+4}{(-3)^{3\,n+2}\,(3\,n+2)} & \text{if } a=-5/2-3\,n,\\ 0 \quad \text{if } a=-7/2-3\,n. \end{cases}$$

$$(\mathrm{v})\; F(a,a+3/2;\;7/2;\;-3)=\begin{cases}\dfrac{-5\cdot 2^{6\,n}}{(6\,n+5)\,(3\,n+1)\,(6\,n-1)} & \text{if } a=-3\,n,\\ \dfrac{5\cdot 2^{6\,n+2}}{(6\,n+7)\,(3\,n+2)\,(6\,n+1)} & \text{if } a=-1-3\,n,\\ 0 \quad \text{if } a=-2-3\,n, \\ \dfrac{5\cdot 2^{6\,n+2}}{(3\,n+4)\,(6\,n+5)\,(3\,n+1)} & \text{if } a=-3/2-3\,n,\\ \dfrac{-5\cdot 2^{6\,n+4}}{(3\,n+5)\,(6\,n+7)\,(3\,n+2)} & \text{if } a=-5/2-3\,n,\\ 0 \quad \text{if } a=-7/2-3\,n. \end{cases}$$

We find (vi)≤(v).

$$(\mathrm{vii})\; F(a,2-a;\;7/2;\;3/4)=\frac{10\sin\left(1/3\,(2\,a+1)\,\pi\right)}{\sqrt{3}\,(a-1)\,(2\,a+1)\,(2\,a-5)}.$$

$$(\mathrm{viii})\; F(a+3/2,7/2-a;\;7/2;\;3/4)=\frac{80}{\sqrt{3}}\,\frac{\sin\left(1/3\,(2\,a+1)\,\pi\right)}{(2\,a+1)\,(2\,a-5)\,(a-1)}.$$

$$(\mathrm{ix})\; F(a,2-a;\;-1/2;\;1/4)=\frac{-2}{3^{3/2}}\,\frac{(2\,a-1)\,(2\,a-3)\sin\left(1/3\,(a+2)\,\pi\right)}{a-1}.$$

$$(\mathrm{x})\; F(a-5/2,-a-1/2;\;-1/2;\;1/4)=\frac{-3}{16}\,\frac{(2\,a-1)\,(2\,a-3)\sin\left(1/3\,(a+2)\,\pi\right)}{a-1}.$$

$$(\mathrm{xi})\; F(a,a-5/2;\;-1/2;\;-1/3)=\frac{-3^{a-3/2}}{2^{2\,a-1}}\,\frac{(2\,a-1)\,(2\,a-3)\sin\left(1/3\,(a+2)\,\pi\right)}{a-1}.$$

We find (xii)≤(xi).

$$(\mathrm{xiii})\; F(a+3/2,7/2-a;\;5/2;\;1/4)=\frac{-32}{\sqrt{3}}\,\frac{\sin\left(1/3\,(a+1/2)\,\pi\right)}{(2\,a-5)\,(2\,a+1)}.$$

(xiv) $F(a-1,1-a;\ 5/2;\ 1/4) = \dfrac{-9\sin(1/3\,(a+1/2)\,\pi)}{(2\,a-5)\,(2\,a+1)}.$

(xv) $F(a-1,a+3/2;\ 5/2;\ -1/3) = \dfrac{-3^{a+1}}{2^{2\,a-2}}\,\dfrac{\sin(1/3\,(a+1/2)\,\pi)}{(2\,a-5)\,(2\,a+1)}.$

We find (xvi)≤(xv), (xvii)≤(i), (xviii)≤(ii), (xix)≤(iii), (xx)≤(iv).

(xxi) $F(a-1,a-5/2;\ -3/2;\ -3)$

$$= \begin{cases} -2^{6\,n+6}\,(6\,n+5)\,(6\,n+7) & \text{if } a=-2-3\,n, \\ 2^{6\,n+1}\,(6\,n+1)\,(6\,n+3) & \text{if } a=-3\,n, \\ 2^{6\,n+3}\,(6\,n+3)\,(6\,n+5) & \text{if } a=-1-3\,n, \\ 2^{6\,n-3}\,(6\,n-2)\,(6\,n-4) & \text{if } a=5/2-3\,n, \\ -2^{6\,n+4}\,(6\,n+4)\,(6\,n+6) & \text{if } a=-3/2-3\,n, \\ -2^{6\,n+6}\,(6\,n+6)\,(6\,n+8) & \text{if } a=-5/2-3\,n. \end{cases}$$

We find (xxii)≤(xxi).

