# Two controller design procedures using SDP and QE for a Power Supply Unit

Matsui, Yoshinobu
Fujitsu Laboratories Ltd

Iwane, Hidenao
Fujitsu Laboratories Ltd

Anai, Hirokazu
Fujitsu Laboratories Ltd | Kyushu University



出版情報：COE Lecture Note. 49, pp.43-51, 2013-08-09. 九州大学マス・フォア・インダストリ研究所
バージョン：
権利関係：

# Two controller design procedures using SDP and QE for a Power Supply Unit

Yoshinobu Matsui[*1,a)] Hidenao Iwane[*1,b)] Hirokazu Anai[*1,*2,c)]

**Abstract:** In this paper, we propose two controller design procedures using semi-definite programing (SDP) and quantifier elimination (QE), respectively. We consider to design controllers for a principal circuit in a power supply unit as an example. In general, a controller design problem is given as a problem finding a controller that satisfies given specifications in the open-loop transfer function's frequency characteristic. This is so-called an open-loop shaping problem in linear control theory. There exist some numerical methods for solving the problem using SDP. We propose an SDP-based controller design method via generalized Kalman-Yakubovich-Popov (GKYP) lemma. These SDP-based methods are effective for finding a feasible controller efficiently, but we cannot describe exact mathematical constraints for the required specifications by these methods.

In order to obtain exact controller's feasible regions for the required specifications, we describe the specifications as exact constraints formulated by sign definite conditions (SDCs) and solve them symbolically using QE.

**Keywords:** Open-loop shaping design problem, Linear matrix inequality, Semi-definite programing, Sign definite condition, Quantifier elimination

## 1. Introduction

The open-loop shaping design problem is a problem finding a controller, in a feedback control system, that satisfies given specifications in the open-loop transfer function's frequency characteristic. The open-loop shaping design problem for a single input and single output linear time-invariant system (SISO-LTI system) is a popular controller design problem in actual control system designs. Many control performances' characteristics are described by the open-loop transfer function's frequency characteristic. These are given as specifications.

Many methods for solving the problem have been proposed. The following methods are typical methods.

- The classical open-loop shaping design procedures in the classical linear control theory.
- The $H^\infty$ mixed sensitivity design procedure [4] and the $H^\infty$ loop shaping design procedure [13] in the $H^\infty$ control theory.
- The design procedure using the generalized Kalman-Yakubovich-Popov (GKYP) lemma [11].
- The mixed sensitivity and Hurwitz stability design procedure using quantifier elimination (QE) [1], [3].

In modern linear control theory, the controller design procedures using semi-definite programing (SDP) have become the mainstream. The typical example is the procedure using the GKYP lemma. However, in the open-loop shaping design problem, we cannot describe exact mathematical constraints for the given specifications by the procedures using SDP, because SDP belongs to the convex programing problem. On the other hand, we may describe exact mathematical constraints by supposing to use QE and get the controller's exact feasible region.

In this paper, we propose two controller design procedures using SDP and QE for the open-loop shaping design problem. We apply these procedures to a controller design problem in a power supply unit, respectively and compare them. The controller design problem is the open-loop shaping design problem and many specifications are given in the open-loop transfer function's frequency characteristic.

We use the design procedure using the GKYP lemma as the SDP procedure. On the other hand, we formulate exact constrains for the required specifications by sign definite conditions (SDCs) and solve them exactly using QE. We note that we use a special QE algorithm for SDCs [1], [8].

We use the following notations. $\mathbb{R}$ denotes the field of real numbers. $\mathbb{C}$ denotes the field of complex numbers. $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}^{n\times m}$ denotes the ring of $n\times m$ matrices, where $n, m \in \mathbb{N}$. $j$ denotes an imaginary unit. $\mathcal{L}[\cdot]$ denotes a Laplace transform. For a square-integrable function $f(t)$,

$$\mathcal{L}[f(t)] := \int_0^\infty f(t)\exp(-st)dt,$$

where $s \in \mathbb{C}$, $t \in \mathbb{R}$. For a matrix $M$, its positive definiteness and transpose and complex conjugate transpose are denoted by $M > 0$, $M^T$ and $M^*$, respectively. For a vector $v \in \mathbb{R}^n$,

*1 FUJITSU LABORATORIES LTD.
*2 Kyushu University.
a) m.yoshinobu@jp.fujitsu.com
b) iwane@jp.fujitsu.com
c) anai@jp.fujitsu.com

the real and imaginary parts are denoted by $\Re v$ and $\Im v$, the transpose is denoted by $v^T$. For matrices $T$ and $S$, $S \otimes T$ denotes the Kronecker product.

We note that we use the following control theory's terms. For a real coefficient rational polynomial $h(s)$, the gain, the phase and the angular frequency are denoted as $|h(s)|$, $\angle h(s)$ and $\frac{d}{dt}\angle h(s)$, respectively. We call $x = 20\log_{10}|h(s)|$ the gain is $x$ dB, $x = \frac{180}{\pi}\angle h(s)$ the phase is $x$ degree and $x = \frac{1}{2\pi}\frac{d}{dt}\angle h(s)$ the angular frequency is $x$ Hz, respectively.

## 2. Power supply unit

A principal circuit in a power supply unit is an AC/DC converter that converts alternating current (AC) input voltage to direct current (DC) output voltage. This mainly consists of the former power factor correction (PFC) circuit and the latter DC/DC converter circuit [5]. In this paper, we focus on the DC/DC back converter.

