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A Lie theoretic proposal on algorithms for spherical harmonic lighting

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1. Introduction

In this article (abstract of the presentation), we would like to provide a group theoretical background of spherical harmonics first, and using this, we propose a possible geometry preserving algebraic/efficient computing, which might accelerate the (numerical and exact) computations slightly for spherical harmonic lighting [3, 6, 1, 5] in some context. A mathematical idea presented here, if it actually works in the rendering process in computer graphics, would not be necessarily limited to the study of computer graphics, whence could be applicable to other fields. Our proposal is based on Lie theory or Representation theory of Lie groups.

Spherical harmonics are orthogonal functions and span rotation invariant spaces on S^2, allowing for efficient, alias-free least squares projection and reconstruction of spherical functions (= functions on the sphere). These properties lead to a number of efficient operations for computing rotations, convolution, and double product integrals [4, 9]. As is well-known, spherical harmonics are used extensively in various fields. They are a basis of the space $L^2(S^2)$ of the square integrable functions on the two dimensional sphere S^2, as the name would suggest. They have been used to solve problems in physics, such as in heat equations, the gravitational and electric fields. They have also been used in quantum chemistry and physics to model the electron configuration in atoms. For the spherical harmonic lighting, in place of the Fourier series expansion on the Euclidean space, one uses the expansion by the spherical harmonics, in other words, replaces exponential functions by the associated Legendre functions (or Legendre spherical functions). From our current point of view, the (usual) Fourier analysis is considered to be based on very simple representation theory of abelian groups $\mathbb{R}^n$, whereas the spherical harmonics is on the representation theory of a non-commutative group $SO(3)$.

To be more explicit, we shall describe certain algebraic treatment for the computation of harmonic expansions, which turns to be a part of the technique at the spherical harmonic lighting, by the framework of harmonic analysis on the sphere $S^2 \cong SO(2)\backslash SO(3)$, $SO(n)$ being the rotation group of order n. More precisely, one considers the irreducible decomposition of the natural action defined by the right translation of $SO(3)$ on $L^2(S^2)$ (i.e. a part of the theory of spherical harmonics) and translate/reformulate the problem into the different Hilbert space using another but unitarily equivalent realization of irreducible representations on the space of polynomials with complex coefficients by the language of the special unitary group $SU(2)$ of degree two (see e.g. [2, 7, 10]). Here the word "unitarily equivalent" means the isometry between two Hilbert spaces.

2. Basic notion for representation theory

One recalls here some of fundamental definitions and facts for representation theory (see e.g. [2, 7]). Please keep patient for a while.

Definition 2.1. A unitary representation of a topological group G is a strongly continuous homomorphism π of G into the group $U(H)$ of unitary operators on a Hilbert space H. Here, a mapping $\pi : G \ni g \mapsto \pi(g) \in U(H)$ is called a homomorphism if it satisfies

$$\pi(gh) = \pi(g)\pi(h) \quad (\forall g, h \in G),$$

and a homomorphism π is called strongly continuous if the mapping $g \mapsto \pi(g)x$ is a continuous mapping of G into H for all $x \in H$. Moreover, one sometimes denotes the representation by a pair of the mapping π and representation space H as (π, H). □

Definition 2.2. Two unitary representations (π_1, H_1) and (π_2, H_2) are called equivalent if there exists an isometry (i.e. a linear mapping preserving the norm) A of H_1 onto H_2 satisfying

$$A\pi_1(g) = \pi_2(g)A \quad (\forall g \in G).$$

In this case, one writes $(\pi_1, H_1) \cong (\pi_2, H_2)$ (or simply $H_1 \cong H_2$). Obviously, the relation "$\cong$" is an equivalence relation. □

Definition 2.3. Let (π, H) be a unitary representation of a group G. A closed linear subspace V of H is called invariant under π if one has

$$\pi(g)V \subset H \quad (\forall g \in G).$$

A unitary representation π is called irreducible if $H \neq \{0\}$ and H and $\{0\}$ are the only invariant subspaces of H.

