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The Power of Orthogonal Duals

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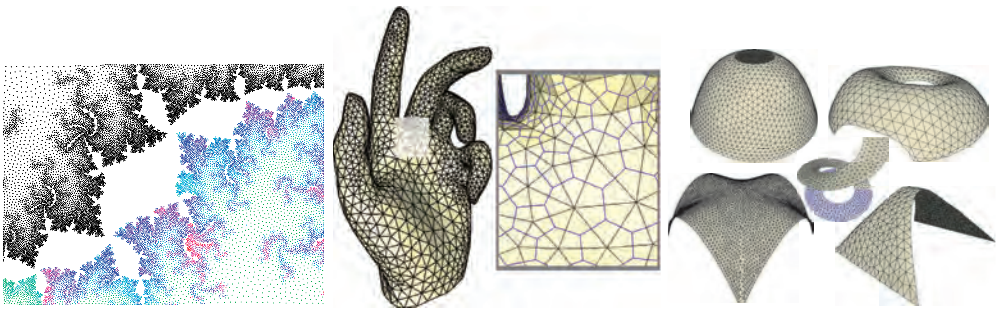
Triangle meshes have found widespread acceptance in computer graphics as a simple, convenient, and versatile representation of surfaces. In particular, computing on such simplicial meshes is a workhorse in a variety of graphics applications. In this context, mesh duals (tied to Poincaré duality and extending the well known relationship between Delaunay triangulations and Voronoi diagrams) are often useful, be it for physical simulation of fluids [5] or parameterization [7]. However, the precise embedding of a dual diagram with respect to its triangulation (i.e., the placement of dual vertices) has mostly remained a matter of taste or a numerical after-thought, and barycentric vs. circumcentric duals are often the only options chosen in practice. In this talk we discuss the notion of *orthogonal dual diagrams*, and show through a series of recent works that exploring the full space of orthogonal dual diagrams to a given simplicial complex is not only powerful and numerically beneficial, but it also reveals (using tools from algebraic topology and computational geometry) discrete analogs to continuous properties.

Starting from a (primal) triangle mesh defined as a simplicial complex (i.e., a piecewise linear approximation of a discrete orientable manifold surface of any topology in



(a) Blue Noise Sampling. (b) Well-centered Meshes. (c) Self-supporting Structures.

Figure 1. Orthogonal dual diagrams to primal simplicial meshes have recently been shown key in a wide variety of applications in geometry processing and graphics, for flat & curved domains of arbitrary topology.

$\mathbb{R}^3$, with or without boundary), a family of intrinsic dual diagrams [9] can be constructed through the addition of a weight per vertex [1]. The resulting diagrams can be intuitively understood as displacements of the canonical (Euclidean) circumentric dual along the gradient of the function defined by the weights, resulting in intrinsically straight dual edges that remain orthogonal to primal edges due to the curl-free nature of any gradient field [4]. For surfaces of non-trivial genus, there are additional displacement fields that are curl-free but are not gradients: they correspond to the so-called harmonic 1-forms (of dimension β_1, the first Betty number of the surface). Therefore, the total space of orthogonal duals is, accounting for the gauge of the gradient, of dimension $\beta_1 + V - 1$ where V denotes the number of vertices in the primal mesh. Note that once a dual diagram is defined through these coordinates, close formulae for the signed measures of the dual elements (dual lengths, dual cell areas) are available.

This simple definition of orthogonal duals is surprisingly versatile in that it offers efficient and foundational solutions to numerous applications, including:

- *Blue noise sampling:* Coined by Ulichney [11], the term *blue noise* refers to an even, isotropic, yet unstructured distribution of points in (typically 2D) Euclidean space. Blue noise was first recognized as crucial in dithering of images since it captures the intensity of an image through its local point density, without introducing artificial structures of its own. It rapidly became prevalent in various scientific fields, especially in computer graphics, where its isotropic properties lead to high-quality sampling of multidimensional signals, and its absence of structure prevents aliasing. However, the generation of high-grade blue-noise importance sampling remains numerically challenging.

  Our orthogonal duals offer a convenient solution to this common requirement. By writing the density requirement as constraints on dual cell areas of the diagram and using optimal transport to formally characterize the isotropy of a point distribution, one ends up with an efficient optimization technique of point distributions via a constrained minimization in the space of orthogonal dual diagrams. In this application, the weights are crucial degrees of freedom to exactly enforce adapted sampling, rendering the formulation not only well-posed but efficient as well. In practice, the resulting blue noise point distributions outachieve previous methods based on both spectral and spatial analyses [2].