### 2.1 DC/DC back converter

The purpose of a DC/DC back converter is a conversion of the voltage level from the high DC input voltage $V_{\text{in}}$ which is the output voltage of the former PFC circuit to a desired low DC output voltage $V_{\text{out}}$. This conversion must be done electrical efficiently in a power supply unit.

Fig. 1 shows a simplified equivalent circuit. This shows operating principles of a DC/DC back converter. $S$ signifies a switch. The switching is done as follows:

$$S(t) = \begin{cases} 1, & kh \le t < (k + d[k])h, \\ 0, & (k + d[k])h \le t < (k+1)h, \end{cases}$$

where $t \in \mathbb{R}$ is continuous time (in seconds), $h \in \mathbb{R}$ is a constant period (in seconds), $k \in \mathbb{N}$ is discrete time, $d[k] \in \mathbb{R}$ is called a duty ratio. When $S$ connects with 1 (we call

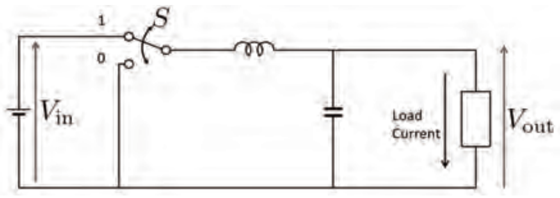


**Fig. 1** Simplified equivalent circuit

this state ON), the output voltage level rises, because the load is connected with $V_{\text{in}}$. On the other hand, when $S$ connects with 0 (OFF), the output voltage falls. The ON time changes at every period. This ON time ratio at each constant period is duty ratio. In order to make the level of $V_{\text{out}}$ follow the desired level, we control the duty ratio. The level of $V_{\text{out}}$ must follow the desired level robustly in some unpredictable situations. For example, PFC influences a DC/DC back converter or the load electrical changes and so on. Therefore, feedback control is used.

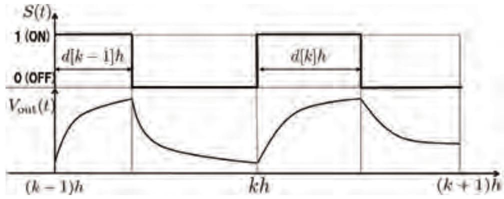


**Fig. 2** Operating principle

### 2.2 Piecewise state space model

In this paper, we employ the normal equivalent circuit in Fig. 3 as an original model of a DC/DC back converter, where $C$ is a condenser (unit is F), $L$ is a coil (unit is H), $I_L$ is an electric current in $L$, $V_C$ is a voltage in $C$, and $R$, $r_C$, $r_q$, $r_d$, $r_L$ are resistances (units are ohm). Here, we define the load electric current as $\frac{V_{\text{out}}}{R}$.

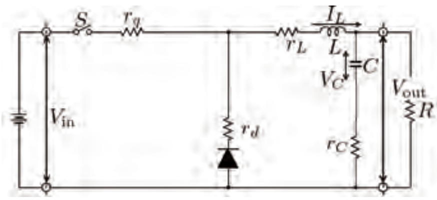


**Fig. 3** Normal equivalent circuit

We get the following piecewise state space model from the normal equivalent circuit by Kirchhoff's law [12].

$$\frac{d}{dt}\xi(t) = \begin{cases} A_1\xi(t) + B_1 V_{\text{in}}, & S(t) = 1, \\ A_2\xi(t), & S(t) = 0, \end{cases}$$

$$V_{\text{out}}(t) = \begin{cases} C_V\xi(t), & S(t) = 1, \\ C_V\xi(t), & S(t) = 0, \end{cases}$$

where $\xi(t) := [I_L(t) \quad V_C(t)]^T$. $A_1 \in \mathbb{R}^{2\times2}$, $A_2 \in \mathbb{R}^{2\times2}$, $B_1 \in \mathbb{R}^{2\times1}$, $C_V \in \mathbb{R}^{1\times2}$ are defined as follows:

$$A_1 := \begin{bmatrix} -\frac{(r_q + r_L + \alpha r_C)}{L} & -\frac{\alpha}{L} \\ \frac{\alpha}{C} & -\frac{\alpha}{CR} \end{bmatrix},$$

$$A_2 := \begin{bmatrix} -\frac{(r_d + r_L + \alpha r_C)}{L} & -\frac{\alpha}{L} \\ \frac{\alpha}{C} & -\frac{\alpha}{CR} \end{bmatrix},$$

$$B_1 := \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad C_V := \begin{bmatrix} \alpha r_C & \alpha \end{bmatrix},$$

where

$$\alpha := \frac{R}{R + r_C}.$$

**Remark 1** Note that the electrical efficiency of a DC/DC back converter is deeply related to $r_q$ and $r_d$. The study about the relationship between the electrical efficiency and the control performance would be one of our future works.

### 2.3 Averaged state space model

In order to design a controller by linear control theory, we need to employ a linear time-invariant system model. Here, we employ the averaged state space model as a linear time-invariant system model. When we design a controller of a DC/DC back converter, the averaged state space model is a popular model.