Non-irreducible unitary representations are "decomposed" into irreducible representations. In a sense, the irreducible representations are the "atoms" of unitary representations. □

Definition 2.4. Let (π_1, H_1) and (π_2, H_2) be two unitary representations of G. A linear operator $T : H_1 \to H_2$ satisfying

$$T\pi_1(g) = \pi_2(g)T \quad (\forall g \in G)$$

is called an intertwiner (or intertwining operator) between H_1 and H_2. □

Proposition 2.1. *Let (π_1, H_1) and (π_2, H_2) be two finite dimensional unitary representations of G and T be an intertwiner between H_1 and H_2. Then, either $T = 0$ or T is a linear isomorphism.* □

Proposition 2.2. *Let $(\pi,\, H)$ be a unitary representations of G. Then, a closed subspace V of H is invariant under π if and only if the orthogonal projection P_V on V commutes with $\pi(g)$ for all $g \in G$. In this case, the orthogonal complement $V^{\perp}$ is also invariant under π.* □

Let $\mathbf{B}(H)$ be the algebra of bounded linear operators on a Hilbert space H and M be a subset of $\mathbf{B}(H)$. The commutant M' of M is defined by

$$M' = \{L \in \mathbf{B}(H) \,|\, LU = UL \ (\forall U \in M)\}.$$

Theorem 2.3. *(Schur's Lemma) Let $(\pi,\, H)$ be a unitary representations of G and $M = \{\pi(g) \,|\, g \in G\}$. Then, π is irreducible if and only if the commutant M' is equal to the set $\mathbb{C}1$ of scalar operators.* □

Theorem 2.4. *Any unitary representation π of a compact group G is a (Hilbert space) direct sum of finite-dimensional irreducible unitary representations. In particular, any irreducible representation of a compact group is finite-dimensional.* □

3. Spherical harmonics

The groups considered in this presentation such as the special unitary group $SU(2)$ (of degree 2) and special orthogonal groups $SO(2)$, $SO(3)$ are compact. Notice that any finite-dimensional representation of a compact group is assumed to be unitary since one may always has the invariant inner product on the representation space by the existence of the Haar measure (invariant measure) [2, 7].

As a matrix group of degree 2, $SU(2)$ acts on the vector space $\mathbb{C}^2$: For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ $(|\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C})$ the action is defined by

$$\mathbb{C}^2 \ni z = (z_1,\, z_2) \mapsto zg = (az_1 + cz_2,\, bz_1 + dz_2).$$

Note that the action satisfies $z(g_1g_2) = (zg_1)g_2\, (\forall g_1, g_2 \in SU(2))$ and $z1 = z$.

Let $\mathbb{C}[z_1, z_2]$ denote the polynomial algebra on $\mathbb{C}^2$. Let $V_m := \mathbb{C}[z_1, z_2]_m$ be the subspace of homogeneous polynomials of degree m of $\mathbb{C}[z_1, z_2]$. Then any polynomial f in V_m can be written uniquely as a linear combination of $m+1$ monomials $z_1^k z_2^{m-k}$ $(0 \leq k \leq m)$. Hence one defines an $m+1$ dimensional representation $(\pi_m,\, V_m)$ of $SU(2)$ by

$$(\pi_m(g)f)(z) := f(zg).$$

The inner product defined by

$$(z_1^k z_2^{m-k},\, z_1^\ell z_2^{m-\ell}) = k!(m-k)!\delta_{k,\ell}$$

is invariant under π_n. In other words, equipped with this inner product, $(\pi_m,\, V_m)$ turns to be a unitary representation of $SU(2)$.

