Non-linear algebraic differential equations satisfied by certain family of elliptic functions

Wakayama, Masato
Faculty of Mathematics, Kyushu University

Yamamoto, Keitaro
Faculty of Mathematics, Kyushu University

<http://hdl.handle.net/2324/14239>

出版情報 : The Ramanujan journal. 30 (2), pp.173-186, 2013-02. Springer US
バージョン :
権利関係 : (C) Springer Science+Business Media, LLC 2012

MI Preprint Series

**Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub**

Non-linear algebraic differential equations satisfied by certain family of elliptic functions

M. Wakayama & K. Yamamoto

MI 2009-17

(Received April 30, 2009)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MASATO WAKAYAMA * AND KEITARO YAMAMOTO

April 27, 2009

Abstract

The paper [KW] studied a family of elliptic functions defined by certain q -series. This family, in particular, contains the Weierstrass $\wp$ -function. In this paper, we prove that elliptic functions in this family satisfy certain non-linear algebraic differential equations whose coefficients are essentially given by rational functions of the first few Eisenstein series of the modular group.

Keywords: elliptic functions, q -series, Weierstrass $\wp$ -function, Eisenstein series, Lambert series.

2000 Mathematical Subject Classification: 11M36, 33E05

1 Introduction

In the paper [KW], Kurokawa and the first author studied the q -series $L_\ell(x)$ ($x \in \mathbb{C}$) defined by

$$L_\ell(x) := \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}(n+x)\ell}}{[n+x]_q^\ell} \quad (\ell = 1, 2, 3, \dots).$$

Here we put $[\beta]_q := \frac{q^\beta - 1}{q - 1}$ for $\beta \in \mathbb{C}$. Throughout the present paper, we assume that $q > 1$. The series $L_\ell(x)$ converges absolutely and uniformly in x , whence defines an elliptic function for the period lattice $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{2\pi i}{\log q}$ when ℓ is even, while $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{4\pi i}{\log q}$ when ℓ is odd. Precisely, L_ℓ has the following properties.

$$\begin{cases} L_\ell(x+1) = L_\ell(x), \\ L_\ell(x + \frac{2\pi i}{\log q}) = (-1)^\ell L_\ell(x), \\ L_\ell(-x) = (-1)^{-\ell} L_\ell(x). \end{cases}$$

It is shown in [KW] that the series $L_2(x)$ gives essentially the Weierstrass $\wp$ -function and hence, in particular, satisfies the non-linear algebraic differential equation. An algebraic differential equation for L_1 is also derived. Beside this fact, it was proved that $L_{2\ell-1} \in \mathbb{C}[L_1]$ and $L_{2\ell} \in \mathbb{C}[L_2]$. In fact, $L_{2\ell-1}$ (resp. $L_{2\ell}$) is expressed as a polynomial in L_1 (resp. L_2) of degree $2\ell - 1$ (resp. ℓ). Moreover, one may replace the fields of the coefficients $\mathbb{C}$ by the ring $\mathbb{Q}(q)$ of Laurent polynomials over $\mathbb{Q}$. From these facts, it was conjectured in [KW] that the elliptic function $L_\ell(x)$ satisfies a non-linear algebraic differential equation for each positive integer ℓ with coefficients in $\mathbb{Q}(q)$.

The purpose of this paper is to prove this conjecture affirmatively and give some precise information about the coefficients of the differential equation. Actually, for instance, we will show that the coefficients in the even ℓ are given by rational functions of $(q-1)^2$, $(\log q)^2$ and the Lambert series $A_\ell(q)$ ($\ell = 1, 3, 5$) over $\mathbb{Q}$. We note that $A_{2\ell+1}(q)$ gives essentially the Fourier series expansion of the Eisenstein series $E_{2\ell}(z)$ for the modular group $SL_2(\mathbb{Z})$ if we put $q = e^{-2\pi iz}$ ($\text{Im}(z) > 0$) [L].

*Work in part supported by Grant-in-Aid for Scientific Research (B) No. 21340011.

2 Basic facts on L_ℓ

Let us recall first the basic facts in [KW], the algebraic differential equations and the Laurent expansion of the functions $L_1(x)$ and $L_2(x)$.

The differential equations for L_1 and L_2 are described as follows.

Lemma 2.1. *Define the Lambert series $B_\ell(q)$ by*

$$B_\ell(q) := \sum_{m=0}^{\infty} \frac{(m + \frac{1}{2})^\ell}{q^{m+\frac{1}{2}} - 1}.$$

Then we have

$$(2.1) \quad (L'_1)^2 = \frac{e_2}{e_1} L_1^4 + e_2 \left(e_3 + \frac{1}{4}\right) L_1^2 + e_1 e_2 e_4,$$

where

$$\begin{cases} e_1 = (q-1)^2, \\ e_2 = (\log q)^2, \\ e_3 = 12B_1(q), \\ e_4 = \frac{9}{4} \left(\frac{1}{12} + 4B_1(q)\right)^2 - \frac{1}{2} \left(\frac{1}{12} + 4B_1(q)\right) - \frac{1}{4} \left(\frac{7}{240} - 16B_3(q)\right). \end{cases}$$

