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Non-convex anisotropic surface energy and zero mean curvature surfaces in the Lorentz-Minkowski space

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Abstract. An anisotropic surface energy functional is the integral of an energy density function over a surface. The energy density depends on the surface normal at each point. The usual area functional is a special case of such a functional. We study stationary surfaces of anisotropic surface energies in the euclidean three-space which are called anisotropic minimal surfaces. For any axisymmetric anisotropic surface energy, we show that, a surface is both a minimal surface and an anisotropic minimal surface if and only if it is a right helicoid. We also construct new examples of anisotropic minimal surfaces, which include zero mean curvature surfaces in the three-dimensional Lorentz-Minkowski space as special cases.

Keywords. anisotropic, mean curvature, minimal surface, zero mean curvature surface, Lorentz-Minkowski space, Wulff shape

1. INTRODUCTION

Let $\gamma: \Omega \rightarrow \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω of the two-dimensional unit sphere $S^2 := \{X \in \mathbf{R}^3; |X| = 1\}$. Let $X: \Sigma \rightarrow \mathbf{R}^3$ be an immersion from a two-dimensional oriented connected compact C^∞ manifold Σ (with or without boundary) to the three-dimensional euclidean space \mathbf{R}^3 . Denote by $\nu = (\nu_1, \nu_2, \nu_3): \Sigma \rightarrow S^2$ the unit normal along X (in other words, the Gauss map of X). If $\nu(\Sigma) \subset \Omega$, we say that X is compatible with γ and we define the following functional.

$$\mathcal{F}[X] = \int_{\Sigma} \gamma(\nu) d\Sigma, \quad (1)$$

where $d\Sigma$ is the area element of X . Such a functional is used to model anisotropic surface energies. Applications can be found in many branches of the physical sciences including metallurgy and crystallography ([14, 15]). We will call $\mathcal{F}[X]$ the anisotropic energy of X , and γ the energy density function.

We call stationary surfaces of (1) for compactly-supported variations γ -minimal surfaces. It is obvious that, for $\gamma \equiv 1$, γ -minimal surfaces are usual minimal surfaces.

Denote by $D\gamma$ and $D^2\gamma$ the gradient and the Hessian of γ on Ω , respectively. Denote by 1 the identity endomorphism field on the tangent space $T_\nu(S^2)$. If the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω , a mapping $Y: \Omega \rightarrow \mathbf{R}^3$ defined by $Y(\nu) = D\gamma + \gamma(\nu)\nu$ is an immersion and Y defines the uniquely determined immersed surface with unit normal ν whose support function coincides with γ , that is $\gamma(\nu) = \langle Y(\nu), \nu \rangle$ holds. We say that Y is the standard body for γ . (As for the terminology “standard body”, we quote

[12].) We will sometimes use the symbol M_γ to represent the mapping Y or the image $Y(\Omega)$ of Y .

We say that $\gamma: \Omega \rightarrow \mathbf{R}_+$ satisfies the *convexity condition*, if the matrix $D^2\gamma + \gamma 1$ is positive definite at each point ν in Ω . In this case, the standard body M_γ for γ is strongly convex (that is, the principal curvatures of M_γ are positive everywhere), and the functional \mathcal{F} appearing in (1) is called a constant coefficient parametric elliptic functional, and stationary surfaces are extensively studied in recent years.

In this paper, we do not assume the convexity condition. By this generalization, we obtain a more variety of important examples. For example, zero mean curvature immersions in the Lorentz-Minkowski space $\mathbf{R}_1^3 := \{(x_1, x_2, x_3) \in \mathbf{R}^3; ds^2 = dx_1^2 + dx_2^2 - dx_3^2\}$ arise as γ -minimal surfaces for a certain simple function γ as follows (cf. §3).

Theorem 1. *Set $\Omega_1 := \{\nu = (\nu_1, \nu_2, \nu_3) \in S^2; |\nu_3| > \sqrt{2}/2\}$, $\Omega_2 := \{\nu \in S^2; |\nu_3| < \sqrt{2}/2\}$. Define a function $\gamma: S^2 \rightarrow \mathbf{R}$ as $\gamma(\nu) = \sqrt{|\nu_3^2 - \nu_1^2 - \nu_2^2|} = \sqrt{|2\nu_3^2 - 1|}$. Then, an immersion $X: \Sigma \rightarrow \mathbf{R}_1^3$ with Gauss image $\nu(\Sigma) \subset \Omega_1 \cup \Omega_2$ is γ -minimal if and only if the mean curvature of X is zero as an immersed surface in \mathbf{R}_1^3 .*

This result indicates that the recent investigations about zero mean curvature surfaces in \mathbf{R}_1^3 changing their causal type across null curves (regular curves whose velocity vector fields are lightlike) or lightlike lines from spacelike zero mean curvature surfaces to timelike zero mean curvature surfaces ([3, 6, 5, 4, 2]) should be very natural and reasonable. Probably the most well-known example of such surfaces is the right helicoid with the timelike axis as its axis, which changes its causal type across a null curve from a spacelike zero mean curvature surface to a timelike zero mean curvature surface ([3, 6]). In §4, we will show a more

general remarkable result as follows.

Theorem 2. *Let $\gamma: \Omega \rightarrow \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω in S^2 . Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω . Assume also that γ is axisymmetric and not a constant function. Let $X: \Sigma \rightarrow \mathbf{R}^3$ be an immersion which is compatible with γ . Then, X is both minimal and γ -minimal if and only if it is a part of either a plane or a right helicoid whose axis is parallel to the axis of γ .*

This result is a generalization of [7, Theorem 4.2] and a refinement of [9, Proposition III.1]. [7, Theorem 4.2] proves that a spacelike plane and the spacelike part of a right helicoid whose axis is parallel to the timelike axis are only both a minimal surface in the euclidean space \mathbf{R}^3 and a spacelike zero mean curvature surface in \mathbf{R}_1^3 . [9, Proposition III.1] proves that a right helicoid is a γ -minimal surface for any axisymmetric γ whose axis is parallel to the axis of the helicoid itself.

Theorem 2 combined with Theorem 1 implies the following:

Corollary 1. *A spacelike plane and the spacelike part of a right helicoid whose axis is parallel to the x_3 -axis are only both a minimal surface in the euclidean space \mathbf{R}^3 and a spacelike zero mean curvature surface in \mathbf{R}_1^3 . Also, a timelike plane and the timelike part of a right helicoid whose axis is parallel to the x_3 -axis are only both a minimal surface in \mathbf{R}^3 and a timelike zero mean curvature surface in \mathbf{R}_1^3 .*

In general, it is not easy to construct examples of γ -minimal surfaces. For any axisymmetric energy density function γ , there exist γ -minimal surfaces which are also symmetric with respect to the same axis as γ . The existence theorem and a certain kind of representation formula of these surfaces were given in [8] and they were called anisotropic catenoid. Although the convexity condition for γ was assumed in [8], the method there works also for non convex γ . In this paper, for certain classes of γ , we will give another type of examples of γ -minimal surfaces which are foliated by parallel circles but are not surfaces of revolution. We will call them γ -minimal surfaces of Riemann-type after Riemann's minimal surfaces in \mathbf{R}^3 .