Theorem 3.1. *For any non-negative integer m, the unitary representation $(\pi_m,\, V_m)$ of $SU(2)$ is irreducible. Moreover, any irreducible unitary representation of $SU(2)$ is equivalent to one of $(\pi_m,\, V_m)$.* □

Theorem 3.2. *Let $\mathbb{C}[z]_m$ be the space of polynomials in z of degree less than or equals m. Then one define the action $\tau_m(g)$ of $g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ on $\mathbb{C}[z]_m$ by*

$$(\tau_m(g)p)(z) = (\overline{\beta}z + \alpha)^m p\Big(\frac{\overline{\alpha}z - \overline{\beta}}{\overline{\beta}z + \alpha}\Big) \quad (p \in \mathbb{C}[z]_m).$$

Equipped with the inner product defined by $(z^k,\, z^\ell)_m = \frac{k!(m-k)!}{(m+1)!}\delta_{k,\ell}$, the representation $(\tau_m,\, \mathbb{C}[z]_m)$ is unitarily equivalent to $(\pi_m,\, V_m)$. □

Remark 3.1. The Hermitian inner product on $\mathbb{C}[z]_m$ defined in the theorem above can be expressed as follows:

$$(p_1,\, p_2)_m = \frac{m+1}{\pi}\int_{\mathbb{C}} p_1(z)\overline{p_2(z)}(1+|z|^2)^{-m-2}dz.$$

The representation theory of the 3-dimensional rotation group $SO(3)$ is derived from that of $SU(2)$ described above, because $SU(2)$ is a (double) covering group of $SO(3)$. Actually, the kernel of the adjoint representation Ad of $SU(2)$ on the three dimensional real Lie algebra $\mathfrak{g} = \mathfrak{su}_2(\mathbb{R})$ spanned by

$$X = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(identified to the tangent space of $SU(2)$ at 1 equipped with the Lie bracket $[X,\, Y] := XY - YX$) defined by $\mathrm{Ad(g)X = gXg^{-1}}$ ($\mathrm{X} \in \mathfrak{g}$) is $\{\pm 1\}$. Then, noting $\{X, Y, Z\}$ is the orthogonal basis with respect to the inner product $\langle A,\, B\rangle := -2\mathrm{Tr(AB)}$, one observes that the map

$$\mathfrak{g} \ni xX + yY + zZ \;\mapsto\; \vec{x} = (x, y, z) \in \mathbb{R}^3$$

is an isometry. From this

$$SU(2)/\{\pm 1\} \cong SO(3).$$

This fact actually gives the following.

Theorem 3.3. *For any non-negative integer ℓ, there exists an irreducible unitary representation ρ_ℓ of $SO(3)$ which is given by*

$$\rho_\ell \circ \mathrm{Ad} = \tau_{2\ell} (\cong \pi_{2\ell}).$$

Any irreducible unitary representation of $SO(3)$ is equivalent to ρ_ℓ for some ℓ. Moreover, if $\ell \neq m$, then ρ_ℓ is not equivalent to ρ_m. □

Since the two dimensional sphere S^2 is realized by a homogeneous space (actually a compact Riemann symmetric space) of $SO(3)$ as $S^2 \cong SO(2)\backslash SO(3) \cong K\backslash SU(2)$, where $K := \{\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \,|\, 0 \leq \theta < 4\pi\}$, the irreducible unitary representations of $SO(3)$ are realized on the space of harmonic polynomials or spherical harmonics (thought as a well matched description).

Define a representation T_ℓ of $SO(3)$ on the space $\mathcal{P}_\ell$ of homogeneous polynomials of degree ℓ in three variables $\vec{x} = (x, y, z)$ by

$$(T_\ell(g)f)(\vec{x}) = f(\vec{x}g).$$

Note that T_ℓ is not irreducible if $\ell \geq 2$, e.g. the space $\mathcal{P}_2$ contains the non-trivial invariant subspace spanned by the quadratic form $x^2 + y^2 + z^2$.

Let

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

be the Laplacian on $\mathbb{R}^3$. Define a space of harmonic polynomials of degree ℓ by

$$\mathcal{H}_\ell = \{f \in P_\ell \,|\, \Delta f = 0\}.$$

Note that $\dim \mathcal{H}_\ell = 2\ell + 1$.

It is important to notice that the Laplacian Δ is the invariant differential operator, i.e. commute with $T_\ell(g)$ for any $g \in SO(3)$: $\Delta \circ T_\ell(g) = T_\ell(g) \circ \Delta$. From this fact, one finds that the space $\mathcal{H}_\ell$ is stable under the representation T_ℓ. Therefore one can define the representation U_ℓ of $SO(3)$ by the restriction of T_ℓ on $\mathcal{H}_\ell$: $U_\ell(g) := T_\ell(g)|\mathcal{H}_\ell$.

Theorem 3.4. *As an irreducible unitary representation of $SO(3)$*

$$U_\ell \cong \rho_\ell.$$

□

Since any element f in $\mathcal{H}_\ell$ is a homogeneous polynomial of degree ℓ, for any $r \geq 0$ one has

$$f(rx, ry, rz) = r^\ell f(x, y, z).$$

Let $\mathcal{K}$ be the restriction of f to S^2. Put

$$\tilde{\mathcal{H}}_\ell := \mathcal{H}_\ell|_{S^2} = \{\mathcal{K}(f) \,|\, f \in \mathcal{H}_\ell\}.$$

Then $\mathcal{K}$ is a linear isomorphism of the vector space $\mathcal{H}_\ell$ onto $\tilde{\mathcal{H}}_\ell$. The space $\mathcal{H}_\ell$ is obviously recovered from $\tilde{\mathcal{H}}_\ell$ by the equation above. The elements of $\tilde{\mathcal{H}}_\ell$ are called spherical harmonics of degree ℓ. Since the space $\tilde{\mathcal{H}}_\ell$ is stable under the action $U_\ell(g)$ ($g \in SO(3)$), $(U_\ell, \tilde{\mathcal{H}}_\ell)$ defines an irreducible unitary representation of $SO(3)$. The inner product on $\tilde{\mathcal{H}}_\ell$ is given by

$$(f,\, g)_{S^2} = \int_{S^2} f(\vec{x})\overline{g(\vec{x})} d\vec{x},$$

where $d\vec{x}$ is the normalized measure on S^2 given by $d\vec{x} = (4\pi)^{-1} \sin\theta \, d\theta d\phi$. Here

$$\vec{x} = (x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \quad (r > 0,\ (\theta, \phi) \in [0, 2\pi] \times [0, \pi])$$

is the polar coordinates of $\mathbb{R}^3$.

Theorem 3.5. *The space $\tilde{\mathcal{H}}_\ell$ has an orthonormal basis $\{Y_\ell^m\}_{-\ell \leq m \leq \ell}$ defined by*

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{(2\ell + 1)(\ell + m)!}{(\ell - m)!}} e^{im\phi} P_\ell^{-m}(\cos\theta),$$

where $P_\ell^m(x)$ is the associated Legendre function defined by

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell.$$

These Y_ℓ^m are the matrix elements of the representation $(U_\ell,\, \tilde{\mathcal{H}}_\ell)$ and called Legendre's spherical functions. □

Since these representations U_ℓ exhaust all (equivalence classes) of irreducible unitary representations of $SO(3)$ one obtains

$$L^2(S^2) = \Sigma_{\ell=0}^{\infty} \oplus \tilde{\mathcal{H}}_\ell \quad \text{(a direct sum)}.$$

This shows that a (square integrable, hence in particular, continuous) function on the unit sphere S^2 can be expanded by the spherical harmonics $Y_\ell^m(\theta,\phi)$ $(-\ell \leq m \leq \ell,\ \ell = 0, 1, \ldots)$. This fact is the consequence of the following Peter-Weyl theorem of compact group:

Theorem 3.6. *Let $\widehat{G} = \{\pi\}$ be the unitary dual of G, the set of all equivalence classes of irreducible unitary representations of G. Take a representative (π, V) of π (using the same letter). Put $d_\pi = \dim_{\mathbb{C}} V$. Then the family $B_G := \{\sqrt{d_\pi}(\pi(g)v_i, v_j) \mid \pi \in \widehat{G},\ 1 \leq i, j \leq d_\pi\}$ is a complete orthonormal family of $L^2(G)$.* □

Example 3.1. The unitary dual $\widehat{SO(3)}$ of $SO(3)$ can be parametrized by the set $\mathbb{Z}_{\geq 0}$ of non-negative integers ℓ. Since $S^2 \cong SO(2)\backslash SO(3)$, the Legendre's spherical function is given by a matrix element $(f, \rho_\ell(g) f_0)$, where f_0 is a $SO(2)$-fixed vector, i.e. $\rho_\ell(k) f_0 = f_0$ for $k \in SO(2)$.