□

Lemma 2.2. *Define the Lambert series $A_\ell(q)$ by*

$$A_\ell(q) := \sum_{m=1}^{\infty} \frac{m^\ell}{q^m - 1} = \sum_{j=1}^{\infty} \sigma_\ell(j) q^{-j} \quad (\sigma_\ell(j) = \sum_{m|j} m^\ell).$$

Then we have

$$(2.2) \quad (L'_2)^2 = \frac{4d_2}{d_1} L_2^3 + d_2(6d_3 + 1) L_2^2 + 4d_1 d_2 d_4 L_2 + d_1^2 d_2 d_5,$$

where

$$\begin{cases} d_1 = (q-1)^2, \\ d_2 = (\log q)^2, \\ d_3 = -4A_1(q), \\ d_4 = -A_1(q) + 12A_1^2(q) - 5A_3(q), \\ d_5 = \frac{1}{432} (1 - 24A_1(q))^3 - \frac{1}{144} (1 - 24A_1(q))(1 + 240A_3(q)) - \frac{1}{216} (1 - 504A_5(q)). \end{cases}$$

□

Remark 2.3. The series L_2 is essentially the Wierstrass $\wp$ -function. Actually, if we define

$$\Omega_2(x) := (q-1)^{-2} L_2(x) + \frac{1}{12} (1 - 24A_1(q)),$$

we see that

$$(\Omega'_2)^2 = \alpha_0 \Omega_2^3 - \alpha_1 \Omega_2 - \alpha_2,$$

where

$$\begin{cases} \alpha_0 = 4(\log q)^2, \\ \alpha_1 = (\log q)^2 \frac{1}{12} (1 + 240A_3(q)), \\ \alpha_2 = (\log q)^2 \frac{1}{216} (1 - 504A_5(q)). \end{cases}$$

Remark 2.4. Note that the Eisenstein series $E_{2\ell}(z)$ (see [L]) is given by

$$E_{2\ell}(z) := \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-2\ell} = 1 + \frac{2}{\zeta(1-2\ell)} A_{2\ell-1}(q) \quad (q = e^{-2\pi iz}, \operatorname{Im}(z) > 0).$$

Also, the series $B_\ell(q)$ is expressed as

$$B_\ell(q) = 2^{-\ell} [A_\ell(q^{\frac{1}{2}}) - 2^{\ell+1} A_\ell(q)].$$

In the proof of the existence of the differential equations satisfied by $L_1(x)$ and $L_2(x)$ in [KW], we have seen the following Laurent expansion of these functions.

Lemma 2.5.

$$(q-1)^{-1} L_1(x) = -2 \sum_{l=0}^{\infty} \frac{B_{2l+1}(q)}{(2l+1)!} (x \log q)^{2l+1} + \frac{1}{x \log q} - \frac{x \log q}{24} + O(x^3),$$

$$(q-1)^{-2} L_2(x) = 2 \sum_{l=0}^{\infty} \frac{A_{2l+1}(q)}{(2l)!} (x \log q)^{2l} + \frac{1}{(x \log q)^2} - \frac{1}{12} + O(x^2).$$

□

Using these Laurent expansions, we have the following relation between $L_1(x)$ and $L_2(x)$ which was suggested by Robin Chapman [C].

Proposition 2.6.

$$L_2(x) = L_1(x)^2 + (q-1)^2 (2A_1(q) + 4B_1(q)).$$

Proof. Put $F(x) := L_2(x) - L_1(x)^2$. Then, from the lemma above one obtains

$$\begin{aligned} (q-1)^{-1} L_1(x) &= \frac{1}{\log q} \frac{1}{x} - (\log q) (2B_1(q) + \frac{1}{24}) x + O(x^3), \\ (q-1)^{-2} L_2(x) &= \frac{1}{(\log q)^2} \frac{1}{x^2} + (2A_1(q) - \frac{1}{12}) + O(x^2). \end{aligned}$$

It follows that

$$(2.3) \quad F(x) = (q-1)^2 (2A_1(q) + 4B_1(q)) + O(x^2).$$

Also, since

$$L_\ell(x+1) = L_\ell(x) \quad \text{and} \quad L_\ell(x + \frac{2\pi i}{\log q}) = (-1)^\ell L_\ell(x),$$

we immediately find that $F(x+1) = F(x)$ and $F(x + \frac{2\pi i}{\log q}) = F(x)$. This implies that $F(x)$ is an (continuous) elliptic function with period $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{2\pi i}{\log q}$. It follows hence from the Liouville theorem that $F(x)$ is equal to a constant $F(0) = (q-1)^2 (2A_1(q) + 4B_1(q))$. This proves the proposition. □

Corollary 2.7. *We have $L_\ell \in \mathbb{C}[L_1]$. If we write $L_\ell = p_\ell(L_1)$, p_ℓ is a monic polynomial of degree ℓ and satisfies $p_\ell(-x) = (-1)^\ell p_\ell(x)$.*

Proof. Since $L_1 \in \mathbb{C}[L_1]$ (of degree $2\ell-1$) and $L_2 \in \mathbb{C}[L_2]$ (of degree ℓ), it follows from Proposition 2.6 that f_ℓ is of degree ℓ . Furthermore, one has

$$\begin{aligned} p_\ell(-L_1(x)) &= p_\ell(L_1(-x)) \\ &= L_\ell(-x) = (-1)^\ell L_\ell(x) \\ &= (-1)^\ell p_\ell(L_1(x)), \end{aligned}$$

whence the desired relation follows immediately. The property that p_ℓ is monic follows from the discussion developed in the proof of Proposition 3.2 and Proposition 3.4 in the subsequent section. □

3 Algebraic Differential equations

3.1 Differential equations for $L_{2\ell}$

We prove the derivatives $L_{2\ell}^{(2k)}$ and $(L_{2\ell}^{(2k+1)})^2$ ($k = 0, 1, 2, \dots$) of $L_{2\ell}$ can be expressed as polynomials in L_2 . To show this, we need the following difference-differential equation satisfied by L_ℓ 's [KW].

