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Ishihara, Yuya  
Faculty of Mathematics, Kyushu University

Kagei, Yoshiyuki  
Faculty of Mathematics, Kyushu University

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Faculty of Mathematics  
Kyushu University  
Fukuoka, JAPAN

# Large time behavior of the semigroup on $L^p$ spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

Yuya Ishihara and Yoshiyuki Kagei\*

## Abstract

Large time behavior of solutions to the linearized compressible Navier-Stokes equation around the motionless state in a cylindrical domain is investigated. The  $L^p$  decay estimates of the associated semigroup are established for all  $1 < p < \infty$ . It is also shown that the time-asymptotic leading part of the semigroup is given by a one dimensional heat semigroup.

**Mathematics Subject Classification (2000).** 35Q30, 76N15.

**Keywords.** Compressible Navier-Stokes equation, large time behavior, cylindrical domain.

## 1. Introduction

This paper studies large time behavior of solutions to the following system of equations

$$(1.1) \quad \partial_t u + Lu = 0,$$

where  $u = {}^T(\phi, v)$ ,  $\phi = \phi(x, t) \in \mathbf{R}$ ,  $v = {}^T(v^1(x, t), v^2(x, t), v^3(x, t)) \in \mathbf{R}^3$ , and  $L$  is an operator defined by

$$L = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & -\nu \Delta - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}$$

with positive constants  $\nu$ ,  $\tilde{\nu}$  and  $\gamma$ . Here  $t \geq 0$  is the time variable and  $x = (x_1, x_2, x_3) \in \mathbf{R}^3$  is the space variable and  ${}^T \cdot$  stands for the transposition.

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\*Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, JAPAN

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In this paper we consider (1.1) in a cylindrical domain

$$\Omega = D \times \mathbf{R} = \{x = (x', x_3); x' = (x_1, x_2) \in D, x_3 \in \mathbf{R}\}$$

under the boundary condition

$$(1.2) \quad v|_{\partial\Omega} = 0.$$

Here  $D$  is a bounded domain in  $\mathbf{R}^2$  with smooth boundary  $\partial D$ .

The system (1.1) arises from the linearization of the compressible Navier-Stokes equation

$$(1.3) \quad \begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \partial_t m + \operatorname{div} \left( \frac{m \otimes m}{\rho} \right) + \nabla P(\rho) &= \mu \Delta \left( \frac{m}{\rho} \right) + (\mu + \mu') \nabla \operatorname{div} \left( \frac{m}{\rho} \right) \end{aligned}$$

around the constant motionless state  $(\rho, m) = (\rho_*, 0)$ , where  $\rho = \rho(x, t)$  is the density;  $m = {}^T(m^1(x, t), m^2(x, t), m^3(x, t))$  is the velocity field; and  $\rho_*$  is a given positive number.

Large time behavior of solutions of (1.3) in unbounded domains has been widely studied, which presents interesting aspects. Concerning the Cauchy problem for (1.3) on the whole space  $\mathbf{R}^3$ , it was shown in [13, 17, 18] that if the initial perturbation  $(\rho(0) - \rho_*, v(0))$  is sufficiently small in  $H^3$ , then there exists a unique global solution to (1.3) and the leading part of the perturbation  $u(t) = (\rho(t) - \rho_*, v(t))$  in large time is given by the solution of the linearized problem, which exhibits a hyperbolic-parabolic aspect of system (1.3). (See [12] for the case of a general class of quasilinear hyperbolic-parabolic systems.) The solution of the linearized problem is approximated in large time by the sum of two terms; one is given by the convolution of the heat kernel and the fundamental solution of the wave equation, the so-called diffusion wave; and the other is the solution of the heat equation. It was found in [3, 4] that hyperbolic and parabolic aspects of the diffusion wave exhibits an interesting interaction phenomena in the decay properties of  $L^p$  norms with  $1 \leq p \leq \infty$ . (See also [16].) Such an interaction phenomena also appears in the exterior domain problem [14, 15] and the half space problem [8, 9]. Furthermore, in the case of the half space problem, some different aspect appears in the decay property of spatial derivatives due to the presence of unbounded boundary.

On the other hand, solutions on the infinite layer  $\mathbf{R}^{n-1} \times (0, 1)$  behave in a different manner from the ones on the domains mentioned above. The

leading part of the solution on the infinite layer is given by a solution of an  $n - 1$  dimensional heat equation [7]. This is due to the fact that the infinite layer has an infinite extent in  $n - 1$  unbounded directions and the remaining one direction has a finite thickness. An analogous result was obtained in [10] for the cylindrical domain  $\Omega$  that has one unbounded direction  $x_3$  and two dimensional bounded cross section  $D$ . In this case, under suitable assumptions on the initial value, the perturbation  $u(t) = (\rho(t) - \rho_*, v(t))$  satisfies

$$\|u(t)\|_{L^2} = O(t^{-1/4}), \quad \|u(t) - u^{(0)}(t)\|_{L^2} = O(t^{-3/4} \log t)$$

as  $t \rightarrow \infty$ . Here  $u^{(0)} = (\phi^{(0)}(x_3, t), 0)$  with  $\phi^{(0)}(x_3, t)$  satisfying

$$(1.4) \quad \partial_t \phi^{(0)} - \kappa \partial_{x_3}^2 \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \frac{1}{|D|} \int_D (\rho_0(x', x_3) - \rho_*) dx',$$

where  $\kappa$  is a positive constant and  $|D|$  denotes the Lebesgue measure of  $D$ . In [10] large time behavior was investigated only in the  $L^2$  space, while in the case of the infinite layer [5, 6, 7] it was investigated in general  $L^p$  spaces. The analysis in  $L^p$  spaces in the case of the infinite layer relies on a solution formula ([5]) whose analogous version seems to be unavailable in the case of cylindrical domains since  $D$  is a general bounded domain of  $\mathbf{R}^2$ .

In this paper we will extend the analysis in the  $L^2$  space in [10] to general  $L^p$  spaces. We here treat only the linearized problem (1.1)–(1.2), since the nonlinear problem (1.3)–(1.2) can be treated as in [7] based on the energy method by Matsumura and Nishida [19] and the analysis of the linearized problem (1.1)–(1.2).

The main result of this paper is summarized as follows. Let  $1 < p < \infty$  and let  $u(t)$  be a solution of (1.1)–(1.2) with  $u|_{t=0} = {}^T(\phi_0, v_0) \in [W^{1,p} \times L^p] \cap L^1$ . Then

$$(1.5) \quad \begin{aligned} \|u(t)\|_{L^p} &= O(t^{-\frac{1}{2}(1-\frac{1}{p})}), \\ \|u(t) - u^{(0)}(t)\|_{L^p} &= O(t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}) \end{aligned}$$

as  $t \rightarrow \infty$ . Here  $u^{(0)} = {}^T(\phi^{(0)}(x_3, t), 0)$  with  $\phi^{(0)}(x_3, t)$  satisfying the equation in (1.4) and  $\phi^{(0)}|_{t=0} = \frac{1}{|D|} \int_D \phi_0(x', x_3) dx'$ .

To prove (1.5) we will consider the Fourier transform of problem (1.1)–(1.2) with respect to  $x_3$  variable which is written in the form

$$(1.6) \quad \begin{aligned} \partial_t \widehat{u} + \widehat{L}_\xi \widehat{u} &= 0, \\ \widehat{v}|_{\partial D} &= 0, \quad \widehat{u}|_{t=0} = \widehat{u}_0. \end{aligned}$$

Here  $\widehat{u} = \widehat{u}(x', \xi, t) = {}^T(\widehat{\phi}(x', \xi, t), \widehat{v}(x', \xi, t))$  ( $x' \in D$ ,  $\xi \in \mathbf{R}$ ,  $t \geq 0$ ) denotes the Fourier transform of  $u(x', x_3, t) = {}^T(\phi(x', x_3, t), v(x', x_3, t))$  with respect to  $x_3$  variable. We investigate problem (1.6) according to the following three cases:

$$(i) |\xi| \ll 1, \quad (ii) |\xi| \gg 1, \quad (iii) r \leq |\xi| \leq M$$

with suitable constants  $0 < r < M < \infty$ . The case (i) is treated similarly as in [6, 10]. We regard problem (1.6) as a perturbation from the one with  $\xi = 0$  and analyze the spectral properties of  $\widehat{L}_\xi$  by applying the analytic perturbation theory. As for the case (ii), we treat it as a perturbation from the problem on the half space and derive necessary estimates for the corresponding part of the resolvent in  $L^p$  spaces by using the Fourier-Multiplier Theorem. As for the case (iii), we derive estimates for the derivatives of  $(\lambda + \widehat{L}_\xi)^{-1}$  with respect to  $\xi$  and then obtain necessary estimates for the resolvent by employing the Riemann-Lebesgue lemma. To investigate the cases (ii) and (iii), we will use the solution formula for the half space problem [8, 9].

This paper is organized as follows. In section 2 we state our main result of this paper. The analysis for the cases (ii) and (iii) are done in sections 3 and 4. Section 5 is devoted to the analysis for the case (i). Based on the analysis in sections 3–5, we prove our main result in section 6.

## 2. Main Result

We first introduce some notation.

For  $1 \leq p \leq \infty$  we denote by  $L^p(\Omega)$  the usual Lebesgue space on  $\Omega$  and its norm is denoted by  $\|\cdot\|_p$ . Let  $\ell$  be a nonnegative integer. The symbol  $W^{\ell,p}(\Omega)$  denotes the  $\ell$ -th order  $L^p$  Sobolev space on  $\Omega$  with norm  $\|\cdot\|_{W^{\ell,p}}$ . When  $p = 2$ , the space  $W^{\ell,2}(\Omega)$  is denoted by  $H^\ell(\Omega)$  and its norm is denoted by  $\|\cdot\|_{H^\ell}$ .  $C_0^{\ell,p}(\Omega)$  stands for the set of all  $C^\ell$  functions which have compact support in  $\Omega$ . We denote by  $W_0^{\ell,p}(\Omega)$  the completion of  $C_0^{\ell,p}(\Omega)$  in  $W^{\ell,p}(\Omega)$ . In particular,  $W_0^{\ell,2}(\Omega)$  is denoted by  $H_0^\ell(\Omega)$ .

We simply denote by  $L^p(\Omega)$  (resp.,  $W^{\ell,p}(\Omega)$ ,  $H^\ell(\Omega)$ ) the set of all vector fields  $v = {}^T(v^1, v^2, v^3)$  on  $\Omega$  with  $v^j \in L^p(\Omega)$  (resp.,  $W^{\ell,p}(\Omega)$ ,  $H^\ell(\Omega)$ ),  $j = 1, 2, 3$ , and its norm is also denoted by  $\|\cdot\|_{L^p}$  (resp.,  $\|\cdot\|_{W^{\ell,p}}$ ,  $\|\cdot\|_{H^\ell}$ ). For  $u = {}^T(\phi, v)$  with  $\phi \in W^{\ell,p}(\Omega)$  and  $v = {}^T(v^1, v^2, v^3) \in W^{j,p}(\Omega)$ , we define  $\|u\|_{W^{\ell,p} \times W^{j,p}} = \|\phi\|_{W^{\ell,p}} + \|v\|_{W^{j,p}}$ . When  $\ell = j$ , we simply write  $\|u\|_{W^{\ell,p}}$  for  $\|u\|_{W^{\ell,p} \times W^{\ell,p}}$ .

Similarly we introduce the function spaces  $L^p(D)$ ,  $W^{\ell,p}(D)$ ,  $H^\ell(D)$  and

$H_0^\ell(D)$ . Their norms are denoted by

$$|\cdot|_p, \quad |\cdot|_{W^{\ell,p}}, \quad |\cdot|_{H^\ell}.$$

For  $u = {}^t(\phi, v)$  with  $\phi \in W^{\ell,p}(D)$  and  $v = {}^T(v^1, v^2, v^3) \in W^{j,p}(D)$ , we denote  $|u|_{W^{\ell,p} \times W^{j,p}} \equiv |\phi|_{W^{\ell,p}} + |v|_{W^{j,p}}$ .

The inner product of  $L^2(D)$  is denoted by

$$(f, g) \equiv \int_D f(x') \overline{g(x')} dx'$$

for  $f, g \in L^2(D)$ . We also denote the inner product of  $L^2(\Omega)$  by the same symbol if no confusion occurs. We define  $\langle \cdot, \cdot \rangle$  by

$$\langle f, g \rangle \equiv \frac{1}{|D|} (f, g)$$

for  $f, g \in L^2(D)$ . In particular, when  $g = 1$ ,  $\langle f, 1 \rangle$  is denoted by  $\langle f \rangle$ , i.e.,

$$\langle f \rangle \equiv \frac{1}{|D|} \int_D f(x') dx'.$$

For a Banach space  $X$ , we denote by  $\mathcal{S}(\mathbf{R}; X)$  the set of all rapidly decreasing functions on  $\mathbf{R}$  with values in  $X$ .

We next introduce some notations about integral operators. We denote the Fourier transform of  $f = f(z)$  ( $z \in \mathbf{R}^k$ ) by

$$[\mathcal{F}_{z \rightarrow \zeta} f](\zeta) = \int_{\mathbf{R}^k} f(z) e^{-i\zeta \cdot z} dz,$$

and the inverse Fourier transform is denoted by

$$[\mathcal{F}_{\zeta \rightarrow z}^{-1} f](z) = (2\pi)^{-k} \int_{\mathbf{R}^k} f(\zeta) e^{i\zeta \cdot z} d\zeta.$$

In particular, the Fourier transform of  $f = f(x_3)$  ( $x_3 \in \mathbf{R}$ ) is denoted by  $\widehat{f}$  or  $\mathcal{F}f$ , i.e.,

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbf{R}} f(x_3) e^{-i\xi \cdot x_3} dx_3,$$

and the inverse Fourier transform is denoted by  $\mathcal{F}^{-1}f$ , i.e.,

$$\mathcal{F}^{-1}f(x_3) = (2\pi)^{-1} \int_{\mathbf{R}} f(\xi) e^{i\xi \cdot x_3} d\xi.$$

For a function  $K(y, z)$  on  $(0, \infty) \times (0, \infty)$  we will denote by  $Kf$  the integral operator  $\int_0^\infty K(y, z)f(z) dz$ .

We denote the resolvent set of a closed operator  $A$  by  $\rho(A)$  and the spectrum by  $\sigma(A)$ . For  $c \in \mathbf{R}$  and  $\theta \in (\frac{\pi}{2}, \pi)$ , we will denote

$$\Sigma(c, \theta) = \{\lambda \in \mathbf{C}; |\arg(\lambda - c)| \leq \theta\}.$$

We denote by  $Q_0$ ,  $\tilde{Q}$  and  $Q'$  the  $4 \times 4$  diagonal matrices

$$Q_0 = \text{diag}(1, 0, 0, 0), \quad \tilde{Q} = \text{diag}(0, 1, 1, 1), \quad Q' = \text{diag}(0, 1, 1, 0),$$

respectively. We then have for  $u = {}^T(\phi, v)$ ,  $v = {}^T(v^1, v^2, v^3)$ ,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ v' \\ 0 \end{pmatrix} \quad (v' = {}^T(v^1, v^2)).$$

We now state our main result. Let  $1 < p < \infty$ . We define an operator  $L$  on  $W^{1,p}(\Omega) \times L^p(\Omega)$  with domain of definition  $D(L)$  by

$$L = \begin{pmatrix} 0 & \gamma \text{div} \\ \gamma \nabla & -\nu \Delta - \tilde{\nu} \nabla \text{div} \end{pmatrix},$$

$$D(L) = W^{1,p}(\Omega) \times [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]$$

with positive constants  $\nu$ ,  $\tilde{\nu}$  and  $\gamma$ .

**Theorem 2.1.** *Let  $1 < p < \infty$ . Then  $-L$  generates an analytic semigroup  $e^{-tL}$  on  $W^{1,p}(\Omega) \times L^p(\Omega)$  and  $e^{-tL}$  has the following properties.*

(i) *There hold the estimates*

$$\|e^{-tL} u_0\|_{W^{1,p} \times L^p} \leq C \|u_0\|_{W^{1,p} \times L^p}$$

and

$$\|\partial_x^\ell \tilde{Q} e^{-tL} u_0\|_p \leq C t^{-\frac{\ell}{2}} \|u_0\|_{W^{1,p} \times L^p} \quad (\ell = 1, 2)$$

for  $0 < t < 1$ .

(ii) *If  $u_0 = {}^T(\phi_0, v_0) \in [W^{1,p}(\Omega) \times L^p(\Omega)] \cap L^1(\Omega)$ , then  $e^{-tL} u_0$  is decomposed as*

$$e^{-tL} u_0 = \mathcal{U}_0(t) u_0 + \mathcal{U}_\infty(t) u_0.$$

Here  $\mathcal{U}_0(t)u_0$  and  $\mathcal{U}_\infty(t)u_0$  satisfy the following (ii-a) and (ii-b).