4. Harmonic expansions by differetiation

In the talk, I will explain the role of the intertwiner A_ℓ (and its inverse) between $\mathbb{C}[w]_\ell$ and $\mathcal{H}_\ell$ given explicitly as

$$(A_\ell p)(\vec{x}) := \frac{2\ell+1}{\pi}\int_{\mathbb{C}} p(w)\overline{H(\vec{x},\, w)}^{\ell}(1+|w|^2)^{-2\ell-2}dw,$$

where $H(\vec{x},\, w) := (x+iy)w^2 + 2z\,w - (x-iy)$. Notice that, for w fixed, as a function of $\vec{x} = (x,y,z) \in \mathbb{R}^3$, one observes immediately that $H(\vec{x},\, w)^\ell \in \mathcal{H}_\ell$. By this description, one may transform the stage of calculations from the one using spherical harmonics to the one using simple monomials z^m. Namely, in some part of the spherical harmonic lighting technique, one might avoid rather complicated recurrence formulas and/or differential equation of the associated Legendre functions P_ℓ^m.

Recall the facts that in terms of the polar coordinate, Δ can be expressed as

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\Delta_{S^2},$$

$$\Delta_{S^2} := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}.$$

Notice that the last expression is actually known to be the image of the Casimir element $\mathcal{C} \in \mathcal{Z}U(\mathfrak{so}_3)$, the center of the universal enveloping algebra $U(\mathfrak{so}_3)(\cong U(\mathfrak{su}_2))$, under the (infinitesimal) right action by $SO(3)$. Therefore, since the representation U_ℓ is irreducible, one finds that $Y_\ell(\theta,\phi)$ is the eigenfunction of Δ_{S^2} with eigenvalue $-\ell(\ell+1)$:

$$\Delta_{S^2} Y_\ell(\theta,\phi) = -\ell(\ell+1) Y_\ell(\theta,\phi).$$

At the harmonic expansion of a spherical function f, one practically considers the approximation $\tilde{f}_N$ truncated by high frequency irreducible components U_ℓ ($\ell \geq N+1$):

$$f \approx \tilde{f}_N := \sum_{\ell=0}^{N} \sum_{|m|\leq \ell} a_\ell^m Y_\ell^m,$$

where we put $a_\ell^m = (f,\, Y_\ell^m)_{S^2}$ (but not computing here this integral). Define the (projection) operator $P_\ell^N : \sum_{j=0}^N \oplus \mathcal{H}_j \to \mathcal{H}_\ell$ by

$$P_\ell^N := \prod_{\ell'=0,\ell'\neq \ell}^{N} \frac{\Delta_{S^2} + \ell'(\ell'+1)}{-\ell(\ell+1) + \ell'(\ell'+1)}.$$

It follows immediately that

$$P_\ell^N \tilde{f}_N = \sum_{|m|\leq \ell} a_\ell^m Y_\ell^m.$$

Then one has

$$a_\ell^m = (P_\ell^N \tilde{f}_N,\, Y_\ell^m)_{S^2} = (A_\ell^{-1} P_\ell^N \tilde{f}_N,\, z^{\ell-m})_{2\ell}.$$

Now we try to overview how to obtain the inverse isomorphism A_ℓ^{-1} from the space $\mathcal{H}_\ell$ to $\mathbb{C}[w]_m$. Let $F \in \mathcal{H}_\ell$. Write F as

$$F(x,y,z) = g_0(x,y) + g_1(x,y)z + \cdots + g_\ell(x,y)z^\ell.$$

Note the fact that $g_j(x,y) \in \mathbb{C}[x,y]_{\ell-j}$ (i.e., is a polynomial of homogeneous degree $\ell - j$). Put $\Delta_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Then it is immediate that

$$\Delta_{x,y} F = \Delta_{x,y} g_0 + \cdots + \Delta_{x,y} g_\ell z^\ell,$$
$$\frac{\partial^2}{\partial z^2} F = 2g_2 + \cdots + \ell(\ell-1) g_\ell z^{\ell-2}.$$

Since g_ℓ (reps. $g_{\ell-1}$) is a constant (resp. a linear function) and $\Delta = \Delta_{x,y} + \frac{\partial^2}{\partial z^2}$, one observes that F is a harmonic polynomial if and only if the following condition are satisfied:

$$g_k = -\frac{1}{k(k-1)} \Delta_{x,y} g_{k-2} \quad (2 \leq k \leq \ell).$$

Hence one finds that there is a one-to-one correspondence:

$$\begin{aligned} \mathcal{H}_\ell \ni F \mapsto (g_0,\, g_1) \in \mathbb{C}[x,y]_\ell \oplus \mathbb{C}[x,y]_{\ell-1} &\cong \mathbb{C}[w]_\ell \oplus \mathbb{C}[w]_{\ell-1} \\ &\cong \mathbb{C}[w^2]_\ell \oplus w\mathbb{C}[w^2]_{\ell-1} \cong \mathbb{C}[w]_{2\ell} \end{aligned}$$

Notice that the latter three isomorphisms are obviously all algebraic. Since $g_0 = F(x,y,0)$ and $g_1 = \frac{\partial}{\partial z} F(x,y,z)|_{z=0}$, using the reproducing kernel $K_z(w) = K(w,\, z) := (1 + \overline{z}w)^{2\ell}$ (more precisely, the even and odd parts of the reproducing kernels) of the Hilbert space $\mathbb{C}[w]_{2\ell}$, one can essentially construct the inverse of the intertwiner A_ℓ.

This allows us to compute the coefficient a_ℓ^m from $P_\ell^N \tilde{f}_N$ avoiding integration process (via comuting $P_\ell^N \tilde{f}_N(x, y, 0)$ and $\frac{\partial}{\partial z} P_\ell^N \tilde{f}_N(x, y, z)|_{z=0}$. In the presentation, I will explain this mathematical framework more precisely for the expecting future use.

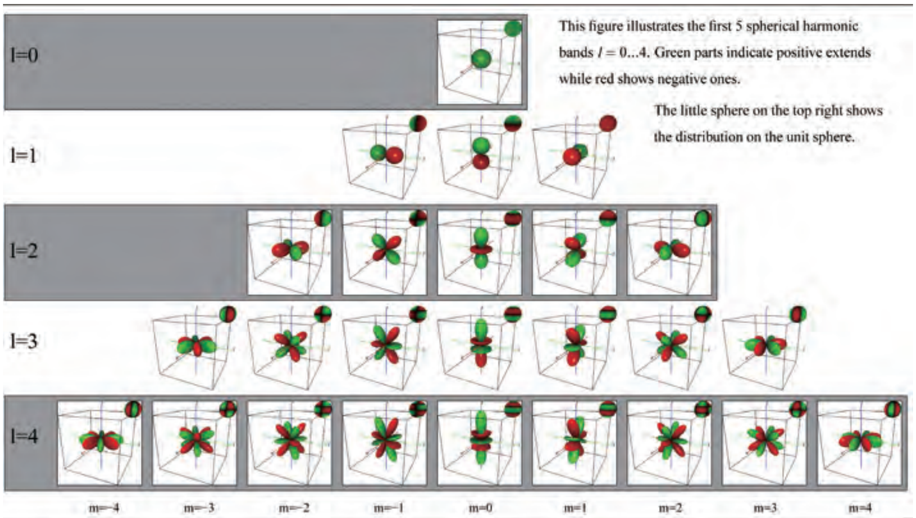

FIGURE 1. Number of the figures $2\ell + 1$ in each horizontal row is the dimension of irreducible representation $(U_\ell, \tilde{\mathcal{H}}_\ell)$. (The figure is from [5]).

Furthermore, implementation of the idea provided here to computers for spherical harmonic lighting would be desirable.

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