Proposition 1. *Let $\gamma: \Omega \rightarrow \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω in S^2 . Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point $\nu \in \Omega$. We also assume that the standard body M_γ for γ is a quadric surface of revolution. Then, there are γ -minimal surfaces of Riemann-type.*

From Theorem 1, we see that spacelike and timelike zero mean curvature surfaces of Riemann-type in \mathbf{R}_1^3 are obtained as special cases of surfaces given by Proposition 1. Actually, for $\gamma|_{\Omega_1}$ in Theorem 1, M_γ is a hyperboloid of two sheets, and for $\gamma|_{\Omega_2}$, M_γ is a hyperboloid of one sheet (§5, Lemma 5).

We should remark that zero mean curvature surfaces of Riemann-type in \mathbf{R}_1^3 were studied also in [10, 11].

In §5, for γ satisfying the assumption in Proposition 1, we will give explicit parameter representations of all γ -

minimal surfaces foliated by circles contained in parallel planes which are orthogonal to the rotation axis of M_γ (Proposition 3). Actually, Proposition 1 is a corollary of Proposition 3.

Some of the results in this article can be generalized to hypersurfaces in \mathbf{R}^{n+1} .

2. PRELIMINARIES

In this section, we give the definitions of the Wulff shape, anisotropic mean curvature, and their fundamental properties and representation formulas. We quote [12, 1, 8] as references.

Let $\gamma: \Omega \rightarrow \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω of the unit sphere S^2 . Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω .

If $\Omega = S^2$, then, for any $V > 0$, there exists a uniquely determined (up to translations in \mathbf{R}^3) convex surface $W(V)$ such that $W(V)$ attains the minimum of \mathcal{F} among all closed piecewise smooth surfaces in \mathbf{R}^3 enclosing the 3-dimensional volume V ([13]). For the special value $V_0 := (1/3) \int_{S^2} \gamma(\nu) dS^2$, $W(V_0)$ is called the Wulff shape for γ , and we will denote it by W . In the special case where $\gamma \equiv 1$, $\mathcal{F}[X]$ is the usual area of the surface X and W is the unit sphere S^2 . In general, W is not smooth. W is a smooth strongly convex surface if and only if γ satisfies the convexity condition (see §1). In this case, W can be parametrized by the smooth mapping

$$Y: S^2 \rightarrow \mathbf{R}^3, \quad Y(\nu) = D\gamma + \gamma(\nu)\nu,$$

where we regard $D\gamma$ at $\nu \in S^2$ as a point in \mathbf{R}^3 in the canonical manner. We remark that the outward unit normal to W at point $Y(\nu)$ coincides with ν . And the function γ coincides with the support function of W , that is $\gamma(\nu) = \langle Y(\nu), \nu \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product in \mathbf{R}^3 . This means that W is the standard body for γ .

Let $X: \Sigma \rightarrow \mathbf{R}^3$ be an immersion. By parallel translation in \mathbf{R}^3 , $D\gamma$ may be considered as a smooth tangent vector field along X . Let $X_\epsilon = X + \epsilon\delta X + \mathcal{O}(\epsilon^2)$ be a smooth, compactly supported variation of X . The anisotropic mean curvature Λ of X is defined by the first variation formula ([8])

$$\delta\mathcal{F} := \partial_\epsilon \mathcal{F}[X_\epsilon]_{\epsilon=0} = - \int_\Sigma \Lambda \langle \delta X, \nu \rangle d\Sigma, \quad (2)$$

$$\Lambda := -\text{trace}_\Sigma(D^2\gamma + \gamma 1)d\nu = -\text{div}_\Sigma D\gamma + 2H\gamma, \quad (3)$$

where H is the mean curvature of X . Hence, γ -minimal surfaces are immersed surfaces whose anisotropic mean curvature Λ vanishes at every point. Since the first variation of the ‘‘enclosed volume’’ $V[X] := (1/3) \int_\Sigma \langle X, \nu \rangle d\Sigma$ satisfies

$$\delta V[X] = \int_\Sigma \langle \delta X, \nu \rangle d\Sigma,$$

the equation $\Lambda \equiv \text{constant}$ characterizes critical points of \mathcal{F} with the enclosed volume constrained to be a constant. If Λ is constant, X is called a surface of constant anisotropic mean curvature. In the case where $\gamma \equiv 1$, $\Lambda = 2H$ holds.

Now we extend the function γ in a homogeneous way to a function $\tilde{\gamma}$ as follows.

- (i) $\tilde{\gamma}(X) = 0$ if and only if $X = \mathbf{0}$.
- (ii) positive homogeneity of degree one:

$$\tilde{\gamma}(rX) = r\gamma(X), \quad \forall r \geq 0, X \in \Omega.$$

In the special case where $\gamma(X) \equiv 1$, $\tilde{\gamma}(X) \equiv |X|$.

Let us consider a surface which is a graph of a C^∞ function $\varphi: \Sigma (\subset \mathbf{R}^2) \rightarrow \mathbf{R}$ as follows:

$$X: \Sigma \rightarrow \mathbf{R}^3, \quad X(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)).$$

The unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to X is given by

$$\nu = \frac{(-\varphi_1, -\varphi_2, 1)}{(1 + |D\varphi|^2)^{1/2}}, \quad (4)$$

where

$$\varphi_1 := \varphi_{x_1}, \quad \varphi_2 := \varphi_{x_2}, \quad D\varphi := (\varphi_1, \varphi_2).$$

Lemma 1. *Set $\varphi_{ij} := \varphi_{x_i x_j}$ for $i, j = 1, 2$. Then*

$$\Lambda = \sum_{i,j=1,2} \tilde{\gamma}_{x_i x_j} \Big|_{X=(-D\varphi, 1)} \varphi_{ij} \quad (5)$$

holds. In the special case where $\tilde{\gamma}(X) \equiv |X|$, the right hand side of (5) is

$$\frac{(1 + \varphi_2^2)\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + (1 + \varphi_1^2)\varphi_{22}}{(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}}, \quad (6)$$

which is the twice of the mean curvature H of X .