We get the following averaged state space model from the piecewise state space model by defining $\eta(t)$ as the average of $\xi(t)$ in a period [12].

$$\begin{cases} \dfrac{d}{dt}\Delta\eta(t) = A\Delta\eta(t) + B\Delta d(t), \\ \Delta V_{\mathrm{out}}(t) = C_V\Delta\eta(t), \end{cases} \tag{1}$$

where $\Delta\eta(t)$, $\Delta V_{\mathrm{out}}(t)$, $\Delta d(t)$ are small perturbation's signals of $\eta(t)$, $V_{\mathrm{out}}(t)$, $d[k]$, respectively. $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$ are defined as follows:

$$A := d_0 A_1 + (1 - d_0)A_2,$$

$$B := (A_1 - A_2)\eta_0 + B_1 V_{\mathrm{in}},$$

where

$$d_0 := \frac{(r_d + r_L + R)V_0}{RV_0 - (r_q - r_d)V_0}, \tag{2}$$

$$\eta_0 := -A^{-1}B_1 V_{\mathrm{in}} d_0, \tag{3}$$

where $V_0$ is the desired level of the output voltage. Let us suppose that the differential function of $\eta(t)$ is always 0 in a steady state, then we get (2) and (3).

We define Laplace transforms of $\Delta V_{\mathrm{out}}(t)$ and $\Delta d(t)$ as follows:

$$\Delta\hat{V}_{\mathrm{out}}(s) := \mathcal{L}[\Delta V_{\mathrm{out}}(t)],$$

$$\Delta\hat{d}(s) := \mathcal{L}[\Delta d(t)].$$

We get the following equation from (1).

$$\Delta\hat{V}_{\mathrm{out}}(s) = P(s)\Delta\hat{d}(s),$$

where

$$P(s) := C_V(sI - A)^{-1}B.$$

This $P(s)$ is a transfer function model for the averaged state space model.

**Remark 2** Note that the averaged state space model is an accurate model in the only low frequency band for the piecewise state space model [14]. A DC/DC back converter its conversion is done electrical efficiently needs also high frequency band model to design the controller. An accurate model in the whole frequency band for the piecewise state space model by sampled-data control theory [15] would be one of our future works.

## 3. Controller design problem

In this section, we show a controller design problem for the normal DC/DC back converter. The original controller design problem is a problem finding a controller, in a feedback control system (shown in §3.1), satisfying the requirement that $V_{\mathrm{out}}$ follows $V_0$ under the following situations.

- **Situation 1** The PFC influences the DC/DC back converter.
- **Situation 2** The load electric current is time-independent.

The control performances for this requirement are described by the specifications in the open-loop transfer function's frequency characteristic as shown in §3.2.

### 3.1 Feedback control system

Fig. 4 shows a feedback control system for $P(s)$, where $r$ is a reference signal, $K(s)$ is a controller to be designed. In this paper, we consider the following one order controller:

$$K(s) = \frac{b_{K_0}s + b_{K_1}}{s + a_{K_1}},$$

where $b_{K_0} \in \mathbb{R}$, $b_{K_1} \in \mathbb{R}$, $a_{K_1} \in \mathbb{R}$ are design parameters for the requirement. We define the open-loop transfer function

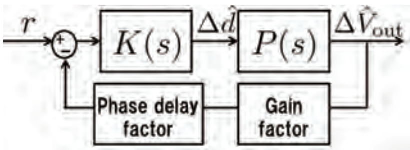


**Fig. 4** Feedback control system

as follows:

$$L(s) := P(s)\times(\text{gain factor})\times(\text{phase delay factor})\times K(s).$$

Here, the open-loop transfer function's frequency characteristic is $L(j\omega)$, where $\omega$ is the angular frequency (unit is Hz). In this paper, we assume that the phase delay factor and the gain factor is given as $\exp(-1.4\times 10^{-5}s)$ and $9.294\times 10^{-2}$, respectively. See Fig. 5. We define $G(\cdot)$ as follows:

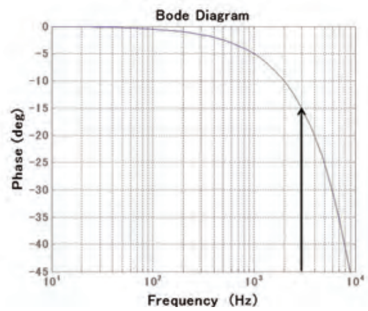


**Fig. 5** Phase delay factor

$$G(\cdot) = P(\cdot)\times(\text{gain factor}).$$

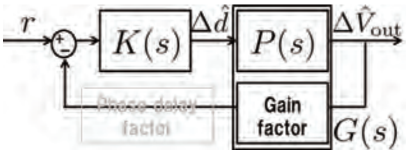


**Fig. 6** Feedback control system for $G(s)$

### 3.2 Open-loop shaping design problem

For $L(j\omega)$, when $\omega$ increases from 0 to $\infty$, we call the trajectory in a complex plain a Nyquist diagram. The stability margin is defined by the separation condition between the trajectory and $-1+0j$ for the gain and the phase, respectively. These stability margins are called the gain margin and the phase margin, respectively (Fig. 7). The closed-loop system is internal stable when the following lemma holds.

**Lemma 1 (Nyquist's satbility criterion [4])** The closed-loop is internal stable as long as the intersection point axis between the trajectory and the negative real axis $> -1$.

In order to satisfy the requirement mentioned above, $L(j\omega)$ must satisfy the following specifications.

- **Specification 0** The closed-loop system is internal stable.
- **Specification 1** The gain$>$ 45 dB when $0 \leq \omega \leq 1$.
- **Specification 2** The gain$>$ 25 dB when $1 \leq \omega \leq 100$.
- **Specification 3** The gain crossover frequency $>$3000.
- **Specification 4** The phase margin (PM) $>$45 degree.
- **Specification 5** The gain margin (GM) $>$7 dB.

Here, we call the problem finding a controller that satisfies these specifications "the open-loop shaping design problem". These specifications are described in a Nyquist diagram of $L(j\omega)$ (Fig. 7).

