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セッション 1
Session 1

数学への応用
Application to mathematics

Solving problems of Goldberg for rational maps on the projective space

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Abstract

In [3], we introduce the generalized Bell representation, and solve a problem of Goldberg that determine the number of equivalence classes of rational maps corresponding to each critical set. In this talk, we solve this problem by using rational maps on the projective space $\mathbb{P}^1(\mathbb{C})$. Symbolic and algebraic computation system is indispensable to determine defining equations of some singular loci.

1 Introduction

In [4], Goldberg suggested a problem that determine the number of equivalence classes of rational maps corresponding to each critical set. This problem is based on her theorem (Theorem 1.3 in [4]), and it is known that this theorem deeply concern with B. and M. Shapiro conjecture (see [1]).

As a joint work with M. Karima (Kabur Univ.) and M. Taniguchi (Nara Women's Univ.), we solved a problem of Goldberg for the generic case when the degree is small (see [2]). Moreover, in [3], we determine several kinds of the non-generic loci for the map from the generalized Bell locus to the space of the sets of critical points explicitly when the degree is small.

In this talk, we solve this problem by using a family of rational maps on the projective space $\mathbb{P}^1(\mathbb{C})$. By this technique, we can obtain the same result as in [3] more simply.

A rational map of degree d is a map with the following form,

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are coprime polynomials with $\max\{\deg P, \deg Q\} = d$.

Definition 1

Two rational maps R_1 and R_2 are said to be *Möbius equivalent* if there is a Möbius transformation $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $R_2 = M \circ R_1$.

Let X_d be the set of all equivalence classes of rational maps of degree d, and $X_d^{(k)}$ be the set of classes of rational maps having critical point at ∞ with multiplicity k, where $k = 0$ means that ∞ is non-critical.

Remark 1

A rational map R of degree d has $2d - 2$ critical points counted including multiplicity. The set of critical points of R is invariant under taking a Möbius conjugate.

For each rational map R of degree d, the multiplicity of critical point at ∞ is at most $d-1$. Therefore, the space X_d is the disjoint union of $X_d^{(0)}, X_d^{(1)}, \cdots, X_d^{(d-1)}$.

Goldberg showed the following theorem.

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Theorem 2 (Goldberg [4])
A $(2d-2)$-tuple B is the critical set of at most $C(d)$ classes in X_d, where $C(d)$ means the d-th Catalan number $\frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}$. The maximal is attained by a Zariski open subset of the space $\hat{\mathbb{C}}^{2d-2}$ of all B.

The map $\Phi_d : X_d \to \widehat{\mathbb{C}}^{2d-2}$ is defined by sending a equivalence class to the set of critical points, and the restriction of Φ_d to $X_d^{(k)}$ is denoted by $\Phi_d^{(k)}$.

Then Goldberg's problem (see [4]) is written as follows:

Problem 1

- *Describe in detail the ramification sets of the maps Φ_d.*
- *Given a critical set $\boldsymbol{\alpha}$, determine the number of points in the preimage $\Phi_d^{-1}(\boldsymbol{\alpha})$.*

The critical set is called *admissible* if every point has multiplicity at most $d-1$. She also asked in [4] whether every admissible set in $\mathbb{C}^{2d-2}$ is attained by some rational map of degree d.

2 Generalized Bell family

In this section, we summarize the results in [3].

Let $CB_d^{(k)}$ $(k = 0, 1, \cdots, d-1)$ be the *generalized Bell locus* consisting of all $H + \hat{P}/Q$, for

$$\begin{aligned} H(z) &= z^{k+1} + c_k z^k + \cdots + c_1 z, \\ \hat{P}(z) &= a_{d-k-2} z^{d-k-2} + \cdots + a_0, \\ Q(z) &= z^{d-k-1} + b_{d-k-2} z^{d-k-2} + \cdots + b_0, \end{aligned}$$

with $\mathrm{Resul}_z(\hat{P}, Q) \neq 0$.

Remark 2
If $k = d-1$, the generalized Bell locus is the family of polynomial maps $CB_d^{(d-1)} = \{z^d + c_{d-1}z^{d-1} + \cdots + c_1 z\}$. If $k = 0$, the generalized Bell locus coincides with the Bell locus; $CB_d^{(0)} = CB_d$ (see [2]).

The following proposition is an extended version of Proposition 5 in [2].

Proposition 3
For every $R \in CB_d^{(k)}$, $[R]$ belongs to $X_d^{(k)}$ for every k, and for each element $[S]$ in $X_d^{(k)}$, there is a unique R in $CB_d^{(k)}$ with $[R] = [S]$.

Hence, each locus $X_d^{(k)}$ has a system of coordinates consisting of coefficients of representatives R in the generalized Bell locus $CB_d^{(k)}$.

Now, consider the map $\Phi_d^{(k)}$ of $CB_d^{(k)}$ to $\mathbb{C}^{2d-2-k}$ defined from the equation

$$\begin{aligned} &\frac{1}{k+1}\Big\{H'(z)Q^2(z) + \hat{P}'(z)Q(z) - \hat{P}(z)Q'(z)\Big\} \\ &\qquad = z^{2d-k-2} + \alpha_{2d-k-3} z^{2d-k-3} + \cdots + \alpha_0 = 0 \end{aligned}$$

by sending

$$(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}) = (c_k, \cdots, c_1,\ a_{d-k-2}, \cdots, a_0,\ b_{d-k-2}, \cdots, b_0)$$

to

$$\boldsymbol{\alpha} = (\alpha_{2d-k-3}, \cdots, \alpha_0).$$

Set

$$R_d^{(k)} = \big\{(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^{2d-2-k} \ :\ \mathrm{Resul}_z(\hat{P}, Q) = 0\big\},$$

which is the locus where $\Phi_d^{(k)}$ is not defined. (In other words, $CB_d^{(k)}$ can be identified with $\mathbb{C}^{2d-2-k} - R_d^{(k)}$.)

Here, we recall the following results in [2].

Proposition 4
The map $\Phi_2^{(0)} : CB_2^{(0)} \to \mathbb{C}^2 - E^{(0)}(2)$ *is bijective, and the exceptional locus* $E^{(0)}(2)$ *is the algebraic curve defined by* $\alpha_1^2 - 4\alpha_0 = 0$*. And the map* $\Phi_2^{(1)} : CB_2^{(1)} \to \mathbb{C}$ *is bijective.*

Now, we recall the following results in [2] and [3].

Proposition 5
The ramification locus of $\Phi_3^{(0)}$ *is* $a_1 = b_1^2 - 4b_0$, $\Phi_3^{(0)}(CB_3^{(0)}) = \mathbb{C}^4 - E^{(0)}(3)$*, and* $\Phi_3^{(0)}$ *is 2-valent on the set of points in* $\mathbb{C}^4 - E^{(0)}(3)$ *satisfying that*

$$\alpha_2^2 - 3\alpha_1\alpha_3 + 12\alpha_0 \neq 0, \quad E_0 \neq 0.$$

Here, the exceptional locus $E^{(0)}(3)$ *is the algebraic variety defined by* $E_0 = E_1 = 0$*. Here*

$$\begin{aligned} E_1 =& 27\alpha_1^2 - 9\alpha_2\alpha_3\alpha_1 + (27\alpha_3^2 - 72\alpha_2)\alpha_0 + 2\alpha_2^3, & (1) \\ E_0 =& -27\alpha_1^4 + (-4\alpha_3^3 + 18\alpha_2\alpha_3)\alpha_1^3 + ((-6\alpha_3^2 + 144\alpha_2)\alpha_0 + \alpha_2^2\alpha_3^2 - 4\alpha_2^3)\alpha_1^2 \\ &+ (-192\alpha_3\alpha_0^2 + (18\alpha_2\alpha_3^3 - 80\alpha_2^2\alpha_3)\alpha_0)\alpha_1 + 256\alpha_0^3 \\ &+ (-27\alpha_3^4 + 144\alpha_2\alpha_3^2 - 128\alpha_2^2)\alpha_0^2 + (-4\alpha_2^3\alpha_3^2 + 16\alpha_2^4)\alpha_0. & (2) \end{aligned}$$

Remark 3
The exceptional locus $E^{(0)}(3)$ *is written as,*

$$\left\{108\alpha_1^2 + (-108\alpha_3\alpha_2 + 27\alpha_3^3)\alpha_1 + 32\alpha_2^3 - 9\alpha_3^2\alpha_2^2 = 0, \quad \textit{and} \quad 3\alpha_3\alpha_1 - \alpha_2^2 - 12\alpha_0 = 0\right\}.$$

In case $d = 3$, there remain the cases that ∞ is a critical point.

Proposition 6
The ramification locus of $\Phi_3^{(1)}$ *is given by* $c_1 - 2b_0 = 0$, $\Phi_3^{(1)}(CB_3^{(1)}) = \mathbb{C}^3 - E^{(1)}(3)$ *and* $\Phi_3^{(1)}$ *is 2-valent on the the set of the points in* $\mathbb{C}^3 - E^{(1)}(3)$ *satisfying that*

$$3\alpha_1 - \alpha_2^2 \neq 0, \quad 4\alpha_1^3 - \alpha_2^2\alpha_1^2 - 18\alpha_0\alpha_2\alpha_1 + 4\alpha_0\alpha_2^3 + 27\alpha_0^2 \neq 0.$$

Here, the exceptional locus $E^{(1)}(3)$ *is the algebraic variety defined by*

$$\left\{3\alpha_1 - \alpha_2^2 = 0, \quad 9\alpha_2\alpha_1 - 2\alpha_2^3 - 27\alpha_0 = 0\right\}.$$

Since the map $\Phi_3^{(2)} : CB_3^{(2)} \to \mathbb{C}^2$ is clearly bijective, we have obtained complete description for the case that $d = 3$.

3 Generalized Bell family on $\mathbb{P}^1(\mathbb{C})$

A rational map R on $\mathbb{P}^1(\mathbb{C})$ is defined by

$$R(z_0, z_1) = \frac{P(z_0, z_1)}{Q(z_0, z_1)}$$

where P and Q are homogeneous polynomial maps of degree d with $Q \not\equiv 0$.

Now, we give the following extended version of Proposition 3. Let PB_d be the family consisting of all $F_{(\boldsymbol{b},\boldsymbol{a})} = \dfrac{P}{Q}$, for

$$\begin{aligned} P(z_0, z_1) &= z_1^d + (1 - b_{d-1})a_{d-1}z_0z_1^{d-1} + (1 - (1 - b_{d-1})b_{d-2})a_{d-2}z_0^2z_1^{d-2} + \cdots \\ &\quad + (1 - (1 - b_{d-1}) \cdots (1 - b_1)b_0)a_0z_0^d, \\ Q(z_0, z_1) &= b_{d-1}z_0z_1^{d-1} + b_{d-2}z_0^2z_1^{d-2} + \cdots + b_0z_0^d, \\ \mathrm{GCD}&(P, Q) \in \mathbb{C}^*, \end{aligned}$$

where

$$\boldsymbol{b} = (b_{d-1} : \cdots : b_0) \in \mathbb{P}^{d-1}(\mathbb{C}),$$
$$\boldsymbol{a} = (1 : a_{d-1} : \cdots : a_0) \in \mathbb{P}^{d}(\mathbb{C}).$$

Remark 4
If the coefficients of Q satisfy $\boldsymbol{b} = (\underbrace{0 : \cdots : 0}_{k} : 1 : b_{d-k-2} : \cdots : b_0)$, then the coefficient of the term $z_0^{k+1} z_1^{d-k-1}$ of numerator P does not depend on a_{d-k-1} and is always zero.

The family PB_d represents the space X_d faithfully.

Theorem 7
For every $F(z_0, z_1) = \dfrac{P(z_0, z_1)}{Q(z_0, z_1)}$ in PB_d, the equivalence class $\left[\dfrac{P(1, z_1)}{Q(1, z_1)}\right]$ belongs to X_d. Conversely, For every $[R]$ in X_d, there is unique rational map $F(z_0, z_1) = \dfrac{P(z_0, z_1)}{Q(z_0, z_1)}$ in PB_d such that $[\tilde{R}] = [R]$, where

$$\tilde{R}\Big(\frac{z_1}{z_0}\Big) = \frac{P(z_0, z_1)}{Q(z_0, z_1)}.$$

Remark 5
For every rational map $F_{(\boldsymbol{b},\boldsymbol{a})}$ in PB_d, $F_{(\boldsymbol{b},\boldsymbol{a})}(0,1) = (0,1)$. The inverse image $F^{-1}_{(\boldsymbol{b},\boldsymbol{a})}(0,1)$ is the set given by

$$\{(z_0, z_1)\,;\, z_0 = 0 \ \ \textit{or} \ \ Q(z_0, z_1) = 0\}.$$

The map $\widehat{\Psi}_d : PB_d \to \mathbb{P}^{2d-2}(\mathbb{C})$ is defined by sending

$$(\boldsymbol{b}, \boldsymbol{a}) = \Big((\underbrace{0 : \cdots : 0}_{k} : 1 : b_{d-k-2} : \cdots : b_0), (1 : a_{d-1} : \cdots : a_{d-k-2} : 0 : a_{d-k} : \cdots : a_0)\Big)$$
$$(k = 0, 1, \cdots, d-1)$$

to

$$\boldsymbol{\alpha} = (\alpha_{2d-2} : \cdots : \alpha_0) \in \mathbb{P}^{2d-2}(\mathbb{C}),$$

where $F_{(\boldsymbol{b},\boldsymbol{a})} = \dfrac{P}{Q} \in PB_d$ and

$$\frac{\partial(Q, P)}{\partial(z_0, z_1)} = \alpha_{2d-2} z_1^{2d-2} + \alpha_{2d-3} z_0 z_1^{2d-3} + \cdots + \alpha_0 z_0^{2d-2}.$$

In the case of $d = 3$, rational map $F_{(\boldsymbol{b},\boldsymbol{a})}$ on $\mathbb{P}^1(\mathbb{C})$ is written by

$$P(z_0, z_1) = z_1^3 + a_2(1 - b_2) z_0 z_1^2 + a_1(1 - (1 - b_2) b_1) z_0^2 z_1 + a_0(1 - (1 - b_2)(1 - b_1) b_0) z_0^3,$$
$$Q(z_0, z_1) = b_2 z_0 z_1^2 + b_1 z_0^2 z_1 + b_0 z_0^3.$$

Now, set $\widehat{R}_3 = \{(\boldsymbol{b}, \boldsymbol{a})\,;\, IP_3 = 0\}$, where

$$
\begin{aligned}
IP_3 =& (a_0b_0b_2 - a_0b_0)b_1^4 + (a_0b_0a_1b_2^4 + (a_0b_0a_2 - 2a_0b_0a_1)b_2^3 + (-2a_0b_0a_2 + a_0b_0a_1)b_2^2 \\
&+ (a_0b_0a_2 + b_0a_1 - a_0b_0)b_2 - b_0a_1 + a_0b_0 - a_0)b_1^3 + (a_0^2b_0^2b_2^5 + (b_0a_1^2 - a_0b_0a_1 \\
&- 2a_0^2b_0^2)b_2^4 + ((b_0a_1 - a_0b_0)a_2 - 2b_0a_1^2 + (3a_0b_0 - a_0)a_1 + a_0^2b_0^2)b_2^3 \\
&+ ((-2b_0a_1 + 2a_0b_0 - a_0)a_2 + b_0a_1^2 + (-2a_0b_0 + a_0)a_1 - 3a_0b_0^2)b_2^2 \\
&+ ((b_0a_1 - a_0b_0 + a_0)a_2 + 3a_0b_0^2)b_2 + b_0a_1)b_1^2 + (-2a_0^2b_0^2b_2^5 + (-2a_0b_0^2a_2 + 4a_0^2b_0^2 \\
&- 2a_0^2b_0)b_2^4 + (4a_0b_0^2a_2 + 2b_0a_1^2 - a_0b_0a_1 - 2a_0^2b_0^2 + 2a_0^2b_0)b_2^3 + ((b_0a_1 - 2a_0b_0^2)a_2 \\
&- 2b_0a_1^2 + (-2b_0^2 + a_0b_0 - a_0)a_1 + 3a_0b_0^2)b_2^2 + ((-b_0a_1 + b_0^2)a_2 + 2b_0^2a_1 \\
&- 3a_0b_0^2 + 3a_0b_0)b_2 - b_0^2a_2)b_1 + a_0^2b_0^2b_2^5 + (2a_0b_0^2a_2 - 2a_0^2b_0^2 + 2a_0^2b_0)b_2^4 \\
&+ (b_0^2a_2^2 + (-4a_0b_0^2 + 2a_0b_0)a_2 + a_0^2b_0^2 - 2a_0^2b_0 + a_0^2)b_2^3 + (-2b_0^2a_2^2 \\
&+ (2a_0b_0^2 - 2a_0b_0)a_2 + b_0a_1^2)b_2^2 + (b_0^2a_2^2 - 2b_0^2a_1)b_2 + b_0^3.
\end{aligned}
$$

Then, we have

Lemma 8
$\widehat{R}_3$ is the locus where $\widehat{\Psi}_3$ is not defined.

Now, Jacobian is given by

$$
\begin{aligned}
J =& \frac{\partial(Q,P)}{\partial(z_0,z_1)} \\
=& 3(b_2z_1^4 + 2b_1z_0z_1^3 + ((-a_1b_2^2 + (-a_2 + a_1)b_2 + a_2)b_1 - a_1b_2 + 3b_0)z_0^2z_1^2 \\
&+ ((2a_0b_0b_2^2 - 2a_0b_0b_2)b_1 - 2a_0b_0b_2^2 + (-2b_0a_2 + 2a_0b_0 - 2a_0)b_2 + 2b_0a_2)z_0^3z_1 \\
&+ ((a_0b_0b_2 - a_0b_0)b_1^2 + ((b_0a_1 - a_0b_0)b_2 - b_0a_1 + a_0b_0 - a_0)b_1 + b_0a_1)z_0^4).
\end{aligned}
$$

Therefore, the map $\widehat{\Psi}_3$ is defined by $(\boldsymbol{b}, \boldsymbol{a}) \mapsto \boldsymbol{\alpha}$, where

$$
\begin{aligned}
&\alpha_4 = b_2, \\
&\alpha_3 = 2b_1, \\
&\alpha_2 = (-a_1b_2^2 + (-a_2 + a_1)b_2 + a_2)b_1 - a_1b_2 + 3b_0, \\
&\alpha_1 = (2a_0b_0b_2^2 - 2a_0b_0b_2)b_1 - 2a_0b_0b_2^2 + (-2b_0a_2 + 2a_0b_0 - 2a_0)b_2 + 2b_0a_2, \\
&\alpha_0 = (a_0b_0b_2 - a_0b_0)b_1^2 + ((b_0a_1 - a_0b_0)b_2 - b_0a_1 + a_0b_0 - a_0)b_1 + b_0a_1.
\end{aligned}
\tag{3}
$$

Eliminating the parameters $\boldsymbol{a}, \boldsymbol{b}$ from $IP_3 = t$ by using (3), we have a quadratic equation $T = 0$, where

$$
\begin{aligned}
T =& 432t^2 + (-216\alpha_4\alpha_1^2 + 72\alpha_3\alpha_2\alpha_1 - 16\alpha_2^3 + 576\alpha_0\alpha_4\alpha_2 - 216\alpha_0\alpha_3^2)t \\
&+ 27\alpha_4^2\alpha_1^4 + (-18\alpha_4\alpha_3\alpha_2 + 4\alpha_3^3)\alpha_1^3 + (4\alpha_4\alpha_2^3 - \alpha_3^2\alpha_2^2 - 144\alpha_0\alpha_4^2\alpha_2 + 6\alpha_0\alpha_4\alpha_3^2)\alpha_1^2 \\
&+ (80\alpha_0\alpha_4\alpha_3\alpha_2^2 - 18\alpha_0\alpha_3^3\alpha_2 + 192\alpha_0^2\alpha_4^2\alpha_3)\alpha_1 - 16\alpha_0\alpha_4\alpha_2^4 \\
&+ 4\alpha_0\alpha_3^2\alpha_2^3 + 128\alpha_0^2\alpha_4^2\alpha_2^2 - 144\alpha_0^2\alpha_4\alpha_3^2\alpha_2 + 27\alpha_0^2\alpha_3^4 - 256\alpha_0^3\alpha_4^3.
\end{aligned}
\tag{4}
$$

There are no rational functions of degree 3 corresponding to $\boldsymbol{\alpha}$ if and only if the equation (4) has 0 as a unique solution for t.

Lemma 9
The exceptional locus $PE(3)$ is given by

$$
\begin{aligned}
PE(3) =& \{108\alpha_4^2\alpha_1^2 + (-108\alpha_4\alpha_3\alpha_2 + 27\alpha_3^3)\alpha_1 + 32\alpha_4\alpha_2^3 - 9\alpha_3^2\alpha_2^2 = 0, \\
&- 3\alpha_3\alpha_1 + \alpha_2^2 + 12\alpha_0\alpha_4 = 0, \\
&27\alpha_4\alpha_1^2 - 27\alpha_3\alpha_2\alpha_1 + 8\alpha_2^3 + 27\alpha_0\alpha_3^2 = 0\}.
\end{aligned}
\tag{5}
$$

Lemma 10
If critical set $\boldsymbol{\alpha}$ satisfies

$$\boldsymbol{\alpha} \notin \{PE_0 = 0\}, \quad \textit{and} \quad \boldsymbol{\alpha} \notin \{Discr(T) = 0\},$$

$\#\widehat{\Psi}_3^{-1}(\boldsymbol{\alpha}) = 2$, *where* PE_0 *is the constant term of* T *for* t *and* $Discr(T) = 3\alpha_3\alpha_1 - \alpha_2^2 - 12\alpha_0\alpha_4$.

Then, we have the following.

Theorem 11
$\widehat{\Psi}_3(PB_3) = \mathbb{P}^4(\mathbb{C}) - PE(3)$ *and* $\widehat{\Psi}_3(PB_3)$ *is 2-valent on the the set of the points in* $\mathbb{P}^4(\mathbb{C}) - PE(3)$ *satisfying that*

$$\mathrm{Discr}(T) \neq 0 \quad \textit{and} \quad PE_0 \neq 0.$$

This theorem corresponds to Proposition 5 and Proposition 6 which are given by using generalized Bell locus. Theorem 11 is obtained without considering the multiplicity of critical point at the point at infinity.

References

[1] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, *Ann. of Math.*, **155** (2002), 105–129.

[2] M. Fujimura, M. Karima, and M. Taniguchi, The Bell locus of rational functions and problems of Goldberg, *Comm. Japan Soc. Symb. Alg. Compt.* **1** (2012), 67–74.

[3] M. Fujimura, M. Karima, and M. Taniguchi, The generalized Bell locus of rational functions and problems of Goldberg, *Proc. of the 19th ICFIDCAA*, (2013), 103–110.

[4] L. Goldberg, Catalan numbers and branched covering by the Riemann sphere, *Adv. Math.*, **85** (1991), 129–144.

[5] I. Scherbak, Rational functions with prescribed critical points, *Geom. Funct. Anal.*, **12** (2002), 1365–1380.