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Abstract

The paper [KW] studied a family of elliptic functions defined by certain q-series. This family, in particular, contains the Weierstrass $\wp$-function. In this paper, we prove that elliptic functions in this family satisfy certain non-linear algebraic differential equations whose coefficients are essentially given by rational functions of the first few Eisenstein series of the modular group.

Keywords: elliptic functions, q-series, Weierstrass $\wp$-function, Eisenstein series, Lambert series.

2000 Mathematical Subject Classification: 11M36, 33E05

1 Introduction

In the paper [KW], Kurokawa and the first author studied the q-series $L_\ell(x)$ $(x \in \mathbb{C})$ defined by

$$L_\ell(x) := \sum_{n\in\mathbb{Z}} \frac{q^{\frac{1}{2}(n+x)\ell}}{[n+x]_q^\ell} \quad (\ell = 1, 2, 3, \ldots).$$

Here we put $[\beta]_q := \frac{q^\beta - 1}{q-1}$ for $\beta \in \mathbb{C}$. Throughout the present paper, we assume that $q > 1$. The series $L_\ell(x)$ converges absolutely and uniformly in x, whence defines an elliptic function for the period lattice $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot \frac{2\pi i}{\log q}$ when ℓ is even, while $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot \frac{4\pi i}{\log q}$ when ℓ is odd. Precisely, L_ℓ has the following properties.

$$\begin{cases} L_\ell(x+1) = L_\ell(x), \\ L_\ell(x + \frac{2\pi i}{\log q}) = (-1)^\ell L_\ell(x), \\ L_\ell(-x) = (-1)^{-\ell} L_\ell(x). \end{cases}$$

It is shown in [KW] that the series $L_2(x)$ gives essentially the Weierstrass $\wp$-function and hence, in particular, satisfies the non-linear algebraic differential equation. An algebraic differential equation for L_1 is also derived. Beside this fact, it was proved that $L_{2\ell-1} \in \mathbb{C}[L_1]$ and $L_{2\ell} \in \mathbb{C}[L_2]$. In fact, $L_{2\ell-1}$ (resp. $L_{2\ell}$) is expressed as a polynomial in L_1 (resp. L_2) of degree $2\ell - 1$ (resp. ℓ). Moreover, one may replace the fields of the coefficients $\mathbb{C}$ by the ring $\mathbb{Q}(q)$ of Laurent polynomials over $\mathbb{Q}$. From these facts, it was conjectured in [KW] that the elliptic function $L_\ell(x)$ satisfies a non-linear algebraic differential equation for each positive integer ℓ with coefficients in $\mathbb{Q}(q)$.

The purpose of this paper is to prove this conjecture affirmatively and give some precise information about the coefficients of the differential equation. Actually, for instance, we will show that the coefficients in the even ℓ are given by rational functions of $(q-1)^2, (\log q)^2$ and the Lambert series $A_\ell(q)$ $(\ell = 1, 3, 5)$ over $\mathbb{Q}$. We note that $A_{2\ell+1}(q)$ gives essentially the Fourier series expansion of the Eisenstein series $E_{2\ell}(z)$ for the modular group $SL_2(\mathbb{Z})$ if we put $q = e^{-2\pi i z}$ $(\mathrm{Im(z)} > 0)$ [L].

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2 Basic facts on L_ℓ

Let us recall first the basic facts in [KW], the algebraic differential equations and the Laurent expansion of the functions $L_1(x)$ and $L_2(x)$.

The differential equations for L_1 and L_2 are described as follows.

Lemma 2.1. *Define the Lambert series $B_\ell(q)$ by*

$$B_\ell(q) := \sum_{m=0}^{\infty} \frac{(m+\frac{1}{2})^\ell}{q^{m+\frac{1}{2}}-1}.$$

Then we have

$$(L_1')^2 = \frac{e_2}{e_1}L_1^4 + e_2(e_3+\frac{1}{4})L_1^2 + e_1e_2e_4, \tag{2.1}$$

where

$$\begin{cases} e_1 = (q-1)^2, \\ e_2 = (\log q)^2, \\ e_3 = 12B_1(q), \\ e_4 = \frac{9}{4}(\frac{1}{12}+4B_1(q))^2 - \frac{1}{2}(\frac{1}{12}+4B_1(q)) - \frac{1}{4}(\frac{7}{240}-16B_3(q)). \end{cases}$$

□

Lemma 2.2. *Define the Lambert series $A_\ell(q)$ by*

$$A_\ell(q) := \sum_{m=1}^{\infty}\frac{m^\ell}{q^m-1} = \sum_{j=1}^{\infty}\sigma_\ell(j)q^{-j} \quad (\sigma_\ell(j) = \sum_{m|\ell} m^j).$$