Lemma 3.1.

$$L_\ell'' = \frac{(\log q)^2}{4} \cdot \ell^2 L_\ell + \frac{(\log q)^2}{(q-1)^2} \cdot \ell(\ell+1) L_{\ell+2}.$$

□

We have then the following.

Proposition 3.2. *There exist constants*

$$a_i(k), b_i(k) \in \mathbb{Q}(d_1, d_2, d_3, d_4, d_5) \quad (a_{\ell+k}(k) \neq 0, b_{2\ell+2k+1}(k) \neq 0)$$

such that the functions $L_{2\ell}^{(2k)}$ and $(L_{2\ell}^{(2k+1)})^2$ are expressed by the polynomials of L_2 as

$$\begin{aligned} L_{2\ell}^{(2k)} &= \sum_{i=0}^{\ell+k} a_i(k) L_2^i, \quad (a_i(0) = 1), \\ (L_{2\ell}^{(2k+1)})^2 &= \sum_{i=0}^{2\ell+2k+1} b_i(k) L_2^i. \end{aligned}$$

Proof. By Lemma 3.1 and (2.1) we easily obtain

$$(3.1) \quad L_\ell'' = \frac{d_2}{4} \cdot \ell^2 L_\ell + \frac{d_2}{d_1} \cdot \ell(\ell+1) L_{\ell+2}$$

and

$$(3.2) \quad (L_2')^2 = \frac{4d_2}{d_1} L_2^3 + d_2(6d_3 + 1) L_2^2 + 4d_1 d_2 d_4 L_2 + d_1^2 d_2 d_5.$$

Differentiating (3.2) we have

$$L_2'' = \frac{6d_2}{d_1} L_2^2 + d_2(6d_3 + 1) L_2 + 2d_1 d_2 d_4.$$

We first prove the former assertion for $k = 0$ by induction on ℓ . Let us assume that the assertion is true for 2ℓ :

$$L_{2\ell} = \sum_{i=0}^{\ell} a_i L_2^i, \quad (a_i \in \mathbb{Q}(d_1, \dots, d_5), a_\ell = 1).$$

Then, since $L_{2\ell}' = \{\sum_{i=1}^{\ell} i a_i L_2^{i-1}\} L_2'$, one gets

$$L_{2\ell}'' = \left\{ \sum_{i=2}^{\ell} i(i-1) a_i L_2^{i-2} \right\} (L_2')^2 + \left\{ \sum_{i=1}^{\ell} i a_i L_2^{i-1} \right\} L_2''.$$

It follows that $L_{2\ell}''$ can be written by a polynomial of L_2 as

$$L_{2\ell}'' = \sum_{i=0}^{\ell+1} b_i L_2^i, \quad (b_i \in \mathbb{Q}(d_1, \dots, d_5))$$

with

$$b_{\ell+1} = \ell(\ell-1)a_\ell \frac{4d_2}{d_1} + \ell a_\ell \frac{6d_2}{d_1} = 2\ell(2\ell+1) \frac{d_2}{d_1} \neq 0.$$

Now replacing ℓ by 2ℓ in (3.1) we see that

$$L_{2\ell}'' = d_2 \ell^2 L_{2\ell} + \frac{d_2}{d_1} 2\ell(2\ell+1) L_{2\ell+2}.$$

Hence we see that

$$L_{2\ell+2} = \sum_{i=0}^{\ell+1} c_i L_2^i, \quad (c_i \in \mathbb{Q}(d_1, \dots, d_5))$$

with

$$c_{\ell+1} = \frac{b_{\ell+1}}{\frac{d_2}{d_1} 2\ell(2\ell+1)} (= a_\ell) = 1.$$

This proves the assertion for $k = 0$.

We next prove the assertion for general k by induction. Suppose that $L_{2\ell}^{(2k)}$ can be written as

$$L_{2\ell}^{(2k)} = \sum_{i=0}^{\ell+k} a_i L_2^i, \quad (a_i \in \mathbb{Q}(d_1, \dots, d_5), a_{\ell+k} \neq 0).$$

Then, taking derivatives twice we have

$$L_{2\ell}^{(2k+2)} = \left\{ \sum_{i=2}^{\ell+k} i(i-1) a_i L_2^{i-2} \right\} (L_2')^2 + \left\{ \sum_{i=1}^{\ell+k} i a_i L_2^{i-1} \right\} L_2''.$$