(xxiii) $F(a-1,1-a;\ -3/2;\ 3/4) = -\,(2\,a-1)\,(2\,a-3)\sin(1/6\,(4\,a-1)\,\pi)\,.$

(xxiv) $F(a-5/2,-a-1/2;\ -3/2;\ 3/4)$
$= -1/8\ (2\,a-1)\,(2\,a-3)\sin(1/6\,(4\,a-1)\,\pi)\,.$

(3,3,6-6), (3,3,6-7), (3,3,6-8), (3,3,6-13), (3,3,6-14), (3,3,6-15), (3,3,6-16)
The special values obtained from (3,3,6-6), (3,3,6,-7), (3,3,6-8), (3,3,6-13), (3,3,6-14), (3,3,6-15) and (3,3,6-16) coincide with those obtained from (3,3,6,5).

$(k,l,m) = (3,4,6)$

In this case, we have

$$(a,b,c,x) = (a,b,2\,a,2). \qquad (3,4,6\text{-}1)$$

(3,4,6-1) The special values obtained from (3,4,6-1) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1).

$(k,l,m) = (3,5,6)$

In this case, we have

$$\begin{cases} (a,b,c,x) = (a,5/3\,a-1/2,2\,a,-8+4\,\sqrt{5}), \\ S^{(n)} = \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{15\,n}(1/3\,a+3/10,n)\,(1/3\,a+7/10,n)}{2^{15\,n}3^{3\,n}\,(1/3\,a+1/6,n)\,(1/3\,a+5/6,n)}, \end{cases} \qquad (3,5,6\text{-}1)$$

$$\begin{cases} (a,b,c,x) = (a,5/3\,a-1/2,2\,a,-8-4\,\sqrt{5}), \\ S^{(n)} = \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(1/3\,a+3/10,n)\,(1/3\,a+7/10,n)}{2^{15\,n}3^{3\,n}\,(1/3\,a+1/6,n)\,(1/3\,a+5/6,n)}, \end{cases} \qquad (3,5,6\text{-}2)$$

$$\begin{cases}(a,b,c,x)=(a,5/3\,a-1/6,2\,a,-8+4\,\sqrt{5}),\\ S^{(n)}=\dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{15\,n}(1/3\,a+17/30,n)\,(1/3\,a+23/30,n)}{2^{15\,n}3^{3\,n}\,(1/3\,a+1/2,n)\,(1/3\,a+5/6,n)},\end{cases} \tag{3,5,6-3}$$

$$\begin{cases}(a,b,c,x)=(a,5/3\,a-1/6,2\,a,-8-4\,\sqrt{5}),\\ S^{(n)}=\dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(1/3\,a+17/30,n)\,(1/3\,a+23/30,n)}{2^{15\,n}3^{3\,n}\,(1/3\,a+1/2,n)\,(1/3\,a+5/6,n)}.\end{cases} \tag{3,5,6-4}$$

(3,5,6-1)

(i) $F(a,5/3\,a-1/2;\;2\,a;\;-8+4\,\sqrt{5})$

$$=\begin{cases}\dfrac{5^{1/4-5/6\,a}\left(\sqrt{5}-1\right)^{3/2-5\,a}\Gamma(2/5)\,\Gamma(4/5)\,\Gamma(1/3\,a+1/6)\,\Gamma(1/3\,a+5/6)}{2^{3/2-5\,a}3^{3/10-a}\Gamma(4/15)\,\Gamma(14/15)\,\Gamma(1/3\,a+3/10)\,\Gamma(1/3\,a+7/10)},\\ \dfrac{7\cdot 5^{5/2\,n-1/2}\left(\sqrt{5}-1\right)^{15\,n+15}(17/10,n)\,(13/10,n)}{2^{15\,n+15}3^{3\,n}\,(11/6,n)\,(7/6,n)} & \text{if } a=-3-3\,n,\\ \dfrac{-5^{5/2\,n}\left(\sqrt{5}-1\right)^{15\,n+7}(31/30,n)\,(19/30,n)}{2^{15\,n+6}3^{3\,n+1}\,(7/6,n)\,(1/2,n)} & \text{if } a=-1-3\,n,\\ \dfrac{-11\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{15\,n+12}(41/30,n)\,(29/30,n)}{2^{15\,n+12}3^{3\,n+3}\,(3/2,n)\,(5/6,n)} & \text{if } a=-2-3\,n,\\ \dfrac{-5^{5/2\,n+1}\left(\sqrt{5}-1\right)^{15\,n+24}(11/5,n)\,(9/5,n)}{2^{15\,n+24}3^{3\,n}\,(7/3,n)\,(5/3,n)} & \text{if } a=-9/2-3\,n,\end{cases}$$