Then we have

$$(L_2')^2 = \frac{4d_2}{d_1}L_2^3 + d_2(6d_3+1)L_2^2 + 4d_1d_2d_4L_2 + d_1^2d_2d_5, \tag{2.2}$$

where

$$\begin{cases} d_1 = (q-1)^2, \\ d_2 = (\log q)^2, \\ d_3 = -4A_1(q), \\ d_4 = -A_1(q) + 12A_1^2(q) - 5A_3(q), \\ d_5 = \frac{1}{432}(1-24A_1(q))^3 - \frac{1}{144}(1-24A_1(q))(1+240A_3(q)) - \frac{1}{216}(1-504A_5(q)). \end{cases}$$

□

Remark 2.3. The series L_2 is essentially the Wierstrass $\wp$-function. Actually, if we define

$$\Omega_2(x) := (q-1)^{-2}L_2(x) + \frac{1}{12}(1-24A_1(q)),$$

we see that

$$(\Omega_2')^2 = \alpha_0\Omega_2^3 - \alpha_1\Omega_2 - \alpha_2,$$

where

$$\begin{cases} \alpha_0 = 4(\log q)^2, \\ \alpha_1 = (\log q)^2\frac{1}{12}(1+240A_3(q)), \\ \alpha_2 = (\log q)^2\frac{1}{216}(1-504A_5(q)). \end{cases}$$

Remark 2.4. Note that the Eisenstein series $E_{2\ell}(z)$ (see [L]) is given by

$$E_{2\ell}(z) := \frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-2\ell} = 1 + \frac{2}{\zeta(1-2\ell)}A_{2\ell-1}(q) \quad (q = e^{-2\pi i z},\ \mathrm{Im(z)} > 0).$$

Also, the series $B_\ell(q)$ is expressed as

$$B_\ell(q) = 2^{-\ell}\big[A_\ell(q^{\frac{1}{2}}) - 2^{\ell+1}A_\ell(q)\big].$$

In the proof of the existence of the differential equations satisfied by $L_1(x)$ and $L_2(x)$ in [KW], we have seen the following Laurent expansion of these functions.

Lemma 2.5.

$$(q-1)^{-1}L_1(x) = -2\sum_{l=0}^{\infty}\frac{B_{2l+1}(q)}{(2l+1)!}(x\log q)^{2l+1} + \frac{1}{x\log q} - \frac{x\log q}{24} + O(x^3),$$

$$(q-1)^{-2}L_2(x) = 2\sum_{l=0}^{\infty}\frac{A_{2l+1}(q)}{(2l)!}(x\log q)^{2l} + \frac{1}{(x\log q)^2} - \frac{1}{12} + O(x^2).$$

□

Using these Laurent expansions, we have the following relation between $L_1(x)$ and $L_2(x)$ which was suggested by Robin Chapman [C].

Proposition 2.6.

$$L_2(x) = L_1(x)^2 + (q-1)^2(2A_1(q) + 4B_1(q)).$$

Proof. Put $F(x) := L_2(x) - L_1(x)^2$. Then, from the lemma above one obtains

$$\begin{aligned}(q-1)^{-1}L_1(x) &= \frac{1}{\log q}\frac{1}{x} - (\log q)(2B_1(q) + \frac{1}{24})x + O(x^3),\\ (q-1)^{-2}L_2(x) &= \frac{1}{(\log q)^2}\frac{1}{x^2} + (2A_1(q) - \frac{1}{12}) + O(x^2).\end{aligned}$$

It follows that

$$F(x) = (q-1)^2(2A_1(q) + 4B_1(q)) + O(x^2). \tag{2.3}$$

Also, since

$$L_\ell(x+1) = L_\ell(x) \quad \text{and} \quad L_\ell(x + \frac{2\pi i}{\log q}) = (-1)^\ell L_\ell(x),$$

we immediately find that $F(x+1) = F(x)$ and $F(x+\frac{2\pi i}{\log q}) = F(x)$. This implies that $F(x)$ is an (continuous) elliptic function with period $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\frac{2\pi i}{\log q}$. It follows hence from the Liouville theorem that $F(x)$ is equal to a constant $F(0) = (q-1)^2(2A_1(q) + 4B_1(q))$. This proves the proposition. □

Corollary 2.7. *We have $L_\ell \in \mathbb{C}[L_1]$. If we write $L_\ell = p_\ell(L_1)$, p_ℓ is a monic polynomial of degree ℓ and satisfies $p_\ell(-x) = (-1)^\ell p_\ell(x)$.*

