Discrete mKdV Flow on Discrete Curves

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Discrete mKdV flow on discrete curves

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1. Plane curves

1.1. **mKdV flow on plane smooth curves.** It is well-known that the plane/space smooth curves admit the isoperimetric motion described by the modified Korteweg-de Vries (mKdV) equation[6, 3]

$$\dot{\kappa} = \frac{3}{2}\kappa^2\kappa' + \kappa''', \tag{1}$$

which is known as one of the most typical integrable systems. Here $\kappa = \kappa(x,t)$, $\dot{}$ and $'$ denote t- and x-derivatives, respectively. Actually, let $\gamma(x,t) \in \mathbb{R}^2$ be an arclength-parametrized curve, x be an arclength and t be a deformation parameter. Note that we have $|\gamma'| = 1$. Consider the Frenet frame

$$\Phi(x,t) = [T(x,t), N(x,t)] \in \mathrm{SO}(2),\ T(x,t) := \gamma'(x,t),\ N(x,t) := R\left(\frac{\pi}{2}\right)T(x,t), \tag{2}$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{3}$$

Then we have the Frenet formula

$$\Phi'(x,t) = \Phi(x,t)L(x,t), \quad L = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}, \tag{4}$$

where $\kappa = \kappa(x,t)$ is the curvature. We consider the isoperimetric motion of the curve given by

$$\dot{\gamma} = \frac{\kappa^2}{2}T + \kappa' N, \tag{5}$$

which can be expressed in terms of the Frenet frame as

$$\dot{\Phi}(x,t) = \Phi(x,t)M(x,t), \quad M = \begin{bmatrix} 0 & \kappa'' + \frac{\kappa^3}{2} \\ -\kappa'' - \frac{\kappa^3}{2} & 0 \end{bmatrix}. \tag{6}$$

The linear system (4) and (6) is known as the auxiliary linear problem of the mKdV equation (1), and the matrices L and M are called the Lax pair. The compatibility condition $\dot{L} - M' = [L, M] = LM - ML$ yields the mKdV equation (1).

1.2. **Discrete mKdV flow on plane discrete curves**[4]**.** Let $\gamma_n^m \in \mathbb{R}^2$ be a plane discrete curve, where n is the index of the vertices and m is the discrete deformation parameter. We put the length of the segment as a_n, namely, $|\gamma_{n+1}^m - \gamma_n^m| = a_n$. The discrete Frenet frame is given by

$$\Phi_n^m = [T_n^m, N_n^m] \in \mathrm{SO}(2),\ T_n^m := \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n},\ N_n^m := R\left(\frac{\pi}{2}\right) T_n^m. \tag{7}$$

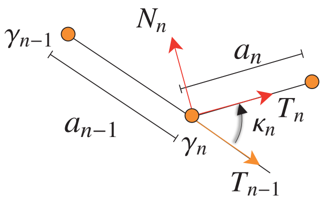

Figure 1.

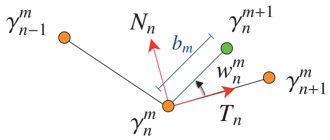

Figure 2.

Then we have the discrete Frenet formula

$$\Phi_{n+1}^m = \Phi_n^m L_n^m, \quad L_n^m = R(\kappa_{n+1}^m), \tag{8}$$

where κ_n^m denotes the angle between T_n^m and T_{n-1}^m (see Figure 1). We consider the isoperimetric motion of the curve given by (see Figure 2)

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = \cos w_n^m\ T_n^m + \sin w_n^m\ N_n^m, \tag{9}$$

where b_m is an arbitrary function of m, and w_n^m is determined by the following equation

$$\tan \frac{w_{n+1}^m + \kappa_{n+1}^m}{2} = \frac{b_m + a_n}{b_m - a_n} \tan \frac{w_n^m}{2}. \tag{10}$$

The motion can be expressed in terms of the discrete Frenet frame as

$$\Phi_n^{m+1} = \Phi_n^m M_n^m, \quad M_n = R(\kappa_{n+1}^m + w_{n+1}^m + w_n^m). \tag{11}$$