(ii-a)  $\mathcal{U}_0(t)u_0$  is written as

$$\mathcal{U}_0(t)u_0 = \mathcal{W}_0(t)u_0 + \mathcal{R}_0(t)u_0.$$

Here  $\mathcal{W}_0(t)u_0$  takes the form

$$\mathcal{W}_0(t)u_0 = \begin{pmatrix} \phi^{(0)}(t) \\ 0 \end{pmatrix},$$

and  $\phi^{(0)}(x_3, t)$  satisfies the following heat equation on  $\mathbf{R}$ :

$$\partial_t \phi^{(0)} - \kappa \partial_{x_3}^2 \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \frac{1}{|D|} \int_D \phi_0(x', x_3) dx'$$

with some positive constant  $\kappa$ .

For  $1 \leq r \leq \infty$  and  $\ell = 0, 1$ , the function  $\mathcal{R}_0(t)u_0$  satisfies the estimate

$$\|\partial_x^\ell \mathcal{R}_0(t)u_0\|_r \leq Ct^{-\frac{1}{2}(1-\frac{1}{r})-\frac{1}{2}} \|u_0\|_1$$

uniformly for  $t \geq 1$ . Furthermore,

$$\mathcal{U}_0(t)\tilde{Q} = \mathcal{R}_0(t)\tilde{Q}$$

and the estimates hold for  $t \geq 1$ :

$$\|\partial_x \mathcal{U}_0(t)\tilde{Q}u_0\|_r \leq Ct^{-\frac{1}{2}(1-\frac{1}{r})-1} \|\tilde{Q}u_0\|_1,$$

$$\|\mathcal{U}_0(t)[\partial_x \tilde{Q}u_0]\|_r \leq Ct^{-\frac{1}{2}(1-\frac{1}{r})-\frac{1}{2}} \|\tilde{Q}u_0\|_1.$$

(ii-b) For  $\ell = 0, 1$ , the function  $\mathcal{U}_\infty(t)u_0$  satisfies the estimate

$$\|\partial_x^\ell \mathcal{U}_\infty(t)u_0\|_p \leq Ce^{-ct} \|u_0\|_{W^{1,p} \times L^p}$$

uniformly for  $t \geq 1$ .

**Remark 2.2.** Since  $\phi^{(0)}$  is a solution of a one dimensional heat equation, we have

$$\|\mathcal{W}_0(t)u_0\|_r \leq Ct^{-\frac{1}{2}(1-\frac{1}{r})} \|u_0\|_1.$$

This implies that

$$\|\mathcal{U}_0(t)u_0\|_r \leq Ct^{-\frac{1}{2}(1-\frac{1}{r})} \|u_0\|_1.$$

The proof of Theorem 2.1 is based on the resolvent problem associated with (1.1)–(1.2):

$$(2.1) \quad (\lambda + L)u = f, \quad v|_{\partial\Omega} = 0.$$

Here  $u = {}^T(\phi, v)$ .

Hereafter we will often write

$$x = {}^t(x', x_3), \quad x' = {}^t(x_1, x_2) \in D, \quad \nabla' = {}^t(\partial_{x_1}, \partial_{x_2}), \quad \Delta' = \partial_{x_1}^2 + \partial_{x_2}^2.$$

We take the Fourier transform of (2.1) with respect to  $x_3$  to obtain

$$(2.2) \quad \begin{cases} \lambda\hat{\phi} + \gamma\nabla' \cdot \hat{v}' + i\gamma\xi\hat{v}^3 = \hat{f}^0, \\ \lambda\hat{v}' - \nu\Delta'\hat{v}' + \nu\xi^2\hat{v}' - \tilde{\nu}\nabla'(\nabla' \cdot \hat{v}' + i\xi\hat{v}^3) + \gamma\nabla'\hat{\phi} = \hat{g}', \\ \lambda\hat{v}^3 - \nu\Delta'\hat{v}^3 + \nu\xi^2\hat{v}^3 - i\tilde{\nu}\xi(\nabla' \cdot \hat{v}' + i\xi\hat{v}^3) + i\gamma\xi\hat{\phi} = \hat{g}^3, \\ \hat{v}|_{\partial D} = 0, \end{cases}$$

which is written in the form

$$(2.3) \quad (\lambda + \hat{L}_\xi)\hat{u} = \hat{f}, \quad \hat{v}|_{\partial D} = 0.$$

Here  $\xi \in \mathbf{R}$  denotes the dual variable; the unknown  $\hat{u} = {}^T(\hat{\phi}, \hat{v}', \hat{v}^3)$  is a function on  $D$  with values in  $\mathbf{C}$ ; and

$$\hat{L}_\xi = \begin{pmatrix} 0 & \gamma^T \nabla' & i\gamma\xi \\ \gamma\nabla' & -\nu\Delta' + \nu\xi^2 - \tilde{\nu}\nabla'^T \nabla' & -i\tilde{\nu}\xi\nabla' \\ i\gamma\xi & -i\tilde{\nu}\xi^T \nabla' & -\nu\Delta' + (\nu + \tilde{\nu})\xi^2 \end{pmatrix}.$$

Problem (2.3) will be investigated according to the cases

$$(i) \ |\xi| \ll 1, \quad (ii) \ |\xi| \gg 1, \quad (iii) \ r \leq |\xi| \leq M$$

with constants  $0 < r < M < \infty$ . We will study the cases (ii) and (iii) in sections 3 and 4, respectively. The case (i) will be studied in section 5. Based on the analysis of problem (2.3), we will prove Theorem 2.1 in section 6.

### 3. Resolvent problem for high frequency part

In this section we establish estimates on  $(\lambda + \hat{L}_\xi)^{-1}$  for  $|\xi| \gg 1$ .

Let  $M > 0$  and set  $\widehat{u}_M = \kappa_M(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}$ . Here  $\kappa_M$  is a  $C^\infty$  function on  $\mathbf{R}$  satisfying

$$0 \leq \kappa_M \leq 1, \quad \kappa_M(\xi) = \begin{cases} 1 & (|\xi| > M), \\ 0 & (|\xi| < \frac{M}{2}). \end{cases}$$

We will show the following estimate.

**Theorem 3.1.** *Let  $1 < p < \infty$  and let  $\kappa_M$  be a function defined as above. Then there exist  $M_0 > 0$ ,  $c_\infty > 0$  and  $\theta_\infty \in (\frac{\pi}{2}, \pi)$  such that if  $M \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ , then  $\mathcal{F}^{-1}[\kappa_M(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}]$  satisfies the estimate*

$$\|\mathcal{F}^{-1}[\kappa_M(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}]\|_{W^{1,p} \times W^{2,p}} \leq C\|f\|_{W^{1,p} \times L^p}$$

uniformly in  $M \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ .

Theorem 3.1 is proved by establishing interior and boundary estimates. We here give a proof of the estimate near the boundary only, since the interior estimate can be proved similarly.

We see from (2.2) that  $\widehat{u}_M = \kappa_M(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}$  satisfies

$$(3.1) \quad \begin{cases} \lambda\widehat{\phi}_M + \gamma\nabla' \cdot \widehat{v}'_M + i\gamma\xi\widehat{v}_M^3 = \kappa_M\widehat{f}^0, \\ \lambda\widehat{v}'_M - \nu\Delta'\widehat{v}'_M + \nu\xi^2\widehat{v}'_M - \tilde{\nu}\nabla'(\nabla' \cdot \widehat{v}'_M + i\xi\widehat{v}_M^3) + \gamma\nabla'\widehat{\phi}_M = \kappa_M\widehat{g}', \\ \lambda\widehat{v}_M^3 - \nu\Delta'\widehat{v}_M^3 + \nu\xi^2\widehat{v}_M^3 - i\tilde{\nu}\xi(\nabla' \cdot \widehat{v}'_M + i\xi\widehat{v}_M^3) + i\gamma\xi\widehat{\phi}_M = \kappa_M\widehat{g}^3, \\ \widehat{v}_M|_{\partial D} = 0. \end{cases}$$

We take a point  $\bar{x}' = (\bar{x}_1, \bar{x}_2) \in \partial D$  and an open neighborhood  $\mathcal{O} \subset \mathbf{R}^2$  of  $\bar{x}'$ . Let  $\chi \in C_0^\infty(\mathcal{O})$ . Then, by (3.1), we have

$$(3.2) \quad \begin{cases} \lambda(\chi\widehat{\phi}_M) + \gamma\nabla' \cdot (\chi\widehat{v}'_M) + i\gamma\xi(\chi\widehat{v}_M^3) = F_M, \\ \lambda(\chi\widehat{v}'_M) + \nu(\xi^2 - \Delta')(\chi\widehat{v}'_M) \\ \quad - \tilde{\nu}\nabla'\{\nabla' \cdot (\chi\widehat{v}'_M) + i\xi(\chi\widehat{v}_M^3)\} + \gamma\nabla'(\chi\widehat{\phi}) = G'_M, \\ \lambda(\chi\widehat{v}_M^3) + \nu(\xi^2 - \Delta')(\chi\widehat{v}_M^3) \\ \quad - i\tilde{\nu}\xi\{\nabla' \cdot (\chi\widehat{v}'_M) + i\xi(\chi\widehat{v}_M^3)\} + i\gamma\xi(\chi\widehat{\phi}_M) = G_M^3, \\ (\chi\widehat{v}_M)|_{\partial D \cap \mathcal{O}} = 0. \end{cases}$$

Here

$$\left\{ \begin{array}{l} F_M = \chi \kappa_M \widehat{f}^0 + (\nabla' \chi) \cdot \widehat{v}'_M, \\ G'_M = \chi \kappa_M \widehat{g}' - \nu(\Delta' \chi) \widehat{v}'_M - 2\nu \nabla' \chi \cdot \nabla' \widehat{v}'_M - \tilde{\nu} \nabla' (\nabla' \chi \cdot \widehat{v}'_M) \\ \quad - \tilde{\nu} \nabla' \chi (\nabla' \cdot \widehat{v}'_M) - i\xi \tilde{\nu} (\nabla' \chi) \widehat{v}_M^3 + \gamma (\nabla' \chi) \widehat{\phi}_M, \\ G_M^3 = \chi \kappa_M \widehat{g}_M^3 - \nu(\Delta' \chi) \widehat{v}_M^3 - 2\nu \nabla' \chi \cdot \nabla' \widehat{v}_M^3 - i\tilde{\nu} \xi (\nabla' \chi) \cdot \widehat{v}'_M. \end{array} \right.$$

For any  $\eta > 0$ , if the diameter of  $\mathcal{O}$  is sufficiently small, then one can find a function  $h$  with the following properties (i)–(iii).

- (i)  $h \in C^\infty(\mathbf{R})$ ,  $\bar{x}_1 = h(\bar{x}_2)$ ,  $h'(\bar{x}_2) = 0$ .
- (ii)  $D \cap \mathcal{O} \subset \{x' = (x_1, x_2); x_1 > h(x_2)\}$ ,  $\partial D \cap \mathcal{O} \subset \{x' = (x_1, x_2); x_1 = h(x_2)\}$ .
- (iii) There are an open neighborhood  $\tilde{\mathcal{O}}$  of the origin of  $\mathbf{R}^2$  and a diffeomorphism  $\omega = {}^T(\omega_1, \omega_2)$  from  $\mathcal{O}$  to  $\tilde{\mathcal{O}}$  such that

$$\left\{ \begin{array}{l} y' = \omega(x') = \begin{pmatrix} x_1 - h(x_2) \\ x_2 - \bar{x}_2 \end{pmatrix} = \begin{pmatrix} \omega_1(x') \\ \omega_2(x') \end{pmatrix}, \quad \omega(\bar{x}') = 0, \\ \omega(D \cap \mathcal{O}) \subset \{y = (y_1, y_2); y_1 > 0\}, \\ \omega(\partial D \cap \mathcal{O}) \subset \{y = (y_1, y_2); y_1 = 0\}, \\ x' = \omega^{-1}(y') = \begin{pmatrix} y_1 + h(y_2 + \bar{x}_2) \\ y_2 + \bar{x}_2 \end{pmatrix}, \\ \sup_{y_2} |h'(y_2 + \bar{x}_2)| < \eta. \end{array} \right.$$

Using the map  $\omega$ , we define  $V_M(y', \xi)$ ,  $\Phi_M(y', \xi)$ ,  $\tilde{F}_M(y', \xi)$ ,  $\tilde{G}'_M(y', \xi)$  and  $\tilde{G}_M^3(y', \xi)$  by

$$\begin{aligned} V_M(y', \xi) &= \chi \widehat{v}_M(\omega^{-1}(y'), \xi), \quad \Phi_M(y', \xi) = \chi \widehat{\phi}_M(\omega^{-1}(y'), \xi), \\ \tilde{F}_M(y', \xi) &= F_M(\omega^{-1}(y'), \xi), \quad \tilde{G}'_M(y', \xi) = G'_M(\omega^{-1}(y'), \xi), \\ \tilde{G}_M^3(y', \xi) &= G_M^3(\omega^{-1}(y'), \xi). \end{aligned}$$

Problem (3.2) is then transformed into the following one on the half space  $\mathbf{R}_+^2 = \{y = (y_1, y_2) \in \mathbf{R}^2; y_1 > 0, y_2 \in \mathbf{R}\}$ :

$$(3.3) \quad \begin{cases} \lambda \Phi_M + \gamma \nabla' \cdot V'_M + i\gamma \xi V_M^3 = \tilde{R}_M^0, \\ \lambda V'_M + \nu(\xi^2 - \Delta') V'_M - \tilde{\nu} \nabla'(\nabla' \cdot V'_M + i\xi V_M^3) + \gamma \nabla' \Phi_M = \tilde{R}'_M, \\ \lambda V_M^3 + \nu(\xi^2 - \Delta') V_M^3 - i\tilde{\nu} \xi(\nabla' \cdot V'_M + i\xi V_M^3) + i\gamma \xi \Phi_M = \tilde{R}_M^3, \\ V_M|_{y_1=0} = 0. \end{cases}$$

Here  $V_M = {}^T(V'_M, V_M^3)$ ,  $V'_M = {}^T(V_M^1, V_M^2)$ ,

$$\tilde{R}_M^0 = \tilde{F}_M + \gamma h' \partial_{y_1} V_M^2,$$

$$\tilde{R}'_M = {}^T(\tilde{R}_M^1, \tilde{R}_M^2),$$

$$\tilde{R}_M^1 = \tilde{G}_M^1 - \nu \{h'' \partial_{y_1} V_M^1 + 2h' \partial_{y_1 y_2} V_M^1 - (h')^2 \partial_{y_1}^2 V_M^1\} - \tilde{\nu} h' \partial_{y_1}^2 V_M^2,$$

$$\begin{aligned} \tilde{R}_M^2 = & \tilde{G}_M^2 - \nu \{h'' \partial_{y_1} V_M^2 + 2h' \partial_{y_1 y_2} V_M^2 - (h')^2 \partial_{y_1}^2 V_M^2\} \\ & - \tilde{\nu} \{h' \partial_{y_1}^2 V_M^1 + h'' \partial_{y_1} V_M^2 + 2h' \partial_{y_1 y_2} V_M^2 - (h')^2 \partial_{y_1}^2 V_M^2 + i\xi h' \partial_{y_1} V_M^3\} \\ & + \gamma h' \partial_{y_1} \Phi_M, \end{aligned}$$

$$\tilde{R}_M^3 = \tilde{G}_M^3 - \nu \{h'' \partial_{y_1} V_M^3 + 2h' \partial_{y_1 y_2} V_M^3 - (h')^2 \partial_{y_1}^2 V_M^3\} - i\tilde{\nu} \xi h' \partial_{y_1} V_M^2.$$

As for problem (3.3), we have the following estimates. In what follows we will write  $y$  for  $(y_1, y_2, x_3) \in \mathbf{R}_+^3$ .

**Proposition 3.2.** *Let  $1 < p < \infty$  and let  $M_0 > 0$  be given. Then there exists a number  $\delta > 0$  such that if  $\text{diam}(\mathcal{O}) \leq \delta$ ,  $M \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ , then the solution  $U_M = {}^T(\Phi_M, V'_M, V_M^3)$  of (3.3) satisfies the following estimates with  $C = C(\chi, \mathcal{O}) > 0$  uniformly for  $M \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ :*

$$(i) \quad \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \leq \frac{C}{M} \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \kappa_M f]\|_{W^{1,p} \times L^p} + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \hat{u}_M]\|_{L^p \times W^{1,p}} \},$$

$$\begin{aligned} & \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} \Phi_M\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \\ (ii) \quad & \leq C \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \kappa_M \hat{f}^0]\|_{W^{1,p}} + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \hat{v}_M]\|_p \\ & + \frac{1}{M} (\|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \kappa_M \hat{g}]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \hat{u}_M]\|_{L^p \times W^{1,p}}) \}, \end{aligned}$$

$$\begin{aligned}
& \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} \Phi_M\|_p + \|\partial_y^2 \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \\
(iii) \quad & \leq C \{ \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \kappa_M \widehat{f}^0]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \kappa_M \widehat{g}]\|_p \\
& + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\chi \widehat{u}_M]\|_{L^p \times W^{1,p}} \}.
\end{aligned}$$

To prove Proposition 3.2, we consider the Fourier transform of (3.3) in  $y_2$  variable. In what follows we will write  $\tilde{\zeta}$  for  ${}^T(\zeta_2, \xi) \in \mathbf{R}^2$ . Then the Fourier transform of (3.3) in  $y_2$  gives

$$(3.4) \quad \begin{cases} (\lambda + \widehat{A}_{\tilde{\zeta}}) \mathcal{F}_{y_2 \rightarrow \zeta_2} U_M = \mathcal{F}_{y_2 \rightarrow \zeta_2} \tilde{R}_M & (y_1 > 0), \\ \mathcal{F}_{y_2 \rightarrow \zeta_2} V_M|_{y_1=0} = 0. \end{cases}$$

Here  $U_M = {}^T(\Phi_M(y_1, y_2, \xi), V_M(y_1, y_2, \xi))$  with  $V'_M = {}^T(V'_M(y_1, y_2, \xi), V_M^3(y_1, y_2, \xi))$  is the solution of (3.3);  $\tilde{R}_M = {}^T(\tilde{R}_M^0(y_1, y_2, \xi), \tilde{R}'_M(y_1, y_2, \xi), \tilde{R}_M^3(y_1, y_2, \xi))$ ; and

$$\widehat{A}_{\tilde{\zeta}} = \begin{pmatrix} 0 & \gamma \partial_{y_1} & i\gamma {}^T \tilde{\zeta} \\ \gamma \partial_{y_1} & \nu(|\tilde{\zeta}|^2 - \partial_{y_1}^2) - \tilde{\nu} \partial_{y_1}^2 & -i\tilde{\nu} {}^T \tilde{\zeta} \partial_{y_1} \\ i\gamma \tilde{\zeta} & -i\tilde{\nu} \tilde{\zeta} \partial_{y_1} & \nu(|\tilde{\zeta}|^2 - \partial_{y_1}^2) I_2 + \tilde{\nu} \tilde{\zeta} {}^T \tilde{\zeta} \end{pmatrix},$$

where  $I_2$  is the  $2 \times 2$  identity matrix.

As for problem (3.4) we make use of some results by [9]. For a given  $f \in C_0^\infty(\overline{\mathbf{R}_+^3}) \times C_0^\infty(\mathbf{R}_+^3)$  let us consider the problem

$$(3.5) \quad \begin{cases} (\lambda + \widehat{A}_{\tilde{\zeta}}) u = \mathcal{F}_{y \rightarrow \tilde{\zeta}} f & (y_1 > 0), \\ v|_{y_1=0} = 0 \end{cases}$$

for the unknown  $u = {}^T(\phi, v)$ . To investigate problem (3.5) we introduce some quantities. We set

$$\begin{aligned}
\lambda_1 &= -\nu |\tilde{\zeta}|^2, \\
\lambda_\pm &= -\frac{\nu_1}{2} |\tilde{\zeta}|^2 \pm \frac{1}{2} \sqrt{\nu_1^2 |\tilde{\zeta}|^4 - 4\gamma^2 |\tilde{\zeta}|^2}
\end{aligned}$$

and

$$\tilde{\lambda}_\pm = -\frac{\tilde{\nu}_1}{2} |\tilde{\zeta}|^2 \pm \frac{1}{2} \sqrt{\tilde{\nu}_1^2 |\tilde{\zeta}|^4 - 4\gamma^2 |\tilde{\zeta}|^2},$$

where  $\nu_1 = \nu + \tilde{\nu}$  and  $\tilde{\nu}_1 = 2\nu + \tilde{\nu}$ . It was shown in [8, 9] that if  $\tilde{\zeta} \neq 0$  and  $\lambda \notin \{\lambda_1, \lambda_\pm, \tilde{\lambda}_\pm, -\gamma^2/\nu_1\}$ , then (3.5) has a unique solution  $u$ . We denote the solution operator for (3.5) by  $\widehat{S}(\lambda, \tilde{\zeta})$ . Then for the solution  $U_M = {}^T(\Phi_M, V_M)$  of (3.3) we have

$$\mathcal{F}_{y_2 \rightarrow \zeta_2} U_M = \widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{y_2 \rightarrow \zeta_2} \tilde{R}_M,$$

and, therefore,

$$\begin{aligned} \mathcal{F}_{\xi \rightarrow x_3}^{-1} U_M &= \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{y_2 \rightarrow \zeta_2} \tilde{R}_M \\ &= \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} [\mathcal{F}_{\xi \rightarrow x_3} \tilde{R}_M]. \end{aligned}$$

As for  $\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}$ , we have the following estimates.

**Lemma 3.3.** *For any  $M_0 > 0$  there exist  $c_\infty > 0$  and  $\theta_0 \in (\frac{\pi}{2}, \pi)$  such that if  $\text{supp}(\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f) \subset \{|\xi| \geq M/2\}$  with  $M \geq 2M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ , then there hold the following estimates uniformly for  $M \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ :*

$$\begin{aligned} (3.6) \quad & \|\partial_y^\alpha Q_0 \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} [\widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]\|_p \\ & \leq C \{ \|\partial_y^\alpha f^0\|_p + \frac{1}{M^{1-|\alpha|}} \|g\|_p \} \quad (|\alpha| = 0, 1), \end{aligned}$$

$$\begin{aligned} (3.7) \quad & \|\partial_y^\alpha \tilde{Q} \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} [\widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]\|_p \\ & \leq C \left\{ \frac{1}{M^{(1-|\alpha|)_+}} \|\partial_y^{(|\alpha|-1)_+} f^0\|_p + \frac{1}{M^{2-|\alpha|}} \|g\|_p \right\} \quad (|\alpha| = 0, 1, 2). \end{aligned}$$

Lemma 3.3 will be proved in a similar argument to that given in [5, Sections 4 and 5], but we here need to pay attention to the dependence on  $M$ . The spectral bound,  $\sup \text{Re } \sigma(\widehat{A}_{\tilde{\zeta}})$ , of  $\widehat{A}_{\tilde{\zeta}}$  satisfies  $\sup \text{Re } \sigma(\widehat{A}_{\tilde{\zeta}}) = O(1) < 0$  as  $|\tilde{\zeta}| \rightarrow \infty$ , and so we in general have  $\|\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} \kappa_M\| = O(1)$  as  $M \rightarrow \infty$ , but we can gain a factor  $M^{-1}$  as in (3.6) and (3.7) which work well to obtain the desired estimate of Theorem 3.1.

To prove Lemma 3.3, we will make use of an integral representation of the solution  $u = \widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f$  of (3.5) given by [9].

We introduce the characteristic roots of the ordinary differential system  $(\lambda + \widehat{A}_{\tilde{\zeta}})u = 0$ , which are given by  $\pm\mu_j(\lambda, \tilde{\zeta})$ ,  $j = 1, 2$ , where

$$\mu_1 = \mu_1(\lambda, |\tilde{\zeta}|^2) = \sqrt{\frac{\lambda + \nu|\tilde{\zeta}|^2}{\nu}},$$

$$\mu_2 = \mu_2(\lambda, |\tilde{\zeta}|^2) = \sqrt{\frac{\lambda^2 + \nu_1|\tilde{\zeta}|^2\lambda + \gamma^2|\tilde{\zeta}|^2}{\nu_1\lambda + \gamma^2}}, \quad \nu_1 = \nu + \tilde{\nu}.$$

We next introduce the Green functions  $g_{\mu_j}^{(+)}(y_1, z_1)$  and  $g_{\mu_j}^{(-)}(y_1, z_1)$  of the equation  $\mu_j^2 w - \partial_{y_1}^2 w = f$  under the Neumann boundary condition and the Dirichlet one at  $y_1 = 0$  respectively. We define  $g_{\mu_j}^{(\pm)}(y_1, z_1)$  by

$$g_{\mu_j}^{(\pm)}(y_1, z_1) = \frac{1}{2\mu_j}(e^{-\mu_j|y_1 - z_1|} \pm e^{-\mu_j(y_1 + z_1)}) \quad (j = 1, 2).$$

We set

$$g_{\mu_1, \mu_2}^{(\pm)}(y_1, z_1) = g_{\mu_1}^{(\pm)}(y_1, z_1) - g_{\mu_2}^{(\pm)}(y_1, z_1).$$

We also define functions  $h_{\mu_j}(y_1)$ ,  $h_{\mu_1, \mu_2}(y_1)$ ,  $\beta_0(z_1)$ ,  $\beta(z_1)$  and  $\mathbf{b}(z_1)$  by

$$h_{\mu_j}(y_1) = \frac{1}{\mu_j}e^{-\mu_j y_1} \quad (j = 1, 2), \quad h_{\mu_1, \mu_2}(y_1) = h_{\mu_1}(y_1) - h_{\mu_2}(y_1),$$

$$\beta_0(z_1) = \frac{\gamma\lambda}{d(\lambda)}e^{-\mu_2 z_1}, \quad \beta(z_1) = |\tilde{\zeta}|^2(e^{-\mu_1 z_1} - e^{-\mu_2 z_1}),$$

$$\mathbf{b}(z_1) = i\tilde{\zeta}\mu_2(e^{-\mu_1 z_1} - e^{-\mu_2 z_1}).$$

Using the functions defined above, we have an integral representation of the solution  $u = \widehat{S}(\lambda, \tilde{\zeta})f$  of (3.5).

**Lemma 3.4.** *If  $\tilde{\zeta} \neq 0$  and  $\lambda \notin \{\lambda_1, \lambda_{\pm}, \tilde{\lambda}_{\pm}, -\gamma^2/\nu_1\}$ , then the solution  $u = \widehat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}f$  of (3.5) is represented as*

$$\widehat{S}(\lambda, \tilde{\zeta})f = \widehat{G}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}f + \widehat{H}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}f,$$

where  $\widehat{G}(\lambda, \tilde{\zeta})$  and  $\widehat{H}(\lambda, \tilde{\zeta})$  are the integral operators given by

$$(\widehat{G}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}f)(y_1) = \int_0^\infty \widehat{G}(\lambda, \tilde{\zeta}, y_1, z_1)\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}f(z_1) dz_1$$

and

$$(\widehat{H}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(y_1) = \int_0^\infty \widehat{H}(\lambda, \tilde{\zeta}, y_1, z_1) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f(z_1) dz_1.$$

Here  $\widehat{G}(\lambda, \tilde{\zeta}, y_1, z_1)$  is a  $4 \times 4$  matrix of the form

$$\widehat{G}(\lambda, \tilde{\zeta}, y_1, z_1) = \frac{\nu_1}{d(\lambda)} \delta(y_1 - z_1) Q_0 + \widehat{G}_1(\lambda, \tilde{\zeta}, y_1, z_1) + \widehat{G}_2(\lambda, \tilde{\zeta}, y_1, z_1),$$

where  $\delta(y_1)$  denotes the Dirac delta function;

$$\widehat{G}_1 = \frac{1}{d(\lambda)} \begin{pmatrix} \frac{\gamma\lambda}{d(\lambda)} g_{\mu_2}^{(+)}(y_1, z_1) & -\partial_{y_1} g_{\mu_2}^{(-)}(y_1, z_1) & -i^T \tilde{\zeta} g_{\mu_2}^{(-)}(y_1, z_1) \\ -\partial_{y_1} g_{\mu_2}^{(+)}(y_1, z_1) & 0 & 0 \\ -i\tilde{\zeta} g_{\mu_2}^{(+)}(y_1, z_1) & 0 & 0 \end{pmatrix};$$

$$\widehat{G}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\nu} g_{\mu_1}^{(-)}(y_1, z_1) - \frac{1}{\lambda} \partial_{y_1}^2 g_{\mu_1, \mu_2}^{(-)}(y_1, z_1) & -\frac{i^T \tilde{\zeta}}{\lambda} \partial_{y_1} g_{\mu_1, \mu_2}^{(-)}(y_1, z_1) \\ 0 & -\frac{i\tilde{\zeta}}{\lambda} \partial_{y_1} g_{\mu_1, \mu_2}^{(-)}(y_1, z_1) & \frac{1}{\nu} g_{\mu_1}^{(-)}(y_1, z_1) I_2 + \frac{\tilde{\zeta}^T \tilde{\zeta}}{\lambda} g_{\mu_1, \mu_2}^{(-)}(y_1, z_1) \end{pmatrix}$$

with  $d(\lambda) = \nu_1 \lambda + \gamma^2$ ; and  $\widehat{H}(\lambda, \tilde{\zeta}, y_1, z_1)$  is a  $4 \times 4$  matrix of the form

$$\widehat{H}(y_1, z_1)$$

$$= \frac{1}{\mu_1 \mu_2 - |\tilde{\zeta}|^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{i\tilde{\zeta}}{\nu} h_{\mu_1}(y_1) \beta_0(z_1) & \frac{1}{\nu} h_{\mu_1}(y_1) \mathbf{b}_0(z_1) & -\frac{1}{\nu} h_{\mu_1}(y_1) \frac{\tilde{\zeta}^T \tilde{\zeta}}{|\tilde{\zeta}|^2} \beta(z_1) \end{pmatrix}$$

$$+ \frac{1}{\mu_1 \mu_2 - |\tilde{\zeta}|^2} \begin{pmatrix} -\frac{\gamma |\tilde{\zeta}|^2}{d(\lambda)} h_{\mu_2}(y_1) \beta_0(z_1) & -\frac{i\gamma \tilde{\zeta}}{d(\lambda)} h_{\mu_2}(y_1) \mathbf{b}(z_1) & \frac{i\gamma^T \tilde{\zeta}}{d(\lambda)} h_{\mu_2}(y_1) \beta(z_1) \\ \frac{|\tilde{\zeta}|^2}{\lambda} \partial_{y_1} h_{\mu_1, \mu_2}(y_1) \beta_0(z_1) & -\frac{i\tilde{\zeta}}{\lambda} \partial_{y_1} h_{\mu_1, \mu_2}(y_1) \mathbf{b}(z_1) & \frac{i^T \tilde{\zeta}}{\lambda} \partial_{y_1} h_{\mu_1, \mu_2}(y_1) \beta(z_1) \\ \frac{i\tilde{\zeta} |\tilde{\zeta}|^2}{\lambda} h_{\mu_1, \mu_2}(y_1) \beta_0(z_1) & \frac{|\tilde{\zeta}|^2}{\lambda} h_{\mu_1, \mu_2}(y_1) \mathbf{b}(z_1) & -\frac{\tilde{\zeta}^T \tilde{\zeta}}{\lambda} h_{\mu_1, \mu_2}(y_1) \beta(z_1) \end{pmatrix}.$$

The solution formula above is given in [9, Section 3]. (See also [8, Section 3 and Appendix] and [5, Theorem 3.8].)

**Remark 3.5.** (i) For  $g_{\mu_j}^{(\pm)}$  ( $j = 1, 2$ ), we have

$$\partial_{y_1}^2 (g_{\mu_j}^{(\pm)} f) = \mu_j^2 g_{\mu_j}^{(\pm)} f - f \quad (j = 1, 2), \quad \partial_{y_1}^2 (g_{\mu_1, \mu_2}^{(\pm)}) = \mu_1^2 g_{\mu_1}^{(\pm)} f - \mu_2^2 g_{\mu_2}^{(\pm)} f.$$

(ii) As for  $\mu_j$  ( $j = 1, 2$ ), an elementary observation shows that  $\mu_1 = \sqrt{\frac{\lambda - \lambda_1}{\nu}}$ ;  $\mu_2 = \sqrt{\frac{(\lambda - \lambda_+)(\lambda - \lambda_-)}{(\nu_1 \lambda + \gamma^2)}}$ ; and if  $|\tilde{\zeta}| < 2\gamma/\nu_1$ , then  $\lambda_- = \overline{\lambda_+}$  and  $\text{Im} \lambda_+ = \gamma|\tilde{\zeta}| \sqrt{1 - \frac{\nu_1^2}{4\gamma^2} |\tilde{\zeta}|^2}$ , while if  $|\tilde{\zeta}| > 2\gamma/\nu_1$ , then  $\lambda_{\pm} \in \mathbf{R}$ . We also have

$$\frac{1}{\mu_1 \mu_2 - |\tilde{\zeta}|^2} = \frac{\nu(\nu_1 \lambda + \gamma^2)(\mu_1 \mu_2 + |\tilde{\zeta}|^2)}{\lambda(\lambda - \tilde{\lambda}_+)(\lambda - \tilde{\lambda}_-)}.$$

Furthermore,

$$\lambda_{\pm} = -\frac{\nu_1}{2} |\tilde{\zeta}|^2 \pm i\gamma|\tilde{\zeta}| + O(|\tilde{\zeta}|^3) \quad \text{as } |\tilde{\zeta}| \rightarrow 0,$$

$$\lambda_+ = -\frac{\gamma^2}{\nu_1} + O(|\tilde{\zeta}|^{-2}), \quad \lambda_- = -\nu_1 |\tilde{\zeta}|^2 + O(1) \quad \text{as } |\tilde{\zeta}| \rightarrow \infty,$$

and similar asymptotics also hold for  $\tilde{\lambda}_{\pm}$ .