From this it follows that

$$L_{2\ell}^{(2k+2)} = \sum_{i=0}^{\ell+k+1} b_i L_2^i, \quad (b_i \in \mathbb{Q}(d_1, \dots, d_5))$$

with

$$b_{\ell+k+1} = 2(\ell+k)(2\ell+2k+1) a_{\ell+k} \frac{d_2}{d_1} \neq 0.$$

This proves the first equation of the lemma for $L_{2\ell}^{(2k)}$. Furthermore, differentiating this equation and taking square we observe

$$(L_{2\ell}^{(2k+1)})^2 = \left(\sum_{i=1}^{\ell+k} i a_i L_2^{i-1} \right)^2 (L_2')^2.$$

From this expression, we see that $(L_{2\ell}^{(2k+1)})^2$ is expressed by a polynomial in L_2 as

$$(L_{2\ell}^{(2k+1)})^2 = \sum_{i=0}^{2\ell+2k+1} b_i L_2^i, \quad (b_i \in \mathbb{Q}(d_1, \dots, d_5))$$

with

$$b_{2\ell+2k+1} = (\ell+k)^2 a_{\ell+k}^2 \frac{4d_2}{d_1} \neq 0.$$

This completes the proof of the proposition. $\square$

We now get a non-linear algebraic differential equation for $L_{2\ell}$.

Theorem 3.3. *The function $L_{2\ell}$ ($\ell \geq 1$) satisfies the following differential equation:*

$$\sum_{i=0}^{\ell} \alpha_i L_{2\ell}^{(2i)} L_{2\ell}^{(4\ell-2i)} + \sum_{i=0}^{3\ell-1} \beta_i L_{2\ell}^{(2i)} + \gamma = 0 \quad (\alpha_i, \beta_i, \gamma \in \mathbb{Q}(d_1, \dots, d_5)).$$

Proof. By Proposition 3.2 one finds that the degree of the polynomial $L_{2\ell}^{(2i)} L_{2\ell}^{(4\ell-2i)}$ in L_2 is calculated as $(\ell + i) + (\ell + 2\ell - i) = 4\ell$ for all $0 \leq i \leq \ell$. In particular, since the degree of the polynomials $L_{2\ell}'' L_{2\ell}^{(4\ell-2)}$, $L_{2\ell} L_{2\ell}^{(4\ell)}$, $L_{2\ell}^{(6\ell-2)}$, $\dots$, $L_{2\ell}$ are 4ℓ , 4ℓ , $4\ell - 1$, $\dots$, ℓ , respectively, if one eliminates the leading coefficients successively one obtains

$$f_1 := L_{2\ell}'' L_{2\ell}^{(4\ell-2)} + a_{1,1} L_{2\ell} L_{2\ell}^{(4\ell)} + a_{1,2} L_{2\ell}^{(6\ell-2)} + \dots + a_{1,3\ell+1} L_{2\ell} = \sum_{i=0}^{\ell-1} b_{1,i} L_2^i,$$

for some coefficients $a_{1,j}, b_{1,i} \in \mathbb{Q}(d_1, \dots, d_5)$. Similarly, for $k = 2, \dots, \ell$, there exist $a_{k,j}, b_{k,i} \in \mathbb{Q}(d_1, \dots, d_5)$ one has

$$f_k := L_{2\ell}^{(2k)} L_{2\ell}^{(4\ell-2k)} + a_{k,1} L_{2\ell} L_{2\ell}^{(4\ell)} + a_{k,2} L_{2\ell}^{(6\ell-2)} + \dots + a_{k,3\ell+1} L_{2\ell} = \sum_{i=0}^{\ell-1} b_{k,i} L_2^i.$$

Write these relations as

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\ell \end{pmatrix} = \begin{pmatrix} b_{1,0} & b_{1,1} & \dots & b_{1,\ell-1} \\ b_{2,0} & \ddots & & b_{2,\ell-1} \\ \vdots & & \ddots & \vdots \\ b_{\ell,0} & \dots & \dots & b_{\ell,\ell-1} \end{pmatrix} \begin{pmatrix} 1 \\ L_2 \\ \vdots \\ L_2^{\ell-1} \end{pmatrix}.$$

Hence, there exists an $\ell \times \ell$ matrix A with coefficients in $\mathbb{Q}(d_1, \dots, d_5)$ such that

$$A \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\ell \end{pmatrix} = \begin{pmatrix} c_{1,0} & 0 & \dots & 0 \\ c_{2,0} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ c_{\ell,0} & \dots & \dots & c_{\ell,\ell-1} \end{pmatrix} \begin{pmatrix} 1 \\ L_2 \\ \vdots \\ L_2^{\ell-1} \end{pmatrix}.$$

Here, it is obvious to see that $c_{i,j} \in \mathbb{Q}(d_1, \dots, d_5)$. Noticing the first row, one has

$$\sum_{i=1}^{\ell} a_i f_i = c_{1,0}. \quad (a_i \in \mathbb{Q}(d_1, \dots, d_5)).$$

Hence, by the definition of f_i , one gets the desired differential equation for $L_{2\ell}$. $\square$

3.2 Differential equations for $L_{2\ell-1}$

A similar procedure developed in the last subsection works also for the odd case. In fact, we have the following.