- *Self-supporting structures:* Masonry structures are arrangements of material blocks, such as bricks or stones, that support their own weight. Constructing curved vaults or domes with compression-only structures of blocks, further prevented from slipping through friction and/or mortar, has been practiced since antiquity. It is therefore no surprise that form finding and stability analysis of self-supporting structures have been an active area of research for years. In particular, it has been shown that equilibrium of a masonry structure is ensured if there exists an inner *thrust* surface which forms a compressive membrane resisting the external loads [6]. Discretizing the continuous balance equations relating the stress field on the thrust surface to the loads has been, however, an open problem for years with no fully satisfactory solution.

  Our primal-dual structures offer, here again, an unexpected approach to this problem: a finite-dimensional formulation of the compressive stress field of these self-supporting membranes represented a triangle meshes can be rigorously derived through homogenization; moreover, equilibrium is guaranteed if (and only if) the dual planar graph induced by vertex weights forms an orthogonal dual diagram—corresponding to the force network at play within the membrane. Therefore, our full characterization of orthogonal duals formally provides discrete (and exact) analogs of continuous properties; in fact, the weight themselves correspond in this context to the Airy stress function, a staple of static continuum mechanics. One can thus derive computational form-finding tools to alter a reference shape into a free standing simplicial structure, which turns out to improve upon previous work in terms of efficiency, accuracy, and scalability [3].

- *Meshing:* Being able not only to choose primal vertex positions, but also a dual diagram, opens up a series of possibilities in the context of meshing, i.e., turning a 2D or 3D domain into a(n often simplicial) complex. A majority of meshing approaches restrict the space of valid meshes to Delaunay triangulations because they are abundantly vetted by theoretical guarantees; in practice, however, Delaunay conditions are often too restrictive to be valuable. Moreover, it is extremely difficult in practice to construct "self-centered" Delaunay triangulations [10] for which each circumcenter lies inside its associated tetrahedron: failure to satisfy this property locally can lead to numerical degeneracies. Recent methods attempting to optimize meshes to avoid this issue remain impractical for complex domains [12]. With the added flexibility offered by weights, one can much more easily optimize a mesh to become well-centered by finding the weight assignment that results in dual vertices closest to each triangle's barycenter. Moreover, as our primal-dual structures are compatible with Discrete Exterior Calculus (DEC, a finite-dimensional calculus inspired by Cartan's exterior calculus), one can also optimize meshes to make a particular discrete operator (e.g., the commonly-used Laplacian) both better conditioned and with smaller error bounds [8].

Orthogonal duals have also a number of connections to other research fields, such as circle packing and discrete conformal structures, which indicates that more results are likely to come out in the next few years.

References

[1] David Glickenstein: Geometric triangulations & discrete Laplacians on manifolds. Preprint at `arXiv.org:math/0508188`, 2005.

[2] Fernando de Goes, Katherine Breeden, Victor Ostromoukhov, and Mathieu Desbrun: Blue noise through optimal transport. *ACM Trans. Graph. (SIGGRAPH Asia), 31(4)*, 171:1–171:11, 2012.

[3] Fernando de Goes, Pierre Alliez, Houman Owhadi, and Mathieu Desbrun: On the equilibrium of simplicial masonry structures.*ACM Trans. Graph. (SIGGRAPH), 32(4)*, 93:1–93:10, 2013.

[4] Fernando de Goes, Patrick Mullen, Pooran Memari, and Mathieu Desbrun: Weighted Triangulations for Geometry Processing. Accepted to *ACM Trans. Graph.*, 2013.

[5] Sharif Elcott, Yiying Tong, Eva Kanso, Peter Schröder, P., Mathieu Desbrun: Stable, circulation-preserving, simplicial fluids. *ACM Trans. Graph. 26 (1)*, 2007.

[6] Jacques Heyman: The stone skeleton. *Int. J. of Solids and Structures 2(2)*, 249–279, 1966.

[7] Christian Mercat: Discrete Riemann surfaces & Ising model. *Comm. Math. Phys. 218*, 177–216, 2001.

[8] Patrick Mullen, Pooran Memari, Fernando de Goes, and Mathieu Desbrun: HOT: Hodge-optimized triangulations. *ACM Trans. Graph. (SIGGRAPH), 30(4)*, 103:1–103:12, 2011.

[9] James R. Munkres: Elements of Algebraic Topology. Addison-Wesley, 1984.

[10] V.T. Rajan: Optimality of Delaunay triangulation in $\mathbb{R}^d$. *Disc. Comp. Geo. 12(1)*, 189–202, 1994.

[11] Robert A. Ulichney: Digital Halftoning. MIT Press, 1987.

[12] Evan VanderZee, Anil N. Hirani, Damrong Guoy, and Edgar Ramos: Well-centered triangulation. *SIAM Journal on Scientific Computing 31(6)*, 4497–4523, 2010.