To estimate  $\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \hat{G}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}$  and  $\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \hat{H}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}$ , we prepare several lemmas. We proceed as in [5, Sections 4 and 5].

**Lemma 3.6 (Fourier Multiplier Theorem).** *Let  $1 < p < \infty$  and let  $s$  be an integer satisfying  $s \geq [k/2] + 1$ . Suppose that  $\Psi(\omega) \in C^s(\mathbf{R}^k - \{0\}) \cap L^\infty(\mathbf{R}^k)$  and that there exists a constant  $C_0 > 0$  such that*

$$|\omega|^{|\alpha|} |\partial_\omega^\alpha \Psi(\omega)| \leq C_0$$

*for all  $\omega \in \mathbf{R}^k - \{0\}$  and  $|\alpha| \leq s$ . Then the operator  $\mathcal{F}_{\omega \rightarrow w}^{-1} [\Psi(\omega) (\mathcal{F}_{w \rightarrow \omega} f)(\omega)]$  is extended to a bounded linear operator on  $L^p(\mathbf{R}^k)$  and there holds the estimate*

$$\|\mathcal{F}_{\omega \rightarrow w}^{-1} [\Psi(\omega) (\mathcal{F}_{w \rightarrow \omega} f)(\omega)]\|_{L^p(\mathbf{R}^k)} \leq C C_0 \|f\|_{L^p(\mathbf{R}^k)}.$$

See, e.g., [2] for the proof of Lemma 3.6.

An elementary observation yields the following lemma.

**Lemma 3.7.** *Let  $g_{\mu_j}^{(1)}(y_1, z_1) = \frac{1}{2\mu_j} e^{-\mu_j|y_1 - z_1|}$  with  $\mu_j = \mu_j(\lambda, |\tilde{\zeta}|^2)$  ( $j = 1, 2$ ). Then*

$$\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \int_0^\infty g_{\mu_j}^{(1)}(y_1, z_1, \tilde{\zeta}) (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(z_1, \tilde{\zeta}) dz_1 \right] = \mathcal{F}_{\zeta \rightarrow y}^{-1} \left[ \frac{1}{\mu_j^2 + \zeta_1^2} \mathcal{F}_{y \rightarrow \zeta}(Ef) \right].$$

Here  $y = (y_1, \tilde{y})$ ,  $\zeta = (\zeta_1, \tilde{\zeta})$  with  $\tilde{\zeta} = (\zeta_2, \xi)$  and

$$(Ef)(y) = \begin{cases} f(y) & (y_1 \geq 0), \\ 0 & (y_1 < 0). \end{cases}$$

The following lemma follows from the boundedness of the Hilbert transform. (See [1, Lemma 2.6].)

**Lemma 3.8.** *Let  $1 < p < \infty$  and set*

$$Tf(y_1) = \int_0^\infty \frac{1}{y_1 + z_1} f(z_1) dz_1, \quad y_1 \in (0, \infty), \quad f \in L^p(0, \infty).$$

*Then there exists a positive constant  $C = C(p) > 0$  such that*

$$\|Tf\|_{L^p(0, \infty)} \leq C \|f\|_{L^p(0, \infty)}.$$

By Remark 3.5 (ii) one can obtain the following estimates. (Cf., [5, Lemma 4.5].) In what follows we will denote

$$\sigma(\lambda, \tilde{\zeta}) = |\lambda| + |\tilde{\zeta}|^2.$$

**Lemma 3.9.** *Let  $M_0 > 0$ . Then there exist  $c_\infty > 0$  and  $\theta_\infty \in (\frac{\pi}{2}, \pi)$  such that if  $|\tilde{\zeta}| \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ , then for any multi-index  $\tilde{\alpha}$  the following estimates hold with some positive constants  $c = c(\tilde{\alpha})$  and  $C = C(\tilde{\alpha})$  uniformly for  $|\tilde{\zeta}| \geq M_0$  and  $\lambda \in \Sigma(-c_\infty, \theta_\infty)$ :*

$$(i) \quad |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \mu_j| \leq C(|\lambda| + |\tilde{\zeta}|^2)^{\frac{1}{2} - \frac{|\tilde{\alpha}|}{2}} \quad (j = 1, 2),$$

$$(ii) \quad |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \frac{1}{\mu_j}| \leq C(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} - \frac{|\tilde{\alpha}|}{2}},$$

- (iii)  $|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}(\mu_1 - \mu_2)| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} - \frac{|\tilde{\alpha}|}{2}},$
- (iv)  $|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}(\mu_1\mu_2 - |\tilde{\zeta}|^2)| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{|\tilde{\alpha}|}{2}},$
- (v)  $|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}(\mu_1\mu_2 - |\tilde{\zeta}|^2)^{-1}| \leq C\frac{1}{|\lambda|}(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{|\tilde{\alpha}|}{2}},$
- (vi)  $|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}e^{-\mu_j y_1}| \leq C(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{|\tilde{\alpha}|}{2}}e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}}y_1} \quad (j = 1, 2),$
- (vii)  $|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}(e^{-\mu_1 y_1} - e^{-\mu_2 y_1})| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-1 - \frac{|\tilde{\alpha}|}{2}}e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}}y_1},$

We are now in a position to prove Lemma 3.3.

**Proof of Lemma 3.3.** Let  $M_0 > 0$  and let  $M \geq 2M_0$ . Suppose that  $\text{supp}(\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f) \subset \{(y_1, \tilde{\zeta}), \tilde{\zeta} = (\zeta_2, \xi); |\xi| \geq M/2\}$ .

We first estimate the  $\widehat{G}(\lambda, \tilde{\zeta})$  part of  $\widehat{S}(\lambda, \tilde{\zeta})$ . We begin with the terms concerning  $g_{\mu_1}^{(-)}$ . We write

$$g_{\mu_1}^{(-)}[\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0](y_1, \tilde{\zeta}) = I_1 - I_2,$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\mu_1} \int_0^\infty e^{-\mu_1|y_1 - z_1|} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(z_1) dz_1, \\ I_2 &= \frac{1}{2\mu_1} \int_0^\infty e^{-\mu_1(y_1 + z_1)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(z_1) dz_1. \end{aligned}$$

As for  $I_1$ , by Lemma 3.7,

$$\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[I_1] = \mathcal{F}_{\zeta \rightarrow y}^{-1} \left[ \frac{1}{\mu_1^2 + \zeta_1^2} \mathcal{F}_{y \rightarrow \zeta}(E f^0) \right].$$

If  $|\xi| \geq M/2$ , then

$$\left| \partial_{\zeta}^{\alpha} \left[ \frac{1}{\mu_1^2 + \zeta_1^2} \right] \right| \leq C_{\alpha} \frac{1}{|\zeta|^2} |\zeta|^{-|\alpha|} \leq \frac{C_{\alpha}}{M^2} |\zeta|^{-|\alpha|}$$

for any  $\alpha$  ( $|\alpha| \geq 0$ ). It then follows from Lemma 3.6 that

$$\|\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[I_1]\|_p \leq \frac{C}{M^2} \|f^0\|_p.$$

Similarly one can obtain

$$\|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[I_1]\|_p \leq \frac{C}{M^{2-|\beta|}} \|f^0\|_p \quad (|\beta| = 1, 2).$$

We next consider  $I_2$ . Let  $|\tilde{\beta}| + \ell = 0, 1, 2$ . By Lemma 3.9, we have

$$\begin{aligned} \left| \partial_{\tilde{\zeta}}^{\tilde{\alpha}} \left[ \tilde{\zeta}^{\tilde{\beta}} \partial_{y_1}^\ell \frac{1}{2\mu_1} e^{-\mu_1(y_1+z_1)} \right] \right| &\leq C_{\tilde{\alpha}} (|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} + \frac{\ell}{2} + \frac{\tilde{\beta}}{2} - \frac{\tilde{\alpha}}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}}(y_1+z_1)} \\ &\leq C_{\tilde{\alpha}} (|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} + \frac{\ell+|\tilde{\beta}|}{2} - \frac{\tilde{\alpha}}{2}} \frac{1}{|\tilde{\zeta}|(y_1+z_1)} \\ &\leq \frac{C_\alpha}{M^{2-\ell-|\tilde{\beta}|}} \cdot \frac{|\tilde{\zeta}|^{-|\tilde{\alpha}|}}{y_1+z_1} \end{aligned}$$

for  $|\xi| \geq M/2$ . It then follows from Lemma 3.6 that

$$\left\| \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \tilde{\zeta}^{\tilde{\beta}} \partial_{y_1}^\ell \frac{1}{2\mu_1} e^{-\mu_1(y_1+z_1)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(z_1) \right] \right\|_{L_{\tilde{y}}^p(\mathbf{R}^2)} \leq \frac{C}{M^{2-\ell-|\tilde{\beta}|}} \cdot \frac{\|f^0(z_1, \tilde{y})\|_{L_{\tilde{y}}^p}}{y_1+z_1},$$

and, therefore, by Minkowski's inequality for integrals, we have

$$\begin{aligned} &\left\| \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \tilde{\zeta}^{\tilde{\beta}} \partial_{y_1}^\ell \frac{1}{2\mu_1} \int_0^\infty e^{-\mu_1(y_1+z_1)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(z_1) dz_1 \right] \right\|_p \\ &\leq \frac{C}{M^{2-\ell-|\tilde{\beta}|}} \left( \int_0^\infty \left( \int_0^\infty \frac{\|f^0\|_{L_{\tilde{y}}^p}}{y_1+z_1} dz_1 \right)^p dy_1 \right)^{\frac{1}{p}}. \end{aligned}$$

Using Lemma 3.8, we see that for  $|\beta| = |\tilde{\beta}| + \ell = 0, 1, 2$ ,

$$\|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[I_2]\|_p = \|\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[\tilde{\zeta}^{\tilde{\beta}} \partial_{y_1}^\ell I_2]\|_p \leq \frac{C}{M^{2-|\beta|}} \|f^0\|_p.$$

From the estimates for  $I_1$  and  $I_2$  obtained above, we conclude

$$\|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[g_{\mu_1}^{(-)}(\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(y_1)]\|_p \leq \frac{C}{M^{2-|\beta|}} \|f^0\|_p \quad (|\beta| = 0, 1, 2).$$

Also, since

$$\begin{aligned} \partial_{y_1}[g_{\mu_j}^{(+)} \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f] &= g_{\mu_j}^{(-)}[\partial_{y_1}[\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]] \quad (j = 1, 2), \\ \tilde{\zeta}[g_{\mu_j}^{(+)} \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f] &= g_{\mu_j}^{(+)} \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}[\partial_{\tilde{y}} f] \quad (j = 1, 2), \end{aligned}$$

one can similarly conclude for  $\partial_{y_1} g_{\mu_2}^{(+)}$  and  $i\tilde{\zeta} g_{\mu_2}^{(+)}$  that

$$\|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} [\partial_{y_1} g_{\mu_2}^{(+)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(y_1)]\|_p \leq \frac{C}{M^{(1-|\beta|)_+}} \|\partial_y^{(|\beta|-1)_+} f^0\|_p \quad (|\beta| = 0, 1, 2),$$

$$\|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} [i\tilde{\zeta} g_{\mu_2}^{(+)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0)(y_1)]\|_p \leq \frac{C}{M^{(1-|\beta|)_+}} \|\partial_y^{(|\beta|-1)_+} f^0\|_p \quad (|\beta| = 0, 1, 2).$$

It remains to estimate the terms concerning  $g_{\mu_1, \mu_2}^{(\pm)}$ . This will be complete if we show the estimates

$$(3.8) \quad \left\| \partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \frac{\tilde{\zeta}^{\tilde{a}}}{\lambda} \partial_{y_1}^b g_{\mu_1, \mu_2}^{(\pm)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(y_1) \right] \right\|_p \leq \frac{C}{M^{2-|\beta|}} \|f\|_p$$

for any  $\tilde{a}$  and  $b$  with  $|\tilde{a}| + b = 2$  and  $|\beta| = 0, 1, 2$ .

Let us prove (3.8). We write

$$\frac{\tilde{\zeta}^{\tilde{a}}}{\lambda} \partial_{y_1}^b g_{\mu_1, \mu_2}^{(\pm)} (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(y_1) = J_1 \pm J_2,$$

where

$$J_1 = \int_0^\infty \frac{\tilde{\zeta}^{\tilde{a}}}{\lambda} \partial_{y_1}^b \left( \frac{1}{2\mu_1} e^{-\mu_1|y_1-z_1|} - \frac{1}{2\mu_2} e^{-\mu_2|y_1-z_1|} \right) (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(z_1) dz_1,$$

$$J_2 = \int_0^\infty \frac{\tilde{\zeta}^{\tilde{a}}}{\lambda} \partial_{y_1}^b \left( \frac{1}{2\mu_1} e^{-\mu_1(y_1+z_1)} - \frac{1}{2\mu_2} e^{-\mu_2(y_1+z_1)} \right) (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(z_1) dz_1.$$

As for  $J_1$ , by Lemma 3.7,

$$\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} [J_1] = \mathcal{F}_{\tilde{\zeta} \rightarrow y}^{-1} \left[ \frac{\tilde{\zeta}^{\tilde{a}}}{\lambda} \partial_{y_1}^b N(\lambda, \zeta) \mathcal{F}_{y \rightarrow \zeta} (Ef) \right].$$

Here

$$N(\lambda, \zeta) = \frac{1}{\mu_1^2 + \zeta_1^2} - \frac{1}{\mu_2^2 + \zeta_1^2}$$

An elementary computation gives

$$N(\lambda, \zeta) = -\frac{\lambda(\tilde{\nu}\lambda + \gamma^2)}{(\lambda + \nu|\zeta|^2)(\lambda^2 + \nu_1|\zeta|^2\lambda + \gamma^2|\zeta|^2)}.$$

In view of Remark 3.5 (ii), one can see that

$$\left| \partial_{\zeta}^{\alpha} \left( \frac{\zeta^j N(\lambda, \zeta)}{\lambda} \right) \right| \leq \frac{C_{\alpha}}{M^2} |\zeta|^{-|\alpha|} \quad (|j| = |\tilde{a}| + b)$$

for  $|\xi| \geq M/2$ . Lemma 3.6 then implies that

$$\|\mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[J_1]\|_p \leq \frac{C}{M^2} \|f\|_p.$$

Similarly we have

$$\|\partial_y^{\beta} \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[J_1]\|_p \leq \frac{C}{M^{2-|\beta|}} \|f\|_p \quad (|\beta| = 1, 2).$$

We next consider  $J_2$ . We set

$$g_{\mu_1, \mu_2}^{(2)}(y_1, z_1) = \frac{1}{2\mu_1} e^{-\mu_1(y_1+z_1)} - \frac{1}{2\mu_2} e^{-\mu_2(y_1+z_1)}.$$

By Lemma 3.9, we have, for  $|\xi| \geq M/2$ ,

$$\begin{aligned} (3.9) \quad & \left| \partial_{\tilde{\zeta}}^{\tilde{\alpha}} \left[ \frac{\tilde{\zeta}^{\tilde{\beta}+\tilde{a}}}{\lambda} \partial_{y_1}^{\ell+b} \frac{1}{2\mu_1} (e^{-\mu_1(y_1+z_1)} - e^{-\mu_2(y_1+z_1)}) \right] \right| \\ & \leq C_{\tilde{\alpha}} (|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} + \frac{\ell+b}{2} + \frac{|\tilde{\beta}|+|\tilde{a}|}{2} - \frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}}(y_1+z_1)} \\ & \leq C_{\tilde{\alpha}} (|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} + \frac{\ell+b+|\tilde{\beta}|+|\tilde{a}|}{2} - \frac{|\tilde{\alpha}|}{2}} \frac{1}{|\tilde{\zeta}|(y_1+z_1)} \\ & \leq \frac{C_{\alpha}}{M^{2-\ell-b-|\tilde{\beta}|-|\tilde{a}|}} \cdot \frac{|\tilde{\zeta}|^{-|\tilde{\alpha}|}}{y_1+z_1} \end{aligned}$$

and

$$(3.10) \quad \left| \partial_{\tilde{\zeta}}^{\tilde{\alpha}} \left[ \frac{\tilde{\zeta}^{\tilde{\beta}+\tilde{a}}}{\lambda} \partial_{y_1}^{\ell+b} \left( \frac{1}{\mu_1} - \frac{1}{\mu_2} \right) e^{-\mu_2(y_1+z_1)} \right] \right| \leq \frac{C_{\alpha}}{M^{2-\ell-b-|\tilde{\beta}|-|\tilde{a}|}} \cdot \frac{|\tilde{\zeta}|^{-|\tilde{\alpha}|}}{y_1+z_1}.$$

Since

$$g_{\mu_1, \mu_2}^{(2)}(y_1, z_1) = \frac{1}{2\mu_1} (e^{-\mu_1(y_1+z_1)} - e^{-\mu_2(y_1+z_1)}) + \frac{1}{2} \left( \frac{1}{\mu_1} - \frac{1}{\mu_2} \right) e^{-\mu_2(y_1+z_1)},$$

we see from Lemma 3.6, (3.9) and (3.10) that

$$\left\| \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \frac{\tilde{\zeta}^{\tilde{\beta} + \tilde{a}}}{\lambda} \partial_{y_1}^{\ell + b} g_{\mu_1, \mu_2}^{(2)}(y_1, z_1) (\mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f)(z_1) \right] \right\|_{L_{\tilde{y}}^p(\mathbf{R}^2)} \leq \frac{C}{M^{2-\ell-b-|\tilde{\beta}|-|\tilde{a}|}} \cdot \frac{\|f\|_{L_{\tilde{y}}^p}}{y_1 + z_1}$$

for  $|\xi| \geq M/2$ . Therefore, by Lemma 3.8 and (3.9), we have, for  $|\beta| = |\tilde{\beta}| + \ell = 0, 1, 2$ ,

$$\left\| \partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} [J_2] \right\|_p = \left\| \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \tilde{\zeta}^{\tilde{\beta}} \partial_{y_1}^\ell J_2 \right] \right\|_p \leq \frac{C}{M^{2-|\beta|}} \|f\|_p.$$

Combining the estimates for  $J_1$  and  $J_2$  we obtain (3.8); and the desired estimates for the  $\hat{G}$ -part are obtained.

We next consider the  $\hat{H}(\lambda, \tilde{\zeta})$  part of  $\hat{S}(\lambda, \tilde{\zeta})$ . By Lemma 3.9, we have

$$(3.11) \quad \begin{cases} |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} h_{\mu_j}(y_1)| \leq C(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{1}{2} - \frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}} y_1}, \\ |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} h_{\mu_1, \mu_2}(y_1)| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{3}{2} - \frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}} y_1}, \\ |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \beta_0(z_1)| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}} z_1}, \\ |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \beta(z_1)| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}} z_1}, \\ |\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \mathbf{b}(z_1)| \leq C|\lambda|(|\lambda| + |\tilde{\zeta}|^2)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}} z_1}. \end{cases}$$

These inequalities yield the desired estimates for the  $\hat{H}$ -part. For example, let us consider the term  $\frac{1}{\mu_1 \mu_2 - |\tilde{\zeta}|^2} \cdot \frac{\tilde{\zeta}}{\nu} h_{\mu_1}(y_1) \beta_0(z_1)$ . By (3.11) and Lemma 3.9 (v), we have

$$\begin{aligned} \left| \partial_{\tilde{\zeta}}^{\tilde{\alpha}} \left[ \tilde{\zeta}^{\tilde{\beta}} \partial_{y_1}^\ell \frac{1}{\mu_1 \mu_2 - |\tilde{\zeta}|^2} \cdot \frac{\tilde{\zeta}}{\nu} h_{\mu_1}(y_1) \beta_0(z_1) \right] \right| &\leq C |\tilde{\zeta}|^{|\tilde{\beta}| + \ell - |\alpha|} e^{-c\sigma(\lambda, \tilde{\zeta})^{\frac{1}{2}} (y_1 + z_1)} \\ &\leq \frac{C}{M^{1-|\tilde{\beta}|-\ell}} \cdot \frac{1}{y_1 + z_1} |\tilde{\zeta}|^{-|\tilde{\alpha}|} \end{aligned}$$

for  $|\xi| \geq M/2$ . As in the estimates for  $I_2$  and  $J_2$  above, we see from Lemmas 3.6 and 3.8 that

$$\left\| \partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1} \left[ \frac{1}{\mu_1 \mu_2 - |\tilde{\zeta}|^2} \cdot \frac{\tilde{\zeta}}{\nu} h_{\mu_1}(y_1) \beta_0(z_1) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f^0 \right] \right\|_p \leq \frac{C}{M^{(1-|\beta|)_+}} \|\partial_y^{(|\beta|-1)_+} f^0\|_p$$

for  $|\beta| = 0, 1, 2$ . Similarly one can obtain

$$\begin{aligned} \|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[Q_0 \hat{H} Q_0 \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]\|_p &\leq C \|\partial_y^\beta f^0\|_p \quad (|\beta| = 0, 1), \\ \|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[\tilde{Q} \hat{H} Q_0 \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]\|_p &\leq \frac{C}{M^{(1-|\beta|)_+}} \|\partial_y^{(|\beta|-1)_+} f^0\|_p \quad (|\beta| = 0, 1, 2), \\ \|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[Q_0 \hat{H} \tilde{Q} \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]\|_p &\leq \frac{C}{M^{1-|\beta|}} \|g\|_p \quad (|\beta| = 0, 1), \\ \|\partial_y^\beta \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[\tilde{Q} \hat{H} \tilde{Q} \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}} f]\|_p &\leq \frac{C}{M^{2-|\beta|}} \|g\|_p \quad (|\beta| = 0, 1, 2). \end{aligned}$$

This completes the proof.  $\square$

We now prove Proposition 3.2.

**Proof of Proposition 3.2.** By Lemma 3.3, if  $\sup_{y_2} |h'(y_2)| \leq \eta$ , then  
(3.12)

$$\begin{aligned} &\|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} \Phi_M\|_p + \|\partial_y^2 \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \\ &\leq C \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_y Q_0 \tilde{R}_M]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{Q} \tilde{R}_M]\|_p \} \\ &\leq C \{ \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{F}_M]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'} V_M]\|_p + \eta \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'}^2 V_M]\|_p \\ &\quad + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{G}_M]\|_p + \eta \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'}^2 V_M]\|_p + \eta \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'} \Phi_M]\|_p \}. \end{aligned}$$

We now take  $\eta > 0$  in such a way that  $C\eta \leq \frac{1}{2}$  and then choose  $\delta > 0$  so small that  $\sup_{y_2} |h'(y_2)| \leq \eta$  whenever  $\text{diam}(\mathcal{O}) \leq \delta$ . It then follows from (3.12) that

$$\begin{aligned} &\|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} \Phi_M\|_p + \|\partial_y^2 \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \\ (3.13) \quad &\leq C \{ \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{F}_M]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{G}_M]\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \}. \end{aligned}$$

Similarly, by Lemma 3.3,

$$\begin{aligned} &\|\mathcal{F}_{\xi \rightarrow x_3}^{-1} \Phi_M\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \\ &\leq C \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [Q_0 \tilde{R}_M]\|_p + \frac{1}{M} \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{Q} \tilde{R}_M]\|_p \} \\ (3.14) \quad &\leq C \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{F}_M]\|_p + \eta \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'} V_M]\|_p \\ &\quad + \frac{1}{M} (\|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{G}_M]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'}^2 V_M]\|_p \\ &\quad + \eta \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'} V_M]\|_p + \eta \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\partial_{y'} \Phi_M]\|_p) \}. \end{aligned}$$

We see from (3.13) and (3.14) that

$$\begin{aligned}
& \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} \Phi_M\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \\
(3.15) \quad & \leq C \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{F}_M]\|_p \\
& \quad + \frac{1}{M} (\|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{G}_M]\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{F}_M]\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p) \}
\end{aligned}$$

by taking  $\eta$  and  $\delta$  smaller if necessary. It then follows from Lemma 3.3, (3.13) and (3.15) that

$$\begin{aligned}
(3.16) \quad & \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \leq \frac{C}{M} \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [Q_0 \tilde{R}_M]\|_p + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{Q} \tilde{R}_M]\|_p \} \\
& \leq \frac{C}{M} \{ \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{F}_M]\|_{W^{1,p}} + \|\mathcal{F}_{\xi \rightarrow x_3}^{-1} [\tilde{G}_M]\|_p + \|\partial_y \mathcal{F}_{\xi \rightarrow x_3}^{-1} V_M\|_p \}.
\end{aligned}$$

Proposition 3.2 now follows from (3.13), (3.15) and (3.16). This completes the proof.  $\square$

We finally prove Theorem 3.1.

**Proof of Theorem 3.1.** For each  $\bar{x}' \in \partial D$  we take  $\mathcal{O}_{\bar{x}'}$  so that the estimates in Proposition 3.2 hold with  $\mathcal{O}$  replaced by  $\mathcal{O}_{\bar{x}'}$ . Since  $D$  is bounded, one can find an open covering  $\{\mathcal{O}_j\}_{j=0}^k$  of  $D$  and  $\{\chi_j\}_{j=0}^k \subset C_0^\infty$  such that  $U_M^{(j)} = \chi_j \hat{u}_M$  ( $j = 1, \dots, k$ ) satisfy the estimates in Proposition 3.2 with  $\mathcal{O}$  replaced by  $\mathcal{O}_j$ . Here  $\{\mathcal{O}_j\}_{j=0}^k$  satisfies  $\overline{\mathcal{O}_0} \subset D$ ,  $\mathcal{O}_j = \mathcal{O}_{\bar{x}_j}$ , for some  $\bar{x}_j \in \partial D$  ( $j = 1, 2, \dots, k$ ), and  $D \subset \cup_{j=0}^k \mathcal{O}_j$ ; and  $\{\chi_j\}_{j=0}^k \subset C^\infty$  is a partition of unity subordinate to  $\{\mathcal{O}_{j=0}^k\}$ , i.e., there hold  $\chi_j \in C_0^\infty(\mathcal{O}_j)$  and  $\sum_{j=0}^k \chi_j \equiv 1$  on  $D$ . One can see that  $U_M^{(0)} = \chi_0 \hat{u}_M$  satisfies similar estimates to those in Proposition 3.2. Furthermore, the constants  $C$  appearing in the estimates for  $U_M^{(j)}$  ( $j = 0, \dots, k$ ) can be taken uniformly in  $j = 0, \dots, k$ .

Proposition 3.2 (i) then yields

$$\begin{aligned}
(3.17) \quad & \|\mathcal{F}^{-1} \hat{v}_M\|_p \leq \sum_{j=0}^k \|\mathcal{F}^{-1} V_M^{(j)}\|_p \\
& \leq \frac{C}{M} \sum_{j=0}^k \{ \|\mathcal{F}^{-1} [\chi_j \kappa_M \hat{f}]\|_{W^{1,p} \times L^p} + \|\mathcal{F}^{-1} [\chi_j \hat{u}_M]\|_{L^p \times W^{1,p}} \} \\
& \leq \frac{C}{M} \{ \|\mathcal{F}^{-1} [\kappa_M \hat{f}]\|_{W^{1,p} \times L^p} + \|\mathcal{F}^{-1} \hat{u}_M\|_{L^p \times W^{1,p}} \}.
\end{aligned}$$

By (3.17) and Proposition 3.2 (ii), we have

$$\begin{aligned}\|\mathcal{F}^{-1}\widehat{u}_M\|_{L^p \times W^{1,p}} &\leq \sum_{j=0}^k \|\mathcal{F}^{-1}U_M^{(j)}\|_{L^p \times W^{1,p}} \\ &\leq C\{\|\mathcal{F}^{-1}[\kappa_M \widehat{f}]\|_{W^{1,p} \times L^p} + \frac{1}{M}\|\mathcal{F}^{-1}\widehat{u}_M\|_{L^p \times W^{1,p}}\}.\end{aligned}$$

Therefore, if  $M > 0$  is taken so large, we obtain

$$(3.18) \quad \|\mathcal{F}^{-1}\widehat{u}_M\|_{L^p \times W^{1,p}} \leq C\|\mathcal{F}^{-1}[\kappa_M \widehat{f}]\|_{W^{1,p}}.$$

It then follows from (3.18) and Proposition 3.2 (iii) that

$$\begin{aligned}\|\mathcal{F}^{-1}\widehat{u}_M\|_{W^{1,p} \times W^{2,p}} &\leq \sum_{j=0}^k \|\mathcal{F}^{-1}U_M^{(j)}\|_{W^{1,p} \times W^{2,p}} \\ &\leq Ck\|\mathcal{F}^{-1}[\kappa_M \widehat{f}]\|_{W^{1,p} \times L^p}.\end{aligned}$$

This completes the proof.  $\square$

#### 4. Resolvent problem for the middle frequency part

Let  $M > 0$  and  $r > 0$ . In this section we establish estimates on  $(\lambda + \widehat{L}_\xi)^{-1}$  for  $r/2 \leq |\xi| \leq M$ .

We begin with estimating  $(\lambda + \widehat{L}_\xi)^{-1}$  for  $\lambda$  in compact sets. We first estimate  $\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f$  for  $f \in W^{1,p}(D) \times L^p(D)$ . Here  $\kappa_{r,M}$  is a function in  $C_0^\infty(\mathbf{R})$  satisfying

$$0 \leq \kappa_{r,M} \leq 1, \quad \kappa_{r,M}(\xi) = \begin{cases} 1 & (r \leq |\xi| \leq \frac{M}{2}) \\ 0 & (|\xi| < \frac{r}{2}, |\xi| > M). \end{cases}$$

Note that here  $f$  is a function of  $x' \in D$  and does not depend on  $\xi$ .

**Proposition 4.1.** *Let  $r$  and  $M$  be numbers satisfying  $0 < r < \frac{M}{2}$  and let  $\Lambda_1 > 0$ . Then there exist constants  $c_1 = c_1(r, M) > 0$  and  $\theta_1 = \theta_1(r, M) \in (\frac{\pi}{2}, \pi)$  such that if  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$ , then for any integer  $k \geq 0$  the function  $\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f$  satisfies the following estimate*

$$|\partial_\xi^k[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f]|_{W^{1,p} \times W^{2,p}} \leq C_k|f|_{W^{1,p} \times L^p}$$

with some constant  $C_k$  uniformly for  $\xi$  and  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$ .

**Proof.** Let  $\tilde{\zeta} = (\zeta_2, \xi) \in \mathbf{R}^2$  and let  $\widehat{S}(\lambda, \tilde{\zeta})$  be the solution operator for problem (3.5) introduced in section 3. We consider the following problem on  $\{y_1 > 0\}$

$$(4.1) \quad \begin{cases} (\lambda + \widehat{A}_{\tilde{\zeta}})w = F & (y_1 > 0), \\ \tilde{Q}w|_{y_1=0} = 0 \end{cases}$$

for the unknown  $w = w(y_1, \tilde{\zeta})$  and a given  $F = F(y_1, \zeta_2)$  with  $\tilde{\zeta} = (\zeta_2, \xi)$  regarded as a parameter. Note that  $F$  does not depend on  $\xi$ . In view of Lemma 3.4 and Remark 3.5 (ii), similarly to the proof of Lemma 3.3, we see that there exist  $c_1 = c_1(r, M) > 0$  and  $\theta_1 = \theta_1(r, M) \in (\frac{\pi}{2}, \pi)$  such that if  $\frac{r}{2} \leq |\xi| \leq M$  and  $\lambda \in \Sigma(-c_1, \theta_1)$ , then (4.1) has a unique solution  $w(y_1, \tilde{\zeta}) = \widehat{S}(\lambda, \tilde{\zeta})F(y_1)$ . Furthermore,  $\mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1}w = \mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1}\widehat{S}(\lambda, \tilde{\zeta})F$ , which is a function of  $y' = (y_1, y_2) \in \mathbf{R}_+^2$  with parameter  $\xi$ , satisfies the estimates

$$(4.2) \quad \begin{aligned} & \|\partial_{y'}^{\alpha'} Q_0 \mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1} [\widehat{S}(\lambda, \tilde{\zeta})F](y', \xi)\|_{L_{y'}^p(\mathbf{R}_+^2)} \\ & \leq C \{ \|\partial_{y'}^{\alpha'} \mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1} [Q_0 F]\|_{L_{y'}^p(\mathbf{R}_+^2)} + \|\mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1} \tilde{Q}F\|_{L_{y'}^p(\mathbf{R}_+^2)} \} \end{aligned}$$

for  $|\alpha'| = 0, 1$ , and,

$$(4.3) \quad \begin{aligned} & \|\partial_{y'}^{\alpha'} \tilde{Q} \mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1} [\widehat{S}(\lambda, \tilde{\zeta})F](y', \xi)\|_{L_{y'}^p(\mathbf{R}_+^2)} \\ & \leq C \{ \|\partial_{y'}^{(|\alpha|-1)} \mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1} [Q_0 F]\|_{L_{y'}^p(\mathbf{R}_+^2)} + \|\mathcal{F}_{\zeta_2 \rightarrow y_2}^{-1} \tilde{Q}F\|_{L_{y'}^p(\mathbf{R}_+^2)} \} \end{aligned}$$

for  $|\alpha'| = 0, 1, 2$ .

We write  $f$  and  $\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f$  as

$$f = {}^T(f^0, g), \quad \kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f = {}^T(\widehat{\phi}_{r,M}, \widehat{v}_{r,M}).$$

Based on (4.2) and (4.3), by using the localization argument as in the proof of Theorem 3.1, one can obtain the estimate

$$(4.4) \quad |\widehat{\phi}_{r,M}|_{W^{1,p}} + |\widehat{v}_{r,M}|_{W^{2,p}} \leq C \{ |f^0|_{W^{1,p}} + |g|_p + |\widehat{\phi}_{r,M}|_p + |\widehat{v}_{r,M}|_{W^{1,p}} \}.$$

Let us prove

$$(4.5) \quad |\widehat{\phi}_{r,M}|_p + |\widehat{v}_{r,M}|_{W^{1,p}} \leq C \{ |f^0|_{W^{1,p}} + |g|_p \}.$$

We will prove (4.5) by a contradiction. Assume that (4.5) does not hold. Then for any  $n \in \mathbf{N}$ , there are  $f_n = {}^T(f_n^0, g_n', g_n^3) \in W^{1,p}(D) \times L^p(D)^3$ ,  $\xi_n \in \mathbf{R}$ ,  $\lambda_n \in \mathbf{C}$  and  $\widehat{u}_n = {}^T(\widehat{\phi}_n, \widehat{v}_n) \in W^{1,p}(D) \times [W^{2,p}(D) \cap W_0^{1,p}(D)]$  satisfying the following (4.6)–(4.8):

$$(4.6) \quad \frac{r}{2} \leq |\xi_n| \leq M, \quad \lambda_n \in \Sigma(-c_1, \theta_1), \quad |\lambda_n| \leq \Lambda_1,$$

$$(4.7) \quad (\lambda + \widehat{L}_{\xi_n})\widehat{u}_n = \kappa_{r,M}(\xi_n)f_n,$$

$$(4.8) \quad |\widehat{\phi}_n|_p + |\widehat{v}_n|_{W^{1,p}} \geq n\{|f_n^0|_{W^{1,p}} + |g_n|_p\}.$$

We may assume that

$$(4.9) \quad |\widehat{\phi}_n|_p + |\widehat{v}_n|_{W^{1,p}} = 1.$$

By (4.3), we have

$$|\widehat{\phi}_n|_{W^{1,p}} + |\widehat{v}_n|_{W^{2,p}} \leq C \left( \frac{1}{n} + 1 \right) \leq 2C.$$

Therefore, we can find a subsequence of  $\{f_n, \xi_n, \lambda_n, \widehat{u}_n\}$ , which we again denote by  $\{f_n, \xi_n, \lambda_n, \widehat{u}_n\}$ , such that, as  $n \rightarrow \infty$ ,

$$\begin{aligned} f_n^0 &\rightarrow 0 \text{ in } W^{1,p}(D), \quad g_n \rightarrow 0 \text{ in } L^p(D), \\ \xi_n &\rightarrow \xi \quad \left( \frac{r}{2} \leq |\xi| \leq M \right), \quad \lambda_n \rightarrow \lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}, \\ \phi_n &\rightharpoonup \phi \text{ in } W^{1,p}(D), \quad \phi_n \rightarrow \phi \text{ in } L^p(D), \\ v_n &\rightharpoonup v \text{ in } W^{2,p}(D), \quad v_n \rightarrow v \text{ in } W_0^{1,p}(D). \end{aligned}$$

Letting  $n \rightarrow \infty$  in (4.7) and (4.9), we have

$$(\lambda + \widehat{L}_\xi)\widehat{u} = 0, \quad \widehat{u} \in W^{1,p}(D) \times [W^{2,p}(D) \cap W_0^{1,p}(D)], \quad |\widehat{u}|_{L^p \times W^{1,p}} = 1.$$

But, by Lemma 4.2 below, if  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$ , then  $\widehat{u} = 0$ , which contradicts  $|\widehat{u}|_{L^p \times W^{1,p}} = 1$ . Therefore, we have (4.5).

It now follows from (4.4) and (4.5) that

$$(4.10) \quad |\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f|_{W^{1,p} \times W^{2,p}} \leq C|f|_{W^{1,p} \times L^p}.$$

We next estimate  $\partial_\xi^k[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f]$ . We set

$$\widehat{u}_{(k)} = \partial_\xi^k[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}f] = {}^T(\widehat{\phi}_{(k)}, \widehat{v}_{(k)}, \widehat{v}_{(k)}^3).$$

Then  $\widehat{u}_{(k)}$  is a solution of the problem

$$\left\{ \begin{array}{l} \lambda \widehat{\phi}_{(k)} + \gamma \nabla' \cdot \widehat{v}_{(k)}' + i\gamma \xi \widehat{v}_{(k)}^3 = (\partial_\xi^k \kappa_1) f^0 + \sum_{j=0}^{k-1} \binom{k-j}{j} \partial_\xi^{k-1} (i\gamma \xi) \widehat{v}_{(j)}^3, \\ \lambda \widehat{v}_{(k)}' - \nu \Delta' \widehat{v}_{(k)}' + \nu \xi^2 \widehat{v}_{(k)}' - \tilde{\nu} \nabla' (\nabla' \cdot \widehat{v}_{(k)}' + i\xi \widehat{v}_{(k)}^3) + \gamma \nabla' \widehat{\phi}_{(k)} \\ \quad = (\partial_\xi^k \kappa_1) g' + \sum_{j=0}^{k-1} \binom{k-j}{j} \{ \partial_\xi^{k-1} (\nu \xi^2) \widehat{v}_{(j)}' - \partial_\xi^{k-1} (\tilde{\nu} i \xi) \nabla' \widehat{v}_{(j)}^3 \}, \\ \lambda \widehat{v}_{(k)}^3 - \nu \Delta' \widehat{v}_{(k)}^3 + \nu \xi^2 \widehat{v}_{(k)}^3 - i\tilde{\nu} \xi (\nabla' \cdot \widehat{v}_{(k)}' + i\xi \widehat{v}_{(k)}^3) + i\gamma \xi \widehat{\phi}_{(k)} \\ \quad = (\partial_\xi^k \kappa_1) g^3 + \sum_{j=0}^{k-1} \binom{k-j}{j} \{ \partial_\xi^{k-1} (\nu \xi^2) \widehat{v}_{(j)}^3 - \partial_\xi^{k-1} (\tilde{\nu} i \xi) \nabla' \cdot \widehat{v}_{(j)}' \\ \quad \quad + \partial_\xi^{k-1} (\tilde{\nu} \xi^2) \widehat{v}_{(j)}^3 + \partial_\xi^{k-1} (i\gamma \xi) \widehat{\phi}_{(j)} \}, \\ \widehat{v}_{(k)}|_{\partial D} = 0. \end{array} \right.$$

By (4.10), we have

$$|\widehat{\phi}_{(k)}|_{W^{1,p}} + |\widehat{v}_{(k)}|_{W^{2,p}} \leq C_k \{ |f^0|_{W^{1,p}} + |g|_p \} + \sum_{j=0}^{k-1} \{ |\widehat{\phi}_{(j)}|_p + |\widehat{v}_{(j)}|_{W^{1,p}} \}.$$

The desired estimate now follows by an induction argument. This completes the proof.  $\square$

**Lemma 4.2.** *Let  $1 < p < \infty$ . If  $u \in W^{1,p}(D) \cap [W^{2,p}(D) \times W_0^{1,p}(D)]$ ,  $(\lambda + \widehat{L}_\xi)u = 0$ ,  $\frac{r}{2} \leq |\xi| \leq M$  and  $\lambda \in \Sigma(-c_1, \theta_1)$ , then  $u = 0$ .*

To prove Lemma 4.2, we prepare some propositions.

**Proposition 4.3.** *Let  $k \in \mathbb{N}$ . If  $\frac{r}{2} \leq |\xi| \leq M$  and  $\lambda \in \Sigma(-c_1, \theta_1)$ , then for any  $f \in H^k(D) \times H^{k-1}(D)$  there exists a unique solution  $u \in H^k(D) \times [H^{k+1}(D) \cap H_0^1(D)]$  of  $(\lambda + \widehat{L}_\xi)u = f$  and  $u$  satisfies the estimate*

$$|u|_{H^k \times H^{k+1}} \leq C |f|_{H^k \times H^{k-1}}.$$

Proposition 4.3 for  $k = 1$  was proved in [10]. (See [10, Proposition 3.14].) The proof for  $k \geq 2$  is done in a similar line to that of [10, Proposition 3.14] by using the Matsumura-Nishida energy method [19]. We here omit the details.

**Remark 4.4.** Proposition 4.3 remains true for the adjoint problem  $(\lambda + \widehat{L}_\xi^*)u = f$ , where

$$\widehat{L}_\xi^* = \begin{pmatrix} 0 & -\gamma^T \nabla' & -i\gamma\xi \\ -\gamma \nabla' & -\nu \Delta' + \nu \xi^2 - \tilde{\nu} \nabla'^T \nabla' & -i\tilde{\nu} \xi \nabla' \\ -i\gamma\xi & -i\tilde{\nu} \xi^T \nabla' & -\nu \Delta' + (\nu + \tilde{\nu}) \xi^2 \end{pmatrix}.$$

**Proposition 4.5.** *Let  $2 \leq q < \infty$ . If  $\frac{r}{2} \leq |\xi| \leq M$  and  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$ , then for any  $f \in W^{1,q}(D) \times L^q(D)$  there exists a unique solution  $u^* \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$  of  $(\lambda + \widehat{L}_\xi^*)u^* = f$ .*

**Proof.** Let  $f \in C^\infty(\bar{D}) \times C_0^\infty(D)$ . Then, by Remark 4.4, there exists a unique solution  $u^*$  of  $(\lambda + \widehat{L}_\xi^*)u^* = f$ , which belongs to  $H^k(D) \times [H^{k+1}(D) \cap H_0^1(D)]$  for any  $k \in \mathbf{N}$ . By the Sobolev embedding theorem, we have  $u^* \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$ . Similarly to the proof of Proposition 4.1, we can obtain the estimate

$$(4.11) \quad |u^*|_{W^{1,q} \times W^{2,p}} \leq C|f|_{W^{1,q} \times L^q},$$

if we show that  $(\lambda + \widehat{L}_\xi^*)u = 0$  and  $u \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$  implies that  $u = 0$ . But, since  $q \geq 2$ , we have  $W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)] \subset H^1(D) \times [H^2(D) \cap H_0^1(D)]$ ; and, hence, by Remark 4.4,  $u = 0$ . We thus obtain (4.11).

We next assume that  $f \in W^{1,q}(D) \times L^q(D)$ . Then there exists  $\{f^{(n)}\}_{n=1}^\infty \subset C^\infty(\bar{D}) \times C_0^\infty(D)$  such that

$$f^{(n)} \rightarrow f \text{ in } W^{1,q}(D) \times L^q(D).$$

By the preceding argument, for each  $n$ , there exists  $u^{*(n)} \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$  such that

$$(\lambda + \widehat{L}_\xi^*)u^{*(n)} = f^{(n)}$$

and

$$|u^{*(n)} - u^{*(m)}|_{W^{1,q} \times W^{2,q}} \leq C|f^{(n)} - f^{(m)}|_{W^{1,q} \times L^q}.$$

Therefore,  $\{u^{*(n)}\}$  is a Cauchy sequence in  $W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$ , and we can find a function  $u^* \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$  such that

$$|u^{*(n)} - u^*|_{W^{1,q} \times W^{2,q}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Letting  $n \rightarrow \infty$  in  $(\lambda + \widehat{L}_\xi^*)u^{*(n)} = f^{(n)}$ , we obtain

$$(\lambda + \widehat{L}_\xi^*)u^* = f.$$

The uniqueness of  $u^*$  follows from Remark 4.4 since  $q \geq 2$ . This completes the proof.  $\square$

We now prove Lemma 4.2.

**Proof of Lemma 4.2.** It suffices to prove Lemma 4.2 for  $1 < p < 2$ . Let  $q \in (2, \infty)$  be the Hölder conjugate to  $p$ . Assume that  $u \in W^{1,p}(D) \cap [W^{2,p}(D) \times W_0^{1,p}(D)]$  satisfies  $(\lambda + \widehat{L}_\xi^*)u = 0$ . By Proposition 4.5, for any  $f \in C^\infty(\bar{D}) \times C_0^\infty(D)$ , there exists a unique solution  $u^* \in W^{1,q}(D) \cap (W^{2,q}(D) \times W_0^{1,q}(D))$  of  $(\bar{\lambda} + \widehat{L}_\xi^*)u^* = f$ . By integration by parts,

$$(u, f) = (u, (\bar{\lambda} + \widehat{L}_\xi^*)u^*) = ((\lambda + \widehat{L}_\xi)u, u^*) = 0,$$

which implies  $u = 0$ . This completes the proof.  $\square$

We now establish the estimate on  $\mathcal{F}^{-1}[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}]$  for  $f \in W^{1,p}(\Omega) \times L^p(\Omega)$ .

**Theorem 4.6.** *Let  $f \in W^{1,p}(\Omega) \times L^p(\Omega)$ . If  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$ , there holds the estimate*

$$\|\mathcal{F}^{-1}[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}]\|_{W^{1,p} \times W^{2,p}} \leq C\|f\|_{W^{1,p} \times L^p}$$

*uniformly for  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$ .*

It suffices to prove Theorem 4.6 for  $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$ . In fact, since  $f \in W^{1,p}(\Omega) \times L^p(\Omega)$  can be approximated by elements in  $\mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$ , Theorem 4.6 immediately follows from the following proposition.

**Proposition 4.7.** *Let  $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leq \Lambda_1\}$  and set  $\widehat{K}(\lambda, \xi) = \kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}$ . Define  $K(\lambda, x_3)$  by*

$$K(\lambda, x_3)F = \mathcal{F}^{-1}[\widehat{K}(\lambda, \xi)F] = \frac{1}{2\pi} \int_{\mathbf{R}} e^{ix_3\xi} \widehat{K}(\lambda, \xi) F d\xi$$

for  $F \in W^{1,p}(D) \times L^p(D)$ . Then for  $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$ , the function  $u = \mathcal{F}^{-1}[\widehat{K}(\lambda, \xi) \widehat{f}(\xi)]$  satisfies  $u = K(\lambda, \cdot) * f$  and the estimate

$$\|u\|_{W^{1,p} \times W^{2,p}} \leq C \|f\|_{W^{1,p} \times L^p}.$$

Here  $*$  means the convolution in  $x_3$ .

**Proof.** We first show

$$\widehat{K}(\lambda, \cdot) \widehat{f}(\xi) = \mathcal{F}[K(\lambda, \cdot) * f]$$

for any  $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$ . Since

$$e^{ix_3\xi} = \frac{1}{(ix_3)^k} \partial_\xi^k e^{ix_3\xi},$$

we see from Proposition 4.1 that for  $F \in W^{1,p}(D) \times L^p(D)$

$$\begin{aligned} & |\partial_{x_3}^\ell K(\lambda, x_3) F|_{W^{1,p} \times W^{2,p}} \\ & \leq \left| \frac{1}{2\pi} \int_{\mathbf{R}} (i\xi)^\ell e^{ix_3\xi} \widehat{K}(\lambda, \xi) F d\xi \right|_{W^{1,p} \times W^{2,p}} \\ & \leq \left| \frac{1}{2\pi} \frac{(-1)^k}{(ix_3)^k} \int_{\mathbf{R}} e^{ix_3\xi} \partial_\xi^k [(i\xi)^\ell \widehat{K}(\lambda, \xi)] F d\xi \right|_{W^{1,p} \times W^{2,p}} \\ & \leq C(1+M)^\ell |x_3|^{-k} \sum_{j=0}^k \int_{\{r/2 \leq |\xi| \leq M\}} |\partial_\xi^j \widehat{K}(\lambda, \xi) F|_{W^{1,p} \times W^{2,p}} d\xi \\ & \leq C(1+M)^\ell |x_3|^{-k} \int_{\{r/2 \leq |\xi| \leq M\}} |F|_{W^{1,p} \times L^p} d\xi \\ & \leq C_{r,M} |x_3|^{-k} |F|_{W^{1,p} \times L^p} \quad (\ell = 0, 1, 2). \end{aligned}$$

It then follows that

$$(4.12) \quad |\partial_{x_3}^\ell K(\lambda, x_3)|_{\mathcal{L}(W^{1,p}(D) \times L^p(D), W^{1,p}(D) \times W^{2,p}(D))} \leq \frac{C}{1 + |x_3|^2}.$$

Here  $|T|_{\mathcal{L}(X,Y)}$  denotes the operator norm of a bounded operator  $T : X \rightarrow Y$ . By (4.12), for any  $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$ , there hold the estimates

$$|K(\lambda, x_3) f(y_3)|_{W^{1,p} \times W^{2,p}} \leq \frac{C}{1 + |x_3|^2} |f(y_3)|_{W^{1,p} \times L^p} \in L^1(\mathbf{R}_{x_3} \times \mathbf{R}_{y_3})$$

and

$$\begin{aligned} & |K(\lambda, z_3 - y_3)f(y_3)|_{W^{1,p} \times W^{2,p}} \\ & \leq \frac{C}{1 + |z_3 - y_3|^2} |f(y_3)|_{W^{1,p} \times L^p} \in L^1(\mathbf{R}_{z_3} \times \mathbf{R}_{y_3}). \end{aligned}$$

Therefore, by Fubini's theorem, we have  $\widehat{K}(\lambda, \xi)\widehat{f}(\xi) = \mathcal{F}[K(\lambda, \cdot) * f]$ , which implies  $u = K(\lambda, \cdot) * f$ .

Furthermore, we see from (4.12) that

$$\begin{aligned} & \|Q_0 K(\lambda, \cdot) * f\|_{W^{1,p}}^p \\ & = \sum_{\ell+k \leq 1} \|\partial_{x_3}^\ell Q_0 K(\lambda, \cdot) * f\|_{L^p(\mathbf{R}; W^{k,p}(D))}^p \\ & = \sum_{\ell+k \leq 1} \int_{\mathbf{R}} \left| \partial_{x_3}^\ell \int_{\mathbf{R}} Q_0 K(\lambda, x_3 - y_3) f(\cdot, y_3) dy_3 \right|_{W^{k,p}}^p dx_3 \\ & \leq C \sum_{\ell+k \leq 1} \int_{\mathbf{R}} \left( \int_{\mathbf{R}} \frac{1}{1 + |x_3 - y_3|^2} |f(\cdot, y_3)|_{W^{1,p} \times L^p} dy_3 \right)^p dx_3 \\ & \leq C \|f\|_{W^{1,p} \times L^p}^p. \end{aligned}$$

Similarly one can estimate  $\|\tilde{Q}K(\lambda, \cdot) * f\|_{W^{2,p}}$  and the desired estimate is obtained. This completes the proof.  $\square$

We next consider estimates on  $\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}$  for large  $|\lambda|$ , which can be obtained by a similar argument as in section 3.

Let  $f \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\Omega)$ . Then  $\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f} = {}^T(\widehat{\phi}_{r,M}, \widehat{v}_{r,M})$  is a solution of (3.1) with  $\kappa_M$  replaced by  $\kappa_{r,M}$ .

Similarly to the proof of Lemma 3.3, one can prove the following estimate (cf., [5, Sections 4 and 5]).

**Lemma 4.8.** *There are  $\tilde{\Lambda} > 0$  and  $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$  such that if  $\lambda \in \Sigma(\tilde{\Lambda}, \tilde{\theta})$ , then there hold the estimate*

$$\|\partial_y^\alpha \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[\widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}^{-1} f]\|_p \leq C \left\{ \frac{\|\partial_y^\alpha f^0\|_p}{|\lambda|} + \frac{\|g\|_p}{|\lambda|^{1-\frac{|\alpha|}{2}}} \right\} \quad (|\alpha| = 0, 1)$$

and

$$\|\partial_y^2 \mathcal{F}_{\tilde{\zeta} \rightarrow \tilde{y}}^{-1}[\tilde{Q}\widehat{S}(\lambda, \tilde{\zeta}) \mathcal{F}_{\tilde{y} \rightarrow \tilde{\zeta}}^{-1} f]\|_p \leq C \{\|\partial_y f^0\|_p + \|g\|_p\}.$$

Based on Lemma 4.8 and the localization argument as in the proof of Theorem 3.1, we have the following estimate (by taking  $\tilde{\Lambda}$  larger if necessary).

**Theorem 4.9.** *There are  $\tilde{\Lambda} > 0$  and  $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$  such that if  $\lambda \in \Sigma(\tilde{\Lambda}, \tilde{\theta})$ , then there hold the estimate*

$$\|\mathcal{F}^{-1}[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}]\|_{W^{1,p} \times W^{2,p}} \leq C\|f\|_{W^{1,p} \times L^p}.$$

Combining Theorems 4.6 and 4.9, we obtain the following estimate for  $\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}$ .

**Theorem 4.10.** Let  $r$  and  $M$  be numbers satisfying  $0 < r < \frac{M}{2}$ . Then there are constants  $\tilde{\Lambda} > 0$ ,  $c_1 > 0$  and  $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$  such that If  $\lambda \in \Sigma(\tilde{\Lambda}, \tilde{\theta}) \cup \{\operatorname{Re}\lambda \geq -c_1\}$ , then

$$\|\mathcal{F}^{-1}[\kappa_{r,M}(\xi)(\lambda + \widehat{L}_\xi)^{-1}\widehat{f}]\|_{W^{1,p} \times W^{2,p}} \leq C\|f\|_{W^{1,p} \times L^p}.$$

## 5. Spectral properties of low frequency part

In this section we investigate spectral properties of  $-\widehat{L}_\xi$  for  $|\xi| \ll 1$ . This case is treated as a perturbation from the case  $\xi = 0$ .

We begin with some spectral properties of  $-\widehat{L}_0$ . We set  $\xi = 0$  in (2.2) to obtain

$$(5.1) \quad (\lambda + \widehat{L}_0)\widehat{u} = \widehat{f}, \quad \widehat{v}|_{\partial D} = 0,$$

where  $\widehat{u} = {}^T(\widehat{\phi}, \widehat{v}', \widehat{v}^3)$ ,  $\widehat{f} = {}^T(\widehat{f}^0, \widehat{g}', \widehat{g}^3)$  and

$$\widehat{L}_0 = \begin{pmatrix} 0 & \gamma^T \nabla' & 0 \\ \gamma \nabla' & -\nu \Delta' - \tilde{\nu} \nabla'^T \nabla' & 0 \\ 0 & 0 & -\nu \Delta' \end{pmatrix}.$$

We decompose  $\widehat{\phi}$  and  $\widehat{f}^0$  into

$$\widehat{\phi} = \widehat{\phi}_0 + \widehat{\phi}_1, \quad \widehat{\phi}_0 \equiv \frac{1}{|D|} \int_D \widehat{\phi}(x') dx',$$

$$\widehat{f}^0 = \widehat{f}_0^0 + \widehat{f}_1^0, \quad \widehat{f}_0^0 \equiv \frac{1}{|D|} \int_D \widehat{f}(x') dx',$$

respectively. This gives an orthogonal decomposition in  $L^2(D)$ , and we have

$$|\widehat{\phi}|_2^2 = |\widehat{\phi}_0|_2^2 + |\widehat{\phi}_1|_2^2.$$

Furthermore, since  $\widehat{\phi}_1$ -component has vanishing mean value, by the Poincaré inequality, there holds the estimate

$$|\widehat{\phi}_1|_p \leq C |\partial_{x'} \widehat{\phi}_1|_p = C |\partial_{x'} \widehat{\phi}|_p.$$

In terms of this decomposition, problem (5.1) is reduced to the following problem (5.2)–(5.5):

$$(5.2) \quad \lambda \widehat{\phi}_0 = \widehat{f}_0^0,$$

$$(5.3) \quad \lambda \widehat{\phi}_1 + \gamma \nabla' \cdot \widehat{v}' = \widehat{f}_1^0,$$

$$(5.4) \quad \lambda \widehat{v}' - \nu \Delta' \widehat{v}' - \tilde{\nu} \nabla' (\nabla' \cdot \widehat{v}') + \gamma \nabla' \widehat{\phi}_1 = \widehat{g}', \quad \widehat{v}'|_{\partial D} = 0,$$

$$(5.5) \quad \lambda \widehat{v}^3 - \nu \Delta' \widehat{v}^3 = \widehat{g}^3, \quad \widehat{v}^3|_{\partial D} = 0.$$

As for the solvability of (5.2)–(5.5) we have the following facts.

It is clear that (5.2) is uniquely solvable if and only if  $\lambda \neq 0$ , and in this case the solution is given by  $\widehat{\phi}_0 = \frac{1}{\lambda} \widehat{f}_0^0$ . It is also easy to see that  $\lambda = 0$  is a simple eigenvalue with eigenfunction  $\widehat{\phi}_0 = 1$ .

As for (5.5), it is well known that there are  $\{\lambda_j\}_{j=1}^\infty$  ( $\lambda_j < 0$ ,  $|\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq \dots \rightarrow \infty$ ) such that each  $\lambda_j$  is a semi-simple eigenvalue and, for  $\lambda \notin \{\lambda_j\}_{j=1}^\infty$ , (5.5) has a unique solution  $\widehat{v}^3 \in W^{2,p}(D) \cap W_0^{1,p}(D)$ . Furthermore, if  $|\arg(\lambda - \frac{1}{2}\lambda_1)| \leq \pi - \varepsilon$  ( $\varepsilon > 0$ ), then the solution  $\widehat{v}^3$  satisfies the estimate

$$|\lambda| |\widehat{v}^3|_p + |\lambda|^{\frac{1}{2}} |\partial_{x'} \widehat{v}^3|_p + |\partial_{x'}^2 \widehat{v}^3|_p \leq C_\varepsilon |\widehat{g}^3|_p.$$

As for the solvability of (5.3)–(5.4), we have the following result.

**Proposition 5.1.** *Let  $1 < p < \infty$ . Then there exist constants  $c_0 > 0$ ,  $\Lambda > 0$  and  $\theta \in (\frac{\pi}{2}, \pi)$  such that if  $\lambda \in \Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re} \lambda \geq -2c_0\}$ , then for any*

$T(\widehat{f}_1^0, \widehat{g}') \in W^{1,p}(D) \times L^p(D)$  with  $\int_D \widehat{f}_1^0 dx' = 0$ , there exists a unique solution  $T(\widehat{\phi}_1, \widehat{v}') \in W^{1,p}(D) \times [W^{2,p}(D) \cap W_0^{1,p}(D)]$  with  $\int_D \widehat{\phi}_1 dx' = 0$  of (5.3)–(5.4), which satisfies the estimate

$$|\lambda| \{ |\widehat{\phi}_1|_{W^{1,p}} + |\widehat{v}'|_p \} + |\lambda|^{\frac{1}{2}} |\partial_{x'} \widehat{v}'|_p + |\partial_{x'}^2 \widehat{v}'|_p \leq C \{ |\widehat{f}_1^0|_{W^{1,p}} + |\widehat{g}'|_p \}.$$

Proposition 5.1 was proved by [21]. (See also [20].) We summarize the spectral properties of  $-\widehat{L}_0$  obtained above.

**Proposition 5.2.** *There are constants  $c_0 > 0$ ,  $\Lambda_0 > 0$  and  $\theta_0 \in (\frac{\pi}{2}, \pi)$  such that*

$$(\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re} \lambda \geq -2c_0\}) \cap \{|\lambda| \geq c_0\} \subset \rho(-\widehat{L}_0)$$

and

$$\sigma(-\widehat{L}_0) \cap \{|\lambda| < c_0\} = \{0\}.$$

If  $\lambda \in (\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re} \lambda \geq -2c_0\}) \cap \{|\lambda| \geq c_0\}$ , then

$$\begin{aligned} |(\lambda + \widehat{L}_0)^{-1} f|_{W^{1,p} \times L^p} &\leq \frac{C}{|\lambda|+1} |f|_{W^{1,p} \times L^p}, \\ |\partial_{x'}^\ell \tilde{Q}(\lambda + \widehat{L}_0)^{-1} f|_p &\leq \frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}} |f|_{W^{1,p} \times L^p} \quad (\ell = 1, 2). \end{aligned}$$

Furthermore, 0 is a simple eigenvalue and the associated eigenprojection  $\widehat{P}^{(0)}$  is given by

$$\widehat{P}^{(0)} u = {}^T(\langle \phi \rangle, 0) \quad \text{for } u = {}^T(\phi, v).$$

Based on Proposition 5.2, one can obtain the following result by a perturbation argument as in the proof of [10, Propositions 4.3 and 4.4].

**Theorem 5.3.** *There exists a positive constant  $r_1 > 0$  such that the following assertions hold.*

(i) *If  $|\xi| \leq r_1$ , then*

$$(\Sigma(\Lambda_0, \theta_0) \cup \{\operatorname{Re} \lambda \geq -c_0\}) \cap \left\{ |\lambda| \geq \frac{c_0}{2} \right\} \subset \rho(-\widehat{L}_\xi).$$

(ii) If  $\lambda \in (\Sigma(\Lambda_0, \theta_0) \cup \{\operatorname{Re} \lambda \geq -c_0\}) \cap \{|\lambda| \geq \frac{c_0}{2}\}$ , then

$$\begin{aligned} |(\lambda + \widehat{L}_\xi)^{-1} f|_{W^{1,p} \times L^p} &\leq \frac{C}{|\lambda|+1} |f|_{W^{1,p} \times L^p}, \\ |\partial_{x'}^\ell \tilde{Q}(\lambda + \widehat{L}_\xi)^{-1} f|_p &\leq \frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}} |f|_{W^{1,p} \times L^p} \quad (\ell = 1, 2). \end{aligned}$$

(iii) If  $|\xi| \leq r_1$ , then

$$\sigma(-\widehat{L}_\xi) \cap \left\{ |\lambda| < \frac{c_0}{2} \right\} = \{\lambda_0(\xi)\}.$$

Here  $\lambda_0(\xi)$  is a simple eigenvalue of  $-\widehat{L}_\xi$ , which satisfies

$$\lambda_0(\xi) = -\frac{a_1 \gamma}{\nu} |\xi|^2 + O(|\xi|^4) \quad (|\xi| \rightarrow 0).$$

for some constant  $a_1 > 0$ .

We next give an estimate for the eigenprojection  $\widehat{P}(\xi)$  associated with the eigenvalue  $\lambda_0(\xi)$ . For this purpose we write  $\widehat{L}_\xi$  as

$$\widehat{L}_\xi = \widehat{L}_0 + \xi \widehat{L}^{(1)} + \xi^2 \widehat{L}^{(2)}.$$

Here

$$\widehat{L}^{(1)} = \begin{pmatrix} 0 & 0 & i\gamma \\ 0 & 0 & -i\tilde{\nu} \nabla' \\ i\gamma & -i\tilde{\nu}^T \nabla' & 0 \end{pmatrix}, \quad \widehat{L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu I & 0 \\ 0 & 0 & \nu + \tilde{\nu} \end{pmatrix}.$$

We begin with following

**Proposition 5.4.** *Let  $R > 0$ . Then the following estimate holds for  $\lambda \in (\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re} \lambda \geq -2c_0\}) \cap \{c_0 \leq |\lambda| \leq R\}$ :*

$$|(\lambda + \widehat{L}_0)^{-1} \widehat{f}|_{H^3 \times H^4} \leq C_R |\widehat{f}|_{H^3 \times H^2}.$$

**Proof.** We here give an outline of the proof. As was shown in [10], an application of the Matsumura-Nishida energy method [19] to (5.3)–(5.4) gives

$$|\widehat{v}'|_{H^2} + |\widehat{\phi}_1|_{H^1} \leq C_R (|\widehat{f}_1^0|_{H^1} + |\widehat{g}'|_2)$$

for some constant  $C = C(R) > 0$ . Then higher order derivatives can also be estimated by the Matsumura-Nishida energy method to obtain

$$(5.6) \quad |\widehat{v}'|_{H^4} + |\widehat{\phi}_1|_{H^3} \leq C_R(|\widehat{f}_1^0|_{H^3} + |\widehat{g}'|_{H^2}).$$

Applying the elliptic regularity estimate to (5.5), we have

$$(5.7) \quad |\widehat{v}^3|_{H^4} \leq C_R|\widehat{g}^3|_{H^2}.$$

Proposition 5.4 now follows from (5.2), (5.6) and (5.7). This completes the proof.  $\square$

**Lemma 5.5.** *There holds the following estimates*

$$|L^{(j)}(\lambda + \widehat{L}_0)^{-1}\widehat{f}|_{H^3} \leq C|\widehat{f}|_{H^3} \quad (j = 1, 2)$$

for  $\lambda \in (\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re}\lambda \geq -2c_0\}) \cap \{c_0 \leq |\lambda| \leq R\}$ .

**Proof.** Let  ${}^T(\widehat{\phi}, \widehat{v}) = (\lambda + \widehat{L}_0)^{-1}\widehat{f}$ . It follows from Proposition 5.4 that

$$\begin{aligned} |L^{(1)}(\lambda + \widehat{L}_0)^{-1}\widehat{f}|_{H^3} = |L^{(1)}\widehat{u}|_{H^3} &\leq C\{|\widehat{v}^3|_{H^3} + |\nabla'\widehat{v}^3|_{H^3} + |\widehat{\phi}|_{H^3} + |\nabla' \cdot \widehat{v}'|_{H^3}\} \\ &\leq C|\widehat{f}|_{H^3 \times H^2} \end{aligned}$$

and

$$|L^{(2)}(\lambda + \widehat{L}_0)^{-1}\widehat{f}|_{H^3} = |L^{(2)}\widehat{u}|_{H^3} \leq C\{|\widehat{v}'|_{H^3} + |\widehat{v}^3|_{H^3}\} \leq C|\widehat{f}|_{H^3 \times H^2}.$$

This completes the proof.  $\square$

We now estimate the integral kernel of the eigenprojection  $\widehat{P}(\xi)$  associated with the eigenvalue  $\lambda_0(\xi)$  for  $|\xi| \ll 1$ .

**Theorem 5.6.** *There exists  $r_2 > 0$  such that if  $|\xi| \leq r_2$ , then the following assertions hold.*

- (i) *The eigenprojection  $\widehat{P}(\xi)$  associated with the eigenvalue  $\lambda_0(\xi)$  is written in the form*

$$\begin{aligned} \widehat{P}(\xi) &= \widehat{P}^{(0)} + \xi\widehat{P}^{(1)} + \widehat{P}^{(2)}(\xi), \\ \widehat{P}^{(j)}u &= \int_D \widehat{P}^{(j)}(x', y')u(y')dy', \quad j = 0, 1, \\ \widehat{P}^{(2)}u &= \int_D \widehat{P}^{(2)}(\xi, x', y')u(y')dy'. \end{aligned}$$

Here  $\widehat{P}^{(0)} = \frac{1}{|D|}Q_0$ ; and  $\widehat{P}^{(1)}(x', y')$  and  $\widehat{P}^{(2)}(\xi, x', y')$  satisfy

$$\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \widehat{P}^{(1)}(x', y'), \partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \widehat{P}^{(2)}(\xi, x', y') \in L^\infty(D \times D)$$

for  $|\alpha'| \leq 1$  and  $|\beta'| \leq 1$ . Furthermore, for any  $\alpha \geq 0$ ,  $\widehat{P}^{(2)}(\xi, x', y')$  satisfies the estimate

$$|\partial_\xi^\alpha \partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \widehat{P}^{(2)}(\xi, \cdot, \cdot)|_{L^\infty(D \times D)} \leq C_\alpha |\xi|^{2-\alpha}.$$

(ii)  $\lambda_0(\xi)$  is a simple eigenvalue of the adjoint operator  $-\widehat{L}_\xi^*$  and the associated eigenprojection  $\widehat{P}^*(\xi)$  is written as

$$\widehat{P}^*(\xi) = \widehat{P}^{(0)*} + \xi \widehat{P}^{(1)*} + \widehat{P}^{(2)*}(\xi),$$

$$\widehat{P}^{(j)*} u = \int_D \widehat{P}^{(j)*}(x', y') u^*(y') dy', \quad j = 0, 1,$$

$$\widehat{P}^{(2)*}(\xi) u = \int_D \widehat{P}^{(2)*}(\xi, x', y') u^*(y') dy'.$$

Here  $\widehat{P}^{(j)*}(x', y')$  ( $j = 0, 1$ ) and  $\widehat{P}^{(2)*}(\xi, x', y')$  satisfy

$$\widehat{P}^{(0)*} = \widehat{P}^{(0)}, \quad \widehat{P}^{(1)*}(x', y') = \overline{\widehat{P}^{(1)}(y', x')},$$

$$\widehat{P}^{(2)*}(\xi, x', y') = \overline{\widehat{P}^{(2)}(\xi, y', x')}.$$

(iii) There hold the following relations

$$(\widehat{P}(\xi)[\partial_{x'} \tilde{Q} u], u^*) = -(u, \partial_{x'} \tilde{Q} \widehat{P}^*(\xi) u^*),$$

$$(\widehat{P}^{(1)}[\partial_{x'} \tilde{Q} u], u^*) = -(u, \partial_{x'} \tilde{Q} \widehat{P}^{(1)} u^*),$$

$$(\widehat{P}^{(2)}[\partial_{x'} \tilde{Q} u], u^*) = -(u, \partial_{x'} \tilde{Q} \widehat{P}^{(2)} u^*).$$

**Proof.** As for (i), we here give an outline of the proof, since it is similar to that of [6, Theorem 3.3].

Let  $\psi_0 = {}^T(1, 0)$ , which is an eigenfunction for the eigenvalue 0 of  $-\widehat{L}_0$  and  $-\widehat{L}_0^*$ . Clearly,  $|\psi_0|_{H^k} = |D|^{\frac{1}{2}}$  for all  $k \geq 0$ .

We define  $\psi(\xi)$  and  $\psi^*(\xi)$  by

$$\psi(\xi) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda + \widehat{L}_{\xi})^{-1} \psi_0 d\lambda$$

and

$$\psi^*(\xi) = \frac{1}{\langle \psi(\xi), \tilde{\psi}^*(\xi) \rangle} \tilde{\psi}^*(\xi)$$

with

$$\tilde{\psi}^*(\xi) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda + \widehat{L}_{\xi}^*)^{-1} \psi_0 d\lambda.$$

where  $\Gamma = \{|\lambda| = \frac{c_0}{2}\}$ . Then as in the proof of [6, Theorem 3.3], one can see the following estimates on  $\psi(\xi)$  and  $\psi^*(\xi)$ . By Lemma 5.5 and the Neumann series expansion of  $(\lambda + \widehat{L})^{-1}$ ,  $\psi(\xi)$  is expanded as

$$\psi(x', \xi) = \psi_0 + \xi \psi^{(1)}(x') + \psi^{(2)}(x', \xi),$$

and, with the aid of the Sobolev embedding  $H^3 \hookrightarrow W^{1,\infty}$ ,

$$|\psi^{(1)}|_{W^{1,\infty}} \leq C |\psi^{(1)}|_{H^3} \leq C,$$

$$|\partial_{\xi}^{\alpha} \psi^{(2)}(\xi)|_{W^{1,\infty}} \leq C |\partial_{\xi}^{\alpha} \psi^{(2)}(\xi)|_{H^3} \leq C |\xi|^{2-\alpha}.$$

The same expansion also holds for  $\psi^*(\xi)$ :

$$\psi^*(x', \xi) = \psi_0 + \xi \psi^{(1)*}(x') + \psi^{(2)*}(x', \xi),$$

where  $\psi^{(j)*}$  ( $j = 1, 2$ ) satisfy the estimates

$$|\psi^{(1)*}|_{W^{1,\infty}} \leq C \quad \text{and} \quad |\partial_{\xi}^{\alpha} \tilde{\psi}^{(2)*}(\xi)|_{W^{1,\infty}} \leq C |\xi|^{2-\alpha}.$$

In terms of  $\psi(x', \xi)$  and  $\psi^*(x', \xi)$ ,  $\widehat{P}(\xi)$  is given in the form

$$\widehat{P}(\xi)u = \langle u, \psi^*(\xi) \rangle \psi(\xi) = \int_D \widehat{P}(\xi, x', y') u(y') dy'$$

with

$$\begin{aligned} \widehat{P}(\xi, x', y') &= \frac{1}{|D|} \psi(x', \xi)^T \psi^*(y', \xi) \\ &= \frac{1}{|D|} Q_0 + \xi \widehat{P}^{(1)}(x', y') + \widehat{P}^{(2)}(\xi, x', y'). \end{aligned}$$

Here

$$\begin{aligned}
\widehat{P}^{(1)}(x', y') &= \frac{1}{|D|} \{ \psi^{(1)}(x')^T \psi_0 + \psi_0^T \psi^{(1)*}(y') \}, \\
\widehat{P}^{(2)}(\xi, x', y') &= \frac{1}{|D|} \{ \psi_0^T \psi^{(2)*}(y', \xi) + \psi^{(2)}(x', \xi)^T \psi_0, \\
&\quad + \psi^{(2)}(x', \xi)^T \psi^{(2)*}(y', \xi) + \xi \psi^{(1)}(x')^T \psi^{(2)*}(y', \xi) \\
&\quad + \xi \psi^{(2)}(x', \xi)^T \psi^{(1)*}(y') + \xi^2 \psi^{(1)}(x')^T \psi^{(1)*}(y') \}.
\end{aligned}$$

It follows from the estimates for  $\psi(\xi)$  and  $\psi^*(\xi)$  obtained above that

$$|\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \widehat{P}^{(1)}(x', y')|_{L^\infty(D \times D)} \leq C$$

and

$$|\partial_\xi^\alpha \partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \widehat{P}^{(2)}(\xi, x', y')|_{L^\infty(D \times D)} \leq C |\xi|^{2-\alpha}.$$

For the details, see the proof of [6, Theorem 3.3].

Assertion (ii) easily follows from the relation

$$((\lambda + \widehat{L}_\xi)^{-1} u, u^*) = (u, (\bar{\lambda} + \widehat{L}_\xi^*)^{-1} u^*)$$

for  $u, u^* \in W^{1,p}(D) \times L^p(D)$ .

As for (iii), since  $\tilde{Q}(\bar{\lambda} + \widehat{L}_\xi^*)^{-1} u^*|_{\partial D} = 0$ , by integration by parts, we have

$$((\lambda + \widehat{L}_\xi)^{-1} [\partial_{x'} \tilde{Q} u], u^*) = -(u, \partial_{x'} \tilde{Q} (\lambda + \widehat{L}_\xi)^{-1} u^*),$$

which yields the desired results. This completes the proof.  $\square$

## 6. Proof of Theorem 2.1

In this section we give an outline of the proof of Theorem 2.1.

The following proposition implies that  $-L$  generates an analytic semi-group.

**Proposition 6.1.** *There are  $\Lambda > 0$  and  $\theta \in (\frac{\pi}{2}, \pi)$  such that  $\Sigma(\Lambda, \theta) \subset \rho(-L)$  and there hold the following estimates uniformly for  $\lambda \in \Sigma(\Lambda, \theta)$ :*

$$\begin{aligned}
\text{(i)} \quad & \|(\lambda + L)^{-1} f\|_{W^{1,p} \times L^p} \leq \frac{C}{|\lambda|} \|f\|_{W^{1,p} \times L^p}, \\
\text{(ii)} \quad & \|\partial_x^\ell \tilde{Q} (\lambda + L)^{-1} f\|_p \leq \frac{C}{|\lambda|^{1-\frac{\ell}{2}}} \|f\|_{W^{1,p} \times L^p} \quad (\ell = 1, 2).
\end{aligned}$$

**Proof.** We here give an outline of the proof. Let  $\lambda \neq 0$ . By (2.1),

$$(6.1) \quad \phi = \frac{1}{\lambda}(f^0 - \gamma \operatorname{div} v)$$

Substituting this into the second equation of (2.1), we have

$$(6.2) \quad \lambda v - \nu \Delta v - \tilde{\nu} \nabla \operatorname{div} v = F, \quad v|_{\partial\Omega} = 0.$$

Here

$$F \equiv g - \frac{\gamma}{\lambda} \nabla(f^0 - \gamma \operatorname{div} v).$$

Since  $Bv = -\nu \Delta v - \tilde{\nu} \nabla \operatorname{div} v$  is strongly elliptic, it holds that there are  $\Lambda' > 0$  and  $\theta \in (\frac{\pi}{2}, \pi)$  such that for  $\lambda \in \Sigma(\Lambda', \theta)$

$$\begin{aligned} |\lambda| \|v\|_p + |\lambda|^{\frac{1}{2}} \|\partial_x v\|_p + \|\partial_x^2 v\|_p &\leq C \|F\|_p \\ &\leq C \{ \|f\|_{W^{1,p} \times L^p} + \frac{1}{|\lambda|} \|\partial_x^2 v\|_p \}. \end{aligned}$$

We take  $\Lambda > 0$  large enough so that  $\frac{C^p}{|\lambda|} \leq \frac{1}{2}$  for  $\lambda \in \Sigma(\Lambda, \theta)$ . Then

$$|\lambda| \|v\|_p + |\lambda|^{\frac{1}{2}} \|\partial_x v\|_p + \|\partial_x^2 v\|_p \leq 2C \|f\|_{W^{1,p} \times L^p}.$$

This, together with (6.1), yields

$$\|\phi\|_{W^{1,p}} \leq \frac{C}{|\lambda|} (\|f^0\|_{W^{1,p}} + \|\operatorname{div} v\|_{W^{1,p}}) \leq \frac{C}{|\lambda|} \|f\|_{W^{1,p} \times L^p}.$$

This completes the proof.  $\square$

By Proposition 6.1,  $-L$  generates an analytic semigroup  $e^{-tL}$  on  $W^{1,p}(\Omega) \times L^p(\Omega)$ ; and  $e^{-tL}$  is represented as

$$e^{-tL} = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda + L)^{-1} d\lambda.$$

Here  $\Gamma = \{\lambda = \Lambda + se^{\pm\theta}; s \geq 0\}$ .

Using the estimates (i) and (ii) in Proposition 6.1, one can show Theorem 2.1 (i) by a standard argument.

We now give a proof of asymptotic behavior of  $e^{-tL}$  given in Theorem 2.1 (ii).

**Proof of Theorem 2.1 (ii).** The proof is done by a similar argument to that in [6, Section 4]. We here give an outline of the proof.

We decompose  $e^{-tL}$  as

$$e^{-tL} = \mathcal{V}_0(t) + \mathcal{V}_\infty(t).$$

Here

$$\mathcal{V}_0(t) = \mathcal{F}^{-1}[\kappa_0(\xi)e^{-t\widehat{L}_\xi}], \quad \mathcal{V}_\infty(t) = \mathcal{F}^{-1}[(1 - \kappa_0(\xi))e^{-t\widehat{L}_\xi}]$$

where  $\kappa_0$  is a function satisfying

$$\kappa_0(\xi) \in C_0^\infty(\mathbf{R}), \quad \kappa_0(\xi) = \begin{cases} 1 & (|\xi| \leq \frac{r}{2}), \\ 0 & (|\xi| \geq r) \end{cases}$$

and

$$e^{-t\widehat{L}_\xi} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + \widehat{L}_\xi)^{-1} d\lambda$$

with  $\Gamma = \{\lambda = \Lambda + se^{\pm\theta}; s \geq 0\}$ . We here take  $r > 0$  in such a way that  $0 < r \leq \min\{r_1, r_2\}$  with  $r_1$  and  $r_2$  given in Theorems 5.3 and 5.6 respectively.

To prove Theorem 2.1 (ii), we will deform the contour  $\Gamma$  in a suitable way.

We first consider  $\mathcal{V}_0(t)$ . By Theorem 5.3, we can deform  $\Gamma$  into  $\Gamma_0 \cup \tilde{\Gamma}_0$  and a suitable circle around 0, where

$$\Gamma_0 = \{\lambda = -c_0 + is; |s| \leq s_0\}, \quad \tilde{\Gamma}_0 = \{\lambda = \Lambda_0 + se^{\pm i\theta_0} : s \geq \tilde{s}_0\}.$$

Here  $\Lambda_0$  and  $\theta_0$  are the numbers given in Theorem 5.3; and we choose  $s_0$  and  $\tilde{s}_0$  in such a way that  $\Gamma_0$  connects with  $\tilde{\Gamma}_0$  at the end points of  $\Gamma_0$ . It then follows from Theorems 5.3, 5.6 and the residue theorem that  $\mathcal{V}_0(t)$  is written as

$$\mathcal{V}_0(t)u_0 = W^{(0)}(t)u_0 + W^{(1)}(t)u_0,$$

where

$$W^{(j)}(t)u_0 = \mathcal{F}^{-1}[\widehat{W}^{(j)}(t)\widehat{u}_0] \quad (j = 0, 1),$$

$$\widehat{W}^{(0)}(t)\widehat{u}_0 = \kappa_0(\xi)e^{\lambda_0(\xi)t}\widehat{P}(\xi)\widehat{u}_0,$$

$$\widehat{W}^{(1)}(t)\widehat{u}_0 = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \tilde{\Gamma}_0} e^{\lambda t} \kappa_0(\xi) (\lambda + \widehat{L}_\xi)^{-1} \widehat{u}_0 d\lambda.$$

By using Theorems 5.3 and 5.6, one can show that  $W^{(0)}(t)u_0$  is written in the form

$$W^{(0)}(t)u_0 = \mathcal{W}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0,$$

where  $\mathcal{W}^{(0)}(t)u_0$  and  $\mathcal{R}^{(0)}(t)u_0$  have the properties in Theorem 2.1 (ii-a). We here omit the details since it can be shown in the same way as in [6, Section 4]. Also, by using Theorem 5.3, one can show that  $W^{(1)}(t)$  satisfies the estimate

$$\|W^{(1)}(t)u_0\|_{W^{1,p} \times W^{2,p}} \leq Ce^{-c_0 t} \|u_0\|_{W^{1,p} \times L^p}.$$

As for  $\mathcal{V}_\infty(t)$ , by Theorems 3.1 and 4.10, one can deform the contour  $\Gamma$  into  $\Gamma = \Gamma_\infty \cup \tilde{\Gamma}_\infty$ , where

$$\Gamma_\infty = \{\lambda; \lambda = -c_\infty + is \ (|s| \leq s_\infty)\}, \quad \tilde{\Gamma}_\infty = \{\lambda; \lambda = \Lambda_0 + se^{\pm i\theta_0}, s \geq \tilde{s}_\infty\},$$

for some  $c_\infty > 0$ . We here take  $s_\infty$  and  $\tilde{s}_\infty$  so that  $\Gamma_\infty$  connects with  $\tilde{\Gamma}_\infty$  at the end points of  $\Gamma_\infty$ . It then follows from Theorems 3.1 and 4.10 that

$$\|\mathcal{V}_\infty(t)u_0\|_{W^{1,p} \times W^{2,p}} \leq Ce^{-c_\infty t} \|u_0\|_{W^{1,p} \times L^p}.$$

Setting  $\mathcal{U}_\infty(t) = W^{(1)}(t) + \mathcal{V}_\infty(t)$ , we see that  $\mathcal{U}_\infty(t)$  satisfies the estimate in Theorem 2.1 (ii-b). This completes the proof.

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