(ii) $F(a,1/3\,a+1/2;\;2\,a;\;-8+4\,\sqrt{5})$

$$=\begin{cases}\dfrac{5^{1/4-5/6\,a}\left(\sqrt{5}-1\right)^{-a-3/2}\Gamma(2/5)\,\Gamma(4/5)\,\Gamma(1/3\,a+1/6)\,\Gamma(1/3\,a+5/6)}{2^{-a-3/2}3^{3/10-a}\Gamma(4/15)\,\Gamma(14/15)\,\Gamma(1/3\,a+3/10)\,\Gamma(1/3\,a+7/10)},\\ \dfrac{7\cdot 5^{5/2\,n-1/2}\left(\sqrt{5}-1\right)^{3\,n+3}(17/10,n)\,(13/10,n)}{2^{3\,n+3}3^{3\,n}\,(11/6,n)\,(7/6,n)} & \text{if } a=-3-3\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{3\,n-1}(31/30,n)\,(19/30,n)}{2^{3\,n-2}3^{3\,n+1}\,(7/6,n)\,(1/2,n)} & \text{if } a=-1-3\,n,\\ \dfrac{11\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{3\,n}(41/30,n)\,(29/30,n)}{2^{3\,n}3^{3\,n+3}\,(3/2,n)\,(5/6,n)} & \text{if } a=-2-3\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{3\,n}(6/5,n)\,(4/5,n)}{2^{3\,n}3^{3\,n}\,(4/3,n)\,(2/3,n)} & \text{if } a=-3/2-3\,n.\end{cases}$$

(iii) $F(a,1/3\,a+1/2;\;2\,a;\;-8-4\,\sqrt{5})$

$$=\begin{cases}\dfrac{7\cdot 5^{5/2\,n-1/2}\left(\sqrt{5}+1\right)^{3\,n+3}(17/10,n)\,(13/10,n)}{2^{3\,n+3}3^{3\,n}\,(11/6,n)\,(7/6,n)} & \text{if } a=-3-3\,n,\\ \dfrac{-5^{5/2\,n}\left(\sqrt{5}+1\right)^{3\,n-1}(31/30,n)\,(19/30,n)}{2^{3\,n-2}3^{3\,n+1}\,(7/6,n)\,(1/2,n)} & \text{if } a=-1-3\,n,\\ \dfrac{-11\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3\,n}(41/30,n)\,(29/30,n)}{2^{3\,n}3^{3\,n+3}\,(3/2,n)\,(5/6,n)} & \text{if } a=-2-3\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{3\,n}(6/5,n)\,(4/5,n)}{2^{3\,n}3^{3\,n}\,(4/3,n)\,(2/3,n)} & \text{if } a=-3/2-3\,n.\end{cases}$$

(iv) $F(a, 5/3\,a - 1/2;\ 2\,a;\ -8 - 4\sqrt{5})$

$$= \begin{cases} \dfrac{7 \cdot 5^{5/2\,n-1/2}\left(\sqrt{5}+1\right)^{15\,n+15}(17/10,n)\,(13/10,n)}{2^{15\,n+15}3^{3\,n}\,(11/6,n)\,(7/6,n)} & \text{if } a = -3 - 3\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n+7}(31/30,n)\,(19/30,n)}{2^{15\,n+6}3^{3\,n+1}\,(7/6,n)\,(1/2,n)} & \text{if } a = -1 - 3\,n, \\ \dfrac{11 \cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+12}(41/30,n)\,(29/30,n)}{2^{15\,n+12}3^{3\,n+3}\,(3/2,n)\,(5/6,n)} & \text{if } a = -2 - 3\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(3/5,n)\,(1/5,n)}{2^{15\,n}3^{3\,n}\,(11/15,n)\,(1/15,n)} & \text{if } a = 3/10 - 3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+3}(4/5,n)\,(2/5,n)}{2^{15\,n+3}3^{3\,n}\,(14/15,n)\,(4/15,n)} & \text{if } a = -3/10 - 3\,n, \\ 0 \quad \text{if } a = -9/10 - 3\,n, \\ \dfrac{-5^{5/2\,n+1}\left(\sqrt{5}+1\right)^{15\,n+24}(11/5,n)\,(9/5,n)}{2^{15\,n+24}3^{3\,n}\,(7/3,n)\,(5/3,n)} & \text{if } a = -9/2 - 3\,n, \\ 0 \quad \text{if } a = -21/10 - 3\,n. \end{cases}$$

(v) $F(a, 5/3\,a - 1/2;\ 2/3\,a + 1/2;\ 9 - 4\sqrt{5})$

$$= \frac{5^{-5/6\,a}\left(\sqrt{5}-1\right)^{1-5\,a}\Gamma(1/3\,a+1/4)\,\Gamma(1/3\,a+3/4)}{2^{1/2-11/3\,a}\Gamma(1/3\,a+3/10)\,\Gamma(1/3\,a+7/10)}.$$

(vi) $F(1-a, 1/2 - 1/3\,a;\ 2/3\,a + 1/2;\ 9 - 4\sqrt{5})$

$$= \frac{5^{-5/6\,a}\left(\sqrt{5}-1\right)^{a-2}\Gamma(1/3\,a+1/4)\,\Gamma(1/3\,a+3/4)}{2^{-5/3\,a-1/2}\Gamma(1/3\,a+3/10)\,\Gamma(1/3\,a+7/10)}$$

(The above is a generalization of Theorem 14 in [Ek]).

(vii) $F(a, 1-a;\ 2/3\,a + 1/2;\ 1/2 - 1/4\sqrt{5})$

$$= \frac{5^{-5/6\,a}\left(\sqrt{5}-1\right)^{1-2\,a}\Gamma(1/3\,a+1/4)\,\Gamma(1/3\,a+3/4)}{2^{1/2-8/3\,a}\Gamma(1/3\,a+3/10)\,\Gamma(1/3\,a+7/10)}.$$

(viii) $F(1/2 - 1/3\,a, 5/3\,a - 1/2;\ 2/3\,a + 1/2;\ 1/2 - 1/4\sqrt{5})$

$$= \frac{5^{-5/6\,a}\left(\sqrt{5}-1\right)^{-1/2}\Gamma(1/3\,a+1/4)\,\Gamma(1/3\,a+3/4)}{2^{-2\,a}\Gamma(1/3\,a+3/10)\,\Gamma(1/3\,a+7/10)}.$$

(ix) $F(a, 1-a;\ 3/2 - 2/3\,a;\ 1/2 + 1/4\sqrt{5})$

$$
= \begin{cases}
\dfrac{\left(\sqrt{5}-1\right)^{6\,n}\,(5/4,n)\,(3/4,n)}{(-16)^n\,5^{5/2\,n}\,(11/10,n)\,(9/10,n)} & \text{if } a=-3\,n, \\
\dfrac{\left(\sqrt{5}-1\right)^{6\,n+4}\,(19/12,n)\,(13/12,n)}{104\,(-16)^n\,5^{5/2\,n}\,(43/30,n)\,(37/30,n)} & \text{if } a=-1-3\,n, \\
\dfrac{11\,\left(\sqrt{5}-1\right)^{6\,n+6}\,(23/12,n)\,(17/12,n)}{1564\,(-16)^{n+1}\,5^{5/2\,n-1/2}\,(53/30,n)\,(47/30,n)} & \text{if } a=-2-3\,n, \\
\dfrac{5^{5/2\,n}\,\left(\sqrt{5}+1\right)^{6\,n}\,(7/30,n)\,(13/30,n)}{(-256)^n\,(1/12,n)\,(7/12,n)} & \text{if } a=1+3\,n, \\
\dfrac{-5^{5/2\,n+1/2}\,\left(\sqrt{5}+1\right)^{6\,n+2}\,(17/30,n)\,(23/30,n)}{2\,(-256)^n\,(5/12,n)\,(11/12,n)} & \text{if } a=2+3\,n, \\
\dfrac{-5^{5/2\,n+1/2}\,\left(\sqrt{5}+1\right)^{6\,n+6}\,(9/10,n)\,(11/10,n)}{64\,(-256)^n\,(3/4,n)\,(5/4,n)} & \text{if } a=3+3\,n.
\end{cases}
$$

(x) $F(1/3\,a+1/2, 3/2-5/3\,a;\ 3/2-2/3\,a;\ 1/2+1/4\,\sqrt{5})$

$$
= \begin{cases}
\dfrac{(-64)^n\,(7/4,n)\,(5/4,n)}{5^{5/2\,n}\,(8/5,n)\,(7/5,n)} & \text{if } a=-3/2-3\,n, \\
\dfrac{5^{5/2\,n}\,(1/5,n)\,(2/5,n)}{(-64)^n\,(1/20,n)\,(11/20,n)} & \text{if } a=9/10+3\,n, \\
\dfrac{-5^{5/2\,n+1/2}\,(2/5,n)\,(3/5,n)}{2\,(-64)^n\,(1/4,n)\,(3/4,n)} & \text{if } a=3/2+3\,n, \\
\dfrac{5^{5/2\,n+1}\,(3/5,n)\,(4/5,n)}{2\,(-64)^n\,(9/20,n)\,(19/20,n)} & \text{if } a=21/10+3\,n, \\
0 \quad \text{if } a=27/10+3\,n, \\
0 \quad \text{if } a=33/10+3\,n.
\end{cases}
$$

(xi) $F(a, 1/3\,a+1/2;\ 3/2-2/3\,a;\ 9+4\,\sqrt{5})$

$$
= \begin{cases}
\dfrac{2^{5\,n}\,\left(\sqrt{5}+1\right)^{3\,n}\,(5/4,n)\,(3/4,n)}{5^{5/2\,n}\,(11/10,n)\,(9/10,n)} & \text{if } a=-3\,n, \\
\dfrac{-2^{5\,n+4}\,\left(\sqrt{5}+1\right)^{3\,n-1}\,(19/12,n)\,(13/12,n)}{13\cdot 5^{5/2\,n}\,(43/30,n)\,(37/30,n)} & \text{if } a=-1-3\,n, \\
\dfrac{-11\cdot 2^{5\,n+4}\,\left(\sqrt{5}+1\right)^{3\,n}\,(23/12,n)\,(17/12,n)}{391\cdot 5^{5/2\,n-1/2}\,(53/30,n)\,(47/30,n)} & \text{if } a=-2-3\,n, \\
\dfrac{2^{5\,n}\,\left(\sqrt{5}+1\right)^{3\,n}\,(7/4,n)\,(5/4,n)}{5^{5/2\,n}\,(8/5,n)\,(7/5,n)} & \text{if } a=-3/2-3\,n
\end{cases}
$$

(The fourth case is identical to Theorem 15 in [Ek]).

(xii) $F(1-a, 3/2-5/3\,a;\ 3/2-2/3\,a;\ 9+4\,\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(7/30,n)\,(13/30,n)}{2^{11\,n}\,(1/12,n)\,(7/12,n)} & \text{if } a = 1+3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+5}(17/30,n)\,(23/30,n)}{2^{11\,n+2}\,(5/12,n)\,(11/12,n)} & \text{if } a = 2+3\,n, \\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+12}(9/10,n)\,(11/10,n)}{2^{11\,n+8}\,(3/4,n)\,(5/4,n)} & \text{if } a = 3+3\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(1/5,n)\,(2/5,n)}{2^{11\,n}\,(1/20,n)\,(11/20,n)} & \text{if } a = 9/10+3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+3}(2/5,n)\,(3/5,n)}{2^{11\,n+2}\,(1/4,n)\,(3/4,n)} & \text{if } a = 3/2+3\,n, \\ \dfrac{5^{5/2\,n+1}\left(\sqrt{5}+1\right)^{15\,n+6}(3/5,n)\,(4/5,n)}{2^{11\,n+3}\,(9/20,n)\,(19/20,n)} & \text{if } a = 21/10+3\,n, \\ 0 \quad \text{if } a = 27/10+3\,n, \\ 0 \quad \text{if } a = 33/10+3\,n. \end{cases}$$

(xiii) $F(1/2-1/3\,a, 5/3\,a-1/2;\ 2/3\,a+1/2;\ 1/2+1/4\,\sqrt{5})$

$$= \begin{cases} \dfrac{(-64)^n\,(3/4,n)\,(5/4,n)}{5^{5/2\,n}\,(4/5,n)\,(6/5,n)} & \text{if } a = 3/2+3\,n, \\ \dfrac{5^{5/2\,n}\,(3/5,n)\,(1/5,n)}{(-64)^n\,(13/20,n)\,(3/20,n)} & \text{if } a = 3/10-3\,n, \\ \dfrac{-5^{5/2\,n+1/2}\,(4/5,n)\,(2/5,n)}{2\,(-64)^n\,(17/20,n)\,(7/20,n)} & \text{if } a = -3/10-3\,n, \\ 0 \quad \text{if } a = -9/10-3\,n, \\ \dfrac{-5^{5/2\,n+1/2}\,(6/5,n)\,(4/5,n)}{4\,(-64)^n\,(5/4,n)\,(3/4,n)} & \text{if } a = -3/2-3\,n, \\ 0 \quad \text{if } a = -21/10-3\,n. \end{cases}$$

(xiv) $F(a, 1-a;\ 2/3\,a+1/2;\ 1/2+1/4\,\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{6\,n}(7/10,n)\,(3/10,n)}{(-256)^n\,(3/4,n)\,(1/4,n)} & \text{if } a = -3\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{6\,n+4}(31/30,n)\,(19/30,n)}{8\,(-256)^n\,(13/12,n)\,(7/12,n)} & \text{if } a = -1-3\,n, \\ \dfrac{-11\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{6\,n+6}(41/30,n)\,(29/30,n)}{320\,(-256)^n\,(17/12,n)\,(11/12,n)} & \text{if } a = -2-3\,n, \\ \dfrac{\left(\sqrt{5}-1\right)^{6\,n}(7/12,n)\,(13/12,n)}{(-16)^n\,5^{5/2\,n}\,(19/30,n)\,(31/30,n)} & \text{if } a = 1+3\,n, \\ \dfrac{-\left(\sqrt{5}-1\right)^{6\,n+2}(11/12,n)\,(17/12,n)}{22\,(-16)^n\,5^{5/2\,n-1/2}\,(29/30,n)\,(41/30,n)} & \text{if } a = 2+3\,n, \\ \dfrac{-\left(\sqrt{5}-1\right)^{6\,n+6}(5/4,n)\,(7/4,n)}{2240\,(-16)^n\,5^{5/2\,n-1/2}\,(13/10,n)\,(17/10,n)} & \text{if } a = 3+3\,n. \end{cases}$$

(xv) $F(a, 5/3\,a-1/2;\ 2/3\,a+1/2;\ 9+4\,\sqrt{5})$

$$= \begin{cases} \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(7/10,n)\,(3/10,n)}{2^{11\,n}\,(3/4,n)\,(1/4,n)} & \text{if } a=-3\,n,\\ \dfrac{-5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n+7}(31/30,n)\,(19/30,n)}{2^{11\,n+4}\,(13/12,n)\,(7/12,n)} & \text{if } a=-1-3\,n,\\ \dfrac{-11\cdot 5^{5/2\,n-1/2}\left(\sqrt{5}+1\right)^{15\,n+12}(41/30,n)\,(29/30,n)}{2^{11\,n+8}\,(17/12,n)\,(11/12,n)} & \text{if } a=-2-3\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}(3/5,n)\,(1/5,n)}{2^{11\,n}\,(13/20,n)\,(3/20,n)} & \text{if } a=3/10-3\,n,\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+3}(4/5,n)\,(2/5,n)}{2^{11\,n+2}\,(17/20,n)\,(7/20,n)} & \text{if } a=-3/10-3\,n,\\ 0 \quad \text{if } a=-9/10-3\,n, & \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+9}(6/5,n)\,(4/5,n)}{2^{11\,n+5}\,(5/4,n)\,(3/4,n)} & \text{if } a=-3/2-3\,n,\\ 0 \quad \text{if } a=-21/10-3\,n. & \end{cases}$$

(xvi) $F(1-a,1/2-1/3\,a;\;2/3\,a+1/2;\;9+4\sqrt{5})$

$$= \begin{cases} \dfrac{2^{5\,n}\left(\sqrt{5}+1\right)^{3\,n}(7/12,n)\,(13/12,n)}{5^{5/2\,n}\,(19/30,n)\,(31/30,n)} & \text{if } a=1+3\,n,\\ \dfrac{2^{5\,n+2}\left(\sqrt{5}+1\right)^{3\,n+1}(11/12,n)\,(17/12,n)}{11\cdot 5^{5/2\,n-1/2}\,(29/30,n)\,(41/30,n)} & \text{if } a=2+3\,n,\\ \dfrac{-2^{5\,n+4}\left(\sqrt{5}+1\right)^{3\,n}(5/4,n)\,(7/4,n)}{7\cdot 5^{5/2\,n+1/2}\,(13/10,n)\,(17/10,n)} & \text{if } a=3+3\,n,\\ \dfrac{2^{5\,n}\left(\sqrt{5}+1\right)^{3\,n}(3/4,n)\,(5/4,n)}{5^{5/2\,n}\,(4/5,n)\,(6/5,n)} & \text{if } a=3/2+3\,n \end{cases}$$

(The fourth case is identical to Theorem 14 in [Ek]).

(xvii) $F(1-a,1/2-1/3\,a;\;2-2\,a;\;-8+4\sqrt{5})$

$$= \begin{cases} \dfrac{3^{9/10-a}\left(\sqrt{5}+1\right)^{3/2-a}\Gamma(4/5)\,\Gamma(3/5)\,\Gamma(7/6-1/3\,a)\,\Gamma(5/6-1/3\,a)}{2^{3/2-a}5^{3/4-5/6\,a}\Gamma(13/15)\,\Gamma(8/15)\,\Gamma(11/10-1/3\,a)\,\Gamma(9/10-1/3\,a)}, & \\ \dfrac{91\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{3\,n+3}(37/30,n)\,(43/30,n)}{2^{3\,n+3}3^{3\,n+4}\,(7/6,n)\,(3/2,n)} & \text{if } a=4+3\,n,\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{3\,n+1}(17/30,n)\,(23/30,n)}{2^{3\,n}3^{3\,n+1}\,(1/2,n)\,(5/6,n)} & \text{if } a=2+3\,n,\\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{3\,n}(9/10,n)\,(11/10,n)}{2^{3\,n}3^{3\,n+1}\,(5/6,n)\,(7/6,n)} & \text{if } a=3+3\,n,\\ \dfrac{5^{5/2\,n}\left(\sqrt{5}-1\right)^{3\,n}(2/5,n)\,(3/5,n)}{2^{3\,n}3^{3\,n}\,(1/3,n)\,(2/3,n)} & \text{if } a=3/2+3\,n. \end{cases}$$

(xviii) $F(1-a,3/2-5/3\,a;\;2-2\,a;\;-8+4\sqrt{5})$

$$= \begin{cases} \dfrac{3^{9/10-a}\left(\sqrt{5}+1\right)^{9/2-5\,a}\Gamma\left(4/5\right)\Gamma\left(3/5\right)\Gamma\left(7/6-1/3\,a\right)\Gamma\left(5/6-1/3\,a\right)}{2^{9/2-5\,a}5^{3/4-5/6\,a}\Gamma\left(13/15\right)\Gamma\left(8/15\right)\Gamma\left(11/10-1/3\,a\right)\Gamma\left(9/10-1/3\,a\right)}, \\ \dfrac{91\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{15\,n+15}\left(37/30,n\right)\left(43/30,n\right)}{2^{15\,n+15}3^{3\,n+4}\left(7/6,n\right)\left(3/2,n\right)} & \text{if } a=4+3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{15\,n+5}\left(17/30,n\right)\left(23/30,n\right)}{2^{15\,n+4}3^{3\,n+1}\left(1/2,n\right)\left(5/6,n\right)} & \text{if } a=2+3\,n, \\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}-1\right)^{15\,n+12}\left(9/10,n\right)\left(11/10,n\right)}{2^{15\,n+12}3^{3\,n+1}\left(5/6,n\right)\left(7/6,n\right)} & \text{if } a=3+3\,n, \\ \dfrac{5^{5/2\,n+1}\left(\sqrt{5}-1\right)^{15\,n+18}\left(7/5,n\right)\left(8/5,n\right)}{2^{15\,n+18}3^{3\,n}\left(4/3,n\right)\left(5/3,n\right)} & \text{if } a=9/2+3\,n. \end{cases}$$

(xix) $F(1-a,3/2-5/3\,a;\;2-2\,a;\;-8-4\,\sqrt{5})$

$$= \begin{cases} \dfrac{91\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+15}\left(37/30,n\right)\left(43/30,n\right)}{2^{15\,n+15}3^{3\,n+4}\left(7/6,n\right)\left(3/2,n\right)} & \text{if } a=4+3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+5}\left(17/30,n\right)\left(23/30,n\right)}{2^{15\,n+4}3^{3\,n+1}\left(1/2,n\right)\left(5/6,n\right)} & \text{if } a=2+3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{15\,n+12}\left(9/10,n\right)\left(11/10,n\right)}{2^{15\,n+12}3^{3\,n+1}\left(5/6,n\right)\left(7/6,n\right)} & \text{if } a=3+3\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{15\,n}\left(1/5,n\right)\left(2/5,n\right)}{2^{15\,n}3^{3\,n}\left(2/15,n\right)\left(7/15,n\right)} & \text{if } a=9/10+3\,n, \\ \dfrac{5^{5/2\,n+1}\left(\sqrt{5}+1\right)^{15\,n+18}\left(7/5,n\right)\left(8/5,n\right)}{2^{15\,n+18}3^{3\,n}\left(4/3,n\right)\left(5/3,n\right)} & \text{if } a=9/2+3\,n, \\ \dfrac{5^{5/2\,n+1}\left(\sqrt{5}+1\right)^{15\,n+6}\left(3/5,n\right)\left(4/5,n\right)}{2^{15\,n+6}3^{3\,n+1}\left(8/15,n\right)\left(13/15,n\right)} & \text{if } a=21/10+3\,n, \\ 0 \quad \text{if } a=27/10+3\,n, \\ 0 \quad \text{if } a=33/10+3\,n. \end{cases}$$

(xx) $F(1-a,1/2-1/3\,a;\;2-2\,a;\;-8-4\,\sqrt{5})$

$$= \begin{cases} \dfrac{91\cdot 5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3\,n+3}\left(37/30,n\right)\left(43/30,n\right)}{2^{3\,n+3}3^{3\,n+4}\left(7/6,n\right)\left(3/2,n\right)} & \text{if } a=4+3\,n, \\ \dfrac{5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3\,n+1}\left(17/30,n\right)\left(23/30,n\right)}{2^{3\,n}3^{3\,n+1}\left(1/2,n\right)\left(5/6,n\right)} & \text{if } a=2+3\,n, \\ \dfrac{-5^{5/2\,n+1/2}\left(\sqrt{5}+1\right)^{3\,n}\left(9/10,n\right)\left(11/10,n\right)}{2^{3\,n}3^{3\,n+1}\left(5/6,n\right)\left(7/6,n\right)} & \text{if } a=3+3\,n, \\ \dfrac{5^{5/2\,n}\left(\sqrt{5}+1\right)^{3\,n}\left(2/5,n\right)\left(3/5,n\right)}{2^{3\,n}3^{3\,n}\left(1/3,n\right)\left(2/3,n\right)} & \text{if } a=3/2+3\,n. \end{cases}$$

(xxi) $F(a,1/3\,a+1/2;\;3/2-2/3\,a;\;9-4\,\sqrt{5})$

$$= \frac{2^{5/2-5/3\,a}\left(\sqrt{5}-1\right)^{-a-1}\Gamma\left(5/4-1/3\,a\right)\Gamma\left(3/4-1/3\,a\right)}{5^{1-5/6\,a}\Gamma\left(11/10-1/3\,a\right)\Gamma\left(9/10-1/3\,a\right)}$$

(The above is a generalization of Theorem 15 in [Ek]).

(xxii) $F(1-a, 3/2-5/3\,a;\ 3/2-2/3\,a;\ 9-4\sqrt{5})$
$$= \frac{2^{7/2-11/3\,a}\left(\sqrt{5}-1\right)^{5\,a-4}\Gamma\left(5/4-1/3\,a\right)\Gamma\left(3/4-1/3\,a\right)}{5^{1-5/6\,a}\Gamma\left(11/10-1/3\,a\right)\Gamma\left(9/10-1/3\,a\right)}.$$

(xxiii) $F(a, 1-a;\ 3/2-2/3\,a;\ 1/2-1/4\sqrt{5})$
$$= \frac{2^{5/2-8/3\,a}\left(\sqrt{5}-1\right)^{2\,a-1}\Gamma\left(5/4-1/3\,a\right)\Gamma\left(3/4-1/3\,a\right)}{5^{1-5/6\,a}\Gamma\left(11/10-1/3\,a\right)\Gamma\left(9/10-1/3\,a\right)}.$$

(xxiv) $F(1/3\,a+1/2, 3/2-5/3\,a;\ 3/2-2/3\,a;\ 1/2-1/4\sqrt{5})$
$$= \frac{2^{2-2\,a}\left(\sqrt{5}-1\right)^{1/2}\Gamma\left(5/4-1/3\,a\right)\Gamma\left(3/4-1/3\,a\right)}{5^{1-5/6\,a}\Gamma\left(11/10-1/3\,a\right)\Gamma\left(9/10-1/3\,a\right)}.$$

(3,5,6-2), (3,5,6-3), (3,5,6-4) The special values obtained from (3,5,6-2), (3,5,6-3), (3,5,6-4) coincide with those obtained from (3,5,6-1).

$(k, l, m) = (3, 6, 6)$

In this case, we have

$$(a, b, c, x) = (a, b, 2\,a, 2) \qquad \text{(3,6,6-1)}$$

(3,6,6-1) The special values obtained from (3,6,6-1) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1).

$(k, l, m) = (3, 7, 6)$

In this case, there is no admissible quadruple.

$(k, l, m) = (3, 8, 6)$

In this case, we have

$$(a, b, c, x) = (a, b, 2\,a, 2) \qquad \text{(3,8,6-1)}$$

(3,8,6-1) The special values obtained from (3,8,6-1) are evaluated in paragraphs (1,2,2-1) and (0,2,2-1).

$(k, l, m) = (3, 9, 6)$

In this case, we have

$$(a, b, c, x) = (a, 3\,a-1, 2\,a, 1/2+1/2\,i\sqrt{3}), \qquad \text{(3,9,6-1)}$$
$$(a, b, c, x) = (a, 3\,a-1, 2\,a, 1/2-1/2\,i\sqrt{3}). \qquad \text{(3,9,6-2)}$$

(3,9,6-1), (3,9,6-2) The special values obtained from (3,9,6-1) and (3,9,6-2) coincide with those obtained from (1,3,2-1).

References

[AAR] G.E.Andrews, R.Askey and R.Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[AZ] M.Apagodu and D.Zeilberger, *Searching for strange hypergeometric identities by Sheer Brute Force*, Integers, **8** (2008), A36, 6 pp.

[Ba] W.N.Bailey, *Generalized hypergeometric series*, Cambridge Mathematical Tract No. 32, Cambridge University Press, (1935).

[Ek] S.B.Ekhad, *Forty "strange" computer-discovered[and computer-proved(of course)] hypergeometric series evaluations*, The Personal Journal of Ekhad and Zeilberger, http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/strange.html, (2004).

[Erd] A.Erdélyi (editor), *Higher transcendental functions Vol.1*, McGraw-Hill, (1953).

[Ga] C.F.Gauss, *Disquisitiones generales circa seriem infinitam*, Comm. Soc. Reg Gött. II, Werke, **3**, 123–162, (1812).

[Ge] I.Gessel, *Finding Identities with the WZ Method*, J. Symbolic Computation, **20**(1995), 537–566.

[Go] R.W.Gosper, *A letter to D. Stanton*, XEROX Palo Alto Research Center, 21 December 1977.

[GS] I.Gessel and D.Stanton, *Strange evaluations of hypergeometric series*, SIAM J. Math. Anal., **13**(1982), no. **2**, 295–308.

[IKSY] K.Iwasaki, H.Kimura, S.Shimomura and M.Yoshida, *From Gauss to Painlevé —A modern theory of special functions*, Vieweg Verlag, Wiesbaden(1991).

[Ka] P.W.Karlsson, *On two hypergeometric summation formulas conjectured by Gosper*, Simon Stevin **60** (1986), no. 4, 329–337.

[Ko] W.Koepf, *Hypergeometric summation —An algorithmic approach to summation and special function identities*, Advanced lectures in mathematics, Friedr. Vieweg & Sohn, Braunschweig(1998).

[PWZ] M.Petkovšek, H.Wilf and D.Zeilberger, *A=B*, A.K.Peters, Wellesley(1996).

[V] R.Vidunas, *Algebraic transformations of Gauss hypergeometric functions*, Funkcial. Ekvac., **52**(2009), no. **2**, 139–180.