Proof. In the integrals below, we will write $\varphi(u_1, u_2)$, $((u_1, u_2) \in \Sigma)$, in order to avoid confusion. We have

$$\begin{aligned} \mathcal{F}[X] &= \iint_{\Sigma} \gamma(\nu)(1 + \varphi_1^2 + \varphi_2^2)^{1/2} du_1 du_2 \\ &= \iint_{\Sigma} \tilde{\gamma}((-\varphi_1, -\varphi_2, 1)) du_1 du_2. \end{aligned}$$

Let $X_\epsilon = (x_1, x_2, \varphi(\epsilon, x_1, x_2))$ be an arbitrary compactly-supported variation of X . We will compute the first variation of \mathcal{F} . We may suppose that Σ is the support of X_ϵ and $X_\epsilon|_{\partial\Sigma} = X|_{\partial\Sigma}$ holds. We compute

$$\begin{aligned} \delta\mathcal{F} &= \iint_{\Sigma} \left(\tilde{\gamma}((-\varphi_1, -\varphi_2, 1)) \right)_\epsilon du_1 du_2 \\ &= \iint_{\Sigma} \tilde{\gamma}_{x_1} \cdot (-\varphi_{1\epsilon}) + \tilde{\gamma}_{x_2} \cdot (-\varphi_{2\epsilon}) du_1 du_2 \\ &= \iint_{\Sigma} \frac{\partial \tilde{\gamma}_{x_1}}{\partial u_1} \Big|_{(-D\varphi, 1)} \varphi_\epsilon + \frac{\partial \tilde{\gamma}_{x_2}}{\partial u_2} \Big|_{(-D\varphi, 1)} \varphi_\epsilon du_1 du_2 \\ &\quad - \iint_{\Sigma} (\tilde{\gamma}_{x_1} \varphi_\epsilon)_{u_1} + (\tilde{\gamma}_{x_2} \varphi_\epsilon)_{u_2} du_1 du_2. \end{aligned}$$

By the partial differentiation, the last term of the above equation becomes

$$\int_{\partial\Sigma} (-\tilde{\gamma}_{x_2} \varphi_\epsilon du_1 + \tilde{\gamma}_{x_1} \varphi_\epsilon du_2) = 0,$$

because $\varphi_\epsilon = 0$ on $\partial\Sigma$. Therefore,

$$\begin{aligned} \delta\mathcal{F} &= - \iint_{\Sigma} (\tilde{\gamma}_{x_1 x_1} \varphi_{11} + 2\tilde{\gamma}_{x_1 x_2} \varphi_{12} + \tilde{\gamma}_{x_2 x_2} \varphi_{22}) \varphi_\epsilon du_1 du_2 \\ &= - \iint_{\Sigma} \left(\sum_{i,j=1,2} \tilde{\gamma}_{x_i x_j} \Big|_{X=(-D\varphi, 1)} \varphi_{ij} \right) \langle \delta X, \nu \rangle d\Sigma, \quad (7) \end{aligned}$$

here we used (4) and the followings:

$$\delta X = (0, 0, \varphi_\epsilon), \quad d\Sigma = (1 + |D\varphi|^2)^{1/2} du_1 du_2.$$

In view of (2), (7) implies (5). By a direct computation, we obtain (6). \square

We will give another representation of the anisotropic mean curvature. Let $X: \Sigma \rightarrow \mathbf{R}^3$ be an immersion with Gauss map ν . Let $\{e_1, e_2\}$ be a locally defined frame on S^2 such that $(D^2\gamma + \gamma 1)e_i = (1/\mu_i)e_i$. Note that the basis $\{e_1, e_2\}$ at $\nu(p)$ also serves as an orthogonal basis for the tangent plane of X at p . Let $(-w_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$(D^2\gamma + \gamma 1)d\nu = \begin{pmatrix} -w_{11}/\mu_1 & -w_{12}/\mu_1 \\ -w_{21}/\mu_2 & -w_{22}/\mu_2 \end{pmatrix}.$$

This with (3) gives

$$\Lambda = w_{11}/\mu_1 + w_{22}/\mu_2. \quad (8)$$

Note that $D^2\gamma + \gamma 1$ is the inverse of the differential of the Gauss map of M_γ and so its eigenvalues $1/\mu_j$ are the negatives of the reciprocals of the principal curvatures of the standard body M_γ with respect to the outward unit normal.

For an axisymmetric γ , μ_i 's are represented in terms of γ as follows:

Lemma 2. *Let $\gamma: \Omega \rightarrow \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω of the unit sphere S^2 . Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω . Assume also that γ is axisymmetric, say $\gamma(\nu) = \gamma(\nu_3)$. Then the standard body M_γ for γ is also symmetric with respect to the x_3 -axis. Denote by μ_1, μ_2 the principal curvatures of M_γ with respect to the normal $-\nu$. We let μ_1 be the curvature of the generating curve of M_γ . Then*

$$\mu_1^{-1} = (1 - \nu_3^2)\gamma'' + \mu_2^{-1}, \quad \mu_2^{-1} = \gamma - \nu_3\gamma' \quad (9)$$

holds.

Proof. The proof is the same as the proof of the same formulas for the case where γ satisfies the convexity condition which was given in [8, Section 5]. \square

3. PROOF OF THEOREM 1

In this section, we give a proof of Theorem 1 which was given in the introduction.

Denote by $\langle \cdot, \cdot \rangle_L$ the scalar product for the Minkowski metric $dx_1^2 + dx_2^2 - dx_3^2$ in \mathbf{R}_1^3 . Let $X: \Sigma (\subset \mathbf{R}^2) \rightarrow \mathbf{R}_1^3$ be a spacelike or timelike immersed surface. Let (u_1, u_2) be

local coordinates of Σ . Denote by H_L the mean curvature of X . That is, H_L is defined by

$$H_L = \frac{\tilde{h}_{11}\tilde{g}_{22} - 2\tilde{h}_{12}\tilde{g}_{12} + \tilde{h}_{22}\tilde{g}_{11}}{2(\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2)},$$

where $\tilde{g}_{ij} := \langle X_{u_i}, X_{u_j} \rangle_L$, $\tilde{h}_{ij} := \langle X_{u_i u_j}, \nu^L \rangle_L$ for $i, j = 1, 2$, and ν^L is the unit normal vector field along X for the Minkowski metric. Let $A_L[X]$ be the area of X defined by

$$A_L[X] := \int_{\Sigma} d\Sigma_L, \quad (d\Sigma_L := |\det(\tilde{g}_{ij})| du_1 du_2).$$

Let X_{ϵ} be an arbitrary compactly-supported variation of X . We will compute the first variation of A_L . We may suppose that Σ is the support of X_{ϵ} and $X_{\epsilon}|_{\partial\Sigma} = X|_{\partial\Sigma}$ holds. Set the variation vector field as

$$\delta X := \partial_{\epsilon}(X_{\epsilon})_{\epsilon=0} = \xi + f\nu^L, \quad \left(\xi = \sum_{i=1,2} \xi^i X_{u_i} \right).$$

Then we have the following.