Proof. Since $L_1 \in \mathbb{C}[L_1]$ (of degree $2\ell-1$) and $L_2 \in \mathbb{C}[L_2]$ (of degree ℓ), it follows from Proposition 2.6 that f_ℓ is of degree ℓ. Furthermore, one has

$$\begin{aligned}p_\ell(-L_1(x)) &= p_\ell(L_1(-x))\\ &= L_\ell(-x) = (-1)^\ell L_\ell(x)\\ &= (-1)^\ell p_\ell(L_1(x)),\end{aligned}$$

whence the desired relation follows immediately. The property that p_ℓ is monic follows from the discussion developed in the proof of Proposition 3.2 and Proposiotn 3.4 in the subsequent section. □

3 Algebraic Differential equations

3.1 Differential equations for $L_{2\ell}$

We prove the derivatives $L_{2\ell}^{(2k)}$ and $(L_{2\ell}^{(2k+1)})^2$ $(k = 0, 1, 2, \ldots)$ of $L_{2\ell}$ can be expressed as polynomilas in L_2. To show this, we need the following difference-differential equation satisfied by L_ℓ's [KW].

Lemma 3.1.

$$L_\ell'' = \frac{(\log q)^2}{4} \cdot \ell^2 L_\ell + \frac{(\log q)^2}{(q-1)^2} \cdot \ell(\ell+1) L_{\ell+2}.$$

□

We have then the following.

Proposition 3.2. *There exist constants*

$$a_i(k), b_i(k) \in \mathbb{Q}(d_1, d_2, d_3, d_4, d_5) \quad (a_{\ell+k}(k) \neq 0, b_{2\ell+2k+1}(k) \neq 0)$$

such that the functions $L_{2\ell}^{(2k)}$ *and* $(L_{2\ell}^{(2k+1)})^2$ *are expressed by the polynomials of* L_2 *as*

$$\begin{aligned} L_{2\ell}^{(2k)} &= \sum_{i=0}^{\ell+k} a_i(k) L_2^i, \quad (a_i(0) = 1), \\ (L_{2\ell}^{(2k+1)})^2 &= \sum_{i=0}^{2\ell+2k+1} b_i(k) L_2^i. \end{aligned}$$

Proof. By Lemma 3.1 and (2.1) we easily obtain

$$L_\ell'' = \frac{d_2}{4} \cdot \ell^2 L_\ell + \frac{d_2}{d_1} \cdot \ell(\ell+1) L_{\ell+2} \tag{3.1}$$

and

$$(L_2')^2 = \frac{4d_2}{d_1} L_2^3 + d_2(6d_3+1) L_2^2 + 4d_1 d_2 d_4 L_2 + d_1^2 d_2 d_5. \tag{3.2}$$

Differentiating (3.2) we have

$$L_2'' = \frac{6d_2}{d_1} L_2^2 + d_2(6d_3+1) L_2 + 2d_1 d_2 d_4.$$

We first prove the former assertion for $k = 0$ by induction on ℓ. Let us assume that the assertion is true for 2ℓ:

$$L_{2\ell} = \sum_{i=0}^{\ell} a_i L_2^i, \quad (a_i \in \mathbb{Q}(d_1, \cdots, d_5), a_\ell = 1).$$

Then, since $L_{2\ell}' = \{\sum_{i=1}^{\ell} i a_i L_2^{i-1}\} L_2'$, one gets

$$L_{2\ell}'' = \{\sum_{i=2}^{\ell} i(i-1) a_i L_2^{i-2}\} (L_2')^2 + \{\sum_{i=1}^{\ell} i a_i L_2^{i-1}\} L_2''.$$

It follows that $L_{2\ell}''$ can be written by a polynomila of L_2 as

$$L_{2\ell}'' = \sum_{i=0}^{\ell+1} b_i L_2^i, \quad (b_i \in \mathbb{Q}(d_1, \cdots, d_5)$$

with

$$b_{\ell+1} = \ell(\ell-1)a_\ell \frac{4d_2}{d_1} + \ell a_\ell \frac{6d_2}{d_1} = 2\ell(2\ell+1)\frac{d_2}{d_1} \neq 0.$$

Now replacing ℓ by 2ℓ in (3.1) we see that

$$L''_{2\ell} = d_2\ell^2 L_{2\ell} + \frac{d_2}{d_1} 2\ell(2\ell+1)L_{2\ell+2}.$$

Hence we see that

$$L_{2\ell+2} = \sum_{i=0}^{\ell+1} c_i L_2^i, \quad (c_i \in \mathbb{Q}(d_1, \cdots, d_5))$$

with

$$c_{\ell+1} = \frac{b_{\ell+1}}{\frac{d_2}{d_1}2\ell(2\ell+1)}(= a_\ell) = 1.$$

This proves the assertion for $k = 0$.