The compatibility condition of the linear system (8) and (11) $L_n^m M_{n+1}^m = M_n^m L_n^{m+1}$ yields $w_{n+1}^m - w_{n-1}^m = \kappa_n^{m+1} - \kappa_{n+1}^m$ and the discrete mKdV equation

$$\frac{w_{n+1}^{m+1}}{2} - \frac{w_n^m}{2} = \arctan\left(\frac{b_{m+1} + a_n}{b_{m+1} - a_n} \tan \frac{w_n^{m+1}}{2}\right) - \arctan\left(\frac{b_m + a_{n+1}}{b_m - a_{n+1}} \tan \frac{w_{n+1}^m}{2}\right), \tag{12}$$

which is known as an integrable discretization of the mKdV equation.
Remark 1.
The continuous limit of the discrete mKdV equation (12) to the mKdV equation (1) is described as follows[5]. First put

$$a_n = a \text{ (const.)},\ b_m = b \text{ (const.)}\ \delta = \frac{a+b}{2},\ \epsilon = \frac{a-b}{2},\ \frac{s}{\delta} = n + m,\ l = n - m. \tag{13}$$

Taking the limit $\epsilon \to 0$ yields the semi-discrete mKdV equation for $\kappa_l(s) = -w_l(s)$

$$\frac{d\kappa_l}{ds} = \frac{1}{\epsilon}\left(\tan\frac{\kappa_{l+1}}{2} - \tan\frac{\kappa_{l-1}}{2}\right). \tag{14}$$

Next putting

$$x = \epsilon l + s, \quad t = -\frac{\epsilon^2}{6}s, \tag{15}$$

and taking the limit $\epsilon \to 0$, we obtain the mKdV equation (1) for $\kappa(x,t)$. The semi-discrete mKdV equation (14) also describes certain isoperimetric flows on plane/space discrete curves[2, 5].

Remark 2.
By using the standard technique of the theory of the integrable systems, it is possible to construct explicit formulas for γ in terms of the τ function. The τ functions corresponding to the soliton or the breather type solutions are given by determinants.

2. Space curves

2.1. **mKdV flow on space smooth curves**[6]**.** Let x be the arclength, $\gamma(x,t) \in \mathbb{R}^3$ be an arclength-parametrized curve so that $|\gamma'| = 1$, and t be the deformation parameter. The Frenet frame is defined by

$$\Phi(x,t) = [T(x,t), N(x,t), B(x,t)] \in \mathrm{SO}(3),\ T := \gamma',\ N := \frac{\gamma''}{|\gamma''|},\ B := T \times N, \tag{16}$$

where T, N, B are the tangent vector, the normal vector and the binormal vector, respectively. Then we have the Frenet-Serret formula

$$\Phi' = \Phi L, \quad L = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}, \quad \kappa = |\gamma''|, \quad \lambda = -\langle B', N\rangle \tag{17}$$

where κ and λ are curvature and torsion, respectively, and $\langle\cdot,\cdot\rangle$ is the standard inner product. We assume that the torsion λ is a constant. We consider the isoperimetric motion of curves defined by

$$\dot{\gamma} = \left(\frac{\kappa^2}{2} - 3\lambda^2\right)T + \kappa' N - 2\lambda\kappa N, \tag{18}$$

which can be expressed in terms of the Frenet frame as

$$\dot{\Phi} = \Phi M, \quad M = \begin{bmatrix} 0 & -\frac{\kappa^3}{2} + \lambda^2\kappa - \kappa'' & \lambda\kappa \\ \frac{\kappa^3}{2} - \lambda^2\kappa + \kappa'' & 0 & -\frac{\lambda}{2}\kappa^2 + \lambda^3 \\ -\lambda\kappa & \frac{\lambda}{2}\kappa^2 - \lambda^3 & 0 \end{bmatrix} \tag{19}$$

The compatibility condition of the linear system (17) and (19) $\dot{L} - M' = [L, M]$ yields the mKdV equation (1).