Proposition 2. *In the above setting, it holds that*

$$\partial_{\epsilon} A_L[X_{\epsilon}]_{\epsilon=0} = -2 \int_{\Sigma} f H_L d\Sigma_L.$$

Proof. We here give a proof in the case where X is timelike. By a similar argument, we can prove this in the case where X is spacelike. We have

$$A_L[X_{\epsilon}] = \int_{\Sigma} \sqrt{-\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} + (\tilde{g}_{12}^{\epsilon})^2} du_1 du_2,$$

where $\tilde{g}_{ij}^{\epsilon} = \langle (X_{\epsilon})_{u_i}, (X_{\epsilon})_{u_j} \rangle_L$ for $i, j = 1, 2$. Then,

$$\begin{aligned} \partial_{\epsilon} A_L[X_{\epsilon}] &= \int_{\Sigma} \partial_{\epsilon} \left(\sqrt{-\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} + (\tilde{g}_{12}^{\epsilon})^2} \right) du_1 du_2 \\ &= \int_{\Sigma} \frac{\partial_{\epsilon} (-\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} + (\tilde{g}_{12}^{\epsilon})^2)}{2\sqrt{-\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} + (\tilde{g}_{12}^{\epsilon})^2}} du_1 du_2 \\ &= \int_{\Sigma} \frac{\partial_{\epsilon} (\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} - (\tilde{g}_{12}^{\epsilon})^2)}{2(\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} - (\tilde{g}_{12}^{\epsilon})^2)} d\Sigma_L \\ &= \int_{\Sigma} \frac{\tilde{g}_{22}^{\epsilon} \partial_{\epsilon} \tilde{g}_{11}^{\epsilon} + \tilde{g}_{11}^{\epsilon} \partial_{\epsilon} \tilde{g}_{22}^{\epsilon} - 2\tilde{g}_{12}^{\epsilon} \partial_{\epsilon} \tilde{g}_{12}^{\epsilon}}{2(\tilde{g}_{11}^{\epsilon}\tilde{g}_{22}^{\epsilon} - (\tilde{g}_{12}^{\epsilon})^2)} d\Sigma_L \end{aligned}$$

holds. By a direct calculation, we have

$$\partial_{\epsilon} (\tilde{g}_{ij}^{\epsilon})_{\epsilon=0} = \langle \xi_{u_i}, X_{u_j} \rangle_L + \langle X_{u_i}, \xi_{u_j} \rangle_L - 2f\tilde{h}_{ij}$$

for $i, j = 1, 2$. Applying the divergence theorem, it follows that

$$\begin{aligned} \partial_{\epsilon} A_L[X_{\epsilon}]_{\epsilon=0} &= \int_{\Sigma} \sum_{i=1,2} \left(\tilde{g}^{ij} \langle \xi_{u_i}, X_{u_j} \rangle_L - f\tilde{g}^{ij}\tilde{h}_{ij} \right) d\Sigma_L \\ &= \int_{\Sigma} (\operatorname{div} \xi - 2fH_L) d\Sigma_L = -2 \int_{\Sigma} fH_L d\Sigma_L, \end{aligned}$$

where we denote by (\tilde{g}^{ij}) the inverse matrix of (\tilde{g}_{ij}) . \square

Proof of Theorem 1. First we assume that the surface is a graph of a C^{∞} function $\varphi: \Sigma (\subset \mathbf{R}^2) \rightarrow \mathbf{R}$ as follows:

$$X: \Sigma \rightarrow \mathbf{R}^3, \quad X(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)). \quad (10)$$

The area element $d\Sigma_L$ of X is given by

$$d\Sigma_L = |1 - \varphi_1^2 - \varphi_2^2|^{1/2} dx_1 dx_2.$$

On the other hand, the unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to X and the area element $d\Sigma$ of X for the euclidean metric are

$$\nu = \frac{(-\varphi_1, -\varphi_2, 1)}{(1 + \varphi_1^2 + \varphi_2^2)^{1/2}}, \quad d\Sigma = (1 + \varphi_1^2 + \varphi_2^2)^{1/2} du_1 du_2.$$

Hence,

$$d\Sigma_L = \left(\frac{|1 - \varphi_1^2 - \varphi_2^2|}{1 + \varphi_1^2 + \varphi_2^2} \right)^{1/2} d\Sigma = |\nu_3^2 - \nu_1^2 - \nu_2^2|^{1/2} d\Sigma.$$

Therefore, by (2) and Proposition 2, $\Lambda \equiv 0$ if and only if $H_L \equiv 0$.

Next, we consider the case where the considered surface Σ cannot be represented as a graph like (10). It is sufficient to consider the case where the image of the Gauss map of Σ is contained in the equator $\{(x_1, x_2, 0) \in S^2\}$. In this case, Σ is timelike. It is proved that Σ is represented as

$$X(s, t) = (x_1(s), x_2(s), t),$$

where $C(s) := (x_1(s), x_2(s))$ is a smooth plane curve. Denote by κ the curvature of C . Note that γ can be represented as $\gamma(\nu) = \gamma(\nu_3)$. Then from (8), $\Lambda(s, t) = \gamma(0)\kappa(s)$ holds. On the other hand, $H_L = \kappa/2$ holds. Hence, $\Lambda \equiv 0$ if and only if $H_L \equiv 0$. \square

4. PROOF OF THEOREM 2

Let (x_1, x_2, x_3) be the standard coordinates in \mathbf{R}^3 . We assume that γ is symmetric with respect to the x_3 -axis without loss of generality. So we can write $\gamma = \gamma(\nu_3)$. Assume that γ is not a constant function.

Denote by Σ the considered surface. First assume that Σ is represented as $x_3 = \varphi(x_1, x_2)$. As in §2, we will write

$$\varphi_i := \varphi_{x_i}, \quad \varphi_{ij} := \varphi_{x_i x_j}, \quad (i, j = 1, 2).$$

By the formula (5) and a simple but long computation, we have

$$\begin{aligned} \Lambda &= 2H \left(\gamma - \frac{\gamma'}{(1 + \varphi_1^2 + \varphi_2^2)^{1/2}} - \frac{\gamma''}{1 + \varphi_1^2 + \varphi_2^2} \right) \\ &\quad + \frac{\gamma''(\varphi_{11} + \varphi_{22})}{(1 + \varphi_1^2 + \varphi_2^2)^{3/2}}. \end{aligned}$$

Hence, if $\Lambda = H = 0$ holds, then $\gamma''(\varphi_{11} + \varphi_{22}) = 0$ holds. Since

$$0 = H = \frac{(1 + \varphi_2^2)\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + (1 + \varphi_1^2)\varphi_{22}}{2(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}},$$

we obtain

$$\gamma''(\varphi_2^2\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + \varphi_1^2\varphi_{22}) = 0. \quad (11)$$

Consider any contour line $\varphi(x_1(s), x_2(s)) \equiv \text{constant}$, (s is arc length of the curve $C: (x_1(s), x_2(s))$), of Σ . Denote by κ the curvature of C . Then,

Lemma 3.

$$|\varphi_2^2\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + \varphi_1^2\varphi_{22}| = |\kappa|(\varphi_1^2 + \varphi_2^2)^{3/2} \quad (12)$$

holds.