We next prove the assertion for general k by induction. Suppose that $L_{2\ell}^{(2k)}$ can be written as

$$L_{2\ell}^{(2k)} = \sum_{i=0}^{\ell+k} a_i L_2^i, \quad (a_i \in \mathbb{Q}(d_1, \cdots, d_5), a_{\ell+k} \neq 0).$$

Then, taking derivatives twice we have

$$L_{2\ell}^{(2k+2)} = \{\sum_{i=2}^{\ell+k} i(i-1)a_i L_2^{i-2}\}(L'_2)^2 + \{\sum_{i=1}^{\ell+k} i a_i L_2^{i-1}\}L''_2.$$

From this it follows that

$$L_{2\ell}^{(2k+2)} = \sum_{i=0}^{\ell+k+1} b_i L_2^i, \quad (b_i \in \mathbb{Q}(d_1, \cdots, d_5))$$

with

$$b_{\ell+k+1} = 2(\ell+k)(2\ell+2k+1)a_{\ell+k}\frac{d_2}{d_1} \neq 0.$$

This proves the first equation of the lemma for $L_{2\ell}^{(2k)}$. Furthermore, differentiating this equation and taking square we observe

$$(L_{2\ell}^{(2k+1)})^2 = (\sum_{i=1}^{\ell+k} i a_i L_2^{i-1})^2 (L'_2)^2.$$

From this expression, we see that $(L_{2\ell}^{(2k+1)})^2$ is expressed by a polynomial in L_2 as

$$(L_{2\ell}^{(2k+1)})^2 = \sum_{i=0}^{2\ell+2k+1} b_i L_2^i, \quad (b_i \in \mathbb{Q}(d_1, \cdots, d_5))$$

with

$$b_{2\ell+2k+1} = (\ell+k)^2 a_{\ell+k}^2 \frac{4d_2}{d_1} \neq 0.$$

This completes the proof of the proposition. □

We now get a non-linear algebraic differential equation for $L_{2\ell}$.

Theorem 3.3. *The function $L_{2\ell}\,(\ell \geq 1)$ satisfies the following differential equation:*

$$\sum_{i=0}^{\ell} \alpha_i L_{2\ell}^{(2i)} L_{2\ell}^{(4\ell-2i)} + \sum_{i=0}^{3\ell-1} \beta_i L_{2\ell}^{(2i)} + \gamma = 0 \quad (\alpha_i, \beta_i, \gamma \in \mathbb{Q}(d_1, \cdots, d_5)).$$

Proof. By Proposition 3.2 one finds that the degree of the polynomial $L_{2\ell}^{(2i)} L_{2\ell}^{(4\ell-2i)}$ in L_2 is calculated as $(\ell+i)+(\ell+2\ell-i)=4\ell$ for all $0 \leq i \leq \ell$. In particular, since the degree of the polynomials $L''_{2\ell}L_{2\ell}^{(4\ell-2)}$, $L_{2\ell}L_{2\ell}^{(4\ell)},\ L_{2\ell}^{(6\ell-2)},\ \cdots,\ L_{2\ell}$ are $4\ell,\ 4\ell,\ 4\ell-1,\ \cdots,\ \ell$, respectively, if one eliminates the leading coefficients successively one obtains

$$f_1 := L''_{2\ell}L_{2\ell}^{(4\ell-2)} + a_{1,1}L_{2\ell}L_{2\ell}^{(4\ell)} + a_{1,2}L_{2\ell}^{(6\ell-2)} + \cdots + a_{1,3\ell+1}L_{2\ell} = \sum_{i=0}^{\ell-1} b_{1,i}L_2^i,$$

for some coefficients $a_{1,j}, b_{1,i} \in \mathbb{Q}(d_1, \cdots, d_5)$. Similarly, for $k = 2, \ldots, \ell$, there exist $a_{i,j}, b_{k,l} \in \mathbb{Q}(d_1, \cdots, d_5)$ one has

$$f_k := L_{2\ell}^{(2k)}L_{2\ell}^{(4\ell-2k)} + a_{k,1}L_{2\ell}L_{2\ell}^{(4\ell)} + a_{i,2}L_{2\ell}^{(6\ell-2)} + \cdots + a_{k,3\ell+1}L_{2\ell} = \sum_{i=0}^{\ell-1} b_{k,i}L_2^i.$$

Write these relations as

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\ell \end{pmatrix} = \begin{pmatrix} b_{1,0} & b_{1,1} & \cdots & b_{1,\ell-1} \\ b_{2,0} & \ddots & & b_{2,\ell-1} \\ \vdots & & \ddots & \vdots \\ b_{\ell,0} & \cdots & \cdots & b_{\ell,\ell-1} \end{pmatrix} \begin{pmatrix} 1 \\ L_2 \\ \vdots \\ L_2^{\ell-1} \end{pmatrix}.$$

Hence, there exists an $\ell \times \ell$ matrix A with coefficients in $\mathbb{Q}(d_1, \cdots, d_5)$ such that

$$A\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\ell \end{pmatrix} = \begin{pmatrix} c_{1,0} & 0 & \cdots & 0 \\ c_{2,0} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ c_{\ell,0} & \cdots & \cdots & c_{\ell,\ell-1} \end{pmatrix} \begin{pmatrix} 1 \\ L_2 \\ \vdots \\ L_2^{\ell-1} \end{pmatrix}.$$

Here, it is obvious to see that $c_{i,j} \in \mathbb{Q}(d_1, \cdots, d_5)$. Noticing the first row, one has

$$\sum_{i=1}^{\ell} a_i f_i = c_{1,0}. \quad (a_i \in \mathbb{Q}(d_1, \cdots, d_5)).$$

Hence, by the definition of f_i, one gets the desired diffrential equation for $L_{2\ell}$. □

3.2 Differential equations for $L_{2\ell-1}$

A similar procedure developed in the last subsection works also for the odd case. In fact, we have the following.

Proposition 3.4. *There exist constants*

$$a_i(k), b_i(k) \in \mathbb{Q}(e_1, e_2, e_3, e_4) \quad (a_{n+k-1}(k) \neq 0, b_{2n+2k}(k) \neq 0)$$

such that the functions $L_{2\ell-1}^{(2k)}$ *and* $(L_{2\ell-1}^{(2k+1)})^2$ *are expressed by the polynomials of* L_1 *as*

$$\begin{aligned} L_{2\ell-1}^{(2k)} &= \sum_{i=0}^{\ell+k-1} a_i(k) L_1^{2i+1}, \quad (a_i(0)=1), \\ (L_{2\ell-1}^{(2k+1)})^2 &= \sum_{i=0}^{2\ell+2k} b_i(k) L_1^{2i}. \end{aligned}$$

□

Using this proposition one can prove the following result. The proof is the same as Theorem 3.3.

Theorem 3.5. *The function* $L_{2\ell-1}\,(\ell \geq 1)$ *satisfies the following differential equation:*

$$\sum_{i=1}^{2\ell-1} \alpha_i L_{2\ell-1}^{(2i)} L_{2\ell-1}^{(8\ell-4-2i)} + \sum_{i=0}^{4\ell-2} \beta_i L_{2\ell-1} L_{2\ell-1}^{(2i)} + \gamma = 0 \quad (\alpha_i, \beta_i, \gamma \in \mathbb{Q}(e_1, \cdots, e_4)).$$

□

Remark 3.6. One finds that, in the proof of Theorem 3.3, there are certain redundancy in the process for deriving the algebraic differential equation satisfied by L_ℓ. Therefore one can obtain, in principle, much simpler equations, i.e. the number of terms and of derivatives of the equations in Theorem 3.3 and Theorem 3.5 can be reduced.

4 Examples

In this section, we give much simpler and explicit differential equations for L_4 and L_3 explicitly. These are in fact simpler than what we have given in the preceding section. We use Mathematica 6.

4.1 A differential equation for L_4

By Proposition 3.2 one sees that there exist $a_i, b_i \in \mathbb{Q}(d_1, \cdots, d_5)$ such that

$$\begin{aligned} L_4 L_4'' + a_1(L_4')^2 + a_2 L_4^{(4)} + a_3 L_4'' + a_4 L_4 + a_5 &= b_1 L_2 + b_2, \\ L_4 L_4'' + a_1(L_4')^2 + a_6 L_4^2 + a_7 L_4'' + a_8 L_4 + a_9 &= b_3 L_2 + b_4. \end{aligned}$$

From these equations we obtain

$$L_4 L_4'' + a(L_4')^2 + bL_4^{(4)} + cL_4^2 + dL_4'' + eL_4 + f = 0, \quad (a,b,c,d,e,f \in \mathbb{Q}(d_1, \cdots, d_5)). \tag{4.1}$$

By Lemma 3.1 and Lemma 2.2, we have

$$\begin{aligned} L_2'' &= d_2 L_2 + \frac{6d_2}{d_1} L_4, \\ (L_2')^2 &= \frac{4d_2}{d_1} L_2^3 + d_2(6d_3+1)L_2^2 + 4d_1d_2d_4L_2 + d_1^2 d_2 d_5. \end{aligned}$$

It follows that

$$\begin{aligned} L_2'' &= \frac{6d_2}{d_1} L_2^2 + d_2(6d_3+1)L_2 + 2d_1d_2d_4, \\ L_4 &= L_2^2 + d_1d_3L_2 + \frac{1}{3}d_1^2 d_4, \\ L_4' &= 2L_2L_2' + d_1d_3L_2', \\ (L_4')^2 &= (2L_2 + d_1d_3)^2 (L_2')^2 \\ &= \frac{16d_2}{d_1} L_2^5 + 4d_2(10d_3+1)L_2^4 + 4d_1d_2(7d_3^2 + d_3 + 4d_4)L_2^3 \\ &\quad + d_1^2 d_2(6d_3^3 + d_3^2 + 16d_3d_4 + 4d_5)L_2^2 + 4d_1^3 d_2 d_3(d_3d_4 + d_5)L_2 + d_1^4 d_2 d_3^2 d_5, \end{aligned}$$

$$\begin{aligned}
L_4'' &= 2(L_2L_2''+(L_2')^2)+d_1d_3L_2'' \\
&= \frac{20d_2}{d_1}L_2^3+2d_2(15d_3+2)L_2^2+d_1d_2(6d_3^2+d_3+12d_4)L_2+2d_1^2d_2(d_3d_4+d_5), \\
L_4''' &= \Big[\frac{60d_2}{d_1}L_2^2+4d_2(15d_3+2)L_2+d_1d_2(6d_3^2+d_3+12d_4)\Big]L_2', \\
L_4^{(4)} &= \Big[\frac{60d_2}{d_1}L_2^2+4d_2(15d_3+2)L_2+d_1d_2(6d_3^2+d_3+12d_4)\Big]L_2'' \\
&\quad +\Big[\frac{120d_2}{d_1}L_2+4d_2(15d_3+2)\Big](L_2')^2 \\
&= \frac{840d_2^2}{d_1^2}L_2^4+\frac{20d_2^2}{d_1}(84d_3+13)L_2^3+2d_2^2(378d_3^2+111d_3+336d_4+8)L_2^2 \\
&\quad +d_1d_2^2(36d_3^3+12d_3^2+432d_3d_4+d_3+60d_4+120d_5)L_2 \\
&\quad +2d_1^2d_2^2(6d_3^2d_4+d_3d_4+12d_4^2+30d_3d_5+4d_5).
\end{aligned}$$

Subsituting these into (4.1), we have the linear equation for $a,b,\cdots,f$ as follows:

$$\begin{cases}
\frac{20d_2}{d_1}+a\cdot\frac{16d_2}{d_1}=0, \\
2d_2(25d_3+2)+a\cdot 4d_2(10d_3+1)+b\cdot\frac{840d_2^2}{d_1^2}+c\cdot 1=0, \\
d_1d_2(36d_3^2+5d_3+\frac{56}{3}d_4)+a\cdot 4d_1d_2(7d_3^2+d_3+4d_4)+b\cdot\frac{20d_2^2}{d_1}(84d_3+13) \\
+c\cdot 2d_1d_3+d\cdot\frac{20d_2}{d_1}=0, \\
d_1^2d_2(6d_3^3+d_3^2+24d_3d_4+\frac{4}{3}d_4+2d_5)+a\cdot d_1^2d_2(6d_3^3+d_3^2+16d_3d_4+4d_5) \\
+b\cdot 2d_2^2(378d_3^2+111d_3+336d_4+8)+c\cdot d_1^2(d_3^2+\frac{2}{3}d_4)+d\cdot 2d_2(15d_3+2)+e\cdot 1=0, \\
d_1^3d_2(4d_3^2d_4+\frac{1}{3}d_3d_4+4d_4^2+2d_3d_5)+a\cdot 4d_1^3d_2d_3(d_3d_4+d_5) \\
+b\cdot d_1d_2^2(36d_3^3+12d_3^2+432d_3d_4+d_3+60d_4+120d_5)+c\cdot\frac{2}{3}d_1^3d_3d_4 \\
+d\cdot d_1d_2(6d_3^2+d_3+12d_4)+e\cdot d_1d_3=0, \\
\frac{2}{3}d_1^4d_2d_4(d_3d_4+d_5)+a\cdot d_1^4d_2d_3^2d_5+b\cdot 2d_1^2d_2^2(6d_3^2d_4+d_3d_4+12d_4^2+30d_3d_5+4d_5) \\
+c\cdot\frac{1}{9}d_1^4d_4^2+d\cdot 2d_1^2d_2(d_3d_4+d_5)+e\cdot\frac{1}{3}d_1^2d_4+f\cdot 1=0.
\end{cases}$$

Solving the equation we can determine the coefficients of the equation (4.1) as follows:

$$\begin{aligned}
a &= -\frac{5}{4}, \\
b &= \frac{d_1^2(-9d_3^4-6d_3^3-d_3^2+24d_3^2d_4+8d_3d_4-16d_4^2)}{20d_2(20d_3^3+17d_3^2-40d_3d_4+4d_3-16d_4+20d_5)}, \\
c &= \frac{d_2(378d_3^4+272d_3^3+59d_3^2-1008d_3^2d_4-376d_3d_4+4d_3+672d_4^2-16d_4+20d_5)}{20d_3^3+17d_3^2-40d_3d_4+4d_3-16d_4+20d_5}, \\
d &= d_1^2(-12d_3^5+36d_3^4+40d_3^3d_4+24d_3^3-116d_3^2d_4+3d_3^2-12d_3^2d_5-32d_3d_4^2-40d_3d_4 \\
&\quad -24d_3d_5+112d_4^2+16d_4d_5)/12(20d_3^3+17d_3^2-40d_3d_4+4d_3-16d_4+20d_5), \\
e &= d_1^2d_2(1332d_3^6-48d_3^5-549d_3^4-156d_3^3-12d_3^2-5328d_3^4d_4-1472d_3^3d_4-104d_3^2d_4 \\
&\quad -64d_3d_4+2304d_3^2d_4^2+448d_3d_4^2+448d_4^2+5376d_4^3+7200d_3^3d_5+6000d_3^2d_5 \\
&\quad +1200d_3d_5-14400d_3d_4d_5-5600d_4d_5+3600d_5^2)/60(20d_3^3+17d_3^2-40d_3d_4 \\
&\quad +4d_3-16d_4+20d_5), \\
f &= d_1^4d_2(12d_3^2d_4+84d_3^3d_4+45d_3^4d_4-222d_3^5d_4-16d_3d_4^2-1244d_3^2d_4^2-6608d_3^3d_4^2 \\
&\quad -11280d_3^4d_4^2-128d_4^3+8032d_3d_4^3+29760d_3^2d_4^3-15360d_4^4-18d_3^2d_5+1152d_3^3d_5 \\
&\quad +6633d_3^4d_5+9720d_3^5d_5-2016d_3d_4d_5-17568d_3^2d_4d_5-37200d_3^3d_4d_5+8752d_4^2d_5 \\
&\quad +30720d_3d_4^2d_5+720d_3d_5^2+4860d_3^2d_5^2-11280d_4d_5^2) \\
&\quad /180(20d_3^3+17d_3^2-40d_3d_4+4d_3-16d_4+20d_5).
\end{aligned}$$

4.2 A differential equation for L_3

The following differential equation satisfied by L_3 can be obtained by a similar discussion made in the case for L_4.

$$L_3^2L_3'' + aL_3(L_3')^2 + bL_3^3 + cL_3^{(6)} + dL_3(4) + eL_3'' + fL_3 = 0, \quad (a,b,c,d,e,f \in \mathbb{Q}(e_1,\cdots,e_4)). \tag{4.2}$$

The coefficients appeared in the equation above can be calculated as follows:

$$\begin{aligned}
a &= -\frac{4}{3},\\
b &= e_2(-37494e_3 + 16758e_1^2e_3 - 210291e_3^2 + 100275e_1^2e_3^2 + 1216848e_3^3 - 770844e_1^2e_3^3\\
&\quad +10259968e_3^4 - 4157814e_1^2e_3^4 + 17720640e_3^5 - 834840e_1^2e_3^5 + 224964e_4\\
&\quad -100548e_1^2e_4 + 1365696e_3e_4 + 286272e_1^2e_3e_4 - 5531328e_3^2e_4 + 4352904e_1^2e_3^2e_4\\
&\quad -24675840e_3^3e_4 - 11950560e_1^2e_3^3e_4 - 4209408e_4^2 - 2612736e_1^2e_4^2 - 22256640e_3e_4^2\\
&\quad +17418240e_1^2e_3e_4^2)/(12(-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3\\
&\quad +71384e_1^2e_3^3 - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4\\
&\quad -42672e_1^2e_3e_4 + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)),\\
c &= e_1^3(72e_3^2 - 72e_3^3 - 2332e_3^4 - 4725e_3^5 - 864e_3e_4 - 1368e_3^2e_4 + 10800e_3^3e_4 + 2592e_4^2\\
&\quad +10800e_3e_4^2)/(135e_2^2(-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3\\
&\quad +71384e_1^2e_3^3 - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4\\
&\quad -42672e_1^2e_3e_4 + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)),\\
d &= e_1^3(546e_3^2 - 5586e_1^2e_3^2 + 43207e_3^3 - 35287e_1^2e_3^3 + 61307e_3^4 + 233393e_1^2e_3^4\\
&\quad -1118174e_3^5 + 1314554e_1^2e_3^5 - 2930256e_3^6 + 264366e_1^2e_3^6 - 6552e_3e_4 + 67032e_1^2e_3e_4\\
&\quad -399498e_3^2e_4 + 339738e_1^2e_3^2e_4 - 1518936e_3^3e_4 - 123384e_1^2e_3^3e_4 - 1090656e_3^4e_4\\
&\quad +4034796e_1^2e_3^4e_4 + 19656e_4^2 - 201096e_1^2e_4^2 + 696384e_3e_4^2 + 102816e_1^2e_3e_4^2\\
&\quad +2515968e_3^2e_4^2 - 1930608e_1^2e_3^2e_4^2 + 870912e_4^3 - 5225472e_1^2e_4^3)/(1080e_2\\
&\quad (-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3 + 71384e_1^2e_3^3\\
&\quad -154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4 - 42672e_1^2e_3e_4\\
&\quad +88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)),\\
e &= -e_1^3(9282e_3^2 - 27930e_1^2e_3^2 + 272819e_3^3 - 288155e_1^2e_3^3 + 635939e_3^4 + 628805e_1^2e_3^4\\
&\quad -6674273e_3^5 + 11265830e_1^2e_3^5 - 26757152e_3^6 + 16964470e_1^2e_3^6 - 19640880e_3^7\\
&\quad +3061080e_1^2e_3^7 - 111384e_3e_4 + 335160e_1^2e_3e_4 - 1251066e_3^2e_4 + 1698690e_1^2e_3^2e_4\\
&\quad -6930792e_3^3e_4 + 2736360e_1^2e_3^3e_4 - 82630752e_3^4e_4 + 80842140e_1^2e_3^4e_4\\
&\quad -234924480e_3^5e_4 + 58289280e_1^2e_3^5e_4 + 334152e_4^2 - 1005480e_1^2e_4^2 - 4782672e_3e_4^2\\
&\quad +4536000e_1^2e_3e_4^2 - 60303744e_3^2e_4^2 + 23127120e_1^2e_3^2e_4^2 - 133781760e_3^3e_4^2\\
&\quad +169248960e_1^2e_3^3e_4^2 + 14805504e_4^3 - 26127360e_1^2e_4^3 + 26127360e_3e_4^3\\
&\quad +104509440e_1^2e_3e_4^3)/(2160(-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2\\
&\quad -124484e_3^3 + 71384e_1^2e_3^3 - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4\\
&\quad +146352e_3e_4 - 42672e_1^2e_3e_4 + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)),\\
f &= e_1^3e_2(5958e_3^2 - 16758e_1^2e_3^2 + 190245e_3^3 - 239925e_1^2e_3^3 + 1012737e_3^4 - 303117e_1^2e_3^4\\
&\quad -2753838e_3^5 + 8314998e_1^2e_3^5 - 31889088e_3^6 + 36513498e_1^2e_3^6 - 70418016e_3^7\\
&\quad +39563376e_1^2e_3^7 - 36546048e_3^8 + 6752928e_1^2e_3^8 - 71496e_3e_4 + 201096e_1^2e_3e_4\\
&\quad -1108926e_3^2e_4 + 2091726e_1^2e_3^2e_4 - 29384904e_3^3e_4 + 16327704e_1^2e_3^3e_4
\end{aligned}$$

$$
\begin{aligned}
&-140392064e_3^4e_4+89458724e_1^2e_3^4e_4+33067264e_3^5e_4-43487104e_1^2e_3^5e_4\\
&+763931136e_3^6e_4+72474944e_1^2e_3^6e_4+214488e_4^2-603288e_1^2e_4^2+41926464e_3e_4^2\\
&-46345824e_1^2e_3e_4^2+351313920e_3^2e_4^2-124364880e_1^2e_3^2e_4^2-1274706432e_3^3e_4^2\\
&+2125418112e_1^2e_3^3e_4^2-6904461312e_3^4e_4^2-131194368e_1^2e_3^4e_4^2-252730368e_4^3\\
&+254596608e_1^2e_4^3-2993614848e_3e_4^3+990517248e_1^2e_3e_4^3-7851810816e_3^2e_4^3\\
&+6524292096e_1^2e_3^2e_4^3-1170505728e_4^4+7023034368e_1^2e_4^4)/(5760(-4166e_3\\
&+1862e_1^2e_3-36803e_3^2+23555e_1^2e_3^2-124484e_3^3+71384e_1^2e_3^3-154224e_3^4\\
&+13914e_1^2e_3^4+24996e_4-11172e_1^2e_4+146352e_3e_4-42672e_1^2e_3e_4\\
&+88704e_3^2e_4+199176e_1^2e_3^2e_4+48384e_4^2-290304e_1^2e_4^2)).
\end{aligned}
$$

Corrigenda to the paper [KW]: We list here some corrections of the equations in the paper [KW]. We did not, however, use these equations.

- The coefficient of the second term of (2.12) in Corollary 2.2 is incorrect and should be 48.
- The coefficient of $B_3(q)$ in the expression of β_2 in Theorem 2.3 is incorrect and should be $-\frac{16}{3}$.
- The equation (2.30) in p.37 should be read as $M_2''(x) = M_2(x)(6\beta_0^2 L_1(x)^2 + \beta_1)$. Hence, the correct equation (2.32) in Corollary 2.4 (p.38) is

$$M_2''(x)^2 + 4\beta_1 M_2(x) M_2''(x) + (36\beta_0^2\beta_2 - 5\beta_1^2) M_2(x)^2 - 36\beta_0^4 M_2(x)^4 = 0.$$

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