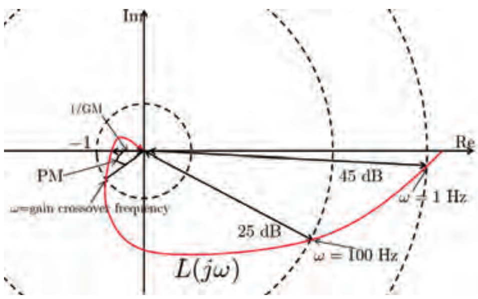


**Fig. 7** Specifications described by a Nyquist diagram

These specifications are specified on the circuit designers experiences. We show what these specifications mean for the original controller design problem, but do not prove them mathematically in this paper.

First, specification 0 must be satisfied so as to make the control system stable. Second, specification 1 must be satisfied so as to make $V_{\text{out}}$ follow $r$. Third, the influence by the PFC becomes smaller when specification 2 is satisfied. Finally, even if the load electric current changes, $V_{\text{out}}$ follows $r$ robustly, when specifications 3,4,5 are satisfied. There is a trade-off between specification 3 and specification 4. The larger the gain crossover frequency and the phase margin are, the better the control performance for the load electric current changing is.

## 4. SDP and QE

In modern linear control theory, the controller design procedures using SDP have become the mainstream [2].

SDP is a convex optimization. On the other hand, QE is a symbolic and algebraic algorithm to deal with first-order formulas over $\mathbb{R}$ and can solve non-convex optimization exactly [3], [9], [10].

For the exact optimal controller design, QE is better, but QE requires enormous computation time.

## 5. Mathematical formulation

In this section, we formulate mathematical constraints for the open-loop shaping deign problem's specifications in two formulations: a linear matrix inequality (LMI) and a sign definite condition (SDC). We propose two procedures to solve the LMI formulation problems by using SDP and the SDC formulation problems by using QE.

An LMI is a matrix inequality that can be come down to the following inequality.

$$F(z) > 0,$$

where $F(z) := F_0 + z_1F_1 + \cdots + z_nF_n$. Each $F_i \in \mathbb{R}^{n\times n}$ is a symmetric matrix, and $z := [z_1, \ldots, z_n]^T$ is a variable vector. A mathematical optimization problem whose constraints are formulated by LMIs and objective functions are linear in $z$, belongs to the class of SDP.

An SDC is defined for a real coefficient rational polynomial $f(x)$ as follows:

$$\forall x \geq 0\,(f(x) > 0).$$

This can be described by a first-order formula.

$$\forall x(x \geq 0 \rightarrow f(x) > 0).$$

We can solve an SDC efficiently by using QE which uses the Strum-Habicht sequence [1], [8].

### 5.1 LMI formulation

The open-loop shaping design problem's specifications can be formulated by LMI constraints using the generalized Kalman-Yakubovich-Popov (GKYP) lemma. Here, we introduce a special case of the GKYP lemma.

**Lemma 2** The following inequality is called a frequency domain inequality (FDI) for $G(s)$.

$$\begin{bmatrix} G(s) & I \end{bmatrix} \Pi \begin{bmatrix} G(s) & I \end{bmatrix}^* < 0, \ \forall s \in \Lambda(\Phi_c, \Psi). \quad (4)$$

$\Pi$ and $\Lambda(\Phi_c, \Psi)$ are defined as follows:

$$\Pi(a_g, b_g, \gamma) := \begin{bmatrix} 0 & a_g - jb_g \\ a_g + jb_g & -2\gamma \end{bmatrix}.$$

$$\Lambda(\Phi_c, \Psi) := \{\lambda \in \mathbb{C} | \sigma(\lambda, \Phi_c) = 0,\ \sigma(\lambda, \Psi) \geq 0\},$$

where

$$\sigma(\lambda, \Phi_c) := \begin{bmatrix} \lambda^* & 1 \end{bmatrix} \Phi_c \begin{bmatrix} \lambda \\ 1 \end{bmatrix}, \Phi_c := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\Psi(\omega_L, \omega_H) := \begin{bmatrix} -1 & j(\omega_L + \omega_H)/2 \\ -j(\omega_L + \omega_H)/2 & -\omega_L \omega_H \end{bmatrix},$$

A necessary and sufficient condition for (4) is given as follows:

There exist a symmetric matrix $P \in \mathbb{R}^{3\times3}$ and a positive definite matrix $Q \in \mathbb{R}^{3\times3}$ such that $\mathcal{G}(\Psi, \Pi) < 0$, where

$$\mathcal{G}(\Psi, \Pi) := W(P, Q) + V,$$

$$W(P, Q) := \begin{bmatrix} A_G & I \\ C_G & 0 \end{bmatrix} (\Phi_c^T \otimes P + \Psi^T \otimes Q) \begin{bmatrix} A_G & I \\ C_G & 0 \end{bmatrix}^T,$$

$$V := \begin{bmatrix} 0 & B_G(a_g - jb_g) \\ B_G^T(a_g + jb_g) & 2a_g D_G - 2\gamma \end{bmatrix},$$

where the set $\{A_G, B_G, C_G, D_G\}$ is the state-space representation of $G(s)$.

**Proof** See [11].

We explain what Lemma 2 means. Equation (4) means the following convex region in a complex plain in which $G(j\omega)$ occurs, that is a feasible region for $G(j\omega)$.

$$a_g \Re G(j\omega) + b_g \Im G(j\omega) < \gamma,\ \omega_L \leq \omega \leq \omega_H.$$

$\Phi$ decides $s = j\omega$, $\Psi$ decides the interval $\omega_L \leq \omega \leq \omega_H$, $\Pi$ decides the convex region.

We define the following matrix inequalities.

$$\mathcal{G}_1 := \{\mathcal{G}(\Psi, \Pi) < 0 | \Psi(L_1, H_1),\ \Pi(-1, 0, -g_1)\},$$

$$\mathcal{G}_2 := \{\mathcal{G}(\Psi, \Pi) < 0 | \Psi(L_2, H_2),\ \Pi(0, 1, -g_2)\},$$

$$\mathcal{G}_3 := \{\mathcal{G}(\Psi, \Pi) < 0 | \Psi(L_3, H_3),\ \Pi(0, 1, -g_3)\} \text{ and}$$

$$\mathcal{G}_4 := \{\mathcal{G}(\Psi, \Pi) < 0 | \Psi(L_4, \infty),\ \Pi(-10, 1, \gamma)\},$$

where, $L_1 := 0$, $H_1 := 1\times2\times\pi$, $L_2 := H_1$, $H_2 := 100\times2\times\pi$, $L_3 := H_2$, $H_3 := 3000 \times 2 \times \pi$, $L_4 := H_3$, $g_1 := 10^{45/20}$, $g_2 := 10^{25/20}$, $g_3 := 1$.

Then the following Lemma 3 holds.

**Lemma 3** When $\gamma < 5 - \sqrt{3}/2 \fallingdotseq 4.134$,

$$\text{Specification 1} \leftarrow \mathcal{G}_1, \quad (5)$$

$$\text{Specification 2} \leftarrow \mathcal{G}_2, \quad (6)$$

$$\text{Specification 3} \leftarrow \mathcal{G}_3 \text{ and} \quad (7)$$

$$\text{Specifications 4, 5} \leftarrow \mathcal{G}_4 \quad (8)$$

hold.

**Proof** (5),...,(7) are obvious by Fig. 7 and Fig. 8. For $\gamma < 5 - \sqrt{3}/2$ and (8), see Fig. 9. Note that we must assure the phase margin > 60 degree, because the phase delay factor is given as Fig. 5 (§3.1). Fig. 5 shows the phase delay is 15 degree in 3 kHz.

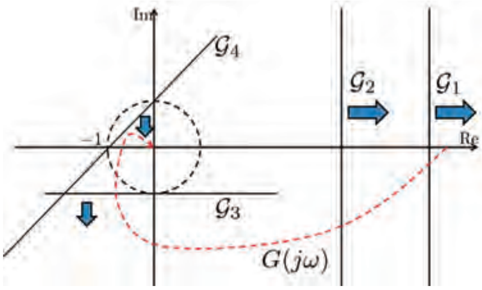


**Fig. 8** Specifications formulated by $\mathcal{G}_1, \ldots, \mathcal{G}_4$

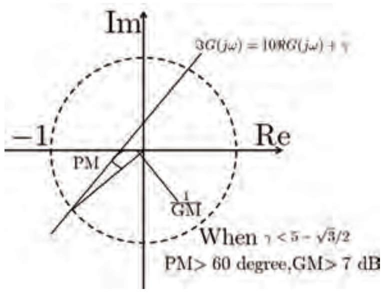


**Fig. 9** Stability margin formulated by $\mathcal{G}_4$

**Remark 3** Note that we can formulate other formulations to the specifications by LMIs using the GKYP lemma. For example we also define $\mathcal{G}_2$ as

$$\mathcal{G}_2 := \{\mathcal{G}(\Psi, \Pi) < 0 | \Psi(L_2, H_2),\ \Pi(1, 1, -\sqrt{2}g_2)\}.$$

However, we cannot express the outside region of a circle by LMIs, because LMIs are based on convex regions.

When we consider parameterizing the controller, $a_{K_1}$, $b_{K_0}$, $b_{K_1}$, $P_i$ and $Q_i$, $i = 1, \ldots, 4$ are parameters. In this case $\mathcal{G}_i < 0$ are not LMIs but are bilinear matrix inequalities (BMIs), because there exists a cross-term between $a_{K_1}$ and $P_i$. BMIs are not SDP. For converting $\mathcal{G}_i < 0$ to LMIs, first we must fix $a_{K_1}$. In this paper, we fix $a_{K_1}$ as 35.34. This $a_{K_1}$ is given by circuit designers experience. Second, we define the following $B_G$

$$B_G := \begin{bmatrix} w \\ Bb_{K_0} \end{bmatrix},$$

where $w$ is a design parameter and $w := b_{K_1} - a_{K_1} b_{K_0}$.

**Remark 4** Note that the controller's pole ($= -35.34$) is not always the best pole for the open-loop shaping design problem. We should consider the case that $a_{K_1}$ is a free parameter and a full-order controller case, but we do not consider the cases, in this paper, it would be one of our future works.

Finally, we can convert $\mathcal{G}_i < 0$ to LMIs $\hat{\mathcal{G}}_i(P_i, Q_i, w, b_{K_0}) < 0$ in case parameterizing the controller and formulate the original open-loop shaping design problem as the following SDP problem, and we can solve it by an interior point method.

**Problem (SDP)** minimize $\gamma$
subject to

$$\begin{bmatrix} \hat{\mathcal{G}}_1(P_1, Q_1, w, b_{K_0}) & & \\ & \ddots & \\ & & \hat{\mathcal{G}}_4(P_4, Q_4, w, b_{K_0}) \end{bmatrix} < 0.$$

We can get $w$, $b_{K_0}$, $P_i$ and $Q_i$ by solving this SDP, and from this $w$, we can parametrize $b_{K_1}$ as $w + a_{K_1} b_{K_0}$. These are optimal controller parameters for the open-loop shaping design problem.

### 5.2 SDC formulation

The open-loop shaping design problem's specifications can be formulated by SDC constraints. We formulate the specifications 2, ... ,6 exactly by SDCs. We define the following SDCs.

$$\mathcal{S}_1 : \forall\omega(L_1 \le \omega \le H_1 \to |G(j\omega)|^2 - g_1^2 > 0), \quad (9)$$

$$\mathcal{S}_2 : \forall\omega(L_2 \le \omega \le H_2 \to |G(j\omega)|^2 - g_2^2 > 0), \quad (10)$$

$$\mathcal{S}_3 : \forall\omega(L_3 \le \omega \le H_3 \to |G(j\omega)|^2 - g_3^2 > 0) \text{ and} \quad (11)$$

$$\mathcal{S}_4 : \forall\omega(L_4 \le \omega \to 3.8\Re G(j\omega) + 1 - \Im G(j\omega) > 0). \quad (12)$$

These show the feasible regions for $G(j\omega)$ as Fig. 10. Note

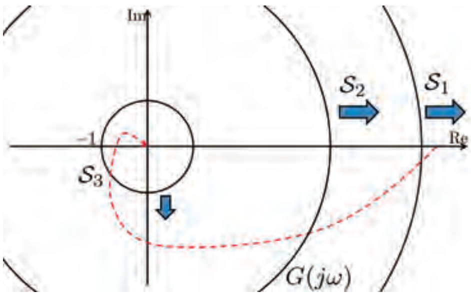


**Fig. 10** Specifications formulated by $\mathcal{S}_1, \dots, \mathcal{S}_3$

that 3.8 in $\mathcal{S}_4$ is given by Lemma 5. Obviously, the following lemma holds.

**Lemma 4** The specifications 2, ... ,6 are formulated by SDCs as follows:

$$\text{Specification } 1 \leftrightarrow \mathcal{S}_1, \quad (13)$$

$$\text{Specification } 2 \leftrightarrow \mathcal{S}_2, \quad (14)$$

$$\text{Specification } 3 \leftrightarrow \mathcal{S}_3 \text{ and} \quad (15)$$

$$\text{Specifications } 4,\ 5 \leftarrow \mathcal{S}_4. \quad (16)$$

**Proof** For specification 1, ... ,3, (13), ... ,(15) obviously hold by Fig. 7 and Fig. 10, respectively.

In order to show why (16) holds, we indicate the following lemma.

**Lemma 5** When the closed-loop system of a feedback system in Fig. 6 (§3.1) is internal stable and $a\Re G(j\omega) + 1 - \Im G(j\omega) > 0$ holds, the phase margin and the gain margin satisfy the followings:

$$\text{PM} \ge 360 \arctan(a)/\pi - 90 \,\text{degree},$$

$$\text{GM} \ge 20 \log_{10}(a) \,\text{dB}. \quad (17)$$

**Proof** When the closed-loop system is internal stable, (17) is obviously holds. See Fig. 11.

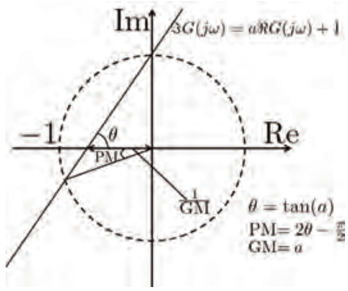


**Fig. 11** Stability margin formulated by $\mathcal{S}_4$

From (17), when $a > 3.8$, the phase margin is over 60 degree and the gain margin is over 7 dB. See Fig. 12. Therefore, when $a > 3.8$, (16) holds.

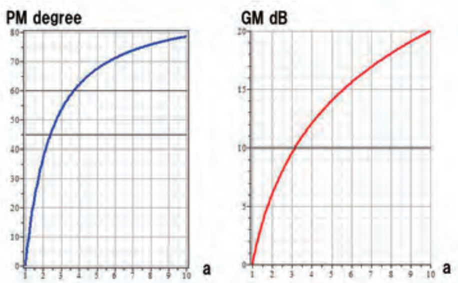


**Fig. 12** $a$ vs PM, GM

We solve the specification 0 algebraically by the Hurwitz stability condition. The following Lemma 6 holds [7].

**Lemma 6** Let a characteristic polynomial for $G(s)$ be $g(s)$. The following two propositions are equivalent.

- The closed-loop system is internal stable.
- All coefficients of $g(s)$ are positive or negative and the all leading principal minor of a Hurwitz matrix for $g(s)$ are positive.

We can get feasible regions of the controller parameters for each specification by using Lemma 6 algebraically and solve $\mathcal{S}_1, \ldots, \mathcal{S}_4$ using QE, and by superposing obtained feasible regions, we can get the feasible regions of the controller for the open-loop shaping design problem's specifications.

## 6. Numerical example

We show a numerical example with the following $P(s)$.

$$P(s) = \frac{4.622 \times 10^7 s + 2.140 \times 10^{12}}{1.128 \times 10^4 s^2 + 1.906 \times 10^8 s + 1.453 \times 10^{12}}.$$

The computational experiments for the following SDP solution (§6.1) was executed on a computer with an Intel (R) Core (TM) i5-2520M CPU 2.5 GHz and 4.0 GByte memory. We solved SDP by LMI control toolbox [6]. The computational experiments for the following QE solution (§6.2) was executed on a computer with an Intel (R) Core (TM) i7-3540M CPU 3.0 GHz and 2.0 GByte memory. We solved QE by our own solver SyNRAC [8].

### 6.1 SDP solution

We get the following optimal controller by solving the SDP problem.

$$K_{\rm gkyp} = \frac{1.944s + 7587}{s + 35.34}, \tag{18}$$

and the optimal $\gamma$,

$$\gamma_{\rm opt} = 3.303. \tag{19}$$

The computing time to obtain the $K_{\rm gkyp}$ is 1.443 seconds.

### 6.2 QE solution

We consider decreasing the degree of the polynomial in $\mathcal{S}_i$ for reducing the computing time.

$$\mathcal{S}_i : \forall\omega(L_i \leq \omega \leq H_i \to |G(j\omega)|^2 - g_i^2 > 0),$$

where $i = 1, 2, 3$. The degrees of the numerator polynomial and the denominator polynomial of $|G(j\omega)|^2 - g_i^2$ are 12 and 12, respectively. We can decrease the degrees to 6 and 6 by substituting $\omega^2$ for $\Omega$, because the numerator and the denominator of $|G(j\omega)|^2 - g_i^2$ are even polynomials. We can decrease the degree of $\mathcal{S}_i$ in $\Omega$ to 6, because the denominator polynomial is always positive.

Feasible regions for $\mathcal{S}_1$ and $\mathcal{S}_2$ are obtained by QE and shown as shaded regions in Fig. 13. We note the feasible region for $\mathcal{S}_2$ is non-convex. The computing time to obtain the feasible regions for $\mathcal{S}_1$ and $\mathcal{S}_2$ are 3.386 seconds and 4.852 seconds, respectively.

The feasible region for $\mathcal{S}_3$ is given as Fig. 14. We note the feasible region for $\mathcal{S}_3$ is non-convex. The computing time to obtain the feasible region for $\mathcal{S}_3$ is 1.794 seconds. In general, a gain-cross over frequency is deeply related to a dead-beat step response. Fig. 14 shows the feasible region for a dead-beat step response is non-convex.

Feasible regions for the Hurwitz stability condition and $\mathcal{S}_4$ are shown in Fig. 15. The superposition of these feasible regions shows a robust stability region that assures PM> 60 and GM> 7 for $G(j\omega)$. The computing time to obtain the feasible region for $\mathcal{S}_4$ is 2.948 seconds.

The superposition of all the feasible regions is given by Fig. 16. This is a feasible region for the open-loop shaping design problem. $K_{\rm gkyp}$ is in the superposition of all the feasible regions.

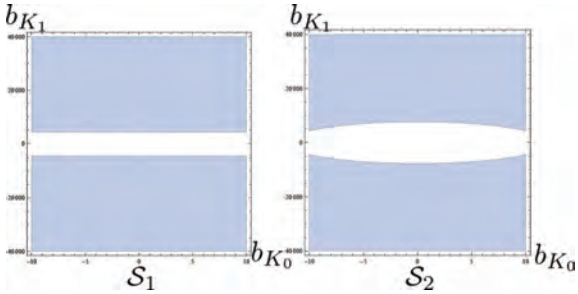


**Fig. 13** Feasible regions for $\mathcal{S}_1$ and $\mathcal{S}_2$

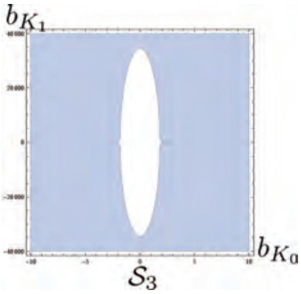


**Fig. 14** Feasible region for $\mathcal{S}_3$

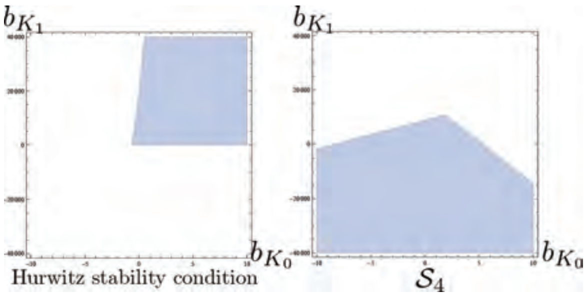


**Fig. 15** Feasible regions for Hurwitz stability and $\mathcal{S}_4$

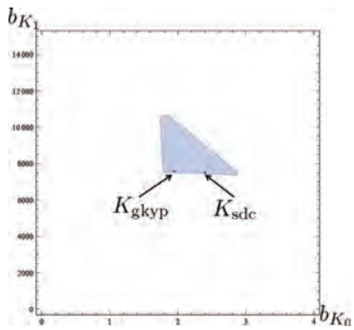


**Fig. 16** Feasible region for the open-loop shaping design problem

### 6.3 Comparison

We select the desired controller from Fig. 16 as follows:

$$K_{\rm sdc} = \frac{2.4s + 7500}{s + 35.34}. \tag{20}$$

The Bode diagram of the open-loop shaped by $K_{\rm sdc}$ and $K_{\rm gkyp}$ are Fig. 17. This shows both controllers designed by the two procedures satisfy the required specifications.

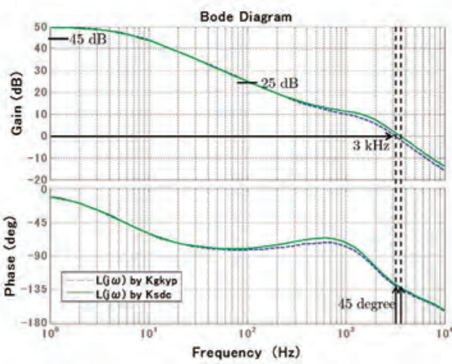


**Fig. 17** Bode diagram

The time response controlled by $K_{\rm sdc}$ and $K_{\rm gkyp}$ are Fig. 18. When the load electric current changing occurs, $V_{\rm out}$ follows $V_0$ robustly. Here, the load electric current changes from 108 to 208, $V_0 = 12$, $K_{\rm gkyp}$ and $K_{\rm sdc}$ are discretized.

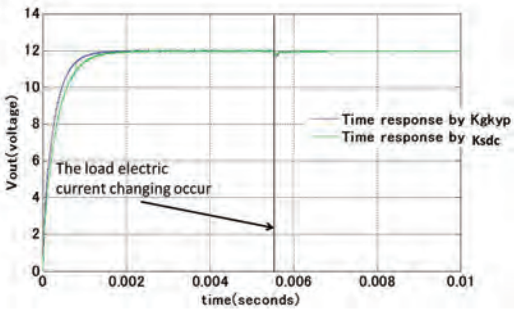


**Fig. 18** Time response

## 7. Conclusion

In this paper, we proposed two controller design procedures using SDP and QE, and compared them by applying to the controller design problem of a normal DC/DC back converter.

We designed controllers that satisfy the desired specifications by the both procedures. We designed an optimal controller by the procedure using SDP numerically. The mathematical constraints and the objective function that we formulated by LMIs were not exact for the desired specifications. We cannot formulate exact (i.e. relaxed expression) mathematical constraints and an objective function by LMIs in principle, because, in the open-loop shaping design problem, many of the desired specifications are non-convex. On the other hand, we can formulate exact mathematical constraints for the many desired specifications by SDCs straight forwardly, and we could get controller's exact feasible regions for the open-loop shaping design problem's specifications by the procedure using a specialized QE. We confirmed the controller's feasible region is non-convex. Therefore, we can design an exact optimal controller for the open-loop shaping problem by the procedure using QE. Hereby, circuit designers can select the best controller for the specifications which they set by their experience even if the specifications are non-convex.

In other words, this means we showed an open-loop shaping design problem for an LTI-system its order is 1/2 and the controller its order is 1/1 can be actually resolved by the procedure using QE as long as the controller's pole is fixed. However, we should consider the case that $a_{K_1}$ is a free parameter and a full-order controller case, respectively as we remarked before. Especially, from the viewpoint of modern linear control theory we should consider the full-order controller case. In modern control theory, the existence of the controller that stabilize the closed-loop in a feedback control system is assured by a full-order controller. Therefore, we should consider at least a full-order controller case for the case that the dynamics of the DC/DC back converter changes significantly.

## References


[1] H. Anai and S. Hara, "A parameter space approach to fixed-order robust controller synthesis by quantifier elimination," *International Journal of Control.*, vol. 79, pp. 1321-1330, 2006.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V.Balakrishnan, "Linear Matrix Inequalities in System and Control Theory," *Studies in Applied Mathematics.*, SIAM, Philadelphia, PA, vol. 15, 1994.

[3] R.F. Caviness and J.R. Johnson (Fds.), "Quantifier Elimination and Cylindrical Algebraic Decomposition, Texts and Monographs in Symbolic Computation," Springer, 1996.

[4] J.C. Doyle, B.A. Francis and A.R. Tannenbaum, "Feedback Control Theory," Prince Hall, 1992.

[5] R.W. Erickson and D. Maksimovic, "Fundamentals of Power Electronics Second Edition," Kluwer Academic Publishers, 2000.

[6] P. Gahinet, A. Nemiroviski, A.J. Laub and M. Chilali, "LMI control toolbox for use with MATLAB,"The Mathworks, 1996.

[7] A. Hurwitz, "Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt," *Mathematische Annalen.*, vol. 46, pp 273-284, 1895.

[8] H. Iwane, H. Higuchi, and H. Anai, "An effective implementation of a special quantifier elimination for a sign definite condition by logical formula simplification," *CASC.*, to appear, 2013.

[9] H. Iwane, H. Yanami, H. Anai, and K. Yokoyama, "An effective implementation of symbolic-numeric cylindrical algebraic decomposition for quantifier elimination," *Theor. Comput. Sci*, vol. 479, pp. 43-69, 2013.

[10] H. Iwane, H. Yanami, and H. Anai, "A Symbolic-Numeric Approach to Multi-Objective Optimization in Manufacturing Design," *Mathematics in Computer Science*, vol. 50, pp. 315-334, 2011.

[11] T. Iwasaki and S. Hara, "Generalized KYP lemma: Unified frequency domain inequalities with design applications," *IEEE Trans. Automat. Control.*, vol. 50, pp. 41-59, 2005.

[12] J.G. Kassakian, M.F. Schelecht and G.C. Verghese, "Princi-

ples of Power Electronics," Prentice Hall, 1991.

[13] D.C. McFarlane and K. Glover, "Robust Controller Design Using Normalized Compromise Factor Plant Description," *Springer-verlag, Berlin, Heidelderg, Lecture Note in Control and Sciences*, no. 138, 1990.

[14] R. Ortega, A. Loria, P.J. Nicklasson and H. SriaRamirez, "Passivity-based Control of Eular-Lagrange Systems," Springer, 1998.

[15] Y. Yamamoto, "A function space approach to sampled-data control systems and tracking problems," *IEEE Trans. Automat. Control.*, vol. 39, pp. 703-713, 1994.