Proof. Denote by “ ’ ” the derivative with respect to s . We differentiate $\varphi(x_1(s), x_2(s)) \equiv \text{constant}$ with respect to s to obtain

$$\varphi_1 x'_1 + \varphi_2 x'_2 = 0. \quad (13)$$

Differentiate (13) again and use $(x''_1, x''_2) = \kappa(-x'_2, x'_1)$ to obtain

$$\varphi_{11}(x'_1)^2 + 2\varphi_{12}x'_1x'_2 + \varphi_{22}(x'_2)^2 = \kappa(\varphi_1x'_2 - \varphi_2x'_1). \quad (14)$$

By using (13), (14), and the fact that $(x'_1)^2 + (x'_2)^2 = 1$, we obtain (12). \square

Now we assume that the surface is not (a part of) a plane. We remark that it is sufficient to prove that the surface is a part of a right helicoid almost everywhere. So we assume that $\nu \neq (0, 0, \pm 1)$ at any point in Σ , that is (φ_1, φ_2) never coincides with $(0, 0)$. Then, (11) combined with (12) shows that $\gamma'' \equiv 0$ or $\kappa \equiv 0$ holds. If $\gamma'' \equiv 0$, then, by Lemma 2, $\mu_1 \equiv \mu_2$ holds. This means that the standard body M_γ for γ is (a part of) a sphere, and hence γ is a constant function, which contradicts the assumption. Hence $\kappa \equiv 0$ holds, and the curve C is a straight line. Therefore, Σ is a ruled surface. Because only planes and right helicoids are ruled surfaces which are minimal, Σ is a right helicoid.

If Σ is represented as $x_3 = \varphi(x_1, x_2)$ in a connected neighborhood U of a point $P_0 \in \Sigma$, then, by the above argument, U is a part of a right helicoid M . Since $\Sigma_1 := \Sigma \cap M$ is an open and closed subset of a connected set Σ , $\Sigma_1 = \Sigma$ must hold. This means that Σ itself is a part of a right helicoid.

If Σ is not represented as a graph $x_3 = \varphi(x_1, x_2)$ at any point, then $\nu(P)$ is in the equator of S^2 for any $P \in \Sigma$. Hence the Gauss curvature K of Σ vanishes at any point. Since $K \equiv 0 \equiv H$, Σ is a plane which is parallel to the x_3 -axis. \square

5. EXAMPLES

Let $\gamma: \Omega \rightarrow \mathbf{R}_+$ be an axisymmetric positive C^∞ function (say, $\gamma(\nu) = \gamma(\nu_3)$) on a nonempty open set Ω in S^2 . Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point $\nu \in \Omega$.

In this section, we study a special type of cyclic surfaces, that is, surfaces foliated by circles in parallel planes which are orthogonal to the x_3 -axis. So our surfaces are represented as follows:

$$X(\theta, t) = (r(t) \cos \theta + f(t), r(t) \sin \theta + g(t), t). \quad (15)$$

As in Lemma 2, we denote by μ_1, μ_2 the principal curvatures of the standard body M_γ with respect to the normal $-\nu$, here μ_1 is the curvature of the generating curve of M_γ .

Lemma 4. *The anisotropic mean curvature of X in (15) is given by*

$$\Lambda = \frac{r(r'' + f'' \cos \theta + g'' \sin \theta) - (f' \sin \theta - g' \cos \theta)^2}{\mu_1 r \{(r' + f' \cos \theta + g' \sin \theta)^2 + 1\}^{\frac{3}{2}}} - \frac{1}{\mu_2 r \sqrt{(r' + f' \cos \theta + g' \sin \theta)^2 + 1}}. \quad (16)$$

Proof. Let ν be the Gauss map of X as usual. Let $\{e_1, e_2\}$ be a locally defined frame on S^2 such that $(D^2\gamma + \gamma 1)e_i = (1/\mu_i)e_i$. Note that the basis $\{e_1, e_2\}$ at $\nu(p)$ also serves as an orthogonal basis for the tangent plane of X at p . As in §2, let $(-w_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$(D^2\gamma + \gamma 1)d\nu = \begin{pmatrix} -w_{11}/\mu_1 & -w_{12}/\mu_1 \\ -w_{21}/\mu_2 & -w_{22}/\mu_2 \end{pmatrix},$$

and

$$\Lambda = w_{11}/\mu_1 + w_{22}/\mu_2 \quad (17)$$

holds. So, we will compute the matrix (w_{ij}) .

Let $\nu^M = (\nu_1^M, \nu_2^M, \nu_3^M)$ be the outward pointing unit normal to M_γ . Since M_γ is a surface of revolution, $D^2\gamma + \gamma 1$ has eigendirections corresponding to

$$E_1 = (0, 0, 1) - \nu_3^M \nu^M, \quad E_2 = \nu^M \times E_1 \quad (18)$$

as long as the normal is not vertical. E_1, E_2 define an orthonormal basis $\{e_1, e_2\}$ on TS^2 as long as X does not intersect with the vertical axis.

Set $g_{11} = \langle X_\theta, X_\theta \rangle$, $g_{12} = g_{21} = \langle X_\theta, X_t \rangle$, $g_{22} = \langle X_t, X_t \rangle$, $h_{11} = \langle X_{\theta\theta}, \nu \rangle$, $h_{12} = h_{21} = \langle X_{\theta t}, \nu \rangle$, $h_{22} = \langle X_{tt}, \nu \rangle$. And set

$$\Delta := r' + f' \cos \theta + g' \sin \theta.$$

Then,

$$\begin{aligned} g_{11} &= r^2, \\ g_{12} &= -r f' \sin \theta + r g' \cos \theta, \\ g_{22} &= (r')^2 + 2r' f' \cos \theta + 2r' g' \sin \theta + (f')^2 + (g')^2 + 1, \\ h_{11} &= \frac{-r}{\sqrt{\Delta^2 + 1}}, \\ h_{12} &= 0, \\ h_{22} &= \frac{r'' + f'' \cos \theta + g'' \sin \theta}{\sqrt{\Delta^2 + 1}}, \end{aligned}$$

and

$$\nu = (\nu_1, \nu_2, \nu_3) := \frac{X_\theta \times X_t}{|X_\theta \times X_t|} = \frac{1}{\sqrt{\Delta^2 + 1}} (\cos \theta, \sin \theta, -\Delta).$$

We have

$$\tilde{E}_1 := (0, 0, 1) - \nu_3 \nu = \frac{1}{\Delta^2 + 1} (\Delta \cos \theta, \Delta \sin \theta, 1).$$

