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Imoto, Yusuke
Graduate School of Mathematics, Kyushu University

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Truncation Error Analysis of SPH Method and it's Applications to Numerical Simulations of Flow Problem

Yusuke Imoto

Graduate School of Mathematics, Kyushu University, Fukuoka, Japan

(joint work with Daisuke Tagami)

Abstract. SPH (Smoothed Particle Hydrodynamics) method is one of the numerical methods for partial differential equations. As the first step of the mathematical analysis on SPH method, we have established truncation error estimates of approximated operators appearing in SPH method. Moreover, we have applied these operators into numerical simulations of flow problems by SPH method, and we have shown some numerical results of the dumbreak problem both in 2D and 3D.

1. Introduction

Numerical methods is computing methods that solve numerically partial differential equations by using computers. Recently, with the development of computer, the numerical simulations have been performed in various fields, and these play an important role in understanding physical phenomena, same as that of theory or physical experiment. We show an image of approximation by SPH method in Figure 1. In this figure, color balls , bell contour form, and h represent particles, a distribution of a weight function, and a smoothing length, respectively. The weight function is continuous one which has non zero value if a distance between analysis point and particle is within the smoothing length. An approximation of differential operators appearing partial differential equations is defined by using a value on the particles and the weight function.

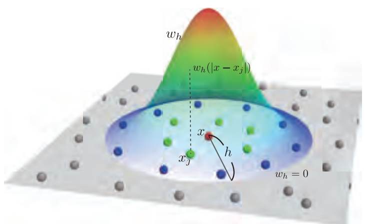

Figure 1. An image of approximation by SPH methods.

In the end of 1970's, SPH method was introduced for compressible Euler equations in Lucy [6] and Gingold–Monaghan [2]. After that, SPH method was applied in compressible or incompressible Navier-Stokes equations. As far as we know, there is not the numerical analysis for SPH method, and there is only the truncation error analysis of approximated gradient operator on MPS(Moving Particle Semi-implicit) method which is called particle-based method same as SPH method by Ishijima–Kimura [4]. Then, we investigate the truncation error analysis on SPH method by using general indicators [5].

This paper is organized as follows. In Section 2, w introduce SPH methodand some approximated operators used in the SPH method. In Section 3, we analyze a truncation errors of approximated operators in SPH method. Moreover, in Section 4, we introduce a discretization for the incompressible Navier-Stokes equations in SPH method. In the Section 5, we show some numerical simulations for incompressible Navier-Stokes equations flow problems by using the operators

2. Approximated Operators of SPH Method

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$, a smoothing length h be a positive number, and N is a finite natural number. We define a particle distribution $\mathcal{P}_{N,h}$ as

$$\mathcal{P}_{N,h} = \{\boldsymbol{x}_j \in \Omega_h \ ; \ j = 1, \ldots, N\},$$
$$\Omega_h := \left\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ {}^{\exists}\boldsymbol{x}' \in \Omega \text{ s.t. } |\boldsymbol{x}' - \boldsymbol{x}| < h\right\} :$$

see in Figure 2.

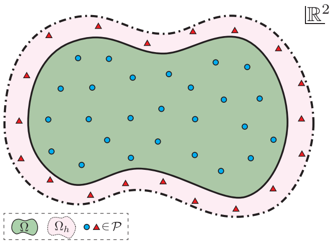

Figure 2. An example of a particle distribution

Next, a reference function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined by a function satisfying the following conditions :

$$w \in C^2\left(\mathbb{R}_0^+\right), \ \operatorname{supp}(w) = [0, 1], \quad \int_{\mathbb{R}^d} w(r) d\boldsymbol{x} = 1 \ (r = |\boldsymbol{x}|).$$

For the reference function, we define a weighting function weight function on SPH method as $w_h(r) := h^{-d} w(r/h)$. Figure 3 show one of the reference function which are a 5-th spline function, called a quintic spline [1].

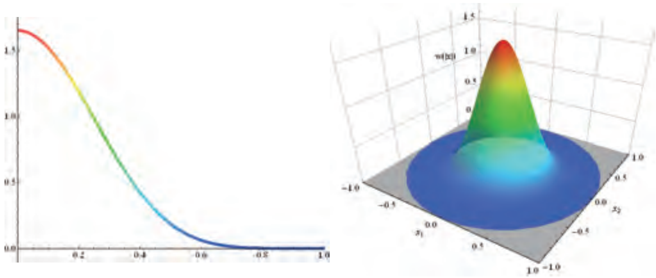

Figure 3. The left figure shows a graph of a reference function w by the quintic spline, and the right figure shows 2D graph of a function $w(|\boldsymbol{x}|)$ $(\boldsymbol{x} \in \mathbb{R}^2)$ for the reference function.