Proposition 3.4. *There exist constants*

$$a_i(k), b_i(k) \in \mathbb{Q}(e_1, e_2, e_3, e_4) \quad (a_{n+k-1}(k) \neq 0, b_{2n+2k}(k) \neq 0)$$

such that the functions $L_{2\ell-1}^{(2k)}$ and $(L_{2\ell-1}^{(2k+1)})^2$ are expressed by the polynomials of L_1 as

$$\begin{aligned} L_{2\ell-1}^{(2k)} &= \sum_{i=0}^{\ell+k-1} a_i(k) L_1^{2i+1}, \quad (a_i(0) = 1), \\ (L_{2\ell-1}^{(2k+1)})^2 &= \sum_{i=0}^{2\ell+2k} b_i(k) L_1^{2i}. \end{aligned}$$

□

Using this proposition one can prove the following result. The proof is the same as Theorem 3.3.

Theorem 3.5. *The function $L_{2\ell-1}$ ($\ell \geq 1$) satisfies the following differential equation:*

$$\sum_{i=1}^{2\ell-1} \alpha_i L_{2\ell-1}^{(2i)} L_{2\ell-1}^{(8\ell-4-2i)} + \sum_{i=0}^{4\ell-2} \beta_i L_{2\ell-1} L_{2\ell-1}^{(2i)} + \gamma = 0 \quad (\alpha_i, \beta_i, \gamma \in \mathbb{Q}(e_1, \dots, e_4)).$$

□

Remark 3.6. One finds that, in the proof of Theorem 3.3, there are certain redundancy in the process for deriving the algebraic differential equation satisfied by L_ℓ . Therefore one can obtain, in principle, much simpler equations, i.e. the number of terms and of derivatives of the equations in Theorem 3.3 and Theorem 3.5 can be reduced.

4 Examples

In this section, we give much simpler and explicit differential equations for L_4 and L_3 explicitly. These are in fact simpler than what we have given in the preceding section. We use Mathematica 6.

4.1 A differential equation for L_4

By Proposition 3.2 one sees that there exist $a_i, b_i \in \mathbb{Q}(d_1, \dots, d_5)$ such that

$$\begin{aligned} L_4 L_4'' + a_1 (L_4')^2 + a_2 L_4^{(4)} + a_3 L_4'' + a_4 L_4 + a_5 &= b_1 L_2 + b_2, \\ L_4 L_4'' + a_1 (L_4')^2 + a_6 L_4^2 + a_7 L_4'' + a_8 L_4 + a_9 &= b_3 L_2 + b_4. \end{aligned}$$

From these equations we obtain

$$(4.1) \quad L_4 L_4'' + a(L_4')^2 + bL_4^{(4)} + cL_4^2 + dL_4'' + eL_4 + f = 0, \quad (a, b, c, d, e, f \in \mathbb{Q}(d_1, \dots, d_5)).$$

By Lemma 3.1 and Lemma 2.2, we have

$$\begin{aligned} L_2'' &= d_2 L_2 + \frac{6d_2}{d_1} L_4, \\ (L_2')^2 &= \frac{4d_2}{d_1} L_2^3 + d_2(6d_3 + 1)L_2^2 + 4d_1 d_2 d_4 L_2 + d_1^2 d_2 d_5. \end{aligned}$$

It follows that

$$\begin{aligned} L_2'' &= \frac{6d_2}{d_1} L_2^2 + d_2(6d_3 + 1)L_2 + 2d_1 d_2 d_4, \\ L_4 &= L_2^2 + d_1 d_3 L_2 + \frac{1}{3} d_1^2 d_4, \\ L_4' &= 2L_2 L_2' + d_1 d_3 L_2', \\ (L_4')^2 &= (2L_2 + d_1 d_3)^2 (L_2')^2 \\ &= \frac{16d_2}{d_1} L_2^5 + 4d_2(10d_3 + 1)L_2^4 + 4d_1 d_2(7d_3^2 + d_3 + 4d_4)L_2^3 \\ &\quad + d_1^2 d_2(6d_3^3 + d_3^2 + 16d_3 d_4 + 4d_5)L_2^2 + 4d_1^3 d_2 d_3(d_3 d_4 + d_5)L_2 + d_1^4 d_2 d_3^2 d_5, \end{aligned}$$

$$\begin{aligned}
L_4'' &= 2(L_2L_2'' + (L_2')^2) + d_1d_3L_2'' \\
&= \frac{20d_2}{d_1}L_2^3 + 2d_2(15d_3 + 2)L_2^2 + d_1d_2(6d_3^2 + d_3 + 12d_4)L_2 + 2d_1^2d_2(d_3d_4 + d_5), \\
L_4''' &= \left[\frac{60d_2}{d_1}L_2^2 + 4d_2(15d_3 + 2)L_2 + d_1d_2(6d_3^2 + d_3 + 12d_4) \right] L_2', \\
L_4^{(4)} &= \left[\frac{60d_2}{d_1}L_2^2 + 4d_2(15d_3 + 2)L_2 + d_1d_2(6d_3^2 + d_3 + 12d_4) \right] L_2'' \\
&\quad + \left[\frac{120d_2}{d_1}L_2 + 4d_2(15d_3 + 2) \right] (L_2')^2 \\
&= \frac{840d_2^2}{d_1^2}L_2^4 + \frac{20d_2^2}{d_1}(84d_3 + 13)L_2^3 + 2d_2^2(378d_3^2 + 111d_3 + 336d_4 + 8)L_2^2 \\
&\quad + d_1d_2^2(36d_3^3 + 12d_3^2 + 432d_3d_4 + d_3 + 60d_4 + 120d_5)L_2 \\
&\quad + 2d_1^2d_2^2(6d_3^2d_4 + d_3d_4 + 12d_4^2 + 30d_3d_5 + 4d_5).
\end{aligned}$$

Substituting these into (4.1), we have the linear equation for $a, b, \dots, f$ as follows:

$$\begin{cases}
\frac{20d_2}{d_1} + a \cdot \frac{16d_2}{d_1} = 0, \\
2d_2(25d_3 + 2) + a \cdot 4d_2(10d_3 + 1) + b \cdot \frac{840d_2^2}{d_1^2} + c \cdot 1 = 0, \\
d_1d_2(36d_3^2 + 5d_3 + \frac{56}{3}d_4) + a \cdot 4d_1d_2(7d_3^2 + d_3 + 4d_4) + b \cdot \frac{20d_2^2}{d_1}(84d_3 + 13) \\
+ c \cdot 2d_1d_3 + d \cdot \frac{20d_2}{d_1} = 0, \\
d_1^2d_2(6d_3^3 + d_3^2 + 24d_3d_4 + \frac{4}{3}d_4 + 2d_5) + a \cdot d_1^2d_2(6d_3^3 + d_3^2 + 16d_3d_4 + 4d_5) \\
+ b \cdot 2d_2^2(378d_3^2 + 111d_3 + 336d_4 + 8) + c \cdot d_1^2(d_3^2 + \frac{2}{3}d_4) + d \cdot 2d_2(15d_3 + 2) + e \cdot 1 = 0, \\
d_1^3d_2(4d_3^2d_4 + \frac{1}{3}d_3d_4 + 4d_4^2 + 2d_3d_5) + a \cdot 4d_1^3d_2d_3(d_3d_4 + d_5) \\
+ b \cdot d_1d_2^2(36d_3^3 + 12d_3^2 + 432d_3d_4 + d_3 + 60d_4 + 120d_5) + c \cdot \frac{2}{3}d_1^3d_3d_4 \\
+ d \cdot d_1d_2(6d_3^2 + d_3 + 12d_4) + e \cdot d_1d_3 = 0, \\
\frac{2}{3}d_1^4d_2d_4(d_3d_4 + d_5) + a \cdot d_1^4d_2d_3^2d_5 + b \cdot 2d_1^2d_2^2(6d_3^2d_4 + d_3d_4 + 12d_4^2 + 30d_3d_5 + 4d_5) \\
+ c \cdot \frac{1}{9}d_1^4d_4^2 + d \cdot 2d_1^2d_2(d_3d_4 + d_5) + e \cdot \frac{1}{3}d_1^2d_4 + f \cdot 1 = 0.
\end{cases}$$

Solving the equation we can determine the coefficients of the equation (4.1) as follows:

$$\begin{aligned}
a &= -\frac{5}{4}, \\
b &= \frac{d_1^2(-9d_3^4 - 6d_3^3 - d_3^2 + 24d_3^2d_4 + 8d_3d_4 - 16d_4^2)}{20d_2(20d_3^3 + 17d_3^2 - 40d_3d_4 + 4d_3 - 16d_4 + 20d_5)}, \\
c &= \frac{d_2(378d_3^4 + 272d_3^3 + 59d_3^2 - 1008d_3^2d_4 - 376d_3d_4 + 4d_3 + 672d_4^2 - 16d_4 + 20d_5)}{20d_3^3 + 17d_3^2 - 40d_3d_4 + 4d_3 - 16d_4 + 20d_5}, \\
d &= \frac{d_1^2(-12d_3^5 + 36d_3^4 + 40d_3^3d_4 + 24d_3^3 - 116d_3^2d_4 + 3d_3^2 - 12d_3^2d_5 - 32d_3d_4^2 - 40d_3d_4 \\
&\quad - 24d_3d_5 + 112d_4^2 + 16d_4d_5)/12(20d_3^3 + 17d_3^2 - 40d_3d_4 + 4d_3 - 16d_4 + 20d_5)}, \\
e &= \frac{d_1^2d_2(1332d_3^5 - 48d_3^5 - 549d_3^4 - 156d_3^3 - 12d_3^2 - 5328d_3^4d_4 - 1472d_3^3d_4 - 104d_3^2d_4 \\
&\quad - 64d_3d_4 + 2304d_3^2d_4^2 + 448d_3d_4^2 + 448d_4^2 + 5376d_4^3 + 7200d_3^3d_5 + 6000d_3^2d_5 \\
&\quad + 1200d_3d_5 - 14400d_3d_4d_5 - 5600d_4d_5 + 3600d_5^2)/60(20d_3^3 + 17d_3^2 - 40d_3d_4 \\
&\quad + 4d_3 - 16d_4 + 20d_5)}, \\
f &= \frac{d_1^4d_2(12d_3^2d_4 + 84d_3^3d_4 + 45d_3^4d_4 - 222d_3^5d_4 - 16d_3d_4^2 - 1244d_3^2d_4^2 - 6608d_3^3d_4^2 \\
&\quad - 11280d_3^4d_4^2 - 128d_4^3 + 8032d_3d_4^3 + 29760d_3^2d_4^3 - 15360d_4^4 - 18d_3^2d_5 + 1152d_3^3d_5 \\
&\quad + 6633d_3^4d_5 + 9720d_3^5d_5 - 2016d_3d_4d_5 - 17568d_3^2d_4d_5 - 37200d_3^3d_4d_5 + 8752d_4^2d_5 \\
&\quad + 30720d_3d_4^2d_5 + 720d_3d_5^2 + 4860d_3^2d_5^2 - 11280d_4d_5^2) \\
&\quad /180(20d_3^3 + 17d_3^2 - 40d_3d_4 + 4d_3 - 16d_4 + 20d_5)}.
\end{aligned}$$