Hence,

$$e_1 = \frac{\tilde{E}_1}{|\tilde{E}_1|} = \frac{1}{\sqrt{\Delta^2 + 1}} (\Delta \cos \theta, \Delta \sin \theta, 1), \quad (19)$$

$$e_2 = \nu \times e_1 = (\sin \theta, -\cos \theta, 0). \quad (20)$$

Now we take a coordinate transformation $\theta(u, v), t(u, v)$ so that, at an arbitrary fixed point (u_0, v_0) ,

$$\frac{\partial X}{\partial u} = e_1, \quad \frac{\partial X}{\partial v} = e_2$$

are satisfied. Then, we have

$$\frac{\partial X}{\partial \theta} \theta_u + \frac{\partial X}{\partial t} t_u = e_1, \quad \frac{\partial X}{\partial \theta} \theta_v + \frac{\partial X}{\partial t} t_v = e_2. \quad (21)$$

Inserting $X_\theta = (-r \sin \theta, r \cos \theta, 0)$, $X_t = (r' \cos \theta + f', r' \sin \theta + g', 1)$, (19), and (20) to (21), we obtain

$$J := \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix} = \begin{pmatrix} \frac{f' \sin \theta - g' \cos \theta}{r\sqrt{\Delta^2 + 1}} & -\frac{1}{r} \\ \frac{1}{\sqrt{\Delta^2 + 1}} & 0 \end{pmatrix}, \quad \det J > 0.$$

Let $(w_{ij}), (\tilde{w}_{ij})$ be the Weingarten mappings for $X(u, v), X(\theta, t)$, respectively. Then,

$$\begin{aligned} (\tilde{w}_{ij}) &= (g_{ij})^{-1}(h_{ij}), \\ (w_{ij}) &= \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix}^{-1} (\tilde{w}_{ij}) \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix} = J^{-1}(\tilde{w}_{ij})J. \end{aligned}$$

Hence, by a computation, we obtain

$$\begin{aligned} w_{11} &= \frac{r(r'' + f'' \cos \theta + g'' \sin \theta) - (f' \sin \theta - g' \cos \theta)^2}{r(\Delta^2 + 1)^{\frac{3}{2}}}, \\ w_{22} &= -\frac{1}{r\sqrt{\Delta^2 + 1}}. \end{aligned}$$

This with (17) gives (16). □

Now we assume that the standard body M_γ for γ is a quadric surface of revolution. Then, by homothety and translation, M_γ is one of the followings:

- (I) a spheroid: $x_1^2 + x_2^2 + \frac{x_3^2}{a^2} = 1$,
- (II) a hyperboloid of two sheets: $x_1^2 + x_2^2 - \frac{x_3^2}{a^2} = -1$,
- (III) a hyperboloid of one sheet: $x_1^2 + x_2^2 - \frac{x_3^2}{a^2} = 1$,
- (IV) a circular paraboloid: $x_3 = a(x_1^2 + x_2^2)$,

where a is a positive constant.

Lemma 5. *The support functions γ of M_γ in the above (I)–(IV) are respectively given by the followings:*

- (I) $\gamma(\nu_3) = \sqrt{1 + b\nu_3^2}$, ($b := a^2 - 1 > -1$),
- (II) $\gamma(\nu_3) = \sqrt{-1 + b\nu_3^2}$, ($b := a^2 + 1 > 1, \frac{1}{\sqrt{b}} < |\nu_3| \leq 1$),

$$(III) \gamma(\nu_3) = \sqrt{1 - b\nu_3^2}, \quad (b := a^2 + 1 > 1, |\nu_3| < \frac{1}{\sqrt{b}}),$$

$$(IV) \gamma(\nu_3) = \frac{-1 + \nu_3^2}{b\nu_3}, \quad (b := 4a > 0, \nu_3 \neq 0).$$

Proof. (I) Represent the upper half of M_γ as

$$Y(x_1, x_2) = (x_1, x_2, a\sqrt{1 - x_1^2 - x_2^2}).$$

The outward pointing unit normal ν to Y is given by

$$\nu = \frac{1}{\sqrt{1 + (a^2 - 1)(x_1^2 + x_2^2)}} (ax_1, ax_2, \sqrt{1 - x_1^2 - x_2^2}).$$

Hence, we obtain

$$\gamma = \langle Y, \nu \rangle = \sqrt{1 + (a^2 - 1)\nu_3^2} = \sqrt{1 + b\nu_3^2},$$

which proves (I).

Similarly, we obtain (II)–(IV). □

Proposition 3. *Let γ be a function given by the above (I)–(IV). Then, there exist γ -minimal surfaces foliated by circles contained in parallel planes which are orthogonal to the x_3 -axis. Up to translations in \mathbf{R}^3 , rotations around the x_3 -axis, and symmetry with respect to a plane $\{x_3 = \text{constant}\}$, they are respectively represented as follows.*

(I) *Catenoid-type:*

$$X(\theta, t) = \left(\frac{\cosh(ct)}{c\sqrt{1+b}} \cos \theta, \frac{\cosh(ct)}{c\sqrt{1+b}} \sin \theta, t \right), \quad c \neq 0. \quad (22)$$

Riemann-type:

$$\begin{aligned} X(\theta, r) &= \left(r \cos \theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 - 1}}, r \sin \theta, \right. \\ &\quad \left. \sqrt{1+b} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 - 1}} \right), \end{aligned} \quad (23)$$

$$c_1 \neq 0, \quad r \geq \left(\frac{-c_2 + \sqrt{c_2^2 + 4c_1^2}}{2c_1^2} \right)^{1/2}. \quad (24)$$

(II) *Catenoid-type:*

$$\begin{aligned} X(\theta, t) &= \left(\frac{\sinh(ct)}{c\sqrt{b-1}} \cos \theta, \frac{\sinh(ct)}{c\sqrt{b-1}} \sin \theta, t \right), \quad (25) \\ c &\neq 0, \quad t \neq 0. \end{aligned} \quad (26)$$

Riemann-type:

$$\begin{aligned} X(\theta, r) &= \left(r \cos \theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, r \sin \theta, \right. \\ &\quad \left. \sqrt{b-1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}} \right), \quad (27) \\ c_1 &\neq 0, \quad c_2 > 2|c_1|, \quad r > 0. \end{aligned} \quad (28)$$

(III) *Catenoid-type:*

$$\begin{aligned} X(\theta, t) &= \left(\frac{\sin(ct)}{c\sqrt{b-1}} \cos \theta, \frac{\sin(ct)}{c\sqrt{b-1}} \sin \theta, t \right), \quad (29) \\ c &\neq 0, \quad \sin(ct) \neq 0. \end{aligned} \quad (30)$$

Riemann-type:

$$X(\theta, r) = \left(r \cos \theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, r \sin \theta, \sqrt{b-1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}} \right), \quad (31)$$

$$c_1 \neq 0, c_2 < -2|c_1|, 0 < r \leq \left(\frac{|c_2| - \sqrt{c_2^2 - 4c_1^2}}{2c_1^2} \right)^{1/2}. \quad (32)$$

(IV) Catenoid-type:

$$X(\theta, t) = (e^{ct} \cos \theta, e^{ct} \sin \theta, t), \quad c \neq 0. \quad (33)$$

Riemann-type:

$$X(\theta, r) = \left(r \cos \theta + \int \frac{c_1 r dr}{\sqrt{c_1^2 r^2 + c_2}}, r \sin \theta, \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2}} \right), \quad (34)$$

$$c_1 \neq 0, c_2 > 0, r > 0. \quad (35)$$



(I) Catenoid-type (I) Riemann-type

Figure 1: γ -minimal surfaces for $\gamma(\nu)$ as in (I) of Lemma 5.



(II) Catenoid-type (II) Riemann-type

Figure 2: γ -minimal surfaces for $\gamma(\nu)$ as in (II) of Lemma 5.



(III) Catenoid-type (III) Riemann-type

Figure 3: γ -minimal surfaces for $\gamma(\nu)$ as in (III) of Lemma 5.

Remark 1. Let γ be a function given in (I), (II), or (III) in Lemma 5. Set $\alpha := \sqrt{b+1}$ for (I), and $\alpha := \sqrt{b-1}$ for (II) and (III). If we take the transformation $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = x_3/\alpha$, then an immersion $X = (x_1, x_2, x_3)$ in \mathbf{R}^3 is γ -minimal if and only if $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a minimal



(IV) Catenoid-type (IV) Riemann-type

Figure 4: γ -minimal surfaces for $\gamma(\nu)$ as in (IV) of Lemma 5.

surface (for (I)), a spacelike zero mean curvature surface in \mathbf{R}_1^3 (for (II)), a timelike zero mean curvature surface in \mathbf{R}_1^3 (for (III)), respectively. This is proved by the same way as Example 4.4 in [8].

Remark 2. γ -minimal surfaces for γ given in (IV) in Lemma 5 are graphs of harmonic functions (cf. [12]).

Proof of Proposition 3. Let μ_1, μ_2 be the principal curvatures of M_γ with respect to the normal $-\nu$ in the same way as Lemma 2.

We represent the surface as

$$X(\theta, t) = (r(t) \cos \theta + f(t), r(t) \sin \theta + g(t), t). \quad (36)$$

Note that X is a surface of revolution around the x_3 -axis if and only if $f \equiv g \equiv 0$ holds.

As in the proof of Lemma 4, we set

$$\Delta := r' + f' \cos \theta + g' \sin \theta.$$

Then, the Gauss map ν of X is

$$\begin{aligned} \nu(\theta, t) &:= \frac{X_\theta \times X_t}{|X_\theta \times X_t|} \\ &= \frac{1}{(\Delta^2 + 1)^{1/2}} (\cos \theta, \sin \theta, -\Delta). \end{aligned}$$

(I) By a simple computation using Lemma 2, we obtain

$$\frac{1}{\mu_1} = \frac{b+1}{(1+b\nu_3^2)^{3/2}}, \quad \frac{1}{\mu_2} = \frac{1}{\sqrt{1+b\nu_3^2}}.$$

Since $\nu_3 = \frac{-\Delta}{(\Delta^2 + 1)^{1/2}}$, we obtain

$$\frac{1}{\mu_1} = (b+1) \left(\frac{1+\Delta^2}{1+(b+1)\Delta^2} \right)^{3/2}, \quad (37)$$

$$\frac{1}{\mu_2} = \left(\frac{1+\Delta^2}{1+(b+1)\Delta^2} \right)^{1/2}. \quad (38)$$

By Lemma 4 with (37) and (38), we see that $\Lambda = 0$ if and only if

$$\begin{aligned} &(b+1)(rf'' - 2r'f') \cos \theta + (b+1)(rg'' - 2r'g') \sin \theta \\ &+ (b+1)(r''r - (r')^2 - (f')^2 - (g')^2) - 1 = 0 \end{aligned}$$

holds. This gives the following system of ordinary differential equations:

$$rf'' - 2r'f' = 0, \quad (39)$$

$$rg'' - 2r'g' = 0, \quad (40)$$

$$(b+1)(r''r - (r')^2 - (f')^2 - (g')^2) - 1 = 0. \quad (41)$$

From (39) and (40), we have

$$f' = c_1 r^2, \quad g' = c_2 r^2. \quad (42)$$

First, assume $f' = g' = 0$. Then, (41) is equivalent to

$$(r''r - (r')^2) - \frac{1}{b+1} = 0. \quad (43)$$

By a standard way, we see that the general solution of (43) is

$$r = \frac{\cosh(c_3(t + c_4))}{c_3 \sqrt{b+1}}, \quad c_3 \neq 0, \quad (44)$$

which gives the formula (22).

Next, we assume that $f' \neq 0$ or $g' \neq 0$ holds. Because of (42), $c_2 f' - c_1 g' = 0$ holds. This implies that $c_2 f - c_1 g = \text{constant}$. Since $(c_1, c_2) \neq (0, 0)$, by rotating the surface around the x_3 -axis if necessary, we may assume that

$$f' = c_1 r^2 \quad (c_1 \neq 0), \quad g(t) \equiv 0 \quad (45)$$

holds. Then, (41) is equivalent to

$$1 + (b+1)((r')^2 - r''r + c_1^2 r^4) = 0.$$

From this, by a standard argument, we obtain

$$\frac{dr}{dt} = \pm \sqrt{c_1^2 r^4 + c_5 r^2 - \frac{1}{b+1}}. \quad (46)$$

Hence,

$$\begin{aligned} t &= \pm \int \frac{dr}{\sqrt{c_1^2 r^4 + c_5 r^2 - \frac{1}{b+1}}} \\ &= \pm \sqrt{b+1} \int \frac{dr}{\sqrt{c_6^2 r^4 + c_7 r^2 - 1}}, \end{aligned} \quad (47)$$

$$c_6 := \sqrt{b+1}c_1 \neq 0, \quad c_7 := (b+1)c_5.$$

By using (45) and (46), we easily obtain

$$f = \pm \int \frac{c_6 r^2 dr}{\sqrt{c_6^2 r^4 + c_7 r^2 - 1}}, \quad c_6 \neq 0. \quad (48)$$

(47) with (48) gives the formula (23). Moreover,

$$c_1^2 r^4 + c_2 r^2 - 1 \geq 0$$

if and only if

$$r \geq \left(\frac{-c_2 + \sqrt{c_2^2 + 4c_1^2}}{2c_1^2} \right)^{1/2}$$

holds, which gives the condition (24).