Here, let a particle volumes V be

$$V := \{V_i \in \mathbb{R}^+ \ ; \ i = 1, \ldots, N\}.$$

For smooth functions $\phi : \Omega_H \to \mathbb{R}$ and $\boldsymbol{\psi} : \Omega_H \to \mathbb{R}^d$, an approximated gradient $\nabla_h \phi$, an approximated divergence $\nabla_h \cdot \boldsymbol{\psi}$, and an approximated Laplace $\Delta_h \phi$ in SPH method are defined by

$$\text{(OP)} \qquad \begin{aligned} \nabla_h \phi(\boldsymbol{x}) &:= \sum_j V_j \left\{\phi(\boldsymbol{x}_j) - \phi(\boldsymbol{x})\right\} \nabla w_h(|\boldsymbol{x} - \boldsymbol{x}_j|), \\ \nabla_h \cdot \boldsymbol{\psi}(\boldsymbol{x}) &:= \sum_j V_j \left\{\boldsymbol{\psi}(\boldsymbol{x}_j) - \boldsymbol{\psi}(\boldsymbol{x})\right\} \cdot \nabla w_h(|\boldsymbol{x} - \boldsymbol{x}_j|), \\ \Delta_h \phi(\boldsymbol{x}) &:= 2 \sum_j V_j \frac{\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_j)}{|\boldsymbol{x} - \boldsymbol{x}_j|^2} (\boldsymbol{x} - \boldsymbol{x}_j) \cdot \nabla w_h(|\boldsymbol{x} - \boldsymbol{x}_j|), \end{aligned}$$

where $\boldsymbol{x} \in \Omega$ and differential operators ∇ and Δ are

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d}\right)^T, \qquad \Delta := \sum_{j=1}^d \frac{\partial^2}{\partial {x_j}^2} \qquad \left(\boldsymbol{x} = (x_1, \ldots, x_d)^T\right).$$

3. Truncation Error Analysis

An indicator $r_c = r_c(\mathcal{P}_{N,h})$ of the particle distribution, called a covering radius, is defined by

$$r_c := \min \left\{ r \ ; \ \bigcup_{\boldsymbol{x}_j \in \mathcal{P}_{N,h}} \overline{B_r(\boldsymbol{x}_j)} \supset \Omega_h \right\},$$

where $B_r(\boldsymbol{x}_j) := \left\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ |\boldsymbol{x} - \boldsymbol{x}_j| < r\right\}$: see in Figure 4.

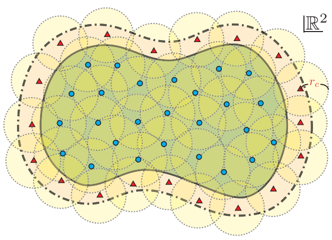

Figure 4. A covering radius of the particle distribution in Figure 2.

Moreover, we define the particle volumes V as volumes by Voronoi decomposition:

$$\begin{aligned} V_j &= |\tilde{V}_j|, \\ \tilde{V}_j &:= \left\{\boldsymbol{x} \in \Omega_h; |\boldsymbol{x} - \boldsymbol{x}_j| < |\boldsymbol{x} - \boldsymbol{x}_k|, {}^\forall \boldsymbol{x}_k (\neq x_j)\right\}. \end{aligned}$$

Theorem. We consider the row of the particle distribution $\{\mathcal{P}_{N,h}\}_{h \downarrow 0}$. If there exists positive constants α, c, c' independent of h and N of the row and the row satisfy

$ch \leq\ r_c^\alpha \leq c'h$, then we have

$$\begin{aligned}
\|\Pi_h u - u\|_{C(\Omega)} &= \mathcal{O}\left({r_c}^{\beta}\right) \qquad u \in C^2(\Omega_H),\\
\|\nabla_h u - \nabla u\|_{C(\Omega)} &= \mathcal{O}\left({r_c}^{\beta}\right) \qquad u \in C^3(\Omega_H),\\
\|\Delta_h u - \Delta u\|_{C(\Omega)} &= \mathcal{O}\left({r_c}^{\gamma}\right) \qquad u \in C^4(\Omega_H),
\end{aligned}$$

where $H = \max\{h\}$, $\beta = \min\{2\alpha, 1-\alpha\}$ $(\beta \leq 2/3)$, $\gamma = \min\{2\alpha, 1-2\alpha\}$ $(\gamma \leq 1/2)$, and $\|\cdot\|_{C(\Omega)}$ is a maximum norm on Ω is defined by $\|\varphi\|_{C(\Omega)} = \max_{\boldsymbol{x}\in\Omega} |\varphi(\boldsymbol{x})|$.
Remark 1. When V has no relations with Voronoi decomposition, if $\mathcal{P}_{N,h}$ and V satisfy a *regular condition* [5], then we have same orders.
Remark 2. For all of the particle distributions whose number of particle is same, the particle distribution minimizing the covering radius has no dense and sparse parts.

4. Discretization of Incompressible Navier-Stokes Equations

In this section, we introduce a numerical scheme of the incompressible Navier-Stokes equations by the projection method [3]. The incompressible Navier-Stokes equations are defined as

$$\text{(NS)} \qquad \begin{cases} \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times I, \\ \dfrac{D}{Dt}\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f} & \text{in } \Omega \times I. \\ +\text{Boundary conditions and Initial conditions} \end{cases}$$

Here, $I := (0, T)$ is time interval, $\mathbf{u} : \Omega \times I \to \mathbb{R}^d, p : \Omega \times I \to \mathbb{R}, \mathbf{f} : \Omega \times I \to \mathbb{R}^d$, and $\sigma : \Omega \times I \to \mathbb{R}^{d\times d}$ are a velocity, a dynamic pressure, an external force, and a stress tensor, respectively, $\nu \in \mathbb{R}$ is a kinematic viscosity, and the Lagrangian derivative is defined by $D/Dt := \partial/\partial t + \mathbf{u} \cdot \nabla$. The external force and the kinematic viscosity are a given function and a given parameter, and the velocity and the dynamic pressure are unknown functions. We discretize (NS) in time and space by SPH method:

$$0 = t^0 < t^1 < \cdots < t^{N_T} = T,\ \Delta t^{n+1} := t^{n+1} - t^n,$$

$$\phi_i^{n+1} := \phi(\mathbf{X}(\boldsymbol{x}_i, t^{n+1}), t^{n+1}),\ \boldsymbol{x}_i \in \Omega_P,$$

$$\mathbf{X}(\boldsymbol{x}, t^{n+1}) = \boldsymbol{x} + \sum_{j=1}^{n} \mathbf{u}\left(\mathbf{X}(\boldsymbol{x}, t^j), t^j\right),\ \mathbf{X}(\boldsymbol{x}, t^0) = \boldsymbol{x},$$

for $n = 0, 1, \ldots, N_T - 1$, for $i = 1, 2, \ldots, N$

First step	(compute $\mathbf{u}_i^*$)	$\mathbf{u}_i^* = \mathbf{u}_i^n + \Delta t^{n+1}\left(\nu \Delta_h \mathbf{u}^n + \mathbf{f}^n\right)$,
Second step	(solve Poisson equation for p_i^n)	$\Delta_h p^n = \dfrac{1}{\Delta t^{n+1}} \nabla_h \cdot \mathbf{u}_i^*$,
Third step	(compute $\mathbf{u}_i^{n+1}$)	$\mathbf{u}_i^{n+1} = -\Delta t^{n+1} \nabla_h p_i^n + \mathbf{u}_i^*$.

The any equations are replaced liner equations by using (OP). So, if there is appropriate the boundary conditions and the initial conditions, by solving through the steps of these, we have unknown functions velocity $\mathbf{u}$ and dynamic pressure p. Particularly, a method solving incompressible Navier-Stokes equations by using projection method and SPH method, we call it ISPH (Incompressible SPH) method. Recently, some ISPH methods which added an effect avoiding dense and sparse parts are suggested, for instance, Asai and et. all. suggest *Stabilized ISPH method* [1], which is ISPH method added a term homogenized particle positions in the Navier Stokes equations.

5. Numerical Computations of Flow Problem

In this section, we introduce some examples of numerical computations by Stabilized ISPH method. The dumbreak problem in 2D and 3D are considered as model problem in [7]. Also, Visualizations of numerical results was performed by *paraview* [8].

5.1. **2D Simulation.** In the 2D simulation, we set the following conditions:

$$\begin{aligned} \text{Given data} : & \ \Omega = (0, 0.2) \times (0, 0.6), \ I = (0, 1.6), \ \nu = 10^{-5}, \\ \text{Initial conditions} : & \ \mathbf{u}(\boldsymbol{x}_i, 0) = 0 \ (\boldsymbol{x}_i \in \mathcal{P}), \\ \text{Boundary conditions} : & \ \mathbf{u}(\boldsymbol{x}_i, t^n) = 0 \ (\text{for } \boldsymbol{x}_i \in \Gamma_w), \\ \text{Discretized parameters} : & \ \Delta \boldsymbol{x} = 0.005, \ N_{\mathcal{P}} = 6,205, h = 3\Delta \boldsymbol{x}, \ \Delta t^{n+1} = 0.1h/\mathbf{u}^n_{\text{max}}. \end{aligned}$$

Moreover, we give the other boundary condition a natural one. Here Γ_w is a wall boundary, $\Delta \boldsymbol{x}$ is a minimum distance between initial particle points, and $\mathbf{u}^n_{\text{max}}$ is the maximum bulk of the velocity at t^n. Figure 5 shows a numerical results by Stabilized ISPH method from $t = 0.0$ to $t = 1.6$, and the particle color represents the pressure (Red denotes high pressure; Blue denotes low pressure).

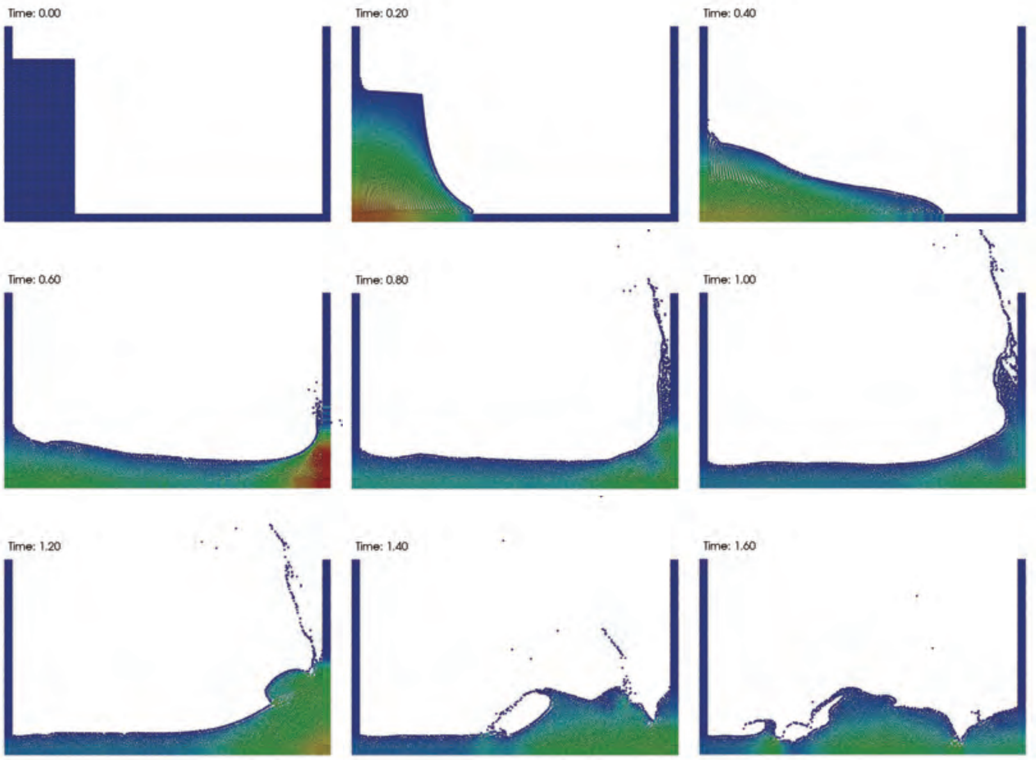

Figure 5. 2D simulation of the dumbreak problem.

5.2. **3D Simulation.** In the 3D simulation, we set the following conditions:

$$\begin{aligned}
\text{Given data} &: \Omega = (0, 1.228) \times (-0.5, 0.5) \times (0, 0.55),\ I = (0, 1.6),\ \nu = 10^{-5},\\
\text{Initial condition} &: \mathbf{u}(\boldsymbol{x}_i, 0) = 0\ (\boldsymbol{x}_i \in \mathcal{P}),\\
\text{Boundary condition} &: \mathbf{u}(\boldsymbol{x}_i, t^n) = 0\ (\text{for } \boldsymbol{x}_i \in \Gamma_w),\\
\text{Discretized parameters} &: \Delta\boldsymbol{x} = 0.025,\ N_{\mathcal{P}} = 10,1991, h = 3\Delta\boldsymbol{x},\ \Delta t^{n+1} = 0.1h/\mathbf{u}^n_{\max}.
\end{aligned}$$

FIGURE 6 shows a numerical results by Stabilized ISPH method from $t = 0.0$ to $t = 1.6$, and the particle color represents the bulk of the velocity (Red denotes high bulk of the velocity; Blue denotes low bulk of the velocity).

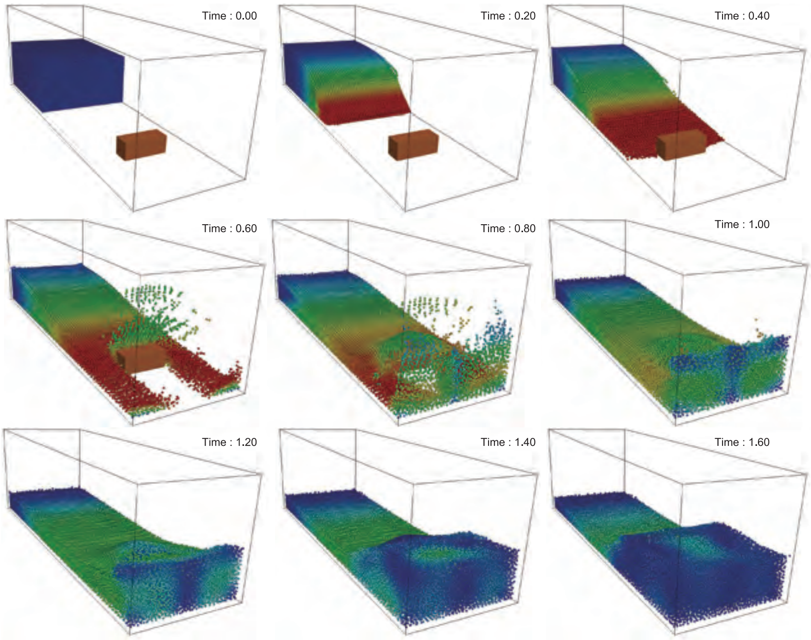

FIGURE 6. 3D simulation of the dumbreak problem.

REFERENCES

[1] Asai, M. and et al. “A Stabilized Incompressible SPH Method by Relaxing the Density Invariance Condition”, *J. Appl. Math.* 2012.

[2] Gingold, R. A. and Monaghan, J. J., “Smoothed particle hydrodynamics - theory and application to non-spherical stars”, *R. Astron. Soc. Ser.*, 181, pp. 375-389, 1977.

[3] Guermond, J. L., Minev, P., and Shen J., “An overview of projection methods for incompressible flows”, *Comput. Methods Appl. Mech. Engrg.*, 195.44, pp. 6011-6045, 2006.

[4] Ishijima, K. and Kimura, M., “Truncation error analysis of finite difference formulae in meshfree particle methods”, *Transactions of the Japan Society for Industrial and Applied Mathematics*, 20, pp. 165-182, 2010.

[5] Imoto, Y. and Tagami, D., “Truncation Error Analysis of Approximated Operators on SPH Method”, in preparation.

[6] Lucy, L. B., “A numerical approach to the testing of the fission hypothesis”, *The Astronomical Journal*, 82, pp. 1013-1024, 1977.

[7] Zhou, Z. Q., J. O. De Kat, and B. Buchner. “A nonlinear 3-D approach to simulate green water dynamics on deck”, *Proceedings of the 7th International Conference on Numerical Ship Hydrodynamics*, 1999.

[8] paraview (open-source), “http://www.paraview.org/”.