4.2 A differential equation for L_3

The following differential equation satisfied by L_3 can be obtained by a similar discussion made in the case for L_4 .

$$(4.2) \quad L_3^2 L_3'' + a L_3 (L_3')^2 + b L_3^3 + c L_3^{(6)} + d L_3(4) + e L_3'' + f L_3 = 0, \quad (a, b, c, d, e, f \in \mathbb{Q}(e_1, \dots, e_4)).$$

The coefficients appeared in the equation above can be calculated as follows:

$$\begin{aligned} a &= -\frac{4}{3}, \\ b &= e_2(-37494e_3 + 16758e_1^2e_3 - 210291e_3^2 + 100275e_1^2e_3^2 + 1216848e_3^3 - 770844e_1^2e_3^3 \\ &\quad + 10259968e_3^4 - 4157814e_1^2e_3^4 + 17720640e_3^5 - 834840e_1^2e_3^5 + 224964e_4 \\ &\quad - 100548e_1^2e_4 + 1365696e_3e_4 + 286272e_1^2e_3e_4 - 5531328e_3^2e_4 + 4352904e_1^2e_3^2e_4 \\ &\quad - 24675840e_3^3e_4 - 11950560e_1^2e_3^3e_4 - 4209408e_4^2 - 2612736e_1^2e_4^2 - 22256640e_3e_4^2 \\ &\quad + 17418240e_1^2e_3e_4^2)/(12(-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3 \\ &\quad + 71384e_1^2e_3^3 - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4 \\ &\quad - 42672e_1^2e_3e_4 + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)), \\ c &= e_1^3(72e_3^2 - 72e_3^3 - 2332e_3^4 - 4725e_3^5 - 864e_3e_4 - 1368e_3^2e_4 + 10800e_3^3e_4 + 2592e_4^2 \\ &\quad + 10800e_3e_4^2)/(135e_2^2(-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3 \\ &\quad + 71384e_1^2e_3^3 - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4 \\ &\quad - 42672e_1^2e_3e_4 + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)), \\ d &= e_1^3(546e_3^2 - 5586e_1^2e_3^2 + 43207e_3^3 - 35287e_1^2e_3^3 + 61307e_3^4 + 233393e_1^2e_3^4 \\ &\quad - 1118174e_3^5 + 1314554e_1^2e_3^5 - 2930256e_3^6 + 264366e_1^2e_3^6 - 6552e_3e_4 + 67032e_1^2e_3e_4 \\ &\quad - 399498e_3^2e_4 + 339738e_1^2e_3^2e_4 - 1518936e_3^3e_4 - 123384e_1^2e_3^3e_4 - 1090656e_3^4e_4 \\ &\quad + 4034796e_1^2e_3^4e_4 + 19656e_4^2 - 201096e_1^2e_4^2 + 696384e_3e_4^2 + 102816e_1^2e_3e_4^2 \\ &\quad + 2515968e_3^2e_4^2 - 1930608e_1^2e_3^2e_4^2 + 870912e_4^3 - 5225472e_1^2e_4^3)/(1080e_2 \\ &\quad (-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3 + 71384e_1^2e_3^3 \\ &\quad - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4 - 42672e_1^2e_3e_4 \\ &\quad + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)), \\ e &= -e_1^3(9282e_3^2 - 27930e_1^2e_3^2 + 272819e_3^3 - 288155e_1^2e_3^3 + 635939e_3^4 + 628805e_1^2e_3^4 \\ &\quad - 6674273e_3^5 + 11265830e_1^2e_3^5 - 26757152e_3^6 + 16964470e_1^2e_3^6 - 19640880e_3^7 \\ &\quad + 3061080e_1^2e_3^7 - 111384e_3e_4 + 335160e_1^2e_3e_4 - 1251066e_3^2e_4 + 1698690e_1^2e_3^2e_4 \\ &\quad - 6930792e_3^3e_4 + 2736360e_1^2e_3^3e_4 - 82630752e_3^4e_4 + 80842140e_1^2e_3^4e_4 \\ &\quad - 234924480e_3^5e_4 + 58289280e_1^2e_3^5e_4 + 334152e_4^2 - 1005480e_1^2e_4^2 - 4782672e_3e_4^2 \\ &\quad + 4536000e_1^2e_3e_4^2 - 60303744e_3^2e_4^2 + 23127120e_1^2e_3^2e_4^2 - 133781760e_3^3e_4^2 \\ &\quad + 169248960e_1^2e_3^3e_4^2 + 14805504e_4^3 - 26127360e_1^2e_4^3 + 26127360e_3e_4^3 \\ &\quad + 104509440e_1^2e_3e_4^3)/(2160(-4166e_3 + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 \\ &\quad - 124484e_3^3 + 71384e_1^2e_3^3 - 154224e_3^4 + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 \\ &\quad + 146352e_3e_4 - 42672e_1^2e_3e_4 + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)), \\ f &= e_1^3e_2(5958e_3^2 - 16758e_1^2e_3^2 + 190245e_3^3 - 239925e_1^2e_3^3 + 1012737e_3^4 - 303117e_1^2e_3^4 \\ &\quad - 2753838e_3^5 + 8314998e_1^2e_3^5 - 31889088e_3^6 + 36513498e_1^2e_3^6 - 70418016e_3^7 \\ &\quad + 39563376e_1^2e_3^7 - 36546048e_3^8 + 6752928e_1^2e_3^8 - 71496e_3e_4 + 201096e_1^2e_3e_4 \\ &\quad - 1108926e_3^2e_4 + 2091726e_1^2e_3^2e_4 - 29384904e_3^3e_4 + 16327704e_1^2e_3^3e_4) \end{aligned}$$

$$\begin{aligned}
& -140392064e_3^4e_4 + 89458724e_1^2e_3^4e_4 + 33067264e_3^5e_4 - 43487104e_1^2e_3^5e_4 \\
& + 763931136e_3^6e_4 + 72474944e_1^2e_3^6e_4 + 214488e_4^2 - 603288e_1^2e_4^2 + 41926464e_3e_4^2 \\
& - 46345824e_1^2e_3e_4^2 + 351313920e_3^2e_4^2 - 124364880e_1^2e_3^2e_4^2 - 1274706432e_3^3e_4^2 \\
& + 2125418112e_1^2e_3^3e_4^2 - 6904461312e_3^4e_4^2 - 131194368e_1^2e_3^4e_4^2 - 252730368e_3^5 \\
& + 254596608e_1^2e_4^3 - 2993614848e_3e_4^3 + 990517248e_1^2e_3e_4^3 - 7851810816e_3^2e_4^3 \\
& + 6524292096e_1^2e_3^2e_4^3 - 1170505728e_4^4 + 7023034368e_1^2e_4^4)/(5760(-4166e_3 \\
& + 1862e_1^2e_3 - 36803e_3^2 + 23555e_1^2e_3^2 - 124484e_3^3 + 71384e_1^2e_3^3 - 154224e_4^3 \\
& + 13914e_1^2e_3^4 + 24996e_4 - 11172e_1^2e_4 + 146352e_3e_4 - 42672e_1^2e_3e_4 \\
& + 88704e_3^2e_4 + 199176e_1^2e_3^2e_4 + 48384e_4^2 - 290304e_1^2e_4^2)).
\end{aligned}$$

Corrigenda to the paper [KW]: We list here some corrections of the equations in the paper [KW]. We did not, however, use these equations.

- The coefficient of the second term of (2.12) in Corollary 2.2 is incorrect and should be 48.
- The coefficient of $B_3(q)$ in the expression of β_2 in Theorem 2.3 is incorrect and should be $-\frac{16}{3}$.
- The equation (2.30) in p.37 should be read as $M_2''(x) = M_2(x)(6\beta_0^2L_1(x)^2 + \beta_1)$. Hence, the correct equation (2.32) in Corollary 2.4 (p.38) is

$$M_2''(x)^2 + 4\beta_1M_2(x)M_2''(x) + (36\beta_0^2\beta_2 - 5\beta_1^2)M_2(x)^2 - 36\beta_0^4M_2(x)^4 = 0.$$

References

- [C] R. Chapman, Private communication, 2006.
- [KW] N. Kurokawa and M. Wakayama, *Certain family of elliptic functions defined by q -series*, The Ramujan J. **10** (2005), 23–41.
- [L] S. Lang, “Elliptic Functions”, Addison Wesley, 1970.

MASATO WAKAYAMA
 Faculty of Mathematics, Kyushu University.
 Hakozaki Higashi-ku, Fukuoka 812-8581, JAPAN.
 wakayama@math.kyushu-u.ac.jp

KEITARO YAMAMOTO
 Graduate School of Mathematics, Kyushu University.
 Hakozaki Higashi-ku, Fukuoka, 812-8581 JAPAN.
Current address:
 JASTEC Co.,Ltd.
 3-5-23, Takanawa, Minato-ku,Tokyo 108-0074 JAPAN

List of MI Preprint Series, Kyushu University

The Grobal COE Program
Math-for-Industry Education & Research Hub

MI

- MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata
- MI2008-2 Eiji ONODERA
The intial value problem for a third-order dispersive flow into compact almost Hermitian manifolds
- MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space
- MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme
- MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in nfinite extensions over a p-adic field
- MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields
- MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited
- MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition
- MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

- MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE
- MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds
- MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator
- MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials
- MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality
- MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions
- MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings
- MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors
- MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization
- MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions
- MI2009-5 Toshiro HIRANOUCI & Yuichiro TAGUCHI
Flat modules and Groebner bases over truncated discrete valuation rings

- MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations
- MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow
- MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization
- MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity
- MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption
- MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve
- MI2009-12 Tetsu MASUDA
Hypergeometric q -functions of the q -Painlevé system of type $E_8^{(1)}$
- MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination
- MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications
- MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions