Improvement of Faugère et al.'s method to solve ECDLP

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Target - ECDLP	Background	Index Calculus Method with Gröbner Basis	Our Contribution
<h2>Improvement of Faugère <i>et al.</i>'s method to solve ECDLP</h2> <p> ◦ Huang Yun-Ju * Christophe Petit * Naoyuki Shinohara † Tsuyoshi Takagi ‡ </p> <p> * Graduate School of Mathematics, Kyushu University * UCL Crypto Group † NICT ‡ Institute of Mathematics for Industry, Kyushu University </p> <p>August 29, 2013</p>			
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Target - ECDLP	Background	Index Calculus Method with Gröbner Basis	Our Contribution
<h2>Abstract</h2>			
<ul style="list-style-type: none"> • Target : ECDLP problem. • Motivation : A new technique for index calculus method algorithm to solve ECDLP proposed by Faugère <i>et al.</i> at Eurocrypt 2012. • Contribution : <ol style="list-style-type: none"> 1. Give a new idea to improve the algorithm proposed by Faugère <i>et al.</i> 2. Implements different strategies solving ECDLP and compares them. 			
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Target - ECDLP			
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Target - ECDLP	Background	Index Calculus Method with Gröbner Basis	Our Contribution
Elliptic Curve Discrete Log Problem (ECDLP)			
Let F_{2^n} is a binary field of prime degree n over F_2 .			
Let $E_{\alpha,\beta} : y^2 + xy = x^3 + \alpha x^2 + \beta$ over field F_{2^n} , where $\alpha, \beta \in F_{2^n}$.			
Given $P \in E_{\alpha,\beta}$, $Q \in \langle P \rangle$,			
Target			
Find smallest non-negative integer k such that $Q = [k]P$			
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Target - ECDLP	Background	Index Calculus Method with Gröbner Basis	Our Contribution
<h2>Known Algorithm</h2>			
<ul style="list-style-type: none"> • Exhaustive Search Time Complexity : $O(2^n)$ • Pollard-rho Method Time Complexity : $O(2^{\frac{n}{2}})$ • Index Calculus Method Time Complexity : claimed to be sub-exponential $O(2^{cn^{2/3} \log n})$ by Petit <i>et al.</i> at Asiacrypto 2012. 			
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<h2>Generic Index Calculus Method</h2>																	
<div style="background-color: #e6f2ff; padding: 10px;"> <h3>Generic Index Calculus Method</h3> <table> <tr> <td>Input :</td> <td>$P, Q \in E_{\alpha, \beta}$</td> </tr> <tr> <td>Output :</td> <td>$k \in \mathbb{N}$ such that $Q = [k]P$</td> </tr> <tr> <td>phase 1:</td> <td>Setup factor base $F_V = \{P_i \in E_{\alpha, \beta} \mid x(P_i) \in V\}$</td> </tr> <tr> <td>phase 2: Relation Search</td> <td>Find sufficient relations $sol_m = \{\sum_{1 \leq j \leq m} P'_j = [a]P + [b]Q\}$ for random $a, b \in \mathbb{N}$, $P'_j \in F_V$.</td> </tr> <tr> <td>phase 3:</td> <td>Transform the relation to matrix M.</td> </tr> <tr> <td>phase 4:</td> <td>Find reduced echelon form M_- of M.</td> </tr> <tr> <td>phase 5:</td> <td>Solve the relation $[a']P + [b']Q = O$ in M_-. $k = \frac{-a'}{b'}$.</td> </tr> </table> <p>$x(P_i)$ means x-coordinate of P_i.</p> </div>				Input :	$P, Q \in E_{\alpha, \beta}$	Output :	$k \in \mathbb{N}$ such that $Q = [k]P$	phase 1:	Setup factor base $F_V = \{P_i \in E_{\alpha, \beta} \mid x(P_i) \in V\}$	phase 2: Relation Search	Find sufficient relations $sol_m = \{\sum_{1 \leq j \leq m} P'_j = [a]P + [b]Q\}$ for random $a, b \in \mathbb{N}$, $P'_j \in F_V$.	phase 3:	Transform the relation to matrix M .	phase 4:	Find reduced echelon form M_- of M .	phase 5:	Solve the relation $[a']P + [b']Q = O$ in M_- . $k = \frac{-a'}{b'}$.
Input :	$P, Q \in E_{\alpha, \beta}$																
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Generic Index Calculus

Points of $E_{\alpha,\beta}$

Generic Index Calculus Method

$$\begin{array}{cccccc}
 P_1 & P_2 & \dots & P_{\#F_V} & P & Q \\
 \left(\begin{array}{cccccc}
 1 & 1 & & 0 & a_1 & b_1 \\
 1 & 0 & & 0 & a_2 & b_2 \\
 \vdots & & \ddots & \vdots & & \vdots \\
 1 & 1 & \dots & 1 & a_t & b_t
 \end{array} \right) \\
 \Downarrow \text{reduced row echelon form} \\
 \left(\begin{array}{cccccc}
 1 & 0 & & 0 & & \\
 0 & 1 & & 0 & & \\
 \vdots & & \ddots & \vdots & & \\
 0 & 0 & \dots & 1 & & \\
 0 & 0 & & 0 & a' & b'
 \end{array} \right)
 \end{array}$$

Semaev's Polynomials[1]

Property - Semaev's summation polynomial

For $R = [a]P + [b]Q$, **Semaev's summation polynomials** s_{m+1} are multivariate polynomials where :

$\forall x_1, \dots, x_m \in F_{2^n}$,

$$s_{m+1}(x_1, x_2, \dots, x_m, x_r) = 0$$

if and only if $\exists P'_j, 1 \leq j \leq m$ such that

$$\sum_{1 \leq j \leq m} P'_j + R = O,$$

where $x_j = x(P'_j)$, $x_r = x(R)$.

The problem to find P'_j s.t. $\sum P'_j = R$ is now reduced to solve $s_{m+1}(x_1, \dots, x_m, x_r) = 0$. x_j is variable and x_r is known value.

Version by Faugère *et al.* (FPPR)

In Eurocrypt 2012, Faugère, Perret, Petit and Renault proposed a new version to solve the Semaev's summation polynomials by Gröbner basis for phase 2 (Relation Search).

Variable Rewritten

We can regard F_{2^n} as the vector space defined by the basis $\{v_0, v_1, \dots, v_{n'-1}\}$.

Variable Substitution

Let $x_j = x(P_j^i)$, $P_j^i \in F_v$, if we rewrite $x_j = \sum_{0 \leq \ell \leq n'-1} c_{j,\ell} v_\ell$, then

$$s_{m+1}(x_1, \dots, x_m, x_r) = s_{m+1}(\sum_{0 \leq \ell \leq n'-1} c_{1,\ell} v_\ell, \dots, \sum_{0 \leq \ell \leq n'-1} c_{m,\ell} v_\ell, \sum_{0 \leq \ell \leq n-1} r_\ell v_\ell)$$

where $c_{j,\ell} \in F_2$ is unknown and r_ℓ is known.

Multivariable Polynomial System

Multivariable Polynomial System

$$s_{m+1}(\sum_{0 \leq \ell \leq n'-1} c_{1,\ell} v_\ell, \dots, \sum_{0 \leq \ell \leq n'-1} c_{m,\ell} v_\ell, \sum_{0 \leq \ell \leq n-1} r_\ell v_\ell) = f_0(c_{j,\ell})v_1 + f_1(c_{j,\ell})v_2 + \dots + f_{n-1}(c_{j,\ell})v_n$$

where $1 \leq j \leq m, 1 \leq \ell \leq n', c_{j,\ell} \in F_2$

$$s_{m+1} = 0 \text{ over } F_{2^n} \iff f_0 = 0, f_1 = 0, f_{n-1} = 0 \text{ over } F_2.$$

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Symmetric function			
Using the fact that Semaev's summation polynomials are symmetric, substitute the polynomials with elementary symmetric functions.			
For example :			
$s_3 = (x_1x_2 + x_1x_r + x_2x_r)^2 + x_1x_2x_r + \beta$ $= (\sigma_2 + \sigma_1x_r)^2 + \sigma_2x_r + \beta$			
where			
$\sigma_1 = x_1 + x_2, \sigma_2 = x_1x_2, \beta$ is the parameter of $E_{\alpha,\beta}$.			
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Rewritten system for symmetric function

- Variables rewritten

$$x_1 = c_{1,0}v_1 + c_{1,1}v_2 + \dots + c_{1,n'-1}v_{n'}$$

$$x_2 = c_{2,0}v_1 + c_{2,1}v_2 + \dots + c_{2,n'-1}v_{n'}$$

$$\dots$$

$$x_m = c_{m,0}v_1 + c_{m,1}v_2 + \dots + c_{m,n'-1}v_{n'}$$
- Symmetric function rewritten

$$\sigma_1 = d_{1,0}v_1 + d_{1,1}v_2 + \dots + d_{1,n-1}v_n$$

$$\sigma_2 = d_{2,0}v_1 + d_{2,1}v_2 + \dots + d_{2,n-1}v_n$$

$$\dots$$

$$\sigma_m = d_{m,0}v_1 + d_{m,1}v_2 + \dots + d_{m,n-1}v_n$$
- Relation of variables and symmetric function

$$d_{1,0} = f_{1,0}(c_{i,j})$$

$$d_{1,1} = f_{1,1}(c_{i,j})$$

$$\dots$$

$$d_{m,n-1} = f_{m,n-1}(c_{i,j})$$

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Symmetric function

- Using symmetric function for s_{m+1} is not a new idea. Gaudry, Diem, Joux and Vitse proposed this in composite extension degree.[2, 3, 4, 5]
- However, in prime extension degree makes the number of variables and number of polynomials grows too large. This makes it impracticable.

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Special factor base V

Let $F_{2^n} = F_2[\omega]/h(\omega)$, where $h(\omega)$ is an irreducible polynomial of prime degree n over F_2 .

Using the special factor base $V = \{1, \omega, \dots, \omega^{n'-1}\}$.

Rewritten system for symmetric function

- Variables rewritten

$$x_1 = c_{1,0} + c_{1,1}\omega + \dots + c_{1,n'-1}\omega^{n'-1}$$

$$x_2 = c_{2,0} + c_{2,1}\omega + \dots + c_{2,n'-1}\omega^{n'-1}$$

...

$$x_m = c_{m,0} + c_{m,1}\omega + \dots + c_{m,n'-1}\omega^{n'-1}$$

- Symmetric function rewritten

$$\sigma_1 = d_{1,0} + d_{1,1}\omega + \dots + d_{1,n'-1}\omega^{n'-1}$$

$$\sigma_2 = d_{2,0} + d_{2,1}\omega + \dots + d_{2,n'-2}\omega^{2n'-2}$$

...

$$\sigma_2 = d_{m,0} + d_{m,1}\omega + \dots + d_{m,n'-m}\omega^{n-m}$$

- Relation of variables and symmetric function

$$d_{1,0} = f_{1,0}(c_{i,j})$$

$$d_{1,1} = f_{1,1}(c_{i,j})$$

...

$$d_{m,n'-m} = f_{m,n'-m}(c_{i,j})$$

Symmetric function with specific vector base V

	s_{m+1}	s'_{m+1}	s'_{m+1} with specific V
#var	mn'	$mn' + mn$	$mn' + (n' - 1)\frac{m(m+1)}{2} + m$
#poly	n	$n + mn$	$n + (n' - 1)\frac{m(m+1)}{2} + m$
$\deg_{reg}$	7 or 6	4 or 3	4 or 3

Table: Comparison for different multivariate polynomial system

The time and memory costs are respectively roughly $\#var^{2*\deg_{reg}}$ and $\#var^{3*\deg_{reg}}$.

Experimental Results

CPU : AMD Opteron 6276*4, 16 cores, 2.3GHz, L3 cache 16MB
 OS : CentOS 6.3
 RAM : 512 GB
 Platform : Magma V2.18-9 64-bit version

Experimental Results

Using Magma to finding one relation $\sum P_i = [a]P + [b]Q$.

	n	n'	sol: yes						
			D_{reg}	var	poly	mono	t_{trans}	t_{groe}	mem
Imp_{FPPR}	23	3	6	9	23	2792.97	5.47	1.06	29.10
Imp_{Ours}	23	3	3	24	38	1079.60	0.91	1.04	15.59
Imp_{FPPR}	53	3	6	9	53	6358.94	12.86	1.03	72.06
Imp_{Ours}	53	3	3	24	68	2348.50	2.12	0.79	24.89
Imp_{FPPR}	23	4	6	12	23	12059.19	21.06	6.83	95.66
Imp_{Ours}	23	4	3	33	44	2173.29	1.83	3.19	29.63
Imp_{FPPR}	53	4	6	12	53	27655.34	50.63	1.86	272.55
Imp_{Ours}	53	4	3	33	74	4701.09	4.19	1.75	40.46

Table: Comparison of the relation search ($m = 3$, $n' = 3, 4$) with two strategies, Imp_{FPPR} and Imp_{Ours} . Units are sec and MB

Experimental Results

Using Magma to finding one relation $\sum P_i = [a]P + [b]Q$.

	n	n'	sol: yes						
			D_{reg}	var	poly	mono	t_{trans}	t_{groe}	mem
Imp_{FPPR}	23	5	7	15	23	40168.90	64.67	70.46	475.55
Imp_{Ours}	23	5	4	42	50	3572.00	3.01	157.86	323.60
Imp_{FPPR}	53	5	6	15	53	91642.50	147.66	80.76	810.08
Imp_{Ours}	53	5	3	42	80	8034.10	6.83	6.68	59.58
Imp_{FPPR}	23	6	7	18	23	107008.67	163.45	3888.70	6656.13
Imp_{Ours}	23	6	4	51	56	5270.00	4.36	5150.12	4791.31
Imp_{FPPR}	53	6	7	18	53	245891.33	366.92	2967.03	7311.44
Imp_{Ours}	53	6	3	51	86	11748.00	10.48	34.82	151.04

Table: Comparison of the relation search ($m = 3$, $n' = 4, 5$) with two strategies, Imp_{FPPR} and Imp_{Ours} . Units are sec and MB

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Experimental Results

Using Magma to solve ECDLP.

n	$\#E_{\alpha,\beta}$	Imp_{FPPR}	Imp_{Ours}
7	4*37	1.574	0.864
11	4*523	8.625	6.702
13	4*2089	49.698	31.058
17	4*32941	2454.470	1364.742
19	4*131431	22474.450	9962.861

Table: Comparison of two ECDLP strategies, Imp_{FPPR} and Imp_{Ours} . The last two columns are computing time in seconds.

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Conlusion

- This work has been accepted by IWSEC2013.
- We give the experimental evidence of our improvements.
- Future work - parallization.

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Thanks

Q & A

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Reference III
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