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Abstract: This paper is concerned with the stability of a parallel flow of the compressible Navier-Stokes equation in a cylindrical domain. Decay estimates on the linearized semigroup is established. It is shown that if the Reynolds and Mach numbers are sufficiently small, then solutions of the linearized problem decay in the L^2 -norm as a one dimensional heat kernel. The proof is given by a variant of the Matsumura-Nishida energy method.

Keywords: Compressible Navier-Stokes equation, Parallel flow, cylindrical domain, linearized semigroup, decay estimates

1 INTRODUCTION

We consider the system of equations for a barotropic motion of viscous compressible gas

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad t > 0, \quad x \in \Omega, \quad (1.1)$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla P(\rho) = \rho g \quad (1.2)$$

in a cylindrical domain $\Omega = D \times \mathbf{R}$:

$$\Omega = \{x = (x', x_3); x' = (x_1, x_2) \in D, x_3 \in \mathbf{R}\}.$$

Here D is a bounded and connected domain in $\mathbf{R}^2$ with a smooth boundary ∂D ; $\rho = \rho(x, t)$ and $v = {}^T(v^1(x, t), v^2(x, t), v^3(x, t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively; $P(\rho)$ is the pressure that is a smooth function of ρ and satisfies

$$P'(\rho_*) > 0$$

for a given positive constant ρ_* ; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \frac{2}{3}\mu + \mu' \geq 0;$$

and g is an external force of the form $g = {}^T(g^1(x'), g^2(x'), g^3(x'))$ with g^1 and g^2 satisfying

$$(g^1(x'), g^2(x')) = (\partial_{x_1} \Phi(x'), \partial_{x_2} \Phi(x')),$$

where Φ and g^3 are given smooth functions of x' . Here and in what follows T stands for the transposition.

The system (1.1)-(1.2) is considered under the boundary condition

$$v|_{\partial D} = 0 \tag{1.3}$$

and the initial condition

$$(\rho, v)|_{t=0} = (\rho_0, v_0). \tag{1.4}$$

One can see that problem (1.1)-(1.3) has the stationary solution $\bar{u}_s = {}^T(\bar{\rho}_s, \bar{v}_s)$; $\bar{\rho}_s$ is determined by

$$\begin{cases} \text{Const.} - \Phi(x') = \int_{\rho_*}^{\bar{\rho}_s} \frac{P'(\eta)}{\eta} d\eta, \\ \int_D \bar{\rho}_s - \rho_* dx' = 0; \end{cases}$$

and $\bar{v}_s$ takes the form

$$\bar{v}_s = {}^T(0, 0, \bar{v}_s^3(x')),$$

where $\bar{v}_s^3(x')$ is the solution of

$$\begin{cases} -\Delta' \bar{v}_s^3 = \frac{\rho_*}{\mu} \bar{\rho}_s g^3, \\ \bar{v}_s^3|_{\partial D} = 0. \end{cases}$$

We are interested in the large time behavior of solutions to problem (1.1)-(1.4) when the initial value $(\rho, v)|_{t=0} = (\rho_0, v_0)$ is sufficiently close to the stationary solution $\bar{u}_s = {}^T(\bar{\rho}_s, \bar{v}_s)$. As a first step of the analysis, we study the linearized problem in this paper and establish decay estimates on solutions of the linearized equation around the parallel flow $\bar{u}_s$.

Let us introduce some known results for stability of a parallel flow. In Kagei-Nagahuchi-Sudou [7], the stability of a plane Poiseuille type flow in an infinite layer of $\mathbf{R}^n$ was considered under the perturbations in some L^2 -Sobolev space on the infinite layer. It was shown in [7] that the low frequency part of the linearized semigroup behaves like $n - 1$ dimensional heat kernel and the high frequency part decays exponentially as $t \rightarrow \infty$, provided that the Reynolds and Mach numbers are sufficiently small and the density of the parallel flow is sufficiently close to the given constant ρ_* . The nonlinear problem was studied by Kagei [6]; and it was proved that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in some L^2 -Sobolev space. Furthermore, the asymptotic behavior of the perturbation is described by an $n - 1$ dimensional heat equation when $n \geq 3$.

When $n = 2$, the asymptotic behavior of the perturbation is no longer described by a linear equation but by a one dimensional viscous Burgers equation.

As for the case of the cylindrical domain Ω , Iooss and Padula [3] studied the linearized stability of a stationary parallel flow in Ω under the perturbations periodic in x_3 . It was shown that the linearized operator generates a C_0 -semigroup in L^2 on the basic periodicity cell under vanishing average condition for the density-component. In particular, if the Reynolds number is suitably small, then the semigroup decays exponentially as time goes to infinity.

On the other hand, in Kagei-Nukumizu [9], the stability of the motionless state $\tilde{u}_s = {}^T(\rho_*, 0)$ was considered under the perturbations in some L^2 -Sobolev space on Ω . It was shown in [9] that the solution of the linearized problem decays in $L^2(\Omega)$ in the order $t^{-\frac{1}{4}}$ and its asymptotic leading parts is given by a solution of a one dimensional heat equation. Furthermore, the asymptotic leading part of the perturbation is given by that for the linearized problem. (See also [4].)

The purpose of this paper is to extend the analysis for the rest state in [9] to the case of the general parallel flow in a cylindrical domain. We will establish decay estimates for solutions of the linearized problem for (1.1)-(1.4), which play an important role in the analysis of the nonlinear problem. To state our result more precisely, consider the non-dimensional linearized problem:

$$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0. \quad (1.5)$$

Here $u = {}^T(\phi, w) = {}^T(\gamma^2(\rho - \rho_s), v - v_s)$, and L denotes the linearized operator on $L^2(\tilde{\Omega})$ defined by

$$L = \begin{pmatrix} v_s \cdot \nabla & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta I_3 - \frac{\nu + \nu'}{\rho_s} \nabla \operatorname{div} + v_s \cdot \nabla \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s I_3 & {}^T(\nabla v_s) \end{pmatrix},$$

with domain

$$D(L) = \{u = {}^T(\phi, w) \in L^2(\tilde{\Omega}); \quad w \in H_0^1(\tilde{\Omega}), \quad Lu \in L^2(\tilde{\Omega})\},$$

where $\tilde{\Omega}$, $\tilde{D}$, $\tilde{\rho}$, $\tilde{v}$, ρ_s , v_s and $\tilde{P}(\rho_s)$ are the non-dimensional form of Ω , D , ρ , v , $\bar{\rho}_s$, $\bar{v}_s$ and $P(\bar{\rho}_s)$ respectively; I_3 denotes the 3×3 identity matrix; ν , ν' and γ are some positive constants. We will prove that the linearized semigroup $u(t) = e^{-tL}u_0$ satisfies

$$\|\partial_{x'}^k \partial_{x_3}^l u(t)\|_{L^2(\tilde{\Omega})} \leq C \left\{ t^{-\frac{1}{4} - \frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(\tilde{D}))} + e^{-dt} \|u_0\|_{H^1(\tilde{\Omega})} \right\} \quad (1.6)$$

for $t \geq 0$ and $0 \leq k + l \leq 1$, provided that the Reynolds number $Re = \frac{1}{\nu}$ and Mach number $Ma = \frac{1}{\gamma}$ are sufficiently small and that $\bar{\rho}_s$ is sufficiently close to ρ_* .

To prove (1.6), we consider the Fourier transform of (1.5) in $x_3 \in \mathbf{R}$ which is written as

$$\partial_t \hat{u} + \hat{L}_\xi \hat{u} = 0, \quad \hat{u}|_{t=0} = \hat{u}_0,$$

where $\xi \in \mathbf{R}$ denotes the dual variable. The operator $\hat{L}_\xi$ has different properties of the cases $|\xi| \ll 1$ and $|\xi| \gg 1$. We thus decompose the semigroup e^{-tL} into two parts:

$e^{-tL} = \mathcal{F}^{-1}(e^{-t\widehat{L}_\xi} |_{|\xi| \leq 1}) + \mathcal{F}^{-1}(e^{-t\widehat{L}_\xi} |_{|\xi| > 1})$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. As for the low frequency part, we take a new approach. A straightforward application of the arguments in [7, 9] seems to yield a more restrictive smallness conditions for the Reynolds and Mach numbers. To overcome this, we combine the arguments in [7, 9] and the energy method in [3]. As in [7, 9], we decompose the low frequency part of the semigroup according to the spectral properties of the linearized operator with zero-frequency. The decay estimate for the L^2 norm is then established with the aid of the energy method in [3] to the decomposed system. Based on the decay estimate for L^2 norm, we obtain the estimate for the L^2 norm of the derivatives. We note that this approach also enables us to improve the decay estimate in [7, 8, Theorem 3.2] which is the one with $t^{-\frac{1}{4}}\|u_0\|_{L^1(\mathbf{R}; L^2(\tilde{D}))}$ in (1.6) replaced by $t^{-\frac{1}{4}}\|u_0\|_{L^1(\mathbf{R}; H^1(\tilde{D}) \times L^2(\tilde{D}))}$. On the other hand, in the case of the high frequency part, we employ the Fourier transformed version of Matsumura-Nishida's energy method as in [7, 9].

This paper is organized as follows. In Section 2 we first rewrite the problem into the system of equations in a non-dimensional form and then present the existence of a stationary solution of parallel flow type. We state our main results in Section 3. We derive the decay estimate of the low frequency part in Section 4, and the high frequency part in Section 5.

2 STATIONARY SOLUTION AND FORMULATION OF THE PROBLEM

We first rewrite the problem into the one in the non-dimensional form. We introduce the following non-dimensional variables:

$$x = \ell \tilde{x}, \quad v = V \tilde{v}, \quad \rho = \rho_* \tilde{\rho}, \quad t = \frac{\ell}{V} \tilde{t},$$

$$P = \rho_* V^2 \tilde{P}, \quad \Phi = \frac{V^2}{\ell} \tilde{\Phi}, \quad g^3 = \frac{V^2}{\ell} \tilde{g}^3,$$

$$V = |\bar{v}_s^3|_{C_*^3(D)} = \sum_{k=0}^3 \sup_{x' \in D} \ell^k |\partial_{x'}^k \bar{v}_s^3(x')|, \quad \ell = \left(\int_D dx' \right)^{\frac{1}{2}}.$$

The problem (1.1)-(1.3) is then transformed into the following non-dimensional problem on $\tilde{\Omega} = \tilde{D} \times \mathbf{R}$:

$$\partial_{\tilde{t}} \tilde{\rho} + \operatorname{div}_{\tilde{x}}(\tilde{\rho} \tilde{v}) = 0, \quad \tilde{t} > 0, \quad (2.1)$$

$$\tilde{\rho}(\partial_{\tilde{t}} \tilde{v} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{v}) - \nu \Delta_{\tilde{x}} \tilde{v} - (\nu + \nu') \nabla_{\tilde{x}} \operatorname{div}_{\tilde{x}} \tilde{v} + \tilde{P}'(\tilde{\rho}) \nabla_{\tilde{x}} \tilde{\rho} = \tilde{\rho} \tilde{g}, \quad (2.2)$$

$$\tilde{v} |_{\partial \tilde{D}} = 0, \quad (2.3)$$

$$(\tilde{\rho}, \tilde{v}) |_{\tilde{t}=0} = (\tilde{\rho}_0, \tilde{v}_0). \quad (2.4)$$

Here $\tilde{D}$ is a bounded and connected domain in $\mathbf{R}^2$; $\tilde{g} = T(\partial_{\tilde{x}_1} \tilde{\Phi}, \partial_{\tilde{x}_2} \tilde{\Phi}, \tilde{g}^3)$; and ν and ν' are non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}.$$

We also introduce a parameter γ :

$$\gamma = \sqrt{\tilde{P}'(1)} = \frac{\sqrt{P'(\rho_*)}}{V}.$$

In what follows, for simplicity, we omit tilde of $\tilde{x}$, $\tilde{t}$, $\tilde{v}$, $\tilde{\rho}$, $\tilde{g}$, $\tilde{P}$, $\tilde{\Phi}$, $\tilde{D}$ and $\tilde{\Omega}$ and write them as x , t , v , ρ , g , P , Φ , D and Ω .

We next introduce some notation which will be used throughout the paper. For a domain X and $1 \leq p \leq \infty$ we denote by $L^p(X)$ the usual Lebesgue space on X and its norm is denoted by $\|\cdot\|_{L^p(X)}$. Let m be a nonnegative integer. $H^m(X)$ denotes the m th order L^2 Sobolev space on X with norm $\|\cdot\|_{H^m(X)}$. $C_0^m(X)$ stands for the set of all C^m functions which have compact support in X . We denote by $H_0^m(X)$ the completion of $C_0^m(X)$ in $H^m(X)$.

We simply denote by $L^p(X)$ (resp., $H^m(X)$) the set of all vector fields $w = {}^T(w^1, w^2, w^3)$ on X and its norm is denoted by $\|\cdot\|_{L^p(X)}$ (resp., $\|\cdot\|_{H^m(X)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(X)$ and $w = {}^T(w^1, w^2, w^3) \in H^m(X)$, we define $\|u\|_{H^k(X) \times H^m(X)}$ by $\|u\|_{H^k(X) \times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^m(X)}$.

In case $X = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as L^p (resp., H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$.

In case $X = D$ we denote the norm of $L^p(D)$ by $|\cdot|_p$. The norm of $H^m(D)$ is denoted by $|\cdot|_{H^m}$, respectively. The inner product of $L^2(D)$ is denoted by

$$(f, g) = \int_D f(x') \overline{g(x')} dx', \quad f, g \in L^2(D).$$

Here $\bar{g}$ denotes the complex conjugate of g . For $u_j = {}^T(\phi_j, w_j)$ ($j = 1, 2$), we also define a weighted inner product $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \frac{1}{\gamma^2} \int_D \phi_1 \bar{\phi}_2 \frac{P'(\rho_s)}{\gamma^2 \rho_s} dx' + \int_D w_1 \cdot \bar{w}_2 \rho_s dx',$$

where $\rho_s = \rho_s(x')$ is the density of the parallel flow u_s . As will be seen in Proposition 2.1 below, $\frac{P'(\rho_s(x'))}{\rho_s(x')}$ is strictly positive in D .

For $f \in L^1(D)$ we denote the mean value of f in D by $\langle f \rangle$:

$$\langle f \rangle = (f, 1) = \int_D f dx'.$$

For $u = {}^T(\phi, w) \in L^1(D)$ with $w = {}^T(w^1, w^2, w^3)$ we define $\langle u \rangle$ by

$$\langle u \rangle = \langle \phi \rangle + \langle w_1 \rangle + \langle w_2 \rangle + \langle w_3 \rangle.$$

We often write $x \in \Omega$ as

$$x = {}^T(x', x_3), \quad x' = {}^T(x_1, x_2) \in D.$$

Partial derivatives of a function u in x , x' , x_3 and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_3} u$ and $\partial_t u$. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

We denote the $n \times n$ identity matrix by I_n . We define 4×4 diagonal matrices $Q_0, \tilde{Q}, Q'$ and Q_3 by

$$\begin{aligned} Q_0 &= \text{diag}(1, 0, 0, 0), & \tilde{Q} &= \text{diag}(0, 1, 1, 1), \\ Q' &= \text{diag}(0, 1, 1, 0), & Q_3 &= \text{diag}(0, 0, 0, 1). \end{aligned}$$

We then have, for $u = {}^T(\phi, w)$ with $w = {}^T(w^1, w^2, w^3)$

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ w^1 \\ w^2 \\ 0 \end{pmatrix}, \quad Q_3 u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w^3 \end{pmatrix}.$$

For a function $f = f(x_3)$ ($x_3 \in \mathbf{R}$), we denote its Fourier transform by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbf{R}} f(x_3) e^{-i\xi x_3} dx_3, \quad \xi \in \mathbf{R}.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x_3) = (2\pi)^{-1} \int_{\mathbf{R}} f(\xi) e^{i\xi x_3} d\xi, \quad x_3 \in \mathbf{R}.$$

Let us state the existence of a stationary solution of Poiseuille flow.

Proposition 2.1. *If $\Phi \in C^3(\overline{D})$ and $g^3 \in H^3(D)$, then (2.1)-(2.3) has a stationary solution $u_s = {}^T(\rho_s, v_s) \in C^3(\overline{D})$; ρ_s is determined by*

$$\begin{cases} \text{Const.} - \Phi(x') = \int_1^{\rho_s(x')} \frac{P'(\eta)}{\eta} d\eta, \\ \int_D \rho_s dx' = 1, \quad \rho_1 < \rho_s(x') < \rho_2 \quad (\rho_1 < 1 < \rho_2), \end{cases}$$

and v_s is a function of the form $v_s = {}^T(0, 0, v_s^3)$ with $v_s^3 = v_s^3(x')$ being the solution of

$$\begin{cases} -\Delta' v_s^3 = \frac{1}{\nu} \rho_s g^3, \\ v_s^3|_{\partial D} = 0. \end{cases}$$

Furthermore, $u_s = {}^T(\rho_s, v_s)$ satisfies the estimates:

$$\begin{aligned} |\rho_s(x') - 1|_{C^3} &\leq C|\Phi|_{C^3}(1 + |\Phi|_{C^3})^3, \\ |v_s^3|_{C^3} &\leq C|v_s^3|_{H^5} \leq C|\Phi|_{C^3}(1 + |\Phi|_{C^3})^3 |g^3|_{H^3}. \end{aligned}$$

Proposition 2.1 can be proved in a similarly manner to the proof of Matsumura-Nishida [10, Lemma 2.1]. We omit the proof.

From now on we simply denote $\nu + \nu'$ by $\tilde{\nu}$:

$$\tilde{\nu} = \nu + \nu'.$$

Setting $\rho = \rho_s + \gamma^{-2}\phi$ and $v = v_s + w$ in (2.1)-(2.4), we arrive at the initial boundary value problem for the disturbance $u = {}^T(\phi, w)$ that is written as follows:

$$\partial_t \phi + v_s^3 \partial_{x_3} \phi + \gamma^2 \operatorname{div}(\rho_s w) = f^0(\phi, w), \quad (2.5)$$

$$\begin{aligned} \partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\ + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi e_3 + v_s^3 \partial_{x_3} w + (w' \cdot \nabla' v_s^3) e_3 = f(\phi, w), \end{aligned} \quad (2.6)$$

$$w|_{\partial\Omega} = 0, \quad (2.7)$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0). \quad (2.8)$$

Here $e^3 = {}^T(0, 0, 1) \in \mathbf{R}$, $\nabla' = {}^T(\partial_{x_1}, \partial_{x_2})$ and $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$;

$$f^0(\phi, w) = -\operatorname{div}(\phi w),$$

$$\begin{aligned} f(\phi, w) = -w \cdot \nabla w + \frac{\nu \phi}{(\phi + \gamma^2 \rho_s) \rho_s} \left(-\Delta w + \frac{1}{\gamma^2 \rho_s} \Delta' v_s \phi \right) - \frac{\tilde{\nu} \phi}{(\phi + \gamma^2 \rho_s) \rho_s} \nabla \operatorname{div} w \\ + \frac{\phi}{\gamma^2 \rho_s} \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) - \frac{1}{2\gamma^4 \rho_s} \nabla \left(P''(\rho_s) \phi^2 \right) + \tilde{P}_3(\rho_s, \phi, \partial_{x'} \phi), \end{aligned}$$

and

$$\begin{aligned} \tilde{P}_3(\rho_s, \phi, \partial_{x'} \phi) = \frac{\phi^3}{\gamma^4 (\phi + \gamma^2 \rho_s) \rho_s^3} \nabla P(\rho_s) - \frac{1}{2\gamma^6 \rho_s} \nabla \left(\phi^3 P_3(\rho_s, \phi) \right) \\ + \frac{\phi}{2\gamma^6 \rho_s^2} \nabla \left(P''(\rho_s) \phi^2 + \frac{1}{\gamma^2} \phi^3 P_3(\rho_s, \phi) \right) \\ - \frac{\phi^2}{\gamma^2 (\phi + \gamma^2 \rho_s) \rho_s^2} \nabla \left(\frac{1}{\gamma^2} P'(\rho_s) \phi + \frac{1}{2\gamma^4} P''(\rho_s) \phi^2 + \frac{1}{2\gamma^6} \phi^3 P_3(\rho_s, \phi) \right), \end{aligned}$$

with

$$P_3(\rho_s, \phi) = \int_0^1 (1 - \theta)^2 P'''(\rho_s + \theta \gamma^{-2} \phi) d\theta.$$

Our main concern in this paper is decay estimates of solutions to the linearized problem, i.e., problem (2.5)-(2.8) with $f^0(\phi, w) = 0$ and $f(\phi, w) = 0$.

3 MAIN RESULTS

Let us consider the linearized problem

$$\partial_t u + Lu = 0, \quad u = {}^T(\phi, w), \quad w|_{\partial D} = 0, \quad u|_{t=0} = u_0. \quad (3.1)$$

Here L is the operator on $L^2(\Omega)$ defined by

$$\begin{aligned} L = \begin{pmatrix} v_s \cdot \nabla & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta I_3 - \frac{\nu + \nu'}{\rho_s} \nabla \operatorname{div} + v_s \cdot \nabla \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s & e_3 \otimes (\nabla v_s^3) \end{pmatrix} \\ := L_1 + L_2 \end{aligned}$$

with domain

$$D(L) = \{u = {}^T(\phi, w) \in L^2(\Omega); w \in H_0^1(\Omega), Lu \in L^2(\Omega)\}.$$

Here, for $\mathbf{a} = {}^T(a_1, a_2, a_3)$ and $\mathbf{b} = {}^T(b_1, b_2, b_3)$, we denote the 3×3 matrix $(a_i b_j)$ by $\mathbf{a} \otimes \mathbf{b}$.

In a similar manner to that in [3], one can show that $-L_1$ generates a C_0 -semigroup on $L^2(\Omega)$. Since $\|L_2 u\|_2 \leq C\|u\|_2$, it follows from the standard perturbation theory that $-L$ generates a C_0 -semigroup e^{-tL} on $L^2(\Omega)$. It is not difficult to prove that if $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$, then

$$\begin{aligned} u(t) &= e^{-tL} u_0 \in C([0, T]; H^1(\Omega) \times H_0^1(\Omega)), \\ Q_0 e^{-tL} u_0 &\in H^1(0, T; L^2(\Omega)), \\ \tilde{Q} e^{-tL} u_0 &\in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \end{aligned} \quad (3.2)$$

for all $T > 0$.

The main results of this paper is stated as follows.

Theorem 3.1. *Suppose that $u_0 = {}^T(\phi_0, w_0) \in (H^1(\Omega) \times H_0^1(\Omega)) \cap L^1(\Omega)$. There exist positive constants ν_0 , γ_0 and ω_0 such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{2\nu + \nu'} \geq \gamma_0$ and $\omega = \|\rho_s - 1\|_{C^3} \leq \omega_0$, then there holds the estimate*

$$\|\partial_x^k \partial_{x_3}^l e^{-tL} u_0\|_{L^2(\Omega)} \leq C \left\{ (1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} + e^{-dt} \|u_0\|_{H^1(\Omega)} \right\}$$

for $t \geq 0$ and $0 \leq k+l \leq 1$ with positive constants C and d .

To prove Theorem 3.1, we consider the Fourier transform of (3.1) in x_3 variable which is written as

$$\partial_t \hat{\phi} + i\xi v_s^3 \hat{\phi} + \gamma^2 \nabla' \cdot (\rho_s \hat{w}') + \gamma^2 i\xi \rho_s \hat{w}^3 = 0, \quad (3.3)$$

$$\partial_t \hat{w}' - \frac{\nu}{\rho_s} (\Delta' - \xi^2) \hat{w}' - \frac{\tilde{\nu}}{\rho_s} \nabla' (\nabla' \cdot \hat{w}' + i\xi \hat{w}^3) + \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \hat{\phi} \right) + i\xi v_s^3 \hat{w}' = 0, \quad (3.4)$$

$$\begin{aligned} \partial_t \hat{w}^3 - \frac{\nu}{\rho_s} (\Delta' - \xi^2) \hat{w}^3 - \frac{\tilde{\nu}}{\rho_s} i\xi (\nabla' \cdot \hat{w}' + i\xi \hat{w}^3) + i\xi \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \hat{\phi} \right) + i\xi v_s^3 \hat{w}^3 \\ + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \hat{\phi} + \hat{w}' \cdot \nabla' v_s^3 = 0, \end{aligned} \quad (3.5)$$

$$\hat{w} \big|_{\partial D} = 0 \quad (3.6)$$

for $t > 0$, and

$${}^T(\hat{\phi}, \hat{w}) \big|_{t=0} = {}^T(\hat{\phi}_0, \hat{w}_0) = \hat{u}_0. \quad (3.7)$$

We thus arrive at the following problem

$$\frac{d}{dt} \hat{u} + \hat{L}_\xi \hat{u} = 0, \quad \hat{u} \big|_{t=0} = \hat{u}_0 \quad (3.8)$$

with a parameter $\xi \in \mathbf{R}$. Here $\hat{u} = {}^T(\hat{\phi}(x', t), \hat{w}(x', t)) \in D(\hat{L}_\xi)$ ($x' \in D, t > 0$), $\hat{u}_0 \in H^1(D) \times H_0^1(D)$, and $\hat{L}_\xi$ is the operator on $L^2(D)$ of the form

$$\hat{L}_\xi = \hat{A}_\xi + \hat{B}_\xi + \hat{C}_0,$$

where

$$\widehat{A}_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s}(\Delta' - |\xi|^2)I_2 - \frac{\tilde{\nu}}{\rho_s}\nabla'\nabla' & -i\frac{\tilde{\nu}}{\rho_s}\xi\nabla' \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}\xi\nabla' & -\frac{\nu}{\rho_s}(\Delta' - |\xi|^2) + \frac{\tilde{\nu}}{\rho_s}|\xi|^2 \end{pmatrix},$$

$$\widehat{B}_\xi = \begin{pmatrix} i\xi v_s^3 & \gamma^2\nabla'(\rho_s\cdot) & i\gamma^2\rho_s\xi \\ \nabla'(\frac{P(\rho_s)}{\gamma^2\rho_s}\cdot) & i\xi v_s^3 I_2 & 0 \\ i\xi\frac{P(\rho_s)}{\gamma^2\rho_s} & 0 & i\xi v_s^3 \end{pmatrix}, \quad \widehat{C}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\nu}{\gamma^2\rho_s^2}\Delta'v_s^3 & T(\nabla'v_s^3) & 0 \end{pmatrix}$$

with domain of definition

$$D(\widehat{L}_\xi) = \{\widehat{u} = {}^T(\widehat{\phi}, \widehat{w}) \in L^2(D); \widehat{w} \in H_0^1(D), \widehat{L}_\xi\widehat{u} \in L^2(D)\}.$$

Note that $D(\widehat{L}_\xi) = D(\widehat{L}_0)$ for all $\xi \in \mathbf{R}$, where $\widehat{L}_0 = \widehat{L}_\xi|_{\xi=0}$.

As in the case of L , following Iooss-Padula [3], one can show that $-\widehat{L}_\xi$ generate a C_0 -semigroup on $L^2(D)$. Furthermore if $\widehat{u}_0 \in H^1(D) \times H_0^1(D)$, then

$$\begin{aligned} e^{-t\widehat{L}_\xi}\widehat{u}_0 &\in C([0, T]; H^1(D) \times H_0^1(D)), \\ Q_0 e^{-t\widehat{L}_\xi}\widehat{u}_0 &\in H^1(0, T; H^1(D)), \\ \widetilde{Q} e^{-t\widehat{L}_\xi}\widehat{u}_0 &\in L^2(0, T; H^2(D)) \cap H^1(0, T; L^2(D)) \end{aligned} \tag{3.9}$$

for any $T > 0$.

To prove Theorem 3.1 we decompose $e^{-tL}u_0$ in the following way. We define $\chi^{(R)}(\xi)$ and $\chi^{(\infty)}(\xi)$ by $\chi^{(R)}(\xi) = 1$ if $|\xi| \leq R$, $\chi^{(R)}(\xi) = 0$ if $|\xi| > R$, and $\chi^{(\infty)}(\xi) = 1 - \chi^{(1)}(\xi)$.

We decompose $e^{-tL}u_0$ as

$$e^{-tL}u_0 = U_1(t)u_0 + U_\infty(t)u_0,$$

where

$$U_j(t)u_0 = \mathcal{F}^{-1}[\chi^{(j)}e^{-t\widehat{L}_\xi}\widehat{u}_0], \quad j = 1, \infty.$$

We can then obtain the following decay estimates for $U_1(t)u_0$ and $U_\infty(t)u_0$.

Theorem 3.2. *There exist positive constants ν_0 , γ_0 , ω_0 and d such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2$ and $\omega = \|\rho_s - 1\|_{C^3} < \omega_0$, then for any $l = 0, 1, \dots$, there exists a positive constant $C = C(l)$ such that the estimates*

$$\|\partial_{x_3}^l U_1(t)u_0\|_{L^2} \leq C(1+t)^{-1/4-l/2}\|u_0\|_{L^1(\mathbf{R}; L^2(D))},$$

$$\|\partial_{x'}\partial_{x_3}^l U_1(t)u_0\|_{L^2} \leq C\{(1+t)^{-1/4-l/2}\|u_0\|_{L^1(\mathbf{R}; L^2(D))} + e^{-dt}(\|u_0\|_{L^2} + \|\partial_{x'}u_0\|_{L^2})\}$$

hold for $t \geq 0$.

Theorem 3.3. *There exist positive constants ν_0 , γ_0 , ω_0 and d such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_0^2$ and $\omega = \|\rho_s - 1\|_{C^3} < \omega_0$, then the estimate*

$$\|U_\infty(t)u_0\|_{H^1} \leq Ce^{-dt}\|u_0\|_{H^1}$$

holds for $t \geq 0$ with a positive constant C .

Theorem 3.1 follows from Theorem 3.2 and Theorem 3.3. In Section 4 we will prove Theorem 3.2 and in Section 5 we will give an outline of the proof of Theorem 3.3.

4 DECAY ESTIMATE OF THE LOW FREQUENCY PART

In this section we give a proof of Theorem 3.2. Theorem 3.2 is a consequence of Proposition 4.12 and Proposition 4.20 below.

For simplicity we omit $\hat{\cdot}$ of $\hat{u}$, $\hat{\phi}$ and $\hat{w}$ in (3.3)-(3.8). In what follows we set

$$\omega = \|\rho_s - 1\|_{C^3}.$$

To prove Theorem 3.2 we decompose $u(t)$ based on a spectral property of $\hat{L}_\xi$ with $\xi = 0$, namely, $\hat{L}_0$.

We introduce the adjoint operator $\hat{L}_\xi^*$ of $\hat{L}_\xi$ with the weighted inner product $\langle \cdot, \cdot \rangle$. We define $\hat{L}_\xi^*$ by

$$\hat{L}_\xi^* = \hat{A}_\xi^* + \hat{B}_\xi^* + \hat{C}_0^*$$

with domain of definition

$$D(\hat{L}_\xi^*) = \{u = {}^T(\phi, w) \in L^2(D); w \in H_0^1(D), \hat{L}_\xi^*u \in L^2(D)\},$$

where

$$\hat{A}_\xi^* = \hat{A}_\xi, \quad \hat{B}_\xi^* = -\hat{B}_\xi$$

and

$$\hat{C}_0^* = \begin{pmatrix} 0 & 0 & \frac{\gamma^2 \nu \Delta' v_s^3}{P'(\rho_s)} \\ 0 & 0 & \nabla' v_s^3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $D(\hat{L}_\xi) = D(\hat{L}_\xi^*)$ for any $\xi \in \mathbf{R}$. It follows that

$$\langle \hat{A}_\xi u, v \rangle = \langle u, \hat{A}_\xi^* v \rangle = \langle u, \hat{A}_\xi v \rangle,$$

$$\langle \hat{B}_\xi u, v \rangle = \langle u, \hat{B}_\xi^* v \rangle = -\langle u, \hat{B}_\xi v \rangle,$$

$$\langle \hat{C}_0 u, v \rangle = \langle u, \hat{C}_0^* v \rangle$$

and

$$\langle \hat{L}_\xi u, v \rangle = \langle u, \hat{L}_\xi^* v \rangle$$

for $u, v \in D(\widehat{L}_\xi)$.

We begin with a lemma on the zero-eigenvalue of $\widehat{L}_0$ and $\widehat{L}_0^*$.

Lemma 4.1. (i) *There exists a constant $\omega_0 > 0$ such that if $\omega \leq \omega_0$, then $\lambda = 0$ is a simple eigenvalue of $\widehat{L}_0$ and $\widehat{L}_0^*$.*

(ii) *The eigenspaces for $\lambda = 0$ of $\widehat{L}_0$ and $\widehat{L}_0^*$ are spanned by $u^{(0)}$ and $u^{(0)*}$, respectively, where*

$$u^{(0)} = T(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = T(0, 0, w^{(0),3})$$

and

$$u^{(0)*} = T(\phi^{(0)*}, 0).$$

Here

$$\phi^{(0)}(x') = \alpha_0 \frac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))}, \quad \alpha_0 = \left(\int_D \frac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))} dx' \right)^{-1};$$

and $w^{(0),3}$ is the solution of the following problem

$$\begin{cases} -\Delta' w^{(0),3} = -\frac{1}{\gamma^2} \Delta' v_s^3 \phi^{(0)}, \\ w^{(0),3} |_{\partial D} = 0; \end{cases}$$

and

$$\phi^{(0)*} = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x').$$

(iii) *The eigenprojections $\widehat{\Pi}^{(0)}$ and $\widehat{\Pi}^{(0)*}$ for $\lambda = 0$ of $\widehat{L}_0$ and $\widehat{L}_0^*$ are given by*

$$\begin{aligned} \widehat{\Pi}^{(0)} u &= \langle u, u^{(0)*} \rangle u^{(0)} = \langle Q_0 u \rangle u^{(0)}, \\ \widehat{\Pi}^{(0)*} u &= \langle u, u^{(0)} \rangle u^{(0)*} \end{aligned}$$

for $u = T(\phi, w)$, respectively.

(iv) *Let $u^{(0)}$ be written as $u^{(0)} = u_0^{(0)} + u_1^{(0)}$, where*

$$u_0^{(0)} = T(\phi^{(0)}, 0), \quad u_1^{(0)} = T(0, w^{(0)}).$$

Then

$$u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$$

and

$$\langle u, u^{(0)} \rangle = \frac{\alpha_0}{\gamma^2} \langle \phi \rangle + (w^3, w^{(0),3} \rho_s)$$

for $u = T(\phi, w^1, w^2, w^3)$.

Remark 4.2. $\phi^{(0)} = O(1)$, $\alpha_0 = O(1)$ and $w^{(0),3} = O(\frac{1}{\gamma^2})$ as $\gamma^2 \rightarrow \infty$.

Proof. Let $\widehat{L}_0 u = 0$ for $u = {}^T(\phi, w', w^3) \in D(\widehat{L}_0)$. Then

$$\begin{cases} \gamma^2 \nabla' \cdot (\rho_s w') = 0, \\ -\frac{\nu}{\rho_s} \Delta' w' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w' + \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) = 0, \\ -\frac{\nu}{\rho_s} \Delta' w^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi + w' \cdot \nabla' v_s^3 = 0, \\ w|_{\partial D} = 0. \end{cases} \quad (4.1)$$

We take the weighted inner product of (4.1) with ${}^T(\phi, w', 0)$ to get

$$\nu |\nabla' w'|_2^2 + \tilde{\nu} |\nabla' \cdot w'|_2^2 = 0.$$

Since $w' \in H_0^1(D)$, we have $w' = 0$. It then follows that

$$\begin{cases} \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) = 0, \\ -\Delta' w^3 = -\frac{1}{\gamma^2 \rho_s} \Delta' v_s^3 \phi, \\ w^3|_{\partial D} = 0. \end{cases}$$

This implies that $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi$ is a constant since D is connected, and we conclude that $\text{Ker}(\widehat{L}_0) = \text{span}\{u^{(0)}\}$. We set this constant $\alpha_0 = \left(\int_D \frac{\gamma^2 \rho_s}{P'(\rho_s)} dx' \right)^{-1}$. Note that $\int_D \phi^{(0)} dx' = 1$.

Let $\widehat{L}_0^* u = 0$ for $u = {}^T(\phi, w', w^3)$. Then

$$\begin{cases} -\gamma^2 \nabla' \cdot (\rho_s w') + \frac{\gamma^2 \nu}{P'(\rho_s)} \Delta' v_s^3 w^3 = 0, \\ -\frac{\nu}{\rho_s} \Delta' w' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w' - \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) + w^3 \nabla' v_s^3 = 0, \\ -\frac{\nu}{\rho_s} \Delta' w^3 = 0, \\ w|_{\partial D} = 0. \end{cases}$$

The third equation, together with $w^3|_{\partial D} = 0$, implies that $w^3 = 0$, and hence,

$$\begin{cases} -\gamma^2 \nabla' \cdot (\rho_s w') = 0, \\ -\frac{\nu}{\rho_s} \Delta' w' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w' - \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) = 0, \\ w'|_{\partial D} = 0. \end{cases}$$

Similaly to the case of $\widehat{L}_0$, one can show that $w' = 0$ and $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi$ is a constant. We set $\phi^{(0)*} = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x')$. Note that $\int_D \phi^{(0)} \overline{\phi^{(0)*}} \frac{P'(\rho_s)}{\gamma^4 \rho_s} dx' = 1$. We thus proved (i), (ii) and (iii) except the simplicity of $\lambda = 0$. The assertion (iv) can be verified by direct computations. It remains to prove the simplicity of $\lambda = 0$. Since we have already proved that $\text{Ker}(\widehat{L}_0) = \text{span}\{u^{(0)}\}$ and $\text{Ker}(\widehat{L}_0^*) = \text{span}\{u^{(0)*}\}$, if we would prove the following lemma, then the proof of the simplicity of $\lambda = 0$ would be complete.

Lemma 4.3. *There exists a constant $\omega_0 > 0$ such that if $\omega \leq \omega_0$, then there hold that*

$$L^2(D) = \text{Ker}(\widehat{L}_0) \oplus R(\widehat{L}_0), \quad L^2(D) = \text{Ker}(\widehat{L}_0^*) \oplus R(\widehat{L}_0^*).$$

To prove Lemma 4.3, we show the following proposition.

Proposition 4.4. *There exists a constant $\omega_0 > 0$ such that if $\omega \leq \omega_0$, then for any $f = {}^T(f^0, g) = {}^T(f^0, g', g^3) \in L^2(D)$ with $\langle f^0 \rangle = 0$, there is a unique solution $u = {}^T(\phi, w) \in D(\widehat{L}_0)$ of the following problem:*

$$\begin{cases} \widehat{L}_0 u = f, \\ \langle \phi \rangle = 0. \end{cases} \quad (4.2)$$

Proof. Let us prove that if $\langle f^0 \rangle = 0$, then (4.2) has a unique solution $u = {}^T(\phi, w) \in L^2(D) \times H_0^1(D)$ with $\langle \phi \rangle = 0$. The problem (4.2) is rewritten as the following system:

$$\begin{cases} \nabla' \cdot w' = F[w'; f^0], \\ -\nu \Delta' w' + \nabla' \phi = G'[\phi, w'; f^0, g'], \\ w' |_{\partial D} = 0, \\ \langle \phi \rangle = 0 \end{cases} \quad (4.3)$$

and

$$\begin{cases} -\nu \Delta' w^3 = G^3[\phi, w'; g^3], \\ w^3 |_{\partial D} = 0, \end{cases} \quad (4.4)$$

where

$$F[w'; f^0] = \nabla' \cdot ((1 - \rho_s)w') + \frac{1}{\gamma^2} f^0,$$

$$\begin{aligned} G'[\phi, w'; f^0, g'] &= \widetilde{\nu} \nabla' F[w'; f^0] + \nabla'((1 - \rho_s)\phi) + \nabla' \rho_s \phi \\ &\quad + \rho_s \nabla' \left(\left(1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) \phi \right) + \rho_s g', \end{aligned}$$

$$G^3[\phi, w'; g^3] = -\frac{\nu}{\gamma^2 \rho_s} \Delta' v_s^3 \phi - \rho_s w' \cdot \nabla' v_s^3 + \rho_s g^3.$$

We define a set X by

$$X = \{(p, v'); p \in L^2(D), v' = {}^T(v^1, v^2) \in H_0^1(D), \langle p \rangle = 0\}.$$

We assume that $(\widetilde{\phi}, \widetilde{w}') \in X$. Let us consider the following problem

$$\begin{cases} \nabla' \cdot w' = F[\widetilde{w}'; f^0], \\ -\nu \Delta' w' + \nabla' \phi = G'[\widetilde{\phi}, \widetilde{w}'; f^0, g'], \\ w' |_{\partial D} = 0. \end{cases} \quad (4.5)$$

It holds that

$$\begin{aligned} F[\widetilde{w}'; f^0] &\in L^2(D), \quad \langle F[\widetilde{w}'; f^0] \rangle = 0, \\ G'[\widetilde{\phi}, \widetilde{w}'; f^0, g'] &\in H^{-1}(D). \end{aligned}$$

In fact, we see from the Poincaré inequality that

$$\begin{aligned} |F[\tilde{w}'; f^0]|_2 &\leq |\nabla' \cdot ((1 - \rho_s)\tilde{w}')|_2 + \frac{1}{\gamma^2}|f^0|_2 \\ &\leq \omega|\tilde{w}'|_{H^1} + \frac{1}{\gamma^2}|f^0|_2 \\ &\leq \omega|\nabla'\tilde{w}'|_2 + \frac{1}{\gamma^2}|f^0|_2, \end{aligned}$$

$$\begin{aligned} |G'[\tilde{\phi}, \tilde{w}'; f^0, g']|_{H^{-1}} &\leq C\{|\nabla' F[\tilde{w}'; f^0]|_{H^{-1}} + |\nabla'((1 - \rho_s)\tilde{\phi})|_{H^{-1}} \\ &\quad + |\nabla' \rho_s \tilde{\phi}|_{H^{-1}} + |\rho_s \nabla'((1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s})\tilde{\phi})|_{H^{-1}} + |g'|_{H^{-1}}\} \\ &\leq C\{|F[\tilde{w}'; f^0]|_2 + \omega|\tilde{\phi}|_2 + |g'|_2\} \\ &\leq C\{\omega(|\tilde{\phi}|_2 + |\nabla'\tilde{w}'|_2) + |g'|_2\}. \end{aligned}$$

From [12, III.1.4, Theorem 1.4.1], we see that there is a unique solution $(\phi, w') \in X$ of (4.5) and there holds the following estimate

$$\begin{aligned} |\phi|_2 + |\nabla' w'|_2 &\leq C\{|F[\tilde{w}'; f^0]|_2 + |G'[\tilde{\phi}, \tilde{w}'; f^0, g']|_{H^{-1}}\} \\ &\leq C\omega\{|\tilde{\phi}|_2 + |\nabla'\tilde{w}'|_2\} + C\{\frac{1}{\gamma^2}|f^0|_2 + |g'|_2\}. \end{aligned} \tag{4.6}$$

Let us define a map $\Gamma : X \rightarrow X$ by

$$\Gamma(\tilde{\phi}, \tilde{w}') = (\phi, w') \quad \text{for } (\tilde{\phi}, \tilde{w}') \in X,$$

where $(\phi, w') \in X$ is a solution of (4.5). We see from (4.6) that

$$|\Gamma(\tilde{\phi}, \tilde{w}')|_X \leq C\omega|(\tilde{\phi}, \tilde{w}')|_X + C\{\frac{1}{\gamma^2}|f^0|_2 + |g'|_2\}.$$

It then follows that

$$\left| \Gamma(\tilde{\phi}_1, \tilde{w}'_1) - \Gamma(\tilde{\phi}_2, \tilde{w}'_2) \right|_X \leq C\omega|(\tilde{\phi}_1, \tilde{w}'_1) - (\tilde{\phi}_2, \tilde{w}'_2)|_X$$

for $(\tilde{\phi}_1, \tilde{w}'_1), (\tilde{\phi}_2, \tilde{w}'_2) \in X$. If we take ω sufficiently small satisfying $C\omega < 1$, then $\Gamma : X \rightarrow X$ is a contraction map. This implies that there is a unique $(\phi, w') \in X$ such that $\Gamma(\phi, w') = (\phi, w')$, i.e., there is a unique solution $(\phi, w') \in X$ of (4.3).

Futhermore, for a solution $(\phi, w') \in X$ of (4.3), since

$$G^3[\phi, w'; g^3] \in L^2(D),$$

there is a unique solution $w^3 \in H_0^1(D)$ of (4.4). Consequently, we have

$$\widehat{L}_0 u = f \quad \text{in the sense of distribution,}$$

where $f = {}^T(f^0, g', g^3) \in L^2(D)$ with $\langle f^0 \rangle = 0$. Since $f \in L^2(D)$, it holds that $\widehat{L}^0 u \in L^2(D)$. It then follows that

$$u \in D(\widehat{L}_0).$$

This completes the proof. $\square$

Proof of Lemma 4.3 We have already proved that

$$\text{Ker}(\widehat{L}^{(0)}) = \widehat{\Pi}^{(0)}L^2(D).$$

To prove $R(\widehat{L}_0) = (I - \widehat{\Pi}^{(0)})L^2(D)$, we first show that

$$u = {}^T(\phi, w) \in (I - \widehat{\Pi}^{(0)})L^2(D) \quad \text{if and only if} \quad \langle \phi \rangle = 0. \quad (4.7)$$

Let us prove (4.7). We can decompose $u = {}^T(\phi, w)$ as

$$u = \langle \phi \rangle u^{(0)} + u_1.$$

Here

$$\langle \phi \rangle u^{(0)} \in \Pi^{(0)}L^2(D), \quad u_1 = {}^T(\phi_1, w_1) \in (I - \Pi^{(0)})L^2(D).$$

This implies that if $\langle \phi \rangle = 0$, then

$$u = \langle \phi \rangle u^{(0)} + u_1 = u_1 \in (I - \widehat{\Pi}^{(0)})L^2(D).$$

On the other hand, if $u = {}^T(\phi, w) \in (I - \widehat{\Pi}^{(0)})L^2(D)$, then there exists $\tilde{u} = {}^T(\tilde{\phi}, \tilde{w}) \in L^2(D)$ such that

$$u = \tilde{u} - \langle \tilde{\phi} \rangle u^{(0)}.$$

It then follows that

$$\langle \phi \rangle = \langle \tilde{\phi} \rangle - \langle \tilde{\phi} \rangle = 0.$$

We thus conclude that (4.7) holds true.

We next show that $R(\widehat{L}_0) = (I - \widehat{\Pi}^{(0)})L^2(D)$. Since $\langle Q_0 \widehat{L}_0 u \rangle = \langle \nabla' \cdot (\rho_s w') \rangle = 0$, we see from (4.7) that $\widehat{L}_0 u \in (I - \widehat{\Pi}^{(0)})L^2(D)$, and, therefore,

$$R(\widehat{L}_0) \subset (I - \Pi^{(0)})L^2(D).$$

On the other hand, if $f = {}^T(f^0, g', g^3) \in (I - \widehat{\Pi}^{(0)})L^2(D)$, then it follows from (4.7) that $\langle f^0 \rangle = 0$. By Proposition 4.4, there exists a unique solution $u = {}^T(\phi, w) \in D(\widehat{L}_0)$ such that $\widehat{L}_0 u = f$ with $\langle \phi \rangle = 0$. This implies that $f \in R(\widehat{L}_0)$, and, thus,

$$(I - \Pi^{(0)})L^2(D) \subset R(\widehat{L}_0).$$

Therefore we see that $R(\widehat{L}_0) = (I - \widehat{\Pi}^{(0)})L^2(D)$. Consequently, we have

$$L^2(D) = \text{Ker}(\widehat{L}_0) \oplus R(\widehat{L}_0).$$

Similarly, one can prove that

$$L^2(D) = \text{Ker}(\widehat{L}_0^*) \oplus R(\widehat{L}_0^*).$$

This completes the proof of Lemma 4.1. □

We are now ready to prove Theorem 3.2. We decompose $u(t)$ as follows

$$\begin{aligned} u(t) &= \sigma(t)u^{(0)} + u_1(t), \\ \sigma(t) &= \langle Q_0 u(t) \rangle = \langle u(t), u^{(0)*} \rangle, \\ u_1(t) &= (I - \widehat{\Pi}^{(0)})u(t). \end{aligned}$$

The density component of u_1 is denoted by ϕ_1 and the velocity component is denoted by w_1 , namely,

$$u_1 = {}^T(\phi_1, w_1).$$

Note that $\langle \phi_1 \rangle = 0$ and $w_1|_{\partial D} = 0$; the latter follows from $u^{(0)} \in D(\widehat{L}_0)$ which implies that $w^{(0),3}|_{\partial D} = 0$.

Remark 4.5. (i) The boundary condition $w_1|_{\partial D} = 0$ implies that the Poincaré inequality holds for w_1 : $|w_1|_2 \leq C|\partial_{x'} w_1|_2$.

(ii) The vanishing mean value condition $\langle \phi_1 \rangle = 0$ implies that the Poincaré inequality holds for ϕ_1 : $|\phi_1|_2 \leq C|\partial_{x'} \phi_1|_2$.

We define $\widetilde{M}_\xi$ by

$$\widetilde{M}_\xi = \widehat{L}_\xi - \widehat{L}_0 = \widetilde{A}_\xi + \widetilde{B}_\xi,$$

where

$$\widetilde{A}_\xi = \widehat{A}_\xi - \widehat{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s} \xi^2 I_2 & -\frac{\widetilde{\nu}}{\rho_s} i \xi \nabla' \\ 0 & -\frac{\widetilde{\nu}}{\rho_s} i \xi \nabla' & \frac{\nu + \widetilde{\nu}}{\rho_s} \xi^2 \end{pmatrix},$$

$$\widetilde{B}_\xi = \widehat{B}_\xi - \widehat{B}_0 = \begin{pmatrix} i \xi v_s^3 & 0 & \gamma^2 i \xi \rho_s \\ 0 & i \xi v_s^3 I_2 & 0 \\ i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} & 0 & i \xi v_s^3 \end{pmatrix}.$$

Decomposing $u(t)$ in (3.8) as $u(t) = \sigma(t)u^{(0)} + u_1(t)$, we obtain

$$\frac{d}{dt}(\sigma u^{(0)} + u_1) + \widehat{L}_0 u_1 + \widetilde{M}_\xi(\sigma u^{(0)} + u_1) = 0.$$

Applying $\Pi^{(0)}$ and $I - \Pi^{(0)}$ to this equation, we have

$$\begin{cases} \frac{d}{dt}\sigma + \langle Q_0 \widetilde{M}_\xi(\sigma u^{(0)} + u_1) \rangle = 0, \\ \frac{d}{dt}u_1 + \widehat{L}_0 u_1 + (I - \widehat{\Pi}^{(0)})\widetilde{M}_\xi(\sigma u^{(0)} + u_1) = 0. \end{cases}$$

Since $\widehat{\Pi}^{(0)}\widetilde{M}_\xi u = \langle Q_0 \widetilde{M}_\xi u \rangle u^{(0)}$ and $Q_0 \widetilde{M}_\xi = Q_0 \widetilde{B}_\xi$, we get

$$\frac{d}{dt}\sigma + \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle = 0, \tag{4.8}$$

$$\frac{d}{dt}u_1 + \widehat{L}_\xi u_1 + \widetilde{M}_\xi(\sigma u^{(0)}) - \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)} = 0, \tag{4.9}$$

$$w_1|_{\partial D} = 0, \quad \sigma(0) = \sigma_0, \quad u_1(0) = u_{1,0}, \tag{4.10}$$

where σ_0 and $u_{1,0}$ are given by

$$\sigma_0 = \langle u_0, u^{(0)*} \rangle, \quad u_{1,0} = (I - \widehat{\Pi}^{(0)})u_0.$$

We see from (3.9) that if $u_0 \in H^1(D) \times H_0^1(D)$, then

$$\begin{aligned} \sigma &\in H^1(0, T), \\ u_1 &\in C([0, T]; H^1(D) \times H_0^1(D)), \\ \phi_1 &\in H^1(0, T; H^1(D)), \\ w_1 &\in L^2(0, T; H^2(D)) \cap H^1(0, T; L^2(D)) \end{aligned}$$

for all $T > 0$.

Lemma 4.6. *For $u_1 = {}^T(\phi_1, w'_1, w_1^3) \in R(I - \widehat{\Pi}^{(0)})$, there hold the estimates:*

- (i) $|\langle Q_0 \widetilde{B}_\xi u^{(0)} \rangle| \leq C|\xi|.$
- (ii) $|\langle Q_0 \widetilde{B}_\xi u_1 \rangle| \leq C|\xi|(|\phi_1|_2 + \gamma^2|w_1^3|_2).$
- (iii) $|\langle Q_0 \widetilde{B}_\xi u_1 \rangle| \leq C(|\xi||\phi_1|_2 + \gamma^2|\nabla' \cdot w'_1 + i\xi w_1^3|_2 + \gamma^2\omega|w'_1|_2).$

Lemma 4.6 can be proved by direct computations. We omit the proof.

We will employ an energy method to obtain the decay estimate on solutions of (4.8)-(4.10). We write (4.9) as:

$$\left\{ \begin{aligned} &\frac{d}{dt}\phi_1 + i\xi v_s^3 \phi_1 + \gamma^2 \nabla' \cdot (\rho_s w'_1) + \gamma^2 i\xi \rho_s w_1^3 \\ &\quad + i\xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i\xi \rho_s \sigma w^{(0),3} - \langle Q_0 \widetilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \phi^{(0)} = 0, \\ &\frac{d}{dt}w'_1 - \frac{\nu}{\rho_s}(\Delta' - \xi^2)w'_1 - \frac{\widetilde{\nu}}{\rho_s} \nabla'(\nabla' \cdot w'_1 + i\xi w_1^3) + \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) + i\xi v_s^3 w'_1 \\ &\quad - \frac{\widetilde{\nu}}{\rho_s} i\xi \nabla'(\sigma w^{(0),3}) = 0, \\ &\frac{d}{dt}w_1^3 - \frac{\nu}{\rho_s}(\Delta' - \xi^2)w_1^3 - \frac{\widetilde{\nu}}{\rho_s} i\xi(\nabla' \cdot w'_1 + i\xi w_1^3) + i\xi \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) + i\xi v_s^3 w_1^3 \\ &\quad + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi_1 + w'_1 \cdot \nabla' v_s^3 + \frac{\nu + \widetilde{\nu}}{\rho_s} \xi^2 \sigma w^{(0),3} + i\xi \alpha_0 \sigma \\ &\quad + i\xi v_s^3 \sigma w^{(0),3} - \langle Q_0 \widetilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle w^{(0),3} = 0. \end{aligned} \right.$$

Before proceeding further we introduce some notations. For $u = {}^T(\phi, w)$ we define $E_0[u]$ by

$$E_0[u] = \frac{1}{\gamma^2} \left| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \phi \right|_2^2 + |\sqrt{\rho_s} w|_2^2.$$

For $w = {}^T(w', w^3)$ with $w' = {}^T(w^1, w^2)$ we define $\widetilde{D}_\xi[w]$ by

$$\widetilde{D}_\xi[w] = \nu(|\nabla' w|_2^2 + |\xi|^2|w|_2^2) + \widetilde{\nu}|\nabla' \cdot w' + i\xi w^3|_2^2.$$

For ϕ we define $\dot{\phi}$ by

$$\dot{\phi} = \frac{d}{dt}\phi + i\xi v_s^3 \phi.$$

Proposition 4.7. *There exists a constant $\nu_0 > 0$ such that if $\nu \geq \nu_0$, then there hold the estimates:*

$$\frac{1}{2} \frac{d}{dt} \left(\frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + \frac{1}{2} \tilde{D}_\xi[w_1] \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) |\phi_1|_2^2 \right\}, \quad (4.11)$$

$$\frac{\nu+\tilde{\nu}}{\gamma^4} |\dot{\phi}_1|_2^2 \leq C \left\{ \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \left(1 + \frac{\nu+\tilde{\nu}}{\nu} \omega^2 \right) \tilde{D}_\xi[w_1] \right\}. \quad (4.12)$$

Proof. Multiplying (4.8) by $\bar{\sigma}(t)$ and taking real part of the resulting equation, we have

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 + \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \bar{\sigma} \} = 0. \quad (4.13)$$

Since $\tilde{B}_\xi^* = -\tilde{B}_\xi$ and $u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$, we see that

$$\begin{aligned} \langle Q_0 \tilde{B}_\xi u_1 \rangle \bar{\sigma} &= \langle \tilde{B}_\xi u_1, \sigma u^{(0)*} \rangle \\ &= -\langle u_1, \tilde{B}_\xi(\sigma u^{(0)*}) \rangle \\ &= -\frac{\gamma^2}{\alpha_0} \langle u_1, \tilde{B}_\xi(\sigma u_0^{(0)}) \rangle, \end{aligned} \quad (4.14)$$

where $u_0^{(0)} = T(\phi^{(0)}, 0)$. On the other hand, since

$$\langle Q_0 \tilde{B}_\xi(\sigma u^{(0)}) \rangle \bar{\sigma} = i\xi |\sigma|^2 \{ \langle v_s^3 \phi_1 \rangle + \langle \gamma^2 \rho_s w_1^3 \rangle \},$$

we have

$$\operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)}) \rangle \bar{\sigma} \} = 0. \quad (4.15)$$

It then follows from (4.13)-(4.15) that

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 - \frac{\gamma^2}{\alpha_0} \operatorname{Re} \langle u_1, \tilde{B}_\xi(\sigma u_0^{(0)}) \rangle = 0. \quad (4.16)$$

We next take the weighted inner product of (4.9) with u_1 . The real part of the resulting equation then gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0[u_1] + \operatorname{Re} \langle \hat{L}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \tilde{M}_\xi(\sigma u^{(0)} + u_1), u_1 \rangle \\ - \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \} = 0. \end{aligned} \quad (4.17)$$

Since $\hat{B}_\xi^* = -\hat{B}_\xi$, we see that $\operatorname{Re} \langle \hat{B}_\xi u_1, u_1 \rangle = 0$. It then follows that

$$\begin{aligned} \operatorname{Re} \langle \hat{L}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \tilde{M}_\xi(\sigma u^{(0)} + u_1), u_1 \rangle \\ = \operatorname{Re} \langle \hat{C}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \hat{A}_\xi u_1, u_1 \rangle + \operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle \\ = \operatorname{Re} \langle \hat{C}_0 u_1, u_1 \rangle + \tilde{D}_\xi[w_1] + \operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle. \end{aligned}$$

This, together with (4.17), gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0[u_1] + \tilde{D}_\xi[w_1] + \operatorname{Re} \langle \hat{C}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle \\ + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle - \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \} = 0. \end{aligned} \quad (4.18)$$

We add $\frac{\alpha_0}{\gamma^2} \times (4.16)$ to (4.18), to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + \tilde{D}_\xi[w_1] + \operatorname{Re} \langle \widehat{C}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle \\ & + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u_1^{(0)}), u_1 \rangle - \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \} = 0, \end{aligned} \quad (4.19)$$

where $u_1^{(0)} = T(0, w^{(0)})$. Here we used the relation

$$-\operatorname{Re} \langle u_1, \tilde{B}_\xi(\sigma u_0^{(0)}) \rangle + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle = \operatorname{Re} \langle \tilde{B}_\xi(\sigma u_1^{(0)}), u_1 \rangle.$$

By the Poincaré inequality we have

$$\begin{aligned} |\operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle| & \leq C \left(\frac{\nu}{\gamma^2} |\xi|^2 |\sigma| |w_1^3|_2 + \frac{\tilde{\nu}}{\gamma^2} |\xi| |\sigma| |\nabla' \cdot w_1' + i\xi w_1^3|_2 \right) \\ & \leq \frac{1}{8} \tilde{D}_\xi[w_1] + C \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2, \end{aligned}$$

$$\begin{aligned} |\operatorname{Re} \langle \tilde{B}_\xi(\sigma u_1^{(0)}), u_1 \rangle| & \leq C |\xi| |\sigma| \left(\frac{1}{\gamma^2} |\phi_1|_2 + \frac{1}{\gamma^2} |w_1^3|_2 \right) \\ & \leq C \left(\frac{1}{\gamma^2} + \frac{1}{\nu \gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^2} |\phi_1|_2^2 + \frac{1}{8} \tilde{D}_\xi[w_1]. \end{aligned}$$

Since $\langle \phi_1 \rangle = 0$, there holds that

$$|\operatorname{Re} \langle u^{(0)}, u_1 \rangle| \leq C \frac{1}{\gamma^2} |w_1^3|_2.$$

Applying Lemma 4.6 and the Poincaré and Hölder inequalities, we thus have the following estimates:

$$\begin{aligned} & |\operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \}| \\ & \leq C |\xi| (|\sigma| + |\phi_1|_2 + \gamma^2 |w_1^3|_2) \frac{1}{\gamma^2} |w_1^3|_2 \\ & \leq \frac{1}{8} \tilde{D}_\xi[w_1] + C \left(\frac{1}{\nu \gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\nu \gamma^4} |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] \right), \\ & |\operatorname{Re} \{ \langle \widehat{C}_0 u_1, u_1 \rangle \}| \leq C \left(\frac{\nu}{\gamma^4} |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] \right). \end{aligned}$$

Therefore we find that there exists a constant $\nu_0 > 0$ such that if $\nu \geq \nu_0$, then

$$\frac{1}{2} \frac{d}{dt} \left(\frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + \frac{1}{2} \tilde{D}_\xi[w_1] \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) |\phi_1|_2^2 \right\}.$$

We next estimate $\dot{\phi}_1$. By the first equation of (4.9) there holds that

$$\begin{aligned} \frac{1}{\gamma^2} \dot{\phi}_1 & = -(\nabla' \cdot (\rho_s w_1') + i\xi \rho_s w_1^3) \\ & \quad - \frac{1}{\gamma^2} \{ i\xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i\xi \rho_s \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \}. \end{aligned}$$

We thus obtain

$$\frac{1}{\gamma^4} |\dot{\phi}_1|_2^2 \leq C \left\{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \left(\frac{1}{\nu + \tilde{\nu}} + \frac{\omega^2}{\nu} \right) \tilde{D}_\xi[w_1] \right\}.$$

Multiplying by $\nu + \tilde{\nu}$ to both sides, we have the desired estimate. This completes the proof. $\square$

Let us estimate $|\phi_1|_2$. We first introduce the Bogovskii lemma.

Proposition 4.8. *Let $\dot{L}^2(D)$ be defined by*

$$\dot{L}^2(D) = \{f \in L^2(D) : \langle f \rangle = 0\}.$$

For any $f \in \dot{L}^2(D)$, there exists a bounded operator $\mathcal{B} : \dot{L}^2(D) \rightarrow H_0^1(D)$ such that

$$-\nabla' \cdot \mathcal{B}f = f,$$

$$|\nabla' \mathcal{B}f|_2 \leq C|f|_2.$$

Proof. See, e.g., [2, III.3, Theorem 3.2]. $\square$

The proof of the following proposition is based on the argument in Iooss-Padula [3].

Proposition 4.9. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$ and $\omega_0 > 0$ such that if $\nu \geq \nu_0$, $\gamma \geq \gamma_0$ and $\omega \leq \omega_0$, then there hold the estimates:*

$$\begin{aligned} & \frac{d}{dt} J_0[u_1] + \frac{1}{2} \frac{1}{\nu+\tilde{\nu}} |\phi_1|_2^2 \\ & \leq C \left\{ \left(\frac{\gamma^2}{\nu(\nu+\tilde{\nu})} + 1 \right) \tilde{D}_\xi[w_1] + \frac{\nu}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] + \left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 \right\}, \end{aligned} \quad (4.20)$$

$$|J_0[u_1]| \leq C \left\{ \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} |w_1|_2^2 + \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} |\phi_1|_2^2 \right\},$$

where

$$J_0[u_1] = \frac{1}{\nu+\tilde{\nu}} (w'_1, \rho_s \psi')$$

with $\psi' = \mathcal{B}\phi_1$.

Proof. Set $\psi' = \mathcal{B}\phi_1$. Taking the inner product of (4.9)₂ with $\rho_s \psi'$, we get

$$(\partial_t w'_1, \rho_s \psi') + \left(\nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right), \rho_s \psi' \right) = I, \quad (4.21)$$

where

$$\begin{aligned} I = & -\nu(\nabla' w'_1, \nabla' \psi') - \nu \xi^2 (w'_1, \psi') + \tilde{\nu}(\nabla' \cdot w'_1, \phi_1) + \tilde{\nu} i \xi (w_1^3, \phi_1) \\ & - i \xi (\rho_s v_s^3 w'_1, \psi') + \tilde{\nu} i \xi (\sigma w^{(0),3}, \phi_1). \end{aligned}$$

We first estimate the first term of the left-hand side of (4.21). It holds that

$$(\partial_t w'_1, \rho_s \psi') = \frac{d}{dt} (w'_1, \rho_s \psi') - (w'_1, \rho_s \partial_t \psi').$$

Since

$$-\nabla' \cdot \partial_t \psi' = \partial_t \phi_1$$

and

$$\begin{aligned} \partial_t \phi_1 = & -\{i \xi v_s^3 \phi_1 + \gamma^2 \nabla' \cdot (\rho_s w'_1) + i \xi \gamma^2 \rho_s w_1^3 \\ & + i \xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i \xi \rho_s \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \phi^{(0)}\}, \end{aligned}$$

we obtain

$$\begin{aligned}
|(w'_1, \rho_s \partial_t \psi')| &\leq C |w'_1|_2 |\partial_t \psi'|_2 \\
&\leq C |w'_1|_2 \{ |\xi| |\phi_1|_2 + \gamma^2 |\nabla' \cdot w'_1|_2 + \gamma^2 |w'_1|_2 + \gamma^2 |\xi|^2 |w'_1|^3|_2 + |\xi| |\sigma| \} \\
&\leq \frac{1}{8} |\phi_1|_2^2 + C \{ (\gamma^2 + \frac{\gamma^2}{\nu+\tilde{\nu}}) |w_1|_2^2 + (1 + \gamma^2) |\xi|^2 |w_1|_2^2 + \gamma^2 |\nabla' \cdot w'_1|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2} |\xi|^2 |\sigma|^2 \} \\
&\leq \frac{1}{8} |\phi_1|_2^2 + C \{ (\frac{1}{\nu} + \frac{\gamma^2}{\nu} + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})}) \tilde{D}_\xi[w_1] + \frac{\nu+\tilde{\nu}}{\gamma^2} |\xi|^2 |\sigma|^2 \}.
\end{aligned}$$

Futhermore, we have

$$(w'_1, \rho_s \psi') \leq C \{ \frac{\gamma^2}{\nu} |w_1|_2^2 + \frac{\nu}{\gamma^2} |\phi_1|_2^2 \}.$$

We next estimate the second term of the left-hand side of (4.21). There exists $\omega_0 > 0$ such that if $\omega \leq \omega_0$, then it holds that

$$\begin{aligned}
(\nabla'(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1), \rho_s \psi') &= \left| \sqrt{\frac{P'(\rho_s)}{\gamma^2}} \phi_1 \right|_2^2 - \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1, (\nabla' \rho_s) \cdot \psi' \right) \\
&\geq C(1 - \omega) \left| \sqrt{\frac{P'(\rho_s)}{\gamma^2}} \phi_1 \right|_2^2 \\
&\geq \frac{3}{4} |\phi_1|_2^2.
\end{aligned}$$

As for I , we have

$$|I| \leq \frac{1}{8} |\phi_1|_2^2 + C \{ (\nu + \frac{\tilde{\nu}^2}{\nu+\tilde{\nu}} + \frac{1}{\nu}) \tilde{D}_\xi[w_1] + \nu |\xi|^2 \tilde{D}_\xi[w_1] + \frac{\tilde{\nu}^2}{\gamma^4} |\xi|^2 |\sigma|^2 \}.$$

Therefore it holds that

$$\begin{aligned}
\frac{d}{dt}(w'_1, \rho_s \psi') + \frac{1}{2} |\phi_1|_2^2 &\leq C \{ (\frac{1}{\nu} + \frac{\gamma^2}{\nu} + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} + \nu + \frac{\tilde{\nu}^2}{\nu+\tilde{\nu}}) \tilde{D}_\xi[w_1] \\
&\quad + \nu |\xi|^2 \tilde{D}_\xi[w_1] + (\frac{\nu+\tilde{\nu}}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}) |\xi|^2 |\sigma|^2 \}.
\end{aligned}$$

Multiplying by $\frac{1}{\nu+\tilde{\nu}}$ to both sides of this inequality, we have the desired estimate. This completes the proof. $\square$

We next derive the estimate for σ . We introduce a notation. Let us define $J_1[u]$ by

$$J_1[u] = i \xi \frac{1}{\nu+\tilde{\nu}} \langle \rho_s (1 + |\xi|^2)^{-1} [\rho_s w_1^3] \rangle \bar{\sigma}$$

for $u = \sigma u^{(0)} + u_1$ with $u_1 = {}^T(\phi_1, w_1^1, w_1^2, w_1^3)$.

Proposition 4.10. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$, $\omega_0 > 0$ and $\tilde{\alpha}_0 > 0$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2$ and $\omega \leq \omega_0$, then there hold the estimates:*

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\frac{\nu}{(\nu+\tilde{\nu})\gamma^2} |\sigma|^2 + J_1[u] \right) + \frac{1}{2} \frac{\tilde{\alpha}_0}{\nu+\tilde{\nu}} |\xi|^2 |\sigma|^2 \\
&\leq C \left\{ \frac{\nu}{(\nu+\tilde{\nu})\gamma^2} |\phi_1|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\phi_1|_2^2 + \frac{\nu^2}{(\nu+\tilde{\nu})\gamma^4} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \right. \\
&\quad \left. + \frac{\gamma^2}{(\nu+\tilde{\nu})\nu} \tilde{D}_\xi[w_1] + (\frac{\tilde{\nu}^2}{(\nu+\tilde{\nu})^2} + \frac{1}{\nu}) |\xi|^2 \tilde{D}_\xi[w_1] \right\}, \tag{4.22}
\end{aligned}$$

$$|J_1[u]| \leq \frac{1}{\gamma^2} |\sigma|^2 + C \frac{\gamma^2}{(\nu+\tilde{\nu})^2} |w_1|_2^2,$$

where $\tilde{\alpha}_0$ is a positive constant.

Proof. Since

$$\langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle = \langle Q_0 \tilde{B}_\xi u^{(0)} \rangle \sigma + \gamma^2 i \xi \langle \rho_s w_1^3 \rangle + i \xi \langle v_s^3 \phi_1 \rangle,$$

(4.8) is written as

$$\partial_t \sigma + \langle Q_0 \tilde{B}_\xi u^{(0)} \rangle \sigma + \gamma^2 i \xi \langle \rho_s w_1^3 \rangle = -i \xi \langle v_s^3 \phi_1 \rangle. \quad (4.23)$$

Set

$$\tilde{B}_\xi^3 = \begin{pmatrix} i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} & 0 & i \xi v_s^3 \end{pmatrix}.$$

Since $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} = \alpha_0$, we have

$$\tilde{B}_\xi^3 u_0^{(0)} = i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} \sigma \phi^{(0)} = i \xi \alpha_0.$$

We thus obtain

$$-(\Delta' - \xi^2) w_1^3 = -\frac{\alpha_0}{\nu} i \xi \sigma \rho_s - \frac{\rho_s}{\nu} \partial_t w_1^3 + I_1^3. \quad (4.24)$$

Here

$$\begin{aligned} I_1^3 = & -\frac{\rho_s}{\nu} \left\{ \hat{C}_0^3 u_1 - \frac{\tilde{\nu}}{\rho_s} i \xi (\nabla' \cdot w_1' + i \xi w_1^3) \right. \\ & \left. + \tilde{B}_\xi^3 u_1 + \sigma \tilde{M}_\xi^3 u_1^{(0)} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle w^{(0),3} \right\}, \end{aligned}$$

where $\hat{C}_0^3$ and $\tilde{M}_\xi^3$ are 1×4 matrix operators defined by

$$\tilde{M}_\xi^3 = \begin{pmatrix} 0 & 0 & \frac{\nu + \tilde{\nu}}{\rho_s} \xi^2 \end{pmatrix} + \tilde{B}_\xi^3, \quad \hat{C}_0^3 = \begin{pmatrix} \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 & {}^T(\nabla' v_s^3) & 0 \end{pmatrix}.$$

Let A be the operator on $L^2(D)$ defined by

$$A\varphi = -\Delta' \varphi \quad \text{for} \quad \varphi \in D(A) = H^2(D) \cap H_0^1(D).$$

It then follows from (4.24) that

$$w_1^3 = -\frac{\alpha_0}{\nu} i \xi \sigma (A + |\xi|^2)^{-1} \rho_s - (A + |\xi|^2)^{-1} \left[\frac{\rho_s}{\nu} \partial_t w_1^3 \right] + (A + |\xi|^2)^{-1} I_1^3.$$

Substituting this into (4.23) we obtain

$$\partial_t \sigma + \langle Q_0 \tilde{B}_\xi u^{(0)} \rangle \sigma + \frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} \rho_s \rangle |\xi|^2 \sigma = I_1^0 - I_2^0, \quad (4.25)$$

where

$$\begin{aligned} I_1^0 &= -\gamma^2 i \xi \langle \rho_s (A + |\xi|^2)^{-1} I_1^3 \rangle - i \xi \langle v_s^3 \phi_1 \rangle, \\ I_2^0 &= \gamma^2 i \xi \langle \rho_s (A + |\xi|^2)^{-1} \left[\frac{\rho_s}{\nu} \partial_t w_1^3 \right] \rangle. \end{aligned}$$

Let us calculate $(4.25) \times \bar{\sigma}$ and take its real part. Since $\text{Re}\{\langle Q_0 \tilde{B}_\xi u^{(0)} \rangle\} = 0$, we have

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 + \frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} \rho_s \rangle |\xi|^2 |\sigma|^2 = \text{Re}(I_1^0 \bar{\sigma}) + \text{Re}(I_2^0 \bar{\sigma}).$$

Since $\langle \rho_s(A + |\xi|^2)^{-1} \rho_s \rangle = |(A + |\xi|^2)^{-\frac{1}{2}} \rho_s|_2^2$ is continuous in ξ and is positive for all $\xi \in \mathbf{R}$, we see that there exists a positive constant $\tilde{\alpha}_0 = \mathcal{O}(|\xi|^{-2})$ as $|\xi| \rightarrow \infty$ such that

$$\frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s(A + |\xi|^2)^{-1} \rho_s \rangle \geq \frac{\tilde{\alpha}_0 \gamma^2}{\nu}$$

for all $\xi \in \mathbf{R}$ with $|\xi| \leq R$. We thus obtain

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 + \frac{\tilde{\alpha}_0 \gamma^2}{\nu} |\xi|^2 |\sigma|^2 \leq \operatorname{Re}(I_1^0 \bar{\sigma}) + \operatorname{Re}(I_2^0 \bar{\sigma}). \quad (4.26)$$

As for the right-hand side of (4.26), we see from

$$|(A + |\xi|^2)^{-1} p|_2 \leq \frac{C}{|\xi|^2 + 1} |p|_2$$

that

$$\begin{aligned} |\operatorname{Re}(I_1^0 \bar{\sigma})| &\leq \frac{\tilde{\alpha}_0 \gamma^2}{\nu} \left(\frac{1}{10} + C \frac{1}{\gamma^2} \right) \min\{1, |\xi|^2\} |\sigma|^2 \\ &\quad + C \left\{ |\phi_1|_2^2 + \left(\frac{\gamma^2}{\nu} + \frac{1}{\nu \gamma^2} \right) |\xi|^2 |\phi_1|_2^2 + \frac{\nu}{\gamma^2} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \right. \\ &\quad \left. + \frac{\gamma^2}{\nu^2} \tilde{D}_\xi[w_1] + \left(\frac{(\nu + \tilde{\nu}) \gamma^2}{\nu^2} + \frac{\gamma^2 \tilde{\nu}^2}{(\nu + \tilde{\nu}) \nu} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.27)$$

We next derive the estimate for $I_2^0 \bar{\sigma}$. There holds that

$$\begin{aligned} I_2^0 \bar{\sigma} &= \gamma^2 i \xi \langle \rho_s(A + |\xi|^2)^{-1} (\frac{\rho_s}{\nu} \partial_t w_1^3) \rangle \bar{\sigma} \\ &= \frac{d}{dt} \left\{ i \xi \frac{\gamma^2}{\nu} \langle \rho_s(A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \bar{\sigma} \right\} - i \xi \frac{\gamma^2}{\nu} \langle \rho_s(A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \partial_t \bar{\sigma}. \end{aligned}$$

Let us estimate the second term of the right-hand side of this equation. We see from (4.8) that

$$\begin{aligned} i \xi \frac{\gamma^2}{\nu} \langle \rho_s(A + |\xi|^2)^{-1} [\rho_s w_1^3] \rangle \partial_t \bar{\sigma} &= i \xi \frac{\gamma^2}{\nu} \langle \rho_s(A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \left\{ -\langle Q_0 \tilde{B}_\xi u^{(0)} \rangle - \gamma^2 i \xi \langle \rho_s w_1^3 \rangle - i \xi \langle v_s^3 \phi_1 \rangle \right\} \\ &\leq C \frac{\gamma^2}{\nu} \frac{|\xi|}{1 + |\xi|^2} |w_1|_2 \left\{ |\xi| |\sigma| + \gamma^2 |\xi| |w_1|_2 + |\xi| |\phi|_2 \right\} \\ &\leq \frac{1}{10} \frac{\tilde{\alpha}_0 \gamma^2}{\nu} \min\{1, |\xi|^2\} |\sigma|^2 + C \left\{ \frac{1}{\nu} |\xi|^2 |\phi_1|_2^2 + \frac{\gamma^4}{\nu^2} \tilde{D}_\xi[w_1] \right\}. \end{aligned}$$

We thus obtain

$$\begin{aligned} |\operatorname{Re}(I_2^0 \bar{\sigma})| &\leq -\frac{d}{dt} \left(\frac{\gamma^2(\nu + \tilde{\nu})}{\nu} J_1[u] \right) + \frac{1}{10} \frac{\tilde{\alpha}_0 \gamma^2}{\nu} \min\{1, |\xi|^2\} |\sigma|^2 \\ &\quad + C \left\{ \frac{1}{\nu} |\xi|^2 |\phi_1|_2^2 + \frac{\gamma^4}{\nu^2} \tilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.28)$$

If $\frac{1}{\nu}$, $\frac{1}{\gamma^2}$ and $\frac{\nu + \tilde{\nu}}{\gamma^2}$ are sufficiently small, it then follows from (4.26), (4.27) and (4.28) that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(|\sigma|^2 + \frac{\gamma^2(\nu + \tilde{\nu})}{\nu} J_1[u] \right) + \frac{1}{2} \frac{\tilde{\alpha}_0 \gamma^2}{\nu} |\xi|^2 |\sigma|^2 \\ &\leq C \left\{ |\phi_1|_2^2 + \frac{\gamma^2}{\nu} |\xi|^2 |\phi_1|_2^2 + \frac{\nu}{\gamma^2} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \right. \\ &\quad \left. + \frac{\gamma^4}{\nu^2} \tilde{D}_\xi[w_1] + \left(\frac{\gamma^2(\nu + \tilde{\nu})}{\nu^2} + \frac{\gamma^2 \tilde{\nu}^2}{\nu(\nu + \tilde{\nu})} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.29)$$

Futhermore we have the estimate

$$\begin{aligned} |J_1[u]| &= \left| i\xi \frac{1}{\nu+\tilde{\nu}} \langle \rho_s(A + |\xi|^2)^{-1} [\rho_s w_1^3] \rangle \bar{\sigma} \right| \\ &\leq \frac{1}{\gamma^2} |\sigma|^2 + C \frac{\gamma^2}{(\nu+\tilde{\nu})^2} |w_1|_2^2. \end{aligned} \quad (4.30)$$

Multiplying to $\frac{\nu}{\gamma^2(\nu+\tilde{\nu})}$ for (4.29) and (4.30), we obtain the desired estimates. This completes the proof. $\square$

From Proposition 4.7, Proposition 4.9 and Proposition 4.10, we get the estimate of $|\sigma|$, $|\phi_1|_2$ and $|w_1|_2$.

Proposition 4.11. *Let $R > 0$. There exist positive constants ν_0 , γ_0 , ω_0 independent of R and an energy functional $E_1[u]$ such that if $\nu \geq \nu_0 R^2$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2 R^2$, $\omega \leq \omega_0$ and $|\xi| \leq R$, then there hold the estimates:*

$$\frac{d}{dt} E_1[u] + \frac{1}{\nu+\tilde{\nu}} (|\xi|^2 |\sigma|^2 + |\phi_1|_2^2) + \tilde{D}_\xi[w_1] \leq 0, \quad (4.31)$$

$$\frac{1}{2} \left(\frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) \leq C E_1[u] \leq \frac{3}{2} \left(\frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right),$$

where C is a positive constant independent of u .

Proof. For a given $R > 0$ we assume that $|\xi| \leq R$. Let $b_1 > 1$ and $b_2 > 1$ be a constants. Define $E_1[u]$ by

$$E_1[u] = b_1 \left(1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \left(\frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + b_2 J_0[u_1] + \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} |\sigma|^2 + J_1[u].$$

Since we have

$$\frac{1}{2} (|\phi_1|_2^2 + |w_1|_2^2) \leq C_0 E_0[u_1] \leq \frac{3}{2} (|\phi_1|_2^2 + |w_1|_2^2),$$

$$|J_0[u_1]| \leq C_1 \left\{ \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} |\phi_1|_2^2 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} |w_1|_2^2 \right\},$$

$$|J_1[u]| \leq \frac{1}{\gamma^2} |\sigma|^2 + C_2 \frac{\gamma^2}{(\nu+\tilde{\nu})^2} |w_1|_2^2,$$

if $\frac{1}{\nu+\tilde{\nu}} < 1$, $\frac{\nu+\tilde{\nu}}{\gamma^2} < 1$ and $b_1 > 8 \max\{C_0 C_1 b_2, C_0 C_2, \alpha_0^{-1}\}$, then there exists a constant $C > 0$ such that

$$\frac{1}{2} \left(\frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) \leq C E_1[u] \leq \frac{3}{2} \left(\frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right). \quad (4.32)$$

Let us compute $b_1 \times \left(1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \times (4.11) + b_2 \times (4.20) + (4.22)$ then

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} E_1[u] + \frac{b_1}{2} \left(1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \tilde{D}_\xi[w_1] + \frac{b_2}{2} \frac{1}{\nu+\tilde{\nu}} |\phi_1|_2^2 + \frac{\tilde{\alpha}_0}{2} \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\sigma|^2 \\ &\leq C_3 \left\{ b_1 \left(1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + b_1 \left(1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) |\phi_1|_2^2 \right. \\ &\quad + b_2 \left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + b_2 \left(1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \tilde{D}_\xi[w_1] + b_2 \frac{\nu}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] \\ &\quad + \frac{\nu}{(\nu+\tilde{\nu})\gamma^2} |\phi_1|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\phi_1|_2^2 + \frac{\nu^2}{(\nu+\tilde{\nu})\gamma^4} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \\ &\quad \left. + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}^2}{(\nu+\tilde{\nu})^2} + \frac{1}{\nu} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \end{aligned}$$

Fix $b_1 > 1$ and $b_2 > 1$ so large that $b_2 \geq 16C_3R^2$ and $b_1 \geq 16 \max\{C_0C_1b_2, C_0C_2, \alpha_0^{-1}, C_3b_2, C_3R^2\}$. We assume that $\nu \geq \nu_0$ and $\gamma \geq \gamma_0$ are so large that $\nu \geq 16C_3b_1 \max\{\tilde{\alpha}_0^{-1}, b_2^{-1}, 1\}$ and $\gamma^2 > 16C_3(1 + \tilde{\alpha}^{-1} + \tilde{\alpha}^{-\frac{1}{2}})(\nu + \tilde{\nu}) \max\{b_1, b_2, b_2^{-1}(1 + R^2)\}$. It then follows that there exists a constant $C > 0$ such that

$$\frac{d}{dt}E_1[u] + C\left\{\frac{1}{\nu+\tilde{\nu}}|\xi|^2|\sigma|^2 + \frac{1}{\nu+\tilde{\nu}}|\phi_1|_2^2 + \tilde{D}_\xi[w_1]\right\} \leq 0. \quad (4.33)$$

We thus obtain the desired estimates. This completes the proof. $\square$

We are now in a position to prove the estimate of the L^2 norm of $U_1(t)u_0$.

Proposition 4.12. *Let $R > 0$. There exist positive constants ν_0 , γ_0 and ω_0 such that if $\nu \geq \nu_0R^2$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2R^2$, and $\omega \leq \omega_0$, then for any $l = 0, 1, \dots$, there exists a constant $C = C(l) > 0$ such that the estimate*

$$\|\partial_{x_3}^l \mathcal{F}^{-1}[\chi^{(R)} e^{-t\hat{L}_\xi} \hat{u}_0]\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} \quad (4.34)$$

holds for $t \geq 0$.

Proof. For a given $R > 0$, we assume that $|\xi| \leq R$. Since

$$|\xi|^2|\sigma|^2 + |\phi_1|_2^2 + \tilde{D}_\xi[w_1] \geq \tilde{d}_0|\xi|^2(|\sigma|^2 + |\phi_1|_2^2 + |w_1|_2^2)$$

for some constant $\tilde{d}_0 = \tilde{d}_0(R) > 0$, we see from (4.31) that there exists a constant $d_0 > 0$ such that

$$\frac{d}{dt}E_1[u](t) + d_0|\xi|^2|u|_2^2 \leq 0.$$

This implies that

$$|e^{-t\hat{L}_\xi} \hat{u}_0(\xi)|_{L^2} \leq C e^{-d_0|\xi|^2 t} |\hat{u}_0(\xi)|_{L^2}. \quad (4.35)$$

We thus obtain the desired estimate. This completes the proof. $\square$

We next estimate derivatives of u . We introduce some notations. We define $J_2^{(0)}[u]$ by

$$J_2^{(0)}[u] = -2\text{Re}\langle \sigma u^{(0)} + u_1, \hat{B}_\xi \tilde{Q} u_1 \rangle \quad \text{for } u = \sigma u^{(0)} + u_1.$$

In addition, we set

$$E_2^{(0)}[u] = \left(1 + \frac{b_3\gamma^2}{\nu}\right) \left(\frac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0[u_1]\right) + \tilde{D}_\xi[w_1],$$

$$\tilde{E}_2^{(0)}[u] = E_2^{(0)}[u] + J_2^{(0)}[u],$$

where b_3 is a positive constant to be determined later. We note that there exists a constant $b_3^* > 0$ such that if $b_3 \geq b_3^*$ and $\gamma^2 \geq 1$ then

$$\frac{1}{2}E_2^{(0)}[u] \leq \tilde{E}_2^{(0)}[u] \leq \frac{3}{2}E_2^{(0)}[u].$$

Taking b_3 suitably large, we have the following estimate for $\tilde{E}_2^{(0)}[u]$.

Proposition 4.13. *There exist constants $b_3 \geq b_3^*$ and $\nu_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2 \geq 1$, then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \widetilde{E}_2^{(0)}[u] + \frac{1}{4} b_3 \frac{\gamma^2}{\nu} \widetilde{D}_\xi[w_1] + \frac{1}{2} |\sqrt{\rho_s} \partial_t w_1|_2^2 \\ & \leq C \left\{ \left(\frac{1}{\nu} + \frac{\nu + \bar{\nu}}{\nu \gamma^2} + \frac{\bar{\nu}^2}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{(\nu + \bar{\nu})^2}{\gamma^4} |\xi|^4 |\sigma|^2 \right. \\ & \quad \left. + \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) |\phi_1|_2^2 + \frac{1}{\gamma^2} |\xi|^2 |\phi_1|_2^2 \right\}. \end{aligned} \quad (4.36)$$

Proof. Since u is a solution of

$$\frac{d}{dt} u + \widehat{L}_\xi u = 0,$$

it holds that

$$\langle \partial_t u, \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{L}_\xi u, \partial_t \widetilde{Q} u_1 \rangle = 0. \quad (4.37)$$

We first consider the first term on the left-hand side of (4.37). Since

$$\partial_t \sigma = -\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle,$$

$$\langle u^{(0)}, \partial_t \widetilde{Q} u_1 \rangle = \langle u_1^{(0)}, \partial_t \widetilde{Q} u_1 \rangle,$$

applying Remark 4.5 and Lemma 4.6, we obtain

$$\begin{aligned} \operatorname{Re} \langle \partial_t u, \partial_t \widetilde{Q} u_1 \rangle &= \operatorname{Re} \{ \langle \partial_t \sigma u^{(0)}, \partial_t \widetilde{Q} u_1 \rangle + \langle \partial_t u_1, \partial_t \widetilde{Q} u_1 \rangle \} \\ &= \operatorname{Re} \{ -\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u_1^{(0)}, \partial_t \widetilde{Q} u_1 \rangle + |\sqrt{\rho_s} \partial_t w_1|_2^2 \} \\ &\geq \frac{7}{8} |\sqrt{\rho_s} \partial_t w_1|_2^2 - C \left\{ \frac{1}{\gamma^4} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + \frac{1}{\nu} \widetilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.38)$$

As for the second term on the left-hand side of (4.37), we see from $\widehat{L}_0 u^{(0)} = 0$ and $\widehat{B}_0 u^{(0)} = 0$ that

$$\begin{aligned} \langle \widehat{L}_\xi u, \partial_t \widetilde{Q} u_1 \rangle &= \langle \widetilde{M}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{L}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle \\ &= \langle \widetilde{A}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1 \rangle \\ &\quad + \langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{C}_0 u_1, \partial_t \widetilde{Q} u_1 \rangle. \end{aligned} \quad (4.39)$$

It follows from (4.37), (4.38) and (4.39) that

$$\begin{aligned} & \frac{7}{8} |\sqrt{\rho_s} \partial_t w_1|_2^2 + \langle \widetilde{A}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1 \rangle \\ & \quad + \langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{C}_0 u_1, \partial_t \widetilde{Q} u_1 \rangle \\ & \leq C \left\{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \frac{1}{\nu} \widetilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.40)$$

Next we show the estimate

$$\begin{aligned} & \operatorname{Re} \{ \langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle \} \\ & \geq \frac{1}{2} \frac{d}{dt} \left(\widetilde{D}_\xi[w_1] + J_2^{(0)}[u] \right) - \epsilon |\sqrt{\rho_s} \partial_t w_1|_2^2 \\ & \quad - C \left\{ \left(\frac{1}{\gamma^2} + \frac{1}{\epsilon \gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^2} |\xi|^2 |\phi_1|_2^2 + \left(\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu} + \frac{1}{\epsilon \nu} \right) \widetilde{D}_\xi[w_1] \right\} \end{aligned} \quad (4.41)$$

for any $\epsilon > 0$ with C independent of ϵ . In fact, it holds by integrating by parts that

$$\operatorname{Re}\langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle = \frac{1}{2} \frac{d}{dt} \widetilde{D}_\xi[w_1]. \quad (4.42)$$

Since $\widehat{B}_\xi^* = -\widehat{B}_\xi$, we see that

$$\langle \widehat{B}_\xi(\sigma u_0^{(0)}), \partial_t \widetilde{Q} u_1 \rangle = -\frac{d}{dt} \langle \sigma u_0^{(0)}, \widehat{B}_\xi \widetilde{Q} u_1 \rangle + \langle \partial_t(\sigma u_0^{(0)}), \widehat{B}_\xi \widetilde{Q} u_1 \rangle. \quad (4.43)$$

By (4.8) we have

$$\partial_t \sigma = -\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle.$$

It then follows from Lemma 4.6 that

$$\begin{aligned} |\langle \partial_t(\sigma u_0^{(0)}), \widehat{B}_\xi \widetilde{Q} u_1 \rangle| &\leq |\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle| |\langle u_0^{(0)}, \widehat{B}_\xi \widetilde{Q} u_1 \rangle| \\ &\leq C \left\{ \frac{1}{\gamma^2} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + \left(\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu} \right) \widetilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.44)$$

Similarly to above, there holds the following equation:

$$\begin{aligned} &\langle \widehat{B}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle \\ &= -\frac{d}{dt} \langle u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle + \langle \partial_t u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \\ &= -\frac{d}{dt} \langle u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle + \langle \partial_t Q_0 u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle + \langle \partial_t \widetilde{Q} u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle. \end{aligned} \quad (4.45)$$

We estimate the second term on the right hand of (4.45). By (4.9) we have

$$\begin{aligned} \partial_t Q_0 u_1 &= -Q_0 \{ \widehat{L}_\xi u_1 + \widetilde{M}_\xi(\sigma u^{(0)}) - \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)} \} \\ &= -Q_0 \widehat{B}_\xi u_1 - Q_0 \widetilde{B}_\xi(\sigma u^{(0)}) + \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)}. \end{aligned}$$

Since $\langle \partial_t Q_0 u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle = \langle \partial_t Q_0 u_1, Q_0 \widehat{B}_\xi \widetilde{Q} u_1 \rangle$, we see from Lemma 4.6 that

$$\begin{aligned} &|\langle \partial_t Q_0 u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle| \\ &\leq C \{ |Q_0 \widehat{B}_\xi u_1|_2 + |Q_0 \widetilde{B}_\xi(\sigma u^{(0)})|_2 \\ &\quad + |\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)}|_2 \} \times \frac{1}{\gamma^2} |Q_0 \widehat{B}_\xi \widetilde{Q} u_1|_2 \\ &\leq C \left\{ \frac{1}{\gamma^2} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + \left(\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu} \right) \widetilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.46)$$

The third term on the right-hand of (4.45) is estimated as

$$\begin{aligned} |\langle \partial_t \widetilde{Q} u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle| &\leq C |\sqrt{\rho_s} \partial_t w_1|_2 (|\nabla' \cdot (\rho_s w_1') + i \xi \rho_s w_1^3|_2 + |\xi| |w_1|_2) \\ &\leq \epsilon |\sqrt{\rho_s} \partial_t w_1|_2^2 + C \frac{1}{\epsilon \nu} \widetilde{D}_\xi[w_1] \end{aligned}$$

for any $\epsilon > 0$ with C independent of ϵ . This, together with (4.45) and (4.46), leads to the inequality

$$\begin{aligned} &\operatorname{Re} \langle \widehat{B}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle \\ &\geq -\frac{d}{dt} \langle u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle - \epsilon |\sqrt{\rho_s} \partial_t w_1|_2^2 \\ &\quad - C \left\{ \frac{1}{\gamma^2} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + \left(\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu} + \frac{1}{\epsilon \nu} \right) \widetilde{D}_\xi[w_1] \right\} \end{aligned} \quad (4.47)$$

for any $\epsilon > 0$ with C independent of ϵ . Furthermore, we have

$$\begin{aligned} |\langle \widehat{B}_\xi(\sigma u_1^{(0)}), \partial_t \widetilde{Q}u_1 \rangle| &\leq C|\sqrt{\rho_s}\partial_t w_1|_2 |i\xi\rho_s\sigma w^{(0),3} + i\xi v_s^3\sigma w^{(0),3}|_2 \\ &\leq \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + C\frac{1}{\epsilon\gamma^4}|\xi|^2|\sigma|^2 \end{aligned} \quad (4.48)$$

for any $\epsilon > 0$ with C independent of ϵ . By (4.43), (4.44), (4.47) and (4.48), we obtain

$$\begin{aligned} &|\langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q}u_1 \rangle| \\ &\leq -\frac{1}{2}\frac{d}{dt}J_2^{(0)}[u] - \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\quad - C\left\{\left(\frac{1}{\gamma^2} + \frac{1}{\epsilon\gamma^4}\right)|\xi|^2|\sigma|^2 + \frac{1}{\gamma^2}|\xi|^2|\phi_1|_2^2 + \left(\frac{1}{\nu\gamma^2} + \frac{\gamma^2}{\nu} + \frac{1}{\epsilon\nu}\right)\widetilde{D}_\xi[w_1]\right\}. \end{aligned}$$

This, together with (4.42), gives (4.41).

The remaining terms on the left-hand side of (4.40) are estimated as

$$\begin{aligned} &|\langle \widetilde{A}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q}u_1 \rangle| \\ &\leq C\{\widetilde{\nu}|\xi||\nabla' w^{(0),3}|_2 + (\nu + \widetilde{\nu})|\xi|^2|w^{(0),3}|_2\}|\sigma||\sqrt{\rho_s}\partial_t w_1|_2 \\ &\leq \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + C\frac{1}{\epsilon}\left\{\frac{\widetilde{\nu}^2}{\gamma^4}|\xi|^2|\sigma|^2 + \frac{(\nu+\widetilde{\nu})^2}{\gamma^4}|\xi|^4|\sigma|^2\right\}, \end{aligned} \quad (4.49)$$

$$\begin{aligned} |\langle \widehat{C}_0 u_1, \partial_t \widetilde{Q}u_1 \rangle| &\leq C\left(\frac{\nu}{\gamma^2}|\phi_1|_2 + |w'_1|_2\right)|\sqrt{\rho_s}\partial_t w_1|_2 \\ &\leq \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + C\frac{1}{\epsilon}\left\{\frac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \frac{1}{\nu}\widetilde{D}_\xi[w_1]\right\}. \end{aligned} \quad (4.50)$$

Here ϵ is an arbitrary positive number and C is a constant independent of ϵ . Taking $\epsilon > 0$ suitably small, we see from (4.40) with (4.41), (4.49) and (4.50) that if $\nu \geq 1$ and $\gamma^2 \geq 1$, then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\widetilde{D}_\xi[w_1] + J_2^{(0)}[u]\right) + \frac{3}{4}|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\leq C_0\left\{\left(\frac{1}{\gamma^2} + \frac{\widetilde{\nu}^2}{\gamma^4}\right)|\xi|^2|\sigma|^2 + \frac{(\nu+\widetilde{\nu})^2}{\gamma^4}|\xi|^4|\sigma|^2\right. \\ &\quad \left.+ \frac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \frac{1}{\gamma^2}|\xi|^2|\phi_1|_2^2 + \frac{\gamma^2}{\nu}\widetilde{D}_\xi[w_1]\right\}. \end{aligned} \quad (4.51)$$

We take b_3 as $b_3 \geq \max\{b_3^*, 4C_0\}$ and then add (4.51) to $b_3\frac{\gamma^2}{\nu} \times (4.11)$, to get (4.36). This completes the proof. $\square$

We next establish the estimate for higher order derivatives near the boundary ∂D . We introduce the local curvilinear coordinate system.

For any $\bar{x}'_0 \in \partial D$, there exist a neighborhood $\widetilde{\mathcal{O}}_{\bar{x}'_0}$ of $\bar{x}'_0$ and a smooth diffeomorphism map $\Psi = (\Psi_1, \Psi_2) : \widetilde{\mathcal{O}}_{\bar{x}'_0} \rightarrow B_1(0) = \{z' = (z_1, z_2) : |z'| < 1\}$ such that

$$\begin{cases} \Psi(\widetilde{\mathcal{O}}_{\bar{x}'_0} \cap D) = \{z' \in B_1(0) : z_1 > 0\}, \\ \Psi(\widetilde{\mathcal{O}}_{\bar{x}'_0} \cap \partial D) = \{z' \in B_1(0) : z_1 = 0\}, \\ \det \nabla_{x'} \Psi \neq 0 \quad \text{on} \quad \overline{\widetilde{\mathcal{O}}_{\bar{x}'_0} \cap D}. \end{cases}$$

By the tubular neighborhood theorem, there exist a neighborhood $\mathcal{O}_{\bar{x}'_0}$ of $\bar{x}'_0$ and a local curvilinear coordinate system $y' = (y_1, y_2)$ on $\mathcal{O}_{\bar{x}'_0}$ defined by

$$x' = y_1 a_1(y_2) + \Psi^{-1}(0, y_2) : \mathcal{R} \rightarrow \mathcal{O}_{\bar{x}'_0}, \quad (4.52)$$

where $\mathcal{R} = \{y' = (y_1, y_2) : |y_1| \leq \tilde{\delta}_1, |y_2| \leq \tilde{\delta}_2\}$ for some $\tilde{\delta}_1, \tilde{\delta}_2 > 0$; $a_1(y_2)$ is the unit inward normal to ∂D that is given by

$$a_1(y_2) = \frac{\nabla_{x'} \Psi_1}{|\nabla_{x'} \Psi_1|}.$$

We set $y_3 = x_3$. It then follows that

$$\nabla_x = e_1(y_2) \partial_{y_1} + J(y') e_2(y_2) \partial_{y_2} + e_3 \partial_{y_3},$$

$$\nabla_y = \begin{pmatrix} {}^T e_1(y_2) \\ \frac{1}{J(y')} {}^T e_2(y_2) \\ {}^T e_3 \end{pmatrix} \nabla_x,$$

where

$$e_1(y_2) = \begin{pmatrix} a_1(y_2) \\ 0 \end{pmatrix}, \quad e_2(y_2) = \begin{pmatrix} a_2(y_2) \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad (4.53)$$

$$J(y') = |\det \nabla_{x'} \Psi|, \quad a_2(y_2) = \frac{-\nabla_{x'}^\perp \Psi_1}{|\nabla_{x'}^\perp \Psi_1|}$$

with $\nabla_{x'}^\perp \Psi_1 = {}^T(-\partial_{x_2} \Psi_1, \partial_{x_1} \Psi_1)$. Note that ∂_{y_1} and ∂_{y_2} are the inward normal derivative and tangential derivative at $x' = \Psi^{-1}(0, y_2) \in \partial D \cap \mathcal{O}_{\bar{x}'_0}$, respectively. We denote the normal and tangential derivatives by ∂_n and ∂ , i.e.,

$$\partial_n = \partial_{y_1}, \quad \partial = \partial_{y_2}.$$

Since ∂D is compact, there are bounded open sets $\mathcal{O}_m$ ($m = 1, \dots, N$) such that $\partial D \subset \cup_{m=1}^N \mathcal{O}_m$ and for each $m = 1, \dots, N$, there exists a local curvilinear coordinate system $y' = (y_1, y_2)$ as defined in (4.52) with $\mathcal{O}_{\bar{x}'_0}$, Ψ and $\mathcal{R}$ replaced by $\mathcal{O}_m$, Ψ^m and $\mathcal{R}_m = \{y' = (y_1, y_2) : |y_1| < \tilde{\delta}_1^m, |y_2| < \tilde{\delta}_2^m\}$ for some $\tilde{\delta}_1^m, \tilde{\delta}_2^m > 0$. At last, we take an open set $\mathcal{O}_0 \subset D$ such that

$$\cup_{m=0}^N \mathcal{O}_m \supset D, \quad \overline{\mathcal{O}_0} \cap \partial D = \emptyset.$$

We set a local coordinate $y' = (y_1, y_2)$ such that $y_1 = x_1, y_2 = x_2$ on $\mathcal{O}_0$.

Note that if $h \in H^2(D)$, then $h|_{\partial D} = 0$ implies that $\partial^k h|_{\partial D \cap \mathcal{O}_m} = 0$ ($k = 0, 1$).

Let us introduce a partition of unity $\{\chi_m\}_{m=0}^N$ subordinate to $\{\mathcal{O}_m\}_{m=0}^N$, satisfying

$$\sum_{m=0}^N \chi_m = 1 \text{ on } D, \quad \chi_m \in C_0^\infty(\mathcal{O}_m) \quad (m = 0, 1, \dots, N).$$

In the following we will denote by $[A, B]$ the commutator of A and B , i.e.,

$$[A, B] = AB - BA.$$

Lemma 4.14. *For $1 \leq m \leq N$ there hold the following estimates.*

- (i) $|[\partial, \partial_{x_j}]h| \leq C|\partial_{x'}h|$ for $h \in H^2(D)$ and $j = 1, 2$.
- (ii) $|(\chi_m[\partial, \partial_{x_j}]h, \chi_m \partial h)| \leq C|\chi_m \partial_{x'}h|_2^2$ for $h \in H^2(D)$ and $j = 1, 2$.
- (iii) $|(\chi_m[\partial, \partial_{x_k} \partial_{x_l}]h, \chi_m \partial h)| \leq \eta|\chi_m \partial_{x'} \partial h|_2^2 + C(1 + \frac{1}{\eta})|\partial_{x'}h|_{L^2(D \cap \mathcal{O}_m)}^2$ for all $\eta > 0$, $h \in H^2(D)$ with $\partial h|_{\partial D \cap \mathcal{O}_m} = 0$ and $k, l = 1, 2$.

Proof. (i) For $x' \in D \cap \mathcal{O}_m$, we set $y' = \Psi^m(x')$, $h(x') = \tilde{h}(y')$. Then there exists a smooth matrix valued function $A_1(y')$ such that $\nabla_{x'} = A_1(y')\nabla_{y'}$. We thus find that

$$[\partial, \partial_{x_j}]h = \partial \partial_{x_j} h - \partial_{x_j} \partial h = \sum_{0 \leq l_1, 0 \leq l_2, l_1 + l_2 = 1} h_{l_1 l_2} \partial^{l_1} \partial_n^{l_2} \tilde{h},$$

where $h_{l_1 l_2} = h_{l_1 l_2}(y')$ are smooth functions depending only on $D \cap \mathcal{O}_m$. Since

$$\frac{1}{C}|\partial_{y'} \tilde{h}| \leq |\partial_{x'} h| \leq C|\partial_{y'} \tilde{h}|$$

for some constant $C > 0$, we have the desired inequality. This completes the proof of (i).

(ii) The estimate in (ii) immediately follows from (i).

(iii) We have $\nabla_{y'} = A_1(y')^{-1} \nabla_{x'}$. We set $A_1(y')^{-1} = (c^{ij}(x'))_{ij}$. There holds that

$$[\partial, \partial_{x_k x_l}]h = - \sum_{j=1}^2 \{ \partial_{x_k} \partial_{x_l} c^{2j} \partial_{x_j} h + \partial_{x_l} c^{2j} \partial_{x_k} \partial_{x_j} h + \partial_{x_k} c^{2j} \partial_{x_l} \partial_{x_j} h \}.$$

It follows from integration by parts that

$$\begin{aligned} & |(\chi_m \partial_{x_l} c^{2j} \partial_{x_k} \partial_{x_j} h, \chi_m \partial h)| \\ &= |(\chi_m \partial_{x_l} c^{2j} \partial_{x_j} h, \chi_m \partial_{x_k} \partial h) + (\chi_m \partial_{x_k} \partial_{x_l} c^{2j} \partial_{x_j} h, \chi_m \partial h) + (\partial_{x_k} \chi_m^2 \partial_{x_l} c^{2j} \partial_{x_j} h, \partial h)| \\ &\leq C \{ |\chi_m \partial_{x_j} h|_2 |\chi_m \partial_{x_k} \partial h|_2 + |\chi_m \partial_{x_j} h|_2 |\chi_m \partial h|_2 + |\partial_{x_j} h|_{L^2(D \cap \mathcal{O}_m)} |\chi_m \partial h|_2 \} \\ &\leq \eta |\chi_m \partial_{x'} \partial h|_2^2 + C(1 + \frac{1}{\eta}) |\partial_{x'} h|_{L^2(D \cap \mathcal{O}_m)}^2. \end{aligned}$$

This complete the proof of (iii). $\square$

We are in a position to estimate higher order derivatives. We first derive the estimate for $\partial \phi_1$.

Proposition 4.15. *For $1 \leq m \leq N$, there exist constants $\nu_0 > 0$ and $b > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_m \partial \dot{\phi}_1|_2^2 \\ & \quad + \frac{1}{2} \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) + \frac{1}{2} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i \xi \partial w_1^3)|_2^2 \\ & \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 \right. \\ & \quad \left. + \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 + \left(\frac{1}{\eta \nu} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\} \end{aligned} \tag{4.54}$$

for any $\eta > 0$ with C independent of η .

Proof. Applying ∂ to (4.9), we have

$$\begin{cases} \partial_t \partial \phi_1 + i\xi v_s^3 \partial \phi_1 + \gamma^2 \nabla' \cdot (\rho_s \partial w_1') + \gamma^2 i\xi \rho_s \partial w_1^3 = \tilde{F}^0, \\ \partial_t \partial w_1' - \frac{\nu}{\rho_s} (\Delta' - |\xi|^2) \partial w_1' - \frac{\tilde{\nu}}{\rho_s} \nabla' (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3) \\ \quad + \nabla' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial \phi_1 \right) + i\xi v_s^3 \partial w_1' = \tilde{G}', \\ \partial_t \partial w_1^3 - \frac{\nu}{\rho_s} (\Delta' - |\xi|^2) \partial w_1^3 - \frac{\tilde{\nu}}{\rho_s} i\xi (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3) \\ \quad + i\xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial \phi_1 + i\xi v_s^3 \partial w_1^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \partial \phi_1 + \partial w_1' \cdot \nabla' v_s^3 = \tilde{G}^3 \end{cases} \quad (4.55)$$

on $D \cap \mathcal{O}_m$ and

$$\partial w_1|_{\partial D \cap \mathcal{O}_m} = 0.$$

Here $\tilde{F}^0 = F_1^0 + F_2^0$, $\tilde{G}' = G_1' + G_2'$ and $\tilde{G}^3 = G_1^3 + G_2^3$, with

$$\begin{aligned} F_1^0 &= -[\partial, i\xi v_s^3] \phi_1 - \gamma^2 [\partial, \nabla' \cdot \rho_s] w_1' - \gamma^2 [\partial, i\xi \rho_s] w_1^3, \\ G_1' &= \nu [\partial, \frac{1}{\rho_s} \Delta'] w_1' - \nu [\partial, \frac{1}{\rho_s} |\xi|^2] w_1' + \tilde{\nu} [\partial, \frac{1}{\rho_s} \nabla' \nabla' \cdot] w_1' + \tilde{\nu} [\partial, \frac{1}{\rho_s} \nabla' (i\xi)] w_1^3 \\ &\quad - [\partial, \nabla' \frac{P'(\rho_s)}{\gamma^2 \rho_s}] \phi_1 - [\partial, i\xi v_s^3] w_1', \\ G_1^3 &= \nu [\partial, \frac{1}{\rho_s} \Delta'] w_1^3 - \nu [\partial, \frac{1}{\rho_s} |\xi|^2] w_1^3 + \tilde{\nu} [\partial, \frac{1}{\rho_s} i\xi \nabla' \cdot] w_1' - \tilde{\nu} [\partial, \frac{1}{\rho_s} |\xi|^2] w_1^3 \\ &\quad - [\partial, \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3] \phi_1 - [\partial, i\xi \frac{P'(\rho_s)}{\gamma^2 \rho_s}] \phi_1 - [\partial, i\xi v_s^3] w_1^3 - [\partial, {}^T(\nabla' v_s^3)] w_1', \\ F_2^0 &= -\{i\xi \sigma \partial (v_s^3 \phi^{(0)}) + \gamma^2 i\xi \sigma \partial (\rho_s w^{(0),3}) - \langle Q_0 \tilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \partial \phi^{(0)}\}, \\ G_2' &= -\{-\tilde{\nu} i\xi \sigma \partial (\frac{1}{\rho_s} \nabla' w^{(0),3})\}, \\ G_2^3 &= -\{(\nu + \tilde{\nu}) \xi^2 \sigma \partial (\frac{1}{\rho_s} w^{(0),3}) + i\xi \sigma \partial (v_s^3 w^{(0),3}) - \langle Q_0 \tilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \partial w^{(0),3}\}. \end{aligned}$$

We set $\tilde{F} = {}^T(\tilde{F}^0, \tilde{G}', \tilde{G}^3)$, $F_1 = {}^T(F_1^0, G_1', G_1^3)$ and $F_2 = {}^T(F_2^0, G_2', G_2^3)$. Taking the weighted inner product of (4.55) with $\chi_m^2 \partial u_1$, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + \left| \chi_m \sqrt{\rho_s} \partial w_1 \right|_2^2 \right) \\ &\quad + \nu \{ |\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2 \} + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2 \\ &= \text{Re} \{ \langle F, \chi_m^2 \partial u_1 \rangle - I \}, \end{aligned} \quad (4.56)$$

where

$$\begin{aligned} I &= \nu (\nabla' \partial w_1, \nabla' (\chi_m^2) \partial w_1) + \tilde{\nu} (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3, \nabla' (\chi_m^2) \cdot \partial w_1') \\ &\quad - \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial \phi_1, \nabla' (\chi_m^2) \cdot \rho_s \partial w_1' \right) + (i\xi v_s^3 \partial w_1, \chi_m^2 \rho_s \partial w_1) \\ &\quad + \left(\frac{\nu}{\gamma^2 \rho_s} \Delta' v_s^3 \partial \phi_1, \chi_m^2 \partial w_1^3 \right) + (\partial w_1' \cdot \nabla' v_s^3, \chi_m^2 \rho_s \partial w_1^3). \end{aligned}$$

Let us estimate the right-hand side of (4.56). By Lemma 4.14 and the Poincaré inequality we have

$$\begin{aligned} &|\text{Re} \langle F_1, \chi_m^2 \partial u_1 \rangle| \\ &\leq (\eta + \frac{C}{\gamma^2}) \eta |\phi_1|_2^2 + (\eta + \frac{C}{\gamma^2}) |\xi|^2 |\phi_1|_2^2 + (\eta + \frac{C}{\gamma^2}) |\partial_{x'} \phi_1|_2^2 \\ &\quad + C \left(\frac{\omega^2}{\nu \eta} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu} \omega^2}{\nu} + \frac{1}{\nu} + 1 \right) \tilde{D}_\xi [w_1] + \frac{1}{8} \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) \\ &\quad + \frac{1}{8} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2, \end{aligned}$$

$$|\operatorname{Re} I| \leq (\eta + \frac{C}{\gamma^2}) |\partial_{x'} \phi_1|_2^2 + C(\frac{\nu}{\gamma^4 \eta} + \frac{1}{\nu \eta} + \frac{1}{\nu} + 1) \tilde{D}_\xi[w_1] \\ + \frac{1}{8} \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) + \frac{1}{8} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i \xi \partial w_1^3)|_2^2$$

for any $\eta > 0$ with C independent of $\eta > 0$. By Lemma 4.6 and the Hölder inequality we deduce that

$$|\operatorname{Re} \langle F_2, \chi_m^2 \partial u_1 \rangle| \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\xi|^2 |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 \right. \\ \left. + \left(\frac{1}{\nu \eta} + \frac{1}{\nu \gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}$$

for any $\eta > 0$ with C independent of $\eta > 0$. Therefore we see from (4.56) that if $\nu \geq 1$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) \\ + \frac{3}{4} \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) + \frac{3}{4} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i \xi \partial w_1^3)|_2^2 \\ \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\xi|^2 |\phi_1|_2^2 \right. \\ \left. + \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 + \left(\frac{1}{\nu \eta} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \quad (4.57)$$

We next estimate $\partial \dot{\phi}_1$. The first equation of (4.55) leads to

$$\frac{1}{\gamma^2} \partial \dot{\phi}_1 = \frac{1}{\gamma^2} (\partial_t \partial \phi_1 + i \xi \partial (v_s^3 \phi_1)) \\ = \frac{1}{\gamma^2} \tilde{F}^0 - \left\{ \frac{1}{\gamma^2} i \xi \partial v_s^3 \phi_1 + \nabla' \cdot (\rho_s \partial w_1') + i \xi \rho_s \partial w_1^3 \right\}.$$

We thus have

$$\frac{1}{\gamma^4} |\chi_m \partial \dot{\phi}_1|_2^2 \leq C \left\{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] + |\chi_m (\nabla' \cdot \partial w_1' + i \xi \partial w_1^3)|_2^2 \right\}.$$

Take $b > 0$ suitably small and add $b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_m \partial \dot{\phi}_1|_2^2$ to (4.57). We thus obtain the desired estimate. This completes the proof. $\square$

We next derive the estimate for $\partial_n \phi_1$.

Proposition 4.16. *For $1 \leq m \leq N$, there exist constants $\nu_0 > 0$ and $b > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq \omega_0$, then there holds the estimate:*

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi_1 \right|_2^2 \right) + \frac{1}{2} \frac{1}{\nu + \tilde{\nu}} \left| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \phi_1 \right|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_m \partial_n \dot{\phi}_1|_2^2 \\ \leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \left(\frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4 (\nu + \tilde{\nu})} \right) |\phi_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] \right. \\ \left. + \frac{\nu}{\nu + \tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] + \frac{\nu^2}{\nu + \tilde{\nu}} (|\chi_m \partial_n \partial w_1|_2^2 + |\chi_m \partial^2 w_1|_2^2) + \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 \right\}. \quad (4.58)$$

Proof. For a scalar field $p(x')$ on $D \cap \mathcal{O}_m$, we set

$$\tilde{p}(y') = p(x') \quad (y' = \Psi^m(x'), x' \in D \cap \mathcal{O}_m).$$

Similarly we transform a vector field $h(x') = {}^T(h^1(x'), h^2(x'), h^3(x'))$ into $\tilde{h}(y') = {}^T(\tilde{h}^1(y'), \tilde{h}^2(y'), \tilde{h}^3(y'))$ as

$$h(x') = E(y')\tilde{h}(y')$$

where $E(y') = (e_1(y_2), e_2(y_2), e_3)$ with $e_1(y_2)$, $e_2(y_2)$ and e_3 given in (4.53). Note that, since $e_3 = {}^T(0, 0, 1)$, the Fourier transform in $x_3 = y_3$ commutes with these transformations. It then follows that $\tilde{\phi}_1(y')$ and $\tilde{w}_1(y') = {}^T(\tilde{w}_1^1(y'), \tilde{w}_1^2(y'), \tilde{w}_1^3(y'))$ are governed by the following system of equations

$$\left\{ \begin{array}{l} \partial_t \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{\phi}_1 + \gamma^2 \widehat{\text{div}}_y(\tilde{\rho}_s \tilde{w}_1) + i\xi \tilde{v}_s^3 \sigma \tilde{\phi}^{(0)} + \gamma^2 i\xi \tilde{\rho}_s \sigma \tilde{w}^{(0),3} \\ \quad - \langle Q_0 \tilde{B}_\xi(\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \tilde{\phi}^{(0)} = 0, \\ \partial_t \tilde{w}_1^1 + \frac{\nu}{\tilde{\rho}_s} (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^1 - \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} (\widehat{\nabla}_y \widehat{\text{div}}_y \tilde{w}_1)^1 + \partial_{y_1} \left(\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \tilde{\phi}_1 \right) \\ \quad + \frac{\nu}{\gamma^2 \tilde{\rho}_s^2} (\Delta_{y'} \tilde{v}_s)^1 \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{w}_1^1 - \frac{\tilde{\nu}}{\tilde{\rho}_s} i\xi \sigma \partial_{y_1} \tilde{w}^{(0),3} = 0, \\ \partial_t \tilde{w}_1^2 + \frac{\nu}{\tilde{\rho}_s} (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^2 - \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} (\widehat{\nabla}_y \widehat{\text{div}}_y \tilde{w}_1)^2 + \frac{1}{J} \partial_{y_2} \left(\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \tilde{\phi}_1 \right) \\ \quad + \frac{\nu}{\gamma^2 \tilde{\rho}_s^2} (\Delta_{y'} \tilde{v}_s)^2 \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{w}_1^2 - \frac{\tilde{\nu}}{\tilde{\rho}_s} i\xi \sigma \frac{1}{J} \partial_{y_2} \tilde{w}^{(0),3} = 0, \\ \partial_t \tilde{w}_1^3 + \frac{\nu}{\tilde{\rho}_s} (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^3 - \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} (\widehat{\nabla}_y \widehat{\text{div}}_y \tilde{w}_1)^3 + i\xi \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \tilde{\phi}_1 \\ \quad + \frac{\nu}{\gamma^2 \tilde{\rho}_s^2} (\Delta_{y'} \tilde{v}_s)^3 \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{w}_1^3 + \tilde{w}_1^1 \partial_{y_1} \tilde{v}_s^3 + \frac{1}{J} \tilde{w}_1^2 \partial_{y_2} \tilde{v}_s^3 + \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} \xi^2 \sigma \tilde{w}^{(0),3} \\ \quad + i\xi \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \sigma \tilde{\phi}^{(0)} + i\xi \tilde{v}_s^3 \sigma \tilde{w}^{(0),3} + \langle Q_0 \tilde{B}_\xi(\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \tilde{w}^{(0),3} = 0 \end{array} \right. \quad (4.59)$$

with $\tilde{\rho}_s(y') = \rho_s(x')$, $\tilde{v}_s^3(y') = v_s^3(x')$ and $\tilde{P}'(\tilde{\rho}_s(y')) = P'(\rho_s(x'))$. Here ∇_y , div_y and rot_y denote the gradient, divergence and rotation in the curvilinear coordinate y which are written for $\tilde{p} = \tilde{p}(y')$ and $\tilde{h} = {}^T(\tilde{h}^1(y'), \tilde{h}^2(y'), \tilde{h}^3(y'))$ as

$$\begin{aligned} \nabla_y \tilde{p} &= e_1 \partial_{y_1} \tilde{p} + \frac{1}{J} e_2 \partial_{y_2} \tilde{p} + e_3 \partial_{y_3} \tilde{p}, \\ \text{div}_y \tilde{h} &= \frac{1}{J} \{ \partial_{y_1} (J \tilde{h}^1) + \partial_{y_2} \tilde{h}^2 + \partial_{y_3} (J \tilde{h}^3) \}, \\ \text{rot}_y \tilde{h} &= (\text{rot}_y \tilde{h})^1 e_1 + (\text{rot}_y \tilde{h})^2 e_2 + (\text{rot}_y \tilde{h})^3 e_3 \end{aligned}$$

with

$$\begin{aligned} (\text{rot}_y \tilde{h})^1 &= \frac{1}{J} \{ \partial_{y_2} \tilde{h}^3 - \partial_{y_3} (J \tilde{h}^2) \}, \\ (\text{rot}_y \tilde{h})^2 &= \partial_{y_3} \tilde{h}^1 - \partial_{y_1} \tilde{h}^3, \\ (\text{rot}_y \tilde{h})^3 &= \frac{1}{J} \{ \partial_{y_1} \tilde{h}^2 - \partial_{y_2} (J \tilde{h}^1) \}, \end{aligned}$$

and, therefore,

$$\begin{aligned} (\text{rot}_y \text{rot}_y \tilde{h})^1 &= \frac{1}{J} \{ \partial_{y_2} (\text{rot}_y \tilde{h})^3 - \partial_{y_3} (\text{rot}_y \tilde{h})^2 \}, \\ (\text{rot}_y \text{rot}_y \tilde{h})^2 &= \partial_{y_3} (\text{rot}_y \tilde{h})^1 - \partial_{y_1} (\text{rot}_y \tilde{h})^3, \\ (\text{rot}_y \text{rot}_y \tilde{h})^3 &= \frac{1}{J} \{ \partial_{y_1} (\text{rot}_y \tilde{h})^2 - \partial_{y_2} (\text{rot}_y \tilde{h})^1 \}; \end{aligned}$$

the Fourier transformed gradient $\widehat{\nabla}_y$ is given by

$$\widehat{\nabla}_y \tilde{p} = e_1 \partial_{y_1} \tilde{p} + \frac{1}{J} e_2 \partial_{y_2} \tilde{p} + e_3 i\xi \tilde{p};$$

and similarly $\widehat{\text{div}}_y$ and $\widehat{\text{rot}}_y$ are obtained from div_y and rot_y by replacing ∂_{y_3} with $i\xi$ respectively. Applying ∂_{y_1} to the first equation of (4.59), we have

$$\begin{aligned} & \partial_t \partial_{y_1} \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \partial_{y_1} \tilde{\phi}_1 + \gamma^2 \tilde{\rho}_s \partial_{y_1} \widehat{\text{div}}_y \tilde{w}_1 \\ &= -\left\{ i\xi \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1 + \gamma^2 \partial_{y_1} (\widehat{\text{div}}_y (\tilde{\rho}_s \tilde{w}_1)) - \gamma^2 \tilde{\rho}_s \partial_{y_1} \widehat{\text{div}}_y \tilde{w}_1 \right. \\ & \quad \left. + i\xi \partial_{y_1} (\tilde{v}_s^3 \sigma \tilde{\phi}^{(0)}) + \gamma^2 i\xi \partial_{y_1} (\tilde{\rho}_s \sigma \tilde{w}^{(0),3}) - \langle Q_0 \tilde{B}_\xi (\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \partial_{y_1} \tilde{\phi}^{(0)} \right\}. \end{aligned} \quad (4.60)$$

To eliminate the term $\partial_{y_1} \partial_{y_1} \tilde{w}_1^1$ in this equation, we consider $\frac{\gamma^2 \tilde{\rho}_s}{\nu + \tilde{\nu}} \times (4.59)_2 + \frac{1}{\tilde{\rho}_s} \times (4.60)$. It then follows that

$$\frac{1}{\tilde{\rho}_s} \partial_t \partial_{y_1} \tilde{\phi}_1 + \frac{\tilde{P}'(\tilde{\rho}_s)}{\nu + \tilde{\nu}} \partial_{y_1} \tilde{\phi}_1 + \frac{1}{\tilde{\rho}_s} i\xi \tilde{v}_s^3 \partial_{y_1} \tilde{\phi}_1 = I, \quad (4.61)$$

where $I = I_1 + I_2$ with

$$\begin{aligned} I_1 &= -\frac{\gamma^2}{\nu + \tilde{\nu}} \left\{ \tilde{\rho}_s \partial_t \tilde{w}_1^1 + \nu (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^1 \right. \\ & \quad \left. + \tilde{\rho}_s \partial_{y_1} \left(\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \right) \tilde{\phi}_1 + \frac{\nu}{\gamma^2} \tilde{\rho}_s (\Delta_{y'} \tilde{v}_s)^1 \tilde{\phi}_1 + i\xi \tilde{\rho}_s \tilde{v}_s^3 \tilde{w}_1^1 \right\} \\ & \quad - \left\{ i\xi \frac{1}{\tilde{\rho}_s} \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1 + \gamma^2 \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\widehat{\text{div}}_y (\tilde{\rho}_s \tilde{w}_1)) - \gamma^2 \partial_{y_1} \widehat{\text{div}}_y \tilde{w}_1 \right\}, \\ I_2 &= -\frac{\gamma^2}{\nu + \tilde{\nu}} (-\tilde{\nu} i\xi \sigma \partial_{y_1} \tilde{w}^{(0),3}) - \left\{ i\xi \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\tilde{v}_s^3 \sigma \tilde{\phi}^{(0)}) \right. \\ & \quad \left. + \gamma^2 i\xi \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\tilde{\rho}_s \sigma \tilde{w}^{(0),3}) - \frac{1}{\tilde{\rho}_s} \langle Q_0 \tilde{B}_\xi (\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \partial_{y_1} \tilde{\phi}^{(0)} \right\}. \end{aligned}$$

Considering $\int_{\Psi^m(D \cap \mathcal{O}_m)} (4.61) \times \tilde{\chi}_m^2 \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^4} \overline{\partial_{y_1} \tilde{\phi}_1} J dy'$ with $\tilde{\chi}_m(y') = \chi_m(x')$, we see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \tilde{\chi}_m \sqrt{\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s}} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right) + \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \\ &= \int_{\Psi^m(D \cap \mathcal{O}_m)} I \times \tilde{\chi}_m^2 \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^4} \overline{\partial_{y_1} \tilde{\phi}_1} J dy'. \end{aligned}$$

Since

$$(\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^1 = \frac{1}{J} \partial_{y_2} \left(\frac{1}{J} \partial_{y_1} (J \tilde{w}_1^2) - \frac{1}{J} \partial_{y_2} \tilde{w}_1^1 \right) - i\xi (i\xi \tilde{w}_1^1 - \partial_{y_1} \tilde{w}_1^3),$$

we obtain

$$\begin{aligned} \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m I_1|_2^2 &\leq C \left\{ \left(\frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{1}{\nu + \tilde{\nu}} |\tilde{\chi}_m \sqrt{\tilde{\rho}_s} \partial_t \tilde{w}_1|_2^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{1}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^4 |\tilde{\chi}_m \tilde{w}_1|_2^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\tilde{\chi}_m \partial_{y'} \partial_{y_2} \tilde{w}_1|_2^2 \right\}, \\ \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m I_2|_2^2 &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\phi}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 + (\nu + \tilde{\nu}) |\tilde{w}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 \right\}. \end{aligned}$$

It then follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \tilde{\chi}_m \sqrt{\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s}} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right) + \frac{3}{4} \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \\ &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \left(\frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\phi}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) |\tilde{w}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 + \frac{1}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^4 |\tilde{\chi}_m \tilde{w}_1|_2^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\tilde{\chi}_m \partial_{y'} \partial_{y_2} \tilde{w}_1|_2^2 \right. \\ & \quad \left. + \frac{1}{\nu + \tilde{\nu}} |\tilde{\chi}_m \sqrt{\tilde{\rho}_s} \partial_t \tilde{w}_1|_2^2 \right\}. \end{aligned} \quad (4.62)$$

We next consider $\partial_{y_1} \dot{\tilde{\phi}}_1$ where $\dot{\tilde{\phi}}_1 = \partial_t \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{\phi}_1$. (4.61) gives that

$$\frac{1}{\gamma^2} \partial_{y_1} \dot{\tilde{\phi}}_1 = \frac{1}{\gamma^2 \tilde{\rho}_s} (I + i\xi \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1 - \frac{\gamma^2}{\nu + \tilde{\nu}} \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \partial_{y_1} \tilde{\phi}_1).$$

This equation leads to the estimate

$$\begin{aligned} \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \partial_{y_1} \dot{\tilde{\phi}}_1|_2^2 &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \tilde{\rho}_s (I + i\xi \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1)|_2^2 + \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right\} \\ &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m I|_2^2 + \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right\}. \end{aligned}$$

Therefore if we take $b > 0$ suitably small and add $b \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \partial_{y_1} \dot{\tilde{\phi}}_1|_2^2$ to (4.62), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \tilde{\chi}_m \sqrt{\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s}} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right) + \frac{1}{2} \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \partial_{y_1} \dot{\tilde{\phi}}_1|_2^2 \\ &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \left(\frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4 (\nu + \tilde{\nu})} \right) |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\phi}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 \right. \\ &\quad + (\nu + \tilde{\nu}) |\tilde{w}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 + \frac{1}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^4 |\tilde{\chi}_m \tilde{w}_1|_2^2 \\ &\quad + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\tilde{\chi}_m \partial_{y'} \partial_{y_2} \tilde{w}_1|_2^2 \\ &\quad \left. + \frac{1}{\nu + \tilde{\nu}} |\tilde{\chi}_m \sqrt{\tilde{\rho}_s} \partial_t \tilde{w}_1|_2^2 \right\}. \end{aligned} \quad (4.63)$$

The desired estimate follows from (4.63) by inverting to the original coordinates x' and noting that $\partial_{y_1} = \partial_n$, $\partial_{y_2} = \partial$. This completes the proof. $\square$

We next derive the interior estimate for the derivative of ϕ_1 .

Proposition 4.17. *There exist constants $\nu_0 > 0$ and $b > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi_1 \right|_2^2 + |\chi_0 \sqrt{\rho_s} \partial_{x'} w_1|_2^2 \right) + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_0 \partial_{x'} \dot{\phi}_1|_2^2 \\ &\quad + \frac{1}{2} \nu (|\chi_0 \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w_1|_2^2) + \frac{1}{2} \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w_1' + i\xi \partial_{x'} w_1^3)|_2^2 \\ &\leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\partial_{x'} \phi_1|_2^2 \right. \\ &\quad \left. + \left(\eta + \frac{1}{\gamma^2} \right) |\xi|^2 |\phi_1|_2^2 + \left(\frac{1}{\eta \nu} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\} \end{aligned} \quad (4.64)$$

for any $\eta > 0$ with C independent of η .

Since $\text{supp}(\chi_0 w_1) \subset D$ we have $\partial_{x'} w_1|_{\partial D \cap \mathcal{O}_0} = 0$. Therefore we can prove this proposition similaly to the proof of Proposition 4.15. We omit the details.

Before proceeding futher we introduce an energy functional. We define $E_3^{(0)}[u_1]$ by

$$\begin{aligned} E_3^{(0)}[u_1] &= \frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi_1 \right|_2^2 + |\chi_0 \sqrt{\rho_s} \partial_{x'} w_1|_2^2 \\ &\quad + b_4 \sum_{m=1}^N \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) + \sum_{m=1}^N \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi_1 \right|_2^2, \end{aligned}$$

where b_4 is a positive constant. Taking b_4 suitably large, we have the following estimate for $E_3^{(0)}[u_1]$.

Proposition 4.18. *There exist constants $\nu_0 > 0$, $b > 0$ and $b_4 > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} E_3^{(0)}[u_1] + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\partial_{x'} \dot{\phi}_1|_2^2 \\
& + \frac{1}{2} \left\{ \nu (|\chi_0 \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\
& + \frac{1}{2} \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_m \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\
& \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left(\eta + \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\phi_1|_2^2 \right. \\
& \quad + \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 \\
& \quad \left. + \left(\frac{1}{\nu \eta} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] + \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 \right\}
\end{aligned} \tag{4.65}$$

for any $\eta > 0$ with C independent of η .

Using Proposition 4.15, Proposition 4.16 and Proposition 4.17, we obtain the estimate of Proposition 4.18.

We next derive a dissipative estimate for $|\partial_{x'}^2 w_1|_2$ and $|\partial_{x'} \phi_1|_2$.

Proposition 4.19. *There exist constants $\nu_0 > 0$ and $\omega_0 > 0$ such that if $\nu \geq \nu_0$, $\omega \leq \omega_0$ and $\gamma^2 \geq 1$, then there holds the estimate:*

$$\begin{aligned}
& \frac{\nu^2}{\nu + \tilde{\nu}} |\partial_{x'}^2 w_1'|_2^2 + \frac{1}{\nu + \tilde{\nu}} |\partial_{x'} \phi_1|_2^2 \\
& \leq C \left\{ \left(\frac{1}{\nu + \tilde{\nu}} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} |\phi_1|_2^2 + \left(\frac{1}{\nu + \tilde{\nu}} + \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\xi|^2 |\phi_1|_2^2 \right. \\
& \quad \left. + \left(\frac{\tilde{\nu}}{\nu} + 1 \right) (1 + |\xi|^2) \tilde{D}_\xi[w_1] + \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4(\nu + \tilde{\nu})} |\dot{\phi}_1|_{H^1}^2 \right\}.
\end{aligned} \tag{4.66}$$

Proof. We first derive the estimate for $\partial_{x'}^2 w_1'$ and $\partial_{x'} \phi_1$. We will employ the following estimate for solutions of Stokes equation. If (p, h') is the solution of

$$\begin{cases} \nabla' \cdot h' = F^0, \\ -\Delta' h' + \frac{1}{\nu} \nabla' p = \frac{1}{\nu} G', \\ h' |_{\partial D} = 0, \end{cases}$$

then there holds

$$|\partial_{x'}^2 h'|_2^2 + \frac{1}{\nu^2} |\partial_{x'} p|_2^2 \leq C \{ |F^0|_{H^1}^2 + \frac{1}{\nu^2} |G'|_2^2 \}. \tag{4.67}$$

(See, e.g., [2, IV.6], [12, III.1.5].) By the first and second equations of (4.9), with the boundary condition of w_1' , we see that (ϕ_1, w_1') satisfies the following Stokes equation

$$\begin{cases} \nabla' \cdot w_1' = F_1^0, \\ -\Delta' w_1' + \frac{1}{\nu} \nabla' \left(\frac{P'(\rho_s)}{\gamma^2} \phi_1 \right) = \frac{1}{\nu} G_1', \\ w_1' |_{\partial D} = 0, \end{cases}$$

where

$$\begin{aligned} F_1^0 &= -\frac{1}{\gamma^2 \rho_s} \{ \partial_t \phi_1 + i\xi v_s^3 \phi_1 + \gamma^2 (\nabla' \rho_s) \cdot w_1' + \gamma^2 i\xi w_1^3 \\ &\quad + i\xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i\xi \rho_s \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \phi^{(0)} \}, \\ G_1' &= -\rho_s \{ \partial_t w_1' + \frac{\nu}{\rho_s} \xi^2 w_1' - \frac{\tilde{\nu}}{\rho_s} (\nabla' \cdot w_1' + i\xi w_1^3) + i\xi v_s^3 w_1' \\ &\quad + \nabla' \left(\frac{1}{\rho_s} \right) \frac{P'(\rho_s)}{\gamma^2} \phi_1 - \frac{\tilde{\nu}}{\rho_s} i\xi \nabla' (\sigma w^{(0),3}) \}. \end{aligned}$$

By Lemma 4.6 and the Poincaré inequality, we have

$$\begin{aligned} |F_1^0|_2^2 &\leq C \{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|_2^2 + \frac{1}{\gamma^4} |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] + \frac{1}{\gamma^4} |\dot{\phi}_1|_2^2 \}, \\ |\partial_{x'} F_1^0|_2^2 &\leq C \{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \frac{1}{\nu} (1 + |\xi|^2) \tilde{D}_\xi[w_1] + \frac{1}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 \}, \\ |G_1'|_2^2 &\leq C \{ \frac{\tilde{\nu}^2}{\gamma^4} |\xi|^2 |\sigma|^2 + (\omega^2 + \frac{\tilde{\nu}^2}{\gamma^4}) |\xi|^2 |\phi_1|_2^2 + (\frac{1}{\nu} + \frac{\tilde{\nu}^2}{\nu}) \tilde{D}_\xi[w_1] \\ &\quad + (\nu + \frac{\tilde{\nu}^2}{\nu}) |\xi|^2 \tilde{D}_\xi[w_1] + \frac{\tilde{\nu}^2}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 + |\sqrt{\rho_s} \partial_t w_1|_2^2 \}. \end{aligned}$$

Since

$$\begin{aligned} \partial_{x'} \left(\frac{P'(\rho_s)}{\gamma^2} \phi_1 \right) &= \frac{P'(\rho_s)}{\gamma^2} \partial_{x'} \phi_1 + \frac{P''(\rho_s) \partial_{x'} \rho_s}{\gamma^2} \phi_1, \\ \frac{P'(\rho_s)}{\gamma^2} &\geq \frac{1}{2}, \end{aligned}$$

and

$$|\phi_1|_2 \leq C |\partial_{x'} \phi_1|_2$$

by the Poincaré inequality, we see that

$$\begin{aligned} |\partial_{x'} \left(\frac{P'(\rho_s)}{\gamma^2} \phi_1 \right)|_2^2 &\geq C \{ |\partial_{x'} \phi_1|_2^2 - \omega^2 |\phi_1|_2^2 \} \\ &\geq C (1 - \omega^2) |\partial_{x'} \phi_1|_2^2 \\ &\geq C |\partial_{x'} \phi_1|_2^2 \end{aligned}$$

for $\omega^2 < \frac{1}{2}$. We thus find the estimate

$$\begin{aligned} &|\partial_{x'}^2 w_1'|_2^2 + \frac{1}{\nu^2} |\partial_{x'} \phi_1|_2^2 \\ &\leq C \frac{1}{\nu^2} \{ \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4} |\phi_1|_2^2 + (\omega^2 + \frac{\nu^2}{\gamma^4} + \frac{\tilde{\nu}^2}{\gamma^4}) |\xi|^2 |\phi_1|_2^2 \\ &\quad + (\nu + \frac{1}{\nu} + \frac{\tilde{\nu}^2}{\nu}) \tilde{D}_\xi[w_1] + (\nu + \frac{\tilde{\nu}^2}{\nu}) |\xi|^2 \tilde{D}_\xi[w_1] + |\sqrt{\rho_s} \partial_t w_1|_2^2 + \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 \}. \end{aligned} \tag{4.68}$$

We next derive the estimate for $\partial_{x'}^2 w_1^3$. The third equation of (4.9), with the boundary condition of w_1^3 , is written as

$$\begin{cases} -\Delta' w_1^3 = G_1^3, \\ w_1^3|_{\partial D} = 0, \end{cases}$$

where

$$\begin{aligned} G_1^3 &= -\frac{\rho_s}{\nu} \{ \partial_t w_1^3 + \frac{\nu}{\rho_s} \xi^2 w_1^3 - \frac{\tilde{\nu}}{\rho_s} i\xi (\nabla' \cdot w_1' + i\xi w_1^3) \\ &\quad + i\xi \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) + i\xi v_s^3 w_1^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi_1 + w_1' \cdot \nabla' v_s^3 \\ &\quad + \frac{\nu + \tilde{\nu}}{\rho_s} \xi^2 \sigma w^{(0),3} + i\xi \alpha_0 \sigma + i\xi v_s^3 \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle w^{(0),3} \}. \end{aligned}$$

We thus obtain

$$|w_1^3|_{H^2}^2 \leq C|G_1^3|_2^2.$$

It then follows that

$$\begin{aligned} |\partial_{x'}^2 w_1^3|_2^2 &\leq C \frac{1}{\nu^2} \left\{ \left(1 + \frac{1}{\gamma^4} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\right) |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4} |\phi_1|_2^2 + \left(1 + \frac{1}{\gamma^4}\right) |\xi|^2 |\phi_1|_2^2 \right. \\ &\quad \left. + \left(\nu + \tilde{\nu} + \frac{1}{\nu}\right) \tilde{D}_\xi[w_1] + \frac{\tilde{\nu}^2}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] + |\sqrt{\rho_s} \partial_t w_1|_2^2 \right\}. \end{aligned} \quad (4.69)$$

Multiplying $\frac{\nu^2}{\nu+\tilde{\nu}}$ to (4.68) + (4.69), we have the desired estimate. This completes the proof. $\square$

We are now in a position to prove Theorem 3.2.

Proposition 4.20. *Let $R > 0$. There exist positive constants $\nu_0, \gamma_0, \omega_0$ and d such that if $\nu \geq \nu_0 R^2$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2 R^2$ and $\omega \leq \omega_0$, then for any $l = 0, 1, \dots$, there exists a constant $C = C(l) > 0$ such that the estimate*

$$\|\partial_{x'} \partial_{x_3}^l \mathcal{F}^{-1}[\chi^{(R)} e^{-t\hat{L}_\xi} \hat{u}_0]\|_{L^2} \leq C \left\{ (1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} + e^{-dt} (\|u_0\|_{L^2} + \|\partial_{x'} u_0\|_{L^2}) \right\} \quad (4.70)$$

holds for $t \geq 0$.

Proof. Let b_5 and b_6 be constants satisfying $b_5, b_6 > 1$. Define $E_4^{(0)}[u]$ by

$$E_4^{(0)}[u] = b_5 \frac{\nu}{\nu+\tilde{\nu}} \tilde{E}_2^{(0)}[u] + b_6 E_3^{(0)}[u_1].$$

If $\gamma^2 \geq 1$, then there exists a constant $C > 0$ such that

$$\begin{aligned} &\frac{1}{2} \left\{ \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] + \frac{1}{\gamma^2} |\partial_{x'} \phi_1|_2^2 + \tilde{D}_\xi[w_1] \right\} \\ &\leq C E_4^{(0)} \leq \frac{3}{2} \left\{ \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] + \frac{1}{\gamma^2} |\partial_{x'} \phi_1|_2^2 + \tilde{D}_\xi[w_1] \right\}. \end{aligned}$$

We compute $b_5 \frac{\nu}{\nu+\tilde{\nu}} \times (4.36) + b_6 \times (4.65) + b b_6 \times (4.12) + (4.66)$. It holds that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} E_4^{(0)}[u] + \frac{\nu^2}{\nu+\tilde{\nu}} |\partial_{x'}^2 w_1|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\partial_{x'} \phi_1|_2^2 \\ &+ \frac{b_3 b_5}{4} \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \tilde{D}_\xi[w_1] + \frac{b_5}{2} \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + b b_6 \frac{\nu+\tilde{\nu}}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 \\ &+ \frac{b_6}{2} \left\{ \nu (|\chi_0 \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\ &+ \frac{b_6}{2} \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_m \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\ &\leq C_4 \left\{ b_5 \frac{\nu}{\nu+\tilde{\nu}} \left(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\nu \gamma^2} + \frac{\tilde{\nu}^2}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + b_5 \frac{\nu}{\nu+\tilde{\nu}} \frac{(\nu+\tilde{\nu})^2}{\gamma^4} |\xi|^4 |\sigma|^2 \right. \\ &+ b_5 \frac{\nu}{\nu+\tilde{\nu}} \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) |\phi_1|_2^2 + b_5 \frac{\nu}{\nu+\tilde{\nu}} \frac{1}{\gamma^2} |\xi|^2 |\phi_1|_2^2 + b_6 \left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 \\ &+ b_6 \left(\eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi_1|_2^2 + b_6 \left(\eta + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 \\ &+ b_6 \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 + b_6 \left(\frac{1}{\nu \eta} + \frac{1}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + b_6 \left(\frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \\ &+ b_6 \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + b b_6 \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + b b_6 \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + b b_6 \left(1 + \frac{\nu+\tilde{\nu}}{\nu} \omega^2 \right) \tilde{D}_\xi[w_1] \\ &+ \left(\frac{1}{\nu+\tilde{\nu}} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} |\phi_1|_2^2 + \left(\frac{1}{\nu+\tilde{\nu}} + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\xi|^2 |\phi_1|_2^2 \\ &+ \left(\frac{\tilde{\nu}}{\nu} + 1 \right) (1 + |\xi|^2) \tilde{D}_\xi[w_1] + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} |\dot{\phi}_1|_{H^1}^2 \left. \right\}. \end{aligned}$$

Fix $b_5 > 1$ and $b_6 > 1$ sufficiently large such that $b_6 \geq \frac{2C_4}{b}$ and $b_5 \geq 8b_6C_4$, respectively. Let us take $\eta > 0$ so small satisfying $\eta \leq \min\{1, \frac{1}{8b_6C_4}\}$. We assume that $\nu \geq \nu_0$ and $\gamma \geq \gamma_0$ are so large that $\nu \geq \nu_0 > 1$ and $\gamma^2 \geq 8b_6C_4(\nu + \tilde{\nu})$. Since we have that

$$\begin{aligned}\tilde{D}_\xi[w_1] &\leq C(1+R)|w_1|_2|\partial_x^2 w_1|_2 \\ &\leq \epsilon|\partial_x^2 w_1|_2^2 + C\frac{1}{\epsilon}(1+R)^2|w_1|_2^2\end{aligned}$$

for any $\epsilon > 0$, if we take ϵ sufficiently small such that $\epsilon < \frac{1}{2}\frac{\nu^2}{\nu+\tilde{\nu}}$, then we get

$$\frac{d}{dt}E_4^{(0)}[u] + d(|\nabla'\phi_1|_2^2 + |\nabla'w_1|_{H^1}^2) \leq C|u|_2^2.$$

Now we decompose $E_4^{(0)}[u]$ as

$$E_4^{(0)}[u] = E_{4,0}^{(0)}[u] + E_{4,1}^{(0)}[u],$$

where

$$\begin{aligned}\frac{1}{2}|u|_2^2 &\leq CE_{4,0}^{(0)}[u] \leq \frac{3}{2}|u|_2^2, \\ \frac{1}{2}(|\nabla'\phi_1|_2^2 + |\nabla'w_1|_{H^1}^2) &\leq CE_{4,1}^{(0)}[u] \leq \frac{3}{2}(|\nabla'\phi_1|_2^2 + |\nabla'w_1|_{H^1}^2).\end{aligned}$$

It then follows that

$$\frac{d}{dt}E_{4,1}^{(0)}[u](t) + d_1E_{4,1}^{(0)}[u] + \frac{d}{2}(|\nabla'\phi_1|_2^2 + |\nabla'w_1|_{H^1}^2) \leq C|u|_2^2 - \frac{d}{dt}E_{4,0}^{(0)}[u](t).$$

We thus obtain

$$\begin{aligned}E_{4,1}^{(0)}[u](t) + \frac{d}{2} \int_0^t e^{-d_1(t-\tau)}(|\nabla'\phi_1|_2^2 + |\nabla'w_1|_{H^1}^2) d\tau \\ \leq e^{-d_1 t} E_{4,1}^{(0)}[u_0] + C \int_0^t e^{-d_1(t-\tau)} |u|_2^2 d\tau - \int_0^t e^{-d_1(t-\tau)} \frac{d}{d\tau} E_{4,0}^{(0)}[u](\tau) d\tau.\end{aligned}$$

Since

$$e^{-d_1(t-\tau)} \frac{d}{d\tau} E_{4,0}^{(0)}[u](\tau) = \frac{d}{d\tau} \{e^{-d_1(t-\tau)} E_{4,0}^{(0)}[u](\tau)\} - \frac{1}{d_1} e^{-d_1(t-\tau)} E_{4,0}^{(0)}[u](\tau)$$

and

$$E_{4,0}^{(0)}[u] \leq C|u|_2^2,$$

we see that

$$E_{4,1}^{(0)}[u](t) \leq e^{-d_1 t} E_{4,1}^{(0)}[u_0] + C \int_0^t e^{-d_1(t-\tau)} |u(\tau)|_2^2 d\tau.$$

From (4.35), we obtain

$$E_{4,1}^{(0)}[u](t) \leq e^{-d_1 t} E_{4,1}^{(0)}[u_0] + C|u_0|_2^2 \int_0^t e^{-d_1(t-\tau)} e^{-d_0|\xi|^2\tau} d\tau.$$

Let us estimate the second term on the right-hand side of this inequality. We have

$$\begin{aligned}
\int_0^{t/2} \exp\{-d_1(t-\tau) - d_0|\xi|^2\tau\} d\tau &\leq \int_0^{t/2} \exp\{-d_1(t-\tau)\} d\tau \\
&\leq \frac{1}{d_1} \exp\left\{-\frac{d_1}{2}t\right\} \\
&\leq \frac{1}{d_1} \exp\left\{-\frac{d_1}{2} \frac{|\xi|^2}{R^2}t\right\}, \\
\int_{t/2}^t \exp\{-d_1(t-\tau) - d_0|\xi|^2\tau\} d\tau &\leq \exp\left\{-\frac{d_0}{2}|\xi|^2t\right\} \int_{t/2}^t \exp\{-d_1(t-\tau)\} d\tau \\
&\leq \frac{1}{d_1} \exp\left\{-\frac{d_0}{2}|\xi|^2t\right\}.
\end{aligned}$$

We set $d_2 = \min\{d_0, \frac{d_1}{R^2}\}$. It then follows that there exist positive constants $\nu_0, \gamma_0, \omega_0, d_1$ and d_2 such that if $\nu \geq \nu_0 R^2, \frac{\gamma^2}{\nu+\nu} \geq \gamma_0^2 R^2$ and $\omega \leq \omega_0$, then

$$E_{4,1}^{(0)}[u](t) \leq C \left\{ e^{-\frac{d_2}{2}|\xi|^2t} |u_0|_2^2 + e^{-d_1t} E_4^{(0)}[u_0] \right\}. \quad (4.71)$$

□

Combining Proposition 4.12 and Proposition 4.20 with $R = 1$ we obtain the desired estimates in Theorem 3.2.

5 DECAY ESTIMATE OF THE HIGHT FREQUENCY PART

In this section we will give a proof of Theorem 3.3 for $|\xi| > R_0$ with a positive constant R_0 . To prove Theorem 3.3 for $|\xi| > R_0$, we will employ an energy method to obtain the estimate on solutions of

$$\partial_t u + \widehat{L}_\xi u = 0, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0$$

similaly to Section 4. The following Propositions 5.1-5.6 can be proved in a similar manner in Section 4. So we give the statements only and omit the proofs. As in the previous section we also set

$$\omega = \|\rho_s - 1\|_{C^3}.$$

Proposition 5.1. *There exists a constant $\nu_0 > 0$ such that if $\nu \geq \nu_0$, then there hold the estimates:*

$$\frac{1}{2} \frac{d}{dt} E_0[u] + \frac{1}{2} \widetilde{D}_\xi[w] \leq C \frac{\nu}{\gamma^4} |\phi|_2^2, \quad (5.1)$$

$$\frac{\nu+\widetilde{\nu}}{\gamma^4} |\dot{\phi}|_2^2 \leq C(1 + \frac{\nu+\widetilde{\nu}}{\nu} \omega^2) \widetilde{D}_\xi[w]. \quad (5.2)$$

We proceed to estimate derivatives of u . We introduce some notations. We define $J_2^{(\infty)}[u]$ by

$$J_2^{(\infty)}[u] = -2\text{Re}\langle u, \widehat{B}_\xi \widetilde{Q}u \rangle.$$

In addition, we set

$$\begin{aligned} E_2^{(\infty)}[u] &= \left(1 + \frac{\widetilde{b}_3 \gamma^2}{\nu}\right) E_0[u] + \widetilde{D}_\xi[w], \\ \widetilde{E}_2^{(\infty)}[u] &= E_2^{(\infty)}[u] + J_2^{(\infty)}[u], \end{aligned}$$

where $\widetilde{b}_3$ is a positive constant to be determined later. We note that there exists a constant $\widetilde{b}_3^* > 0$ such that if $\widetilde{b}_3 \geq \widetilde{b}_3^*$ and $\gamma^2 \geq 1$, then

$$\frac{1}{2} E_2^{(\infty)}[u] \leq \widetilde{E}_2^{(\infty)}[u] \leq \frac{3}{2} E_2^{(\infty)}[u].$$

Taking $\widetilde{b}_3$ suitably large, we have the following estimate for $\widetilde{E}_2^{(\infty)}[u]$.

Proposition 5.2. *There exist constants $\widetilde{b}_3 \geq \widetilde{b}_3^*$ and $\nu_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2 \geq 1$, then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \widetilde{E}_2^{(\infty)}[u] + \frac{1}{4} \widetilde{b}_3 \frac{\gamma^2}{\nu} \widetilde{D}_\xi[w] + \frac{1}{2} |\sqrt{\rho_s} \partial_t w|_2^2 \\ & \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) |\phi|_2^2 + \frac{1}{\gamma^2} |\xi|^2 |\phi|_2^2 \right\}. \end{aligned} \quad (5.3)$$

Proposition 5.3. *For $1 \leq m \leq N$, there exist constants $\nu_0 > 0$ and $b > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w|_2^2 \right) + b \frac{\nu + \widetilde{\nu}}{\gamma^4} |\chi_m \partial \dot{\phi}|_2^2 \\ & \quad + \frac{1}{2} \nu (|\chi_m \nabla' \partial w|_2^2 + |\xi|^2 |\chi_m \partial w|_2^2) + \frac{1}{2} \widetilde{\nu} |\chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3)|_2^2 \\ & \leq C \left\{ \left(\eta + \frac{1}{\gamma^2} \right) |\phi|_2^2 + \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \widetilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi|_2^2 + \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi|_2^2 \right. \\ & \quad \left. + \left(\frac{1}{\eta \nu} + \frac{\nu}{\gamma^4 \eta} + \frac{\widetilde{\nu}}{\nu} + 1 \right) \widetilde{D}_\xi[w] \right\} \end{aligned} \quad (5.4)$$

for any $\eta > 0$ with C independent of η .

Proposition 5.4. *For $1 \leq m \leq N$, there exist constants $\nu_0 > 0$ and $b > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi \right|_2^2 \right) + \frac{1}{2} \frac{1}{\nu + \widetilde{\nu}} \left| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \phi \right|_2^2 + b \frac{\nu + \widetilde{\nu}}{\gamma^4} |\chi_m \partial_n \dot{\phi}|_2^2 \\ & \leq C \left\{ \left(\frac{\omega^2}{\nu + \widetilde{\nu}} + \frac{\nu^2}{\gamma^4 (\nu + \widetilde{\nu})} \right) |\phi|_2^2 + \frac{\nu + \widetilde{\nu}}{\gamma^4} |\xi|^2 |\phi|_2^2 + \left(\frac{\widetilde{\nu}}{\nu} + 1 \right) \widetilde{D}_\xi[w] + \frac{\nu}{\nu + \widetilde{\nu}} |\xi|^2 \widetilde{D}_\xi[w] \right. \\ & \quad \left. + \frac{\nu^2}{\nu + \widetilde{\nu}} (|\chi_m \partial_n \partial w|_2^2 + |\chi_m \partial^2 w|_2^2) + \frac{1}{\nu + \widetilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 \right\}. \end{aligned} \quad (5.5)$$

Proposition 5.5. *There exist constants $\nu_0 > 0$ and $b > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi \right|_2^2 + \left| \chi_0 \sqrt{\rho_s} \partial_{x'} w \right|_2^2 \right) + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left| \chi_0 \partial_{x'} \dot{\phi} \right|_2^2 \\
& + \frac{1}{2} \nu \left(\left| \chi_0 \nabla' \partial_{x'} w \right|_2^2 + \left| \xi \right|^2 \left| \chi_0 \partial_{x'} w \right|_2^2 \right) + \frac{1}{2} \tilde{\nu} \left| \chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3) \right|_2^2 \\
& \leq C \left\{ \left(\eta + \frac{1}{\gamma^2} \right) \left| \phi \right|_2^2 + \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \left| \xi \right|^2 \left| \phi \right|_2^2 + \left(\eta + \frac{1}{\gamma^2} \right) \left| \partial_{x'} \phi \right|_2^2 \right. \\
& \quad \left. + \left(\frac{1}{\eta \nu} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] \right\}
\end{aligned} \tag{5.6}$$

for any $\eta > 0$ with C independent of η .

Before proceeding further we introduce an energy functional. We define $E_3^{(\infty)}[u]$ by

$$\begin{aligned}
E_3^{(\infty)}[u] &= \frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi \right|_2^2 + \left| \chi_0 \sqrt{\rho_s} \partial_{x'} w \right|_2^2 \\
&+ \tilde{b}_4 \sum_{m=1}^N \left(\frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi \right|_2^2 + \left| \chi_m \sqrt{\rho_s} \partial w \right|_2^2 \right) + \sum_{m=1}^N \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi \right|_2^2,
\end{aligned}$$

where $\tilde{b}_4$ is a positive constant. Taking $\tilde{b}_4$ suitably large, we have the following estimate for $E_3^{(\infty)}[u]$.

Proposition 5.6. *There exist constants $\nu_0 > 0$, $b > 0$ and $\tilde{b}_4 > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} E_3^{(\infty)}[u] + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left| \partial_{x'} \dot{\phi} \right|_2^2 \\
& + \frac{1}{2} \left\{ \nu \left(\left| \chi_0 \nabla' \partial_{x'} w \right|_2^2 + \left| \xi \right|^2 \left| \chi_0 \partial_{x'} w \right|_2^2 \right) + \tilde{\nu} \left| \chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3) \right|_2^2 \right\} \\
& + \frac{1}{2} \sum_{m=1}^N \left\{ \nu \left(\left| \chi_m \nabla' \partial w \right|_2^2 + \left| \xi \right|^2 \left| \chi_m \partial w \right|_2^2 \right) + \tilde{\nu} \left| \chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3) \right|_2^2 \right\} \\
& \leq C \left\{ \left(\eta + \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) \left| \phi \right|_2^2 + \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \left| \xi \right|^2 \left| \phi \right|_2^2 \right. \\
& \quad + \left(\eta + \frac{1}{\gamma^2} \right) \left| \partial_{x'} \phi \right|_2^2 + \left(\frac{1}{\nu \eta} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] \\
& \quad \left. + \frac{\nu}{\nu + \tilde{\nu}} \left| \xi \right|^2 \tilde{D}_\xi[w] + \frac{1}{\nu + \tilde{\nu}} \left| \sqrt{\rho_s} \partial_t w \right|_2^2 \right\}
\end{aligned} \tag{5.7}$$

for any $\eta > 0$ with C independent of η .

We do not have the estimate for ϕ such as $|\phi|_2 \leq C |\partial_{x'} \phi|_2$ similar to that for ϕ_1 in Section 4. We thus use the estimate for a solution of the Fourier transformed Stokes equation of the case $|\xi|^2 \gg 1$.

Proposition 5.7. *Assume that $(p, h) \in H^1(D) \times H^2(D)$ is a solution of the following Stokes equation*

$$\begin{cases} \nabla' \cdot h' + i \xi h^3 = F^0, \\ (|\xi|^2 - \Delta') h' + \frac{1}{\nu} \partial_{x'} p = \frac{1}{\nu} G', \\ (|\xi|^2 - \Delta') h^3 + \frac{1}{\nu} i \xi p = \frac{1}{\nu} G^3, \\ h|_{\partial D} = 0. \end{cases}$$

There exists a constant $R_0 = R_0(D) > 0$ such that if $|\xi| \geq R_0$, then there holds the following estimate:

$$\begin{aligned} & \frac{1}{\nu^2}|p|_2^2 + \frac{1}{\nu^2}|\xi|^2|p|_2^2 + \frac{1}{\nu^2}|\partial_{x'}p|_2^2 \\ & + |h|_2^2 + |\xi|^2|h|_2^2 + |\partial_{x'}h|_2^2 + \sum_{j=0}^2 |\xi|^{2j}|\partial_{x'}^{2-j}h|_2^2 \\ & \leq CR_0^2\{|F^0|_2^2 + |\xi|^2|F^0|_2^2 + |\partial_{x'}F^0|_2^2 + \frac{1}{\nu^2}|G|_2^2 + |\partial_{x'}h|_2^2\}, \end{aligned}$$

where C is a positive constant independent of $|\xi|$.

Proposition 5.7 can be proved similarly to the proof of Kagei [5, Lemma6.6] and we omit the proof. Applying Proposition 5.7, we have the following estimate.

Proposition 5.8. *There exist constant $\nu_0 > 0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$ and $\omega \leq 1$, then there holds the estimate:*

$$\begin{aligned} & \frac{1}{\nu+\tilde{\nu}}(|\phi|_2^2 + |\xi|^2|\phi|_2^2 + |\partial_{x'}\phi|_2^2) \\ & + \frac{\nu^2}{\nu+\tilde{\nu}}\left(|w|_2^2 + |\xi|^2|w|_2^2 + |\partial_{x'}w|_2^2 + \sum_{j=0}^2 |\xi|^{2j}|\partial_{x'}^{2-j}w|_2^2\right) \\ & \leq CR_0^2\left\{\left(\frac{\omega^2}{\nu+\tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})}\right)|\phi|_2^2 + \frac{\nu}{\nu+\tilde{\nu}}\tilde{D}_\xi[w] \right. \\ & \quad \left. + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})}(|\dot{\phi}|_2^2 + |\xi|^2|\dot{\phi}|_2^2 + |\partial_{x'}\dot{\phi}|_2^2) + \frac{1}{\nu+\tilde{\nu}}|\sqrt{\rho_s}\partial_t w|_2^2\right\} \end{aligned} \tag{5.8}$$

for $|\xi| \geq R_0$, where R_0 is the constant given in Proposition 5.7 and C is a positive constant independent of $|\xi|$.

Proof. We observe that (ϕ, w) satisfies the following Stokes equation

$$\begin{cases} \nabla' \cdot w' + i\xi w^3 = F^0, \\ (\xi^2 - \Delta')w' + \frac{1}{\nu}\nabla'\left(\frac{P'(\rho_s)}{\gamma^2}\phi\right) = \frac{1}{\nu}G', \\ (\xi^2 - \Delta')w^3 + \frac{1}{\nu}i\xi\frac{P'(\rho_s)}{\gamma^2}\phi = \frac{1}{\nu}G^3, \\ w|_{\partial D} = 0, \end{cases}$$

where

$$\begin{aligned} F^0 &= -\frac{1}{\rho_s}\{\partial_t\phi + i\xi v_s^3\phi + (\nabla'\rho_s) \cdot w'\}, \\ G' &= -\rho_s\{\partial_t w' - \frac{\tilde{\nu}}{\rho_s}\nabla'(\nabla' \cdot w' + i\xi w^3) - \frac{P'(\rho_s)}{\gamma^2\rho_s}\phi\nabla'\rho_s + i\xi v_s^3 w'\}, \\ G^3 &= -\rho_s\{\partial_t w^3 - \frac{\tilde{\nu}}{\rho_s}i\xi(\nabla' \cdot w' + i\xi w^3) + i\xi v_s^3 w^3 + \frac{\nu}{\gamma^2\rho_s}\Delta'v_s^3\phi + w' \cdot \nabla'v_s^3\}. \end{aligned}$$

Therefore we get the desired estimate from Proposition 5.7. This completes the proof. $\square$

We finally prove Theorem 3.3.

Proof of Theorem 3.3 Let $\tilde{b}_5$, $\tilde{b}_6$ and $\tilde{b}_7$ be constants satisfying $\tilde{b}_5, \tilde{b}_6, \tilde{b}_7 > 1$. Define $\tilde{E}_4^{(\infty)}[u]$ by

$$\tilde{E}_4^{(\infty)}[u] = \tilde{b}_5 E_3^{(\infty)}[u] + \frac{\tilde{b}_6}{\nu + \tilde{\nu}} \tilde{E}_2^{(\infty)}[u] + \tilde{b}_7 \left(1 + \frac{\tilde{\nu}}{\nu}\right) (1 + |\xi|^2) E_0[u].$$

We compute (5.8) + $\tilde{b}_5 \times \{(5.7) + b \frac{\nu + \tilde{\nu}}{\gamma^4} (1 + |\xi|^2) |\dot{\phi}|_2^2\} + \frac{\tilde{b}_6}{\nu + \tilde{\nu}} \times (5.3) + \tilde{b}_7 (1 + \frac{\tilde{\nu}}{\nu}) (1 + |\xi|^2) \times (5.1)$ then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \tilde{E}_4^{(\infty)}[u] + \frac{\nu^2}{\nu + \tilde{\nu}} \left(|w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2 + \sum_{j=0}^2 |\xi|^{2j} |\partial_{x'}^j w|_2^2 \right) \\ & + \frac{1}{\nu + \tilde{\nu}} (|\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2) + \tilde{b}_5 \frac{\nu + \tilde{\nu}}{\gamma^4} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\partial_{x'} \dot{\phi}|_2^2) \\ & + \frac{\tilde{b}_5}{2} \left\{ \nu (|\chi_0 \nabla' \partial_{x'} w|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3)|_2^2 \right\} \\ & + \frac{\tilde{b}_5}{2} \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial w|_2^2 + |\xi|^2 |\chi_m \partial w|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3)|_2^2 \right\} \\ & + \frac{\tilde{b}_3 \tilde{b}_6}{4} \frac{\gamma^2}{\nu(\nu + \tilde{\nu})} \tilde{D}_\xi[w] + \frac{\tilde{b}_6}{2} \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \frac{\tilde{b}_7}{2} (1 + \frac{\tilde{\nu}}{\nu}) (1 + |\xi|^2) \tilde{D}_\xi[w] \\ & \leq \tilde{C}_4 \left\{ R_0^2 \left(\frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\phi|_2^2 + R_0^2 \frac{\nu}{\nu + \tilde{\nu}} \tilde{D}_\xi[w] + R_0^2 \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4(\nu + \tilde{\nu})} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\partial_{x'} \dot{\phi}|_2^2) \right. \\ & + R_0^2 \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \tilde{b}_5 \left(\eta + \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\phi|_2^2 + \tilde{b}_5 \left(\eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi|_2^2 \\ & + \tilde{b}_5 \left(\eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi|_2^2 + \tilde{b}_5 \left(\frac{1}{\nu \eta} + \frac{\nu}{\gamma^4 \eta} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] + \tilde{b}_5 \frac{\nu}{\nu + \tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w] \\ & + \tilde{b}_5 \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \tilde{b}_5 \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4 \nu} \right) (1 + |\xi|^2) \tilde{D}_\xi[w] \\ & \left. + \tilde{b}_6 \left(\frac{1}{\gamma^2(\nu + \tilde{\nu})} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\phi|_2^2 + \tilde{b}_6 \frac{1}{\gamma^2(\nu + \tilde{\nu})} |\xi|^2 |\phi|_2^2 + \tilde{b}_7 \frac{\nu + \tilde{\nu}}{\gamma^4} (1 + |\xi|^2) |\phi|_2^2 \right\}. \end{aligned}$$

Fix $\tilde{b}_5 > 1$, $\tilde{b}_6 > 1$ and $\tilde{b}_7 > 1$ so large that $\tilde{b}_5 \geq \frac{2\tilde{C}_4}{b} R_0^2$, $\tilde{b}_6 \geq 8\tilde{C}_4 \max\{R_0^2, \tilde{b}_5\}$ and $\tilde{b}_7 > 20\tilde{C}_4 \max\{R_0^2, \tilde{b}_5 \frac{1}{\eta(\nu + \tilde{\nu})}, \tilde{b}_5, \tilde{b}_5 \tilde{b}_5\}$, respectively. We take $\eta > 0$ and $\omega > 0$ sufficiently small such that $\eta < \frac{1}{20\tilde{C}_4 \tilde{b}_5} \frac{1}{\nu + \tilde{\nu}}$ and $\omega^2 < \frac{1}{20\tilde{C}_4} \min\{\frac{1}{R_0^2}, \frac{1}{\tilde{b}_5}\}$, respectively. We assume that $\nu \geq \nu_0$ and $\gamma \geq \gamma_0$ are large enough such that $\nu \geq \nu_0 > 1$ and $\gamma^2 > 20\tilde{C}_4 \max\{\tilde{b}_6(\nu + \tilde{\nu}), (\tilde{b}_5 \frac{1}{\eta})^{\frac{1}{2}}, \sqrt{\tilde{b}_7}(\nu + \tilde{\nu})\}$. We then arrive at the estimate

$$\begin{aligned} & \frac{d}{dt} \tilde{E}_4^{(\infty)}[u] + \frac{\nu^2}{\nu + \tilde{\nu}} \left(|w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2 + \sum_{j=1}^2 |\xi|^{2j} |\partial_{x'}^j w|_2^2 \right) \\ & + \frac{1}{\nu + \tilde{\nu}} (|\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2) + \frac{\nu + \tilde{\nu}}{\gamma^4} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\dot{\phi}|_{H^1}^2) \\ & + \nu (|\chi_0 \nabla' \partial_{x'} w|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3)|_2^2 \\ & + \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial w|_2^2 + |\xi|^2 |\chi_m \partial w|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3)|_2^2 \right\} \\ & + \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \frac{\nu + \tilde{\nu}}{\nu} (1 + |\xi|^2) \tilde{D}_\xi[w] \\ & \leq 0 \end{aligned}$$

for all $\xi \in \mathbf{R}$ with $|\xi| \geq R_0$. We define $E_4^{(\infty)}[u]$ by

$$E_4^{(\infty)}[u] = |\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2 + |w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2.$$

Since

$$\frac{1}{2} \left\{ \left(1 + \frac{\tilde{b}_3 \gamma^2}{\nu}\right) E_0[u] + \tilde{D}_\xi[w] \right\} \leq \tilde{E}_2^{(\infty)}[u] \leq \frac{3}{2} \left\{ \left(1 + \frac{\tilde{b}_3 \gamma^2}{\nu}\right) E_0[u] + \tilde{D}_\xi[w] \right\},$$

$$\frac{1}{2} \frac{1}{\gamma^2} |\partial_{x'} \phi|_2^2 \leq \tilde{C}_5 E_3^{(\infty)}[u] \leq \frac{3}{2} \left(\frac{1}{\gamma^2} |\partial_{x'} \phi|_2^2 + |\partial_{x'} w|_2^2 \right)$$

for a positive constant $\tilde{C}_5$, we see that

$$\frac{1}{2} E_4^{(\infty)}[u] \leq \tilde{C}_6 \tilde{E}_4^{(\infty)}[u] \leq \frac{3}{2} E_4^{(\infty)}[u]$$

for a positive constant $\tilde{C}_6$. We thus see that there exist positive constants ν_0 , γ_0 , ω_0 and d such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2 R_0^2$ and $\omega \leq \omega_0 R_0^{-2}$, then

$$E_4^{(\infty)}[u](t) \leq C e^{-dt} E_4^{(\infty)}[u_0].$$

This gives the desired estimate in Theorem 3.3 for $|\xi| \geq R_0$. For $1 \leq |\xi| \leq R_0$, we obtain the desired estimate from (4.71) with $R = R_0$. This completes the proof. $\square$

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