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Mathematical Formulation of Motion/Deformation and its Applications

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This talk is intended to give a summary of Lie groups and Lie algebras for computer graphics, including an example from interpolations and blending of motions and deformation. See also the forthcoming SIGGRAPH ASIA 2013 course note.

1. Lie groups and Lie algebras

Lie groups are used for describing the transformations, and Lie algebra is its *linear* approximation which has a main feature of Lie groups.

A *Lie group* is a manifold with a group structure. A typical example of Lie group is a subset of a matrix space $M(N) = M(N, \mathbb{R})$ for some N closed under the product and the inverse. To focus applications to computer graphics, we may restrict Lie groups to be this sub-classes without loss of generality.

An affine transformation of $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ which maps every line to a line. We denote by $\mathrm{Aff}(n)$ the set of affine transformations, and by $\mathrm{Aff}^+(n)$ the set of positive (i.e., reflection-free) affine transformations. In CG, we mainly treat $n = 2$ and $n = 3$, while a part of the theory holds for general n in parallel manner. We see that $\mathrm{Aff}(n)$ is a group and $\mathrm{Aff}^+(n)$ is a subgroup. A well-known homogeneous realization of $\mathrm{Aff}(n)$ is given as a set of block upper-triangular matrices:

$$\mathrm{Aff}(n) = \left\{ A = \begin{pmatrix} \hat{A} & d_A \\ 0 & 1 \end{pmatrix} \middle| \hat{A} \in GL(n), d_A \in \mathbb{R}^n \right\}, \tag{1}$$

where $GL(n) = \{A \in M(n, \mathbb{R}) \mid \det(A) \neq 0\}$. This realization shows that $\mathrm{Aff}(n)$ is a Lie group.

The exponential of a square matrix $X \in M(N)$ is defined to be

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k, \tag{2}$$

motivated by the exactly the same Taylor expansion formula for the scalar exponential function. This infinite series always converges, however, the computation is reduced to the exponential of diagonal matrices by using the formula $\exp(PXP^{-1}) = P \exp(X) P^{-1}$.

Let G be a Lie group realized in $M(N,\mathbb{R})$. The tangent space of G at the origin is denoted by $\mathfrak{g}$, the corresponding German letter, and is called the *Lie algebra* of G. The Lie algebra approximates the Lie group, without loosing the (local) informations, rather surprisingly. The exponential map gives a map from $\mathfrak{g}$ to G. If G is abelian(= commutative) and connected, then the exponential map is surjective. If G is simply-connected, connected, and abelian, then the exponential map is bijective. If G is compact and connected, then the exponential map is surjective. The exponential map is a local isomorphism at the origin, but is not necessarily injective or surjective, e.g. $G = SL(2,\mathbb{R})$. To relax this difficulty on the exponential functions for general Lie groups, we will introduce the decomposition/factorization of matrices.

2. Matrix factorization

In order to understand and analyze a complicated groups, several types decompositions of Lie groups and their generalization are known and used. We here recall some of them.

semi-direct product. The affine transformation group is an example of semi-direct product groups, so that $\mathrm{Aff}(n) = GL(n) \ltimes \mathbb{R}^n$. In general, let G be a group and H_1, H_2 subgroups of G. If the multiplication map

$$H_1 \times H_2 \ni (h_1, h_2) \mapsto h_1 h_2 \in G \tag{3}$$

is bijective and $h_1 h_2 h_1^{-1} \in H_2$ for all $h_1 \in H_1$ and $h_2 \in H_2$, then G is isomorphic to the semi-direct product group $H_1 \ltimes H_2$. Another example of the semi-direct product group is a motion group $SE^+(n) = SO(n) \ltimes \mathbb{R}^n$, or a congruence group $SE(n) = O(n) \ltimes \mathbb{R}^n$. If $h_1 h_2 h_1^{-1} = h_2$ in the above, then G is isomorphic to the direct product group $H_1 \times H_2$.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the group of invertible dual numbers (see [5] Definition 1 for the notation) is

$$\hat{\mathbb{K}}^\times = (\mathbb{K} + \mathbb{K}\varepsilon)^\times = \mathbb{K}^\times + \mathbb{K}\varepsilon = \mathbb{K}^\times \ltimes \mathbb{K}\varepsilon \tag{4}$$

which also gives an example of semi-direct product groups. The set $\hat{\mathbb{K}}_1$ of unit dual numbers is a subgroup of $\hat{\mathbb{K}}^\times$, which is also a semi-direct product group $\mathbb{K}_1 \ltimes \mathbb{K}\varepsilon$.

Diagonalization of symmetric matrix. Every (real) symmetric matrix is diagonalized by the conjugate of an orthogonal matrix. This fact is rephrased as the subjectivity of the multiplication map

$$O(n) \times \mathrm{Diag}(n) \ni (R, D) \mapsto RDR^{-1} \in \mathrm{Sym}(n). \tag{5}$$

Note that the diagonal entries of diagonal matrix D is the set of eigenvalues (with multiplicities) of a given symmetric matrix $X = RDR^{-1}$, so that it is determined by X up to the ordering of eigenvalues. The column vectors of R are the corresponding eigenvectors which form orthonormal frame of $\mathbb{R}^n$. These facts give an algorithm to compute (R, D) from X. As a special case of this decomposition (5), every positive-definite symmetric matrix is diagonalized into diagonal matrices with positive diagonal entries:

$$O(n) \times \mathrm{Diag}^+(n) \ni (R, D) \mapsto RDR^{-1} \in \mathrm{Sym}^+(n). \tag{6}$$

Note that $\mathrm{Sym}^+(n)$ is not a group since the product of two symmetric matrices are not necessarily symmetric. But still the exponential map gives a bijective from $\mathrm{Sym}(n)$ to $\mathrm{Sym}^+(n)$. This enables us to consider the logarithm, which is defined to be the inverse of the exponential, and the fractional power S^t for $t \in \mathbb{R}$ defined to be $\exp(t \log(S))$ for $S \in \mathrm{Sym}^+(n)$. (An example is the square root $S^{1/2}$.) This is a key for interpolations.

Polar decomposition. Every invertible matrix is a product of orthogonal matrix and a positive-definite symmetric matrix. In other words, the multiplication map

$$O(n) \times \mathrm{Sym}^+(n) \ni (R, S) \mapsto RS \in GL(n) \tag{7}$$

is bijective.

Singular value decomposition (SVD). The following multiplication map is surjective:

$$SO(n) \times \mathrm{Diag}^+(n) \times SO(n) \ni (R, D, R') \mapsto RDR' \in GL^+(n). \tag{8}$$

Triangular decomposition. In some setting, almost all matrices can be decomposed. Here is an example form Gaussian elimination of systems of linear equations. Let $N^{\pm}$ be the set of upper/lower triangular matrices whose diagonal entries are 1. Then the multiplication map

$$N^- \times \mathrm{Diag}(n) \times N^+ \ni (U', D, U) \mapsto U'DU \in GL(n) \tag{9}$$

is injective and its image is an open dense subset of $GL(n)$.

Iwasawa decomposition. Some decomposition can treat shears. The multiplication map

$$SO(n) \times \mathrm{Diag}^+(n) \times N^+ \ni (R, D, U) \mapsto RDU \in GL^+(n) \tag{10}$$

gives a bijection.

3. Interpolation and blend

By combining the matrix factorization and the exponential map, we get a straight forward idea for interpolation. Graphically,

$$\begin{array}{ccccc} GL^+(2) & \overset{\sim}{\to} & SO(2) \times \mathrm{Sym}^+(2) & \overset{\log}{\to} & \mathfrak{so}(2) \times \mathfrak{sym}(2) \\ & & & & \downarrow \text{linear interpolation} \\ GL^+(2) & \overset{\sim}{\leftarrow} & SO(2) \times \mathrm{Sym}^+(2) & \overset{\exp}{\leftarrow} & \mathfrak{so}(2) \times \mathfrak{sym}(2). \end{array} \tag{11}$$

Our goal is to give an interpolation on $GL^+(2)$, but this space is not convex subset of $M(2)$. By decomposition, we switch to the spaces whose exponential maps behave well, and then by the exponential map, we move to a linear space where we have a reasonable interpolation. This is a modification of the idea of [1], employed in [3]. Note that the right-most column is a vector space, so we can blend more than two objects just as a linear combination $\sum_{i=1}^{m} w_i S_i$. A further example is given in [2].

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