2.2. **discrete mKdV flow on space discrete curves.** This section contains the main result of our presentation. For some fixed m, let $\gamma_n^m \in \mathbb{R}^3$ be an space discrete curve, and ϵ_n be the length of the segment, namely, $\epsilon_n = |\gamma_{n+1}^m - \gamma_n^m|$. We introduce the discrete Frenet frame by

$$\Phi_n^m = [T_n^m, N_n^m, B_n^m] \in \mathrm{SO}(3), \tag{20}$$

where the tangent vector T_n^m, the normal vector N_n^m and the binormal vector B_n^m are defined by

$$T_n^m := \frac{\gamma_{n+1}^m - \gamma_n^m}{\epsilon_n}, \quad B_n^m := \frac{T_{n-1}^m \times T_n^m}{\left|T_{n-1}^m \times T_n^m\right|}, \quad N_n^m := B_n^m \times T_n^m, \tag{21}$$

respectively. Note that the normal vector is chosen as $N_n^m \in \mathrm{span}\{T_{n-1}^m, T_n^m\}$ (see Figure 3).

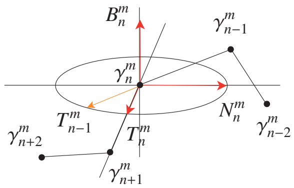

Figure 3.

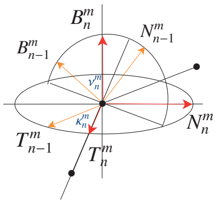

Figure 4.

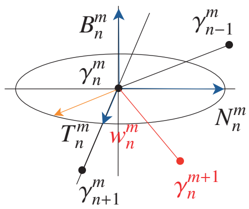

Figure 5.

We define κ_n^m and ν_n^m by (see Figure 4)

$$\langle T_n^m, T_{n-1}^m\rangle = \cos\kappa_n^m, \quad \langle B_n^m, B_{n-1}^m\rangle = \cos\nu_n^m, \quad \langle B_n^m, N_{n-1}^m\rangle = \sin\nu_n^m. \tag{22}$$

Then we see that the Frenet frame satisfies the discrete Frenet-Serret formula

$$\Phi_{n+1}^m = \Phi_n^m L_n^m, \quad L_n^m = R_1(-\nu_{n+1}^m) R_3(\kappa_{n+1}^m), \tag{23}$$

where $R_1(\theta)$ and $R_3(\theta)$ are the rotation matrices given by

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{24}$$

respectively. We define the function λ_n by

$$\lambda_n = \frac{\sin\nu_{n+1}}{\epsilon_n}, \tag{25}$$

which we refer to as the torsion of γ. In the following, we assume that $\lambda_n = \lambda$ (const.).

Now we formulate the discrete flow of the space discrete curve γ_n^m described by the discrete mKdV equation (12):

Theorem 1. Let γ^0 be a space discrete curve with constant torsion λ. We introduce a_n, δ_m and α_n^m by

$$\epsilon_n = \frac{a_n}{1 + \left(\frac{a_n\lambda}{2}\right)^2}, \quad \delta_m = \frac{b_m}{1 + \left(\frac{b_m\lambda}{2}\right)^2}, \tag{26}$$

$$\alpha_n^m = \frac{b_m + a_n}{b_m - a_n}, \tag{27}$$

respectively, where b_m is an arbitrary function in m. For an arbitrary sequence w_0^m we determine the function w_n^m by

$$w_{n+1}^m = 2\arctan\left(\alpha_n^m \tan\frac{w_n^m}{2}\right) - \kappa_{n+1}^m, \tag{28}$$

and define the sequence of space discrete curves $\gamma^1, \gamma^2, \ldots$ by (see Figure 5)

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{\delta_m} = \cos w_n^m \ T_n^m + \sin w_n^m \ N_n^m. \tag{29}$$

Then we have the following:

(1) For any m, γ^m satisfies $|\gamma_{n+1}^m - \gamma_n^m| = \epsilon_n$, and the torsion of γ^m is λ. Namely, (29) gives a torsion preserving isoperimetric deformation.

(2) The Frenet frame Φ_n^m satisfies

$$\Phi_{n+1}^m = \Phi_n^m L_n^m, \quad L_n^m = R_1(-\nu_{n+1}) R_3(\kappa_{n+1}^m), \tag{30}$$

$$\Phi_n^{m+1} = \Phi_n^m M_n^m, \quad M_n^m = R_3(w_n^m) R_1(-\mu_m) R_3(w_{n+1}^m + \kappa_{n+1}^m), \tag{31}$$

where

$$\nu_n = 2\arctan\frac{a_n\lambda}{2}, \tag{32}$$

$$\mu_m = 2\arctan\frac{b_m\lambda}{2}, \tag{33}$$

Compatibility condition of the linear system (30) and (31) gives $w_{n+1}^m - w_{n-1}^m = \kappa_n^{m+1} - \kappa_{n+1}^m$ which yields the the discrete mKdV equation (12).

(3) Conversely, for a constant λ and sequences a_n, b_m, w_n^0 and w_0^m, we determine the function ν, μ, w and κ by (32), (33), (12) and (28), respectively. Let Φ be a solution to the linear system (30) and (31). Transforming Φ to SU(2)-valued function ϕ by the SU(2)-SO(3) correspondence, Sym's formula

$$\gamma_n^m = -\left(\frac{\partial}{\partial\lambda}\phi_n^m\right)(\phi_n^m)^{-1} \in \mathfrak{su}(2) \simeq \mathbb{R}^3 \tag{34}$$

recovers the sequence of isoperimetric deformation of the space discrete curves $\ldots, \gamma^{-1}, \gamma^0, \gamma^1, \ldots$ with constant torsion λ.

Remark 3.
The space curve motions recovers the plane curve motions by setting $\lambda = 0$ for both smooth and discrete curves.

Remark 4.
One can see from (26) that the lattice intervals ϵ, δ for the space discrete curve and

a, b for the discrete mKdV are different. This is an essential feature of the discrete motion of space discrete curves. They coincide by setting $\lambda = 0$ (case of plane curves).

Remark 5. (Discrete sine-Gordon flow)
Space discrete curves also admits the deformation described by the discrete sine-Gordon equation. Actually, for a space discrete curve γ^m and a sequence c_m we put

$$\delta_m = \frac{c_m}{c_m^2 + \left(\frac{\lambda}{2}\right)^2}, \quad \alpha_n^m = \frac{1 + a_n c_m}{1 - a_n c_m}, \tag{35}$$

and define the discrete flow of γ by (28) and (29). Then it follows that the Frenet frame Φ satisfies
(36)

$$\Phi_n^{m+1} = \Phi_n^m M_n^m, \quad M_n^m = R_3(w_n^m) R_1(-\mu_m) R_3(-w_{n+1}^m - \kappa_n^m), \quad \mu_m = -2 \arctan \frac{2c_m}{\lambda}.$$

Consistency of the discrete flow is guaranteed by $w_{n+1}^m - w_{n-1}^m = -\kappa_n^{m+1} - \kappa_{n+1}^m$, from which we obtain the discrete sine-Gordon equation

$$\frac{w_{n+1}^{m+1}}{2} + \frac{w_n^m}{2} = \arctan\left(\alpha_n^{m+1} \tan \frac{w_n^{m+1}}{2}\right) + \arctan\left(\alpha_{n+1}^m \tan \frac{w_{n+1}^m}{2}\right), \tag{37}$$

or in terms of the potential function defined by $w_n^m = -\frac{\theta_n^{m+1} + \theta_{n+1}^m}{2}$

$$\sin \frac{\theta_{n+1}^{m+1} - \theta_n^{m+1} - \theta_{n+1}^m + \theta_n^m}{4} = a_n c_m \sin \frac{\theta_{n+1}^{m+1} + \theta_n^{m+1} + \theta_{n+1}^m + \theta_n^m}{4}. \tag{38}$$

Remark 6. (Discrete K-surface)
The sequence of isoperimetric deformations of space discrete curves $\gamma = \gamma_n^m$ described in Theorem 1 and Remark 5 both form discrete surfaces with the constant negative Gaussian curvature (discrete K-surfaces)[1]. Conversely, any discrete K-surface can be constructed as Theorem 1 or Remark 5. This gives a new method of construction of discrete K-surfaces.

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