(II) The proof is similar to the proof of (I). We have

$$\begin{aligned} \frac{1}{\mu_1} &= \frac{-(b-1)}{(-1 + b\nu_3^2)^{3/2}} \\ &= -(b-1) \left(\frac{1 + \Delta^2}{-1 + (b-1)\Delta^2} \right)^{3/2}, \end{aligned}$$

$$\frac{1}{\mu_2} = \frac{-1}{\sqrt{-1 + b\nu_3^2}} = - \left(\frac{1 + \Delta^2}{-1 + (b-1)\Delta^2} \right)^{1/2}.$$

We see that $\Lambda = 0$ if and only if

$$\begin{aligned} (b-1)(rf'' - 2r'f') \cos \theta + (b-1)(rg'' - 2r'g') \sin \theta \\ + (b-1)\{r''r - (r')^2 - (f')^2 - (g')^2\} + 1 = 0 \end{aligned}$$

holds. This gives the following system of ordinary differential equations:

$$rf'' - 2r'f' = 0, \quad (49)$$

$$rg'' - 2r'g' = 0, \quad (50)$$

$$(b-1)(r''r - (r')^2 - (f')^2 - (g')^2) + 1 = 0. \quad (51)$$

First, assume $f' = g' = 0$. Then, (51) is equivalent to

$$(r''r - (r')^2) + \frac{1}{b-1} = 0. \quad (52)$$

We have the following three types of general solutions of (52):

$$r = \frac{1}{\sqrt{b-1}}t + c, \quad (53)$$

$$r = \frac{\sinh(c_1(t + c_2))}{c_1 \sqrt{b-1}}, \quad c_1 \neq 0, \quad (54)$$

$$r = \frac{\sin(c_1(t + c_2))}{c_1 \sqrt{b-1}}, \quad c_1 \neq 0. \quad (55)$$

By a suitable translation, the corresponding surfaces are given by

$$X(\theta, t) = \left(\frac{t}{\sqrt{b-1}} \cos \theta, \frac{t}{\sqrt{b-1}} \sin \theta, t \right), \quad (56)$$

$$X(\theta, t) = \left(\frac{\sinh(ct)}{c\sqrt{b-1}} \cos \theta, \frac{\sinh(ct)}{c\sqrt{b-1}} \sin \theta, t \right), \quad c \neq 0, \quad (57)$$

$$X(\theta, t) = \left(\frac{\sin(ct)}{c\sqrt{b-1}} \cos \theta, \frac{\sin(ct)}{c\sqrt{b-1}} \sin \theta, t \right), \quad c \neq 0, \quad (58)$$

respectively. Later, we will show that in order that the surface is compatible with γ , the surface must be given by (57), which gives the formula (25).

Next, we assume that $f' \neq 0$ or $g' \neq 0$ holds. By rotating the surface around the x_3 -axis if necessary, we may assume that

$$f' = c_1 r^2 \quad (c_1 \neq 0), \quad g(t) \equiv 0 \quad (59)$$

holds. Then, (51) is equivalent to

$$1 - (b-1)((r')^2 - r''r + c_1^2 r^4) = 0. \quad (60)$$

From this, by a standard argument, we obtain

$$\frac{dr}{dt} = \pm \sqrt{c_1^2 r^4 + c_2 r^2 + \frac{1}{b-1}}. \quad (61)$$

Hence,

$$\begin{aligned} t &= \pm \sqrt{b-1} \int \frac{dr}{\sqrt{c_3^2 r^4 + c_4 r^2 + 1}}, \\ c_3 &:= \sqrt{b-1}c_1 \neq 0, \quad c_4 := (b-1)c_2. \end{aligned} \quad (62)$$

By using $f' = c_1 r^2$ and (61), we obtain

$$f = \pm \int \frac{c_3 r^2 dr}{\sqrt{c_3^2 r^4 + c_4 r^2 + 1}}, \quad c_3 \neq 0. \quad (63)$$

(62) with (63) gives the formula (27).

Next, we will check whether the surface is compatible with γ or not. Note that the surface X is compatible with γ if and only if its Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies $b\nu_3^2 - 1 > 0$ for all θ .

For the surface X given by (56), the Gauss map ν is

$$\begin{aligned} \nu &:= \frac{X_\theta \times X_t}{|X_\theta \times X_t|} \\ &= b^{-1/2} \left(\sqrt{b-1} \cos \theta, \sqrt{b-1} \sin \theta, -1 \right). \end{aligned}$$

This shows that $b\nu_3^2 - 1 \equiv 0$, and hence X is not compatible with γ .

For the surface X given by (57),

$$\nu = \left(1 + \frac{\cosh^2(ct)}{b-1} \right)^{-1/2} \left(\cos \theta, \sin \theta, -\frac{\cosh(ct)}{\sqrt{b-1}} \right).$$

This shows that $b\nu_3^2 - 1 \geq 0$ always holds, and that $b\nu_3^2 - 1 > 0$ for all θ if and only if $t \neq 0$.

For the surface X given by (58),

$$\nu = \left(1 + \frac{\cos^2(ct)}{b-1} \right)^{-1/2} \left(\cos \theta, \sin \theta, -\frac{\cos(ct)}{\sqrt{b-1}} \right). \quad (64)$$

This shows that $b\nu_3^2 - 1 \leq 0$ always holds, and hence X is not compatible with γ .

For the surface X given by (27), the Gauss map ν is

$$\nu := \frac{X_\theta \times X_r}{|X_\theta \times X_r|}, \quad (65)$$

$$\begin{aligned} X_\theta \times X_r &= \left(\frac{\sqrt{b-1}r \cos \theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}}, \frac{\sqrt{b-1}r \sin \theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}}, \right. \\ &\quad \left. -r - \frac{c_1 r^3 \cos \theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}} \right). \end{aligned} \quad (66)$$

This shows that, $b\nu_3^2 - 1 > 0$ for all θ if and only if $c_2 > 2|c_1|$ holds.

(III) The proof is again similar to the proof of (I). We see that the condition $\Lambda = 0$ is equivalent to the condition that the system of ordinary differential equations (49), (50) and (51) holds.

First, assume $f' = g' = 0$. Then, (51) is equivalent to (52). Note that $b-1 > 0$. The general solutions of (52) are given by (53), (54), and (55), and corresponding surfaces are given by (56), (57), and (58). Note that the surface X is compatible with γ if and only if its Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies $1 - b\nu_3^2 > 0$ for all θ . As we have seen above, for the surfaces (56) and (57), $b\nu_3^2 - 1 \geq 0$ holds at every point. Hence, they are not compatible with γ . Therefore, only the possibility is the case (58), which is the same as the formula (29). The Gauss map ν for this

surface is given by (64), which shows that, $1 - b\nu_3^2 > 0$ for all θ if and only if $\sin(ct) \neq 0$.

Next, we assume that $f' \neq 0$ or $g' \neq 0$ holds. By rotating the surface around the x_3 -axis if necessary, we may assume that f and g satisfy (59). Then, (51) is equivalent to (60). Hence, we obtain the formula (31).

We will check whether the surface is compatible with γ or not. The Gauss map ν is given by (65), (66) as in the case (II). This shows that, $1 - b\nu_3^2 > 0$ for all θ if and only if both $c_2 < -2|c_1|$ and

$$0 < r \leq \left(\frac{|c_2| - \sqrt{c_2^2 - 4c_1^2}}{2c_1^2} \right)^{1/2}$$

hold, which gives the condition (32).

(IV) The proof is again similar to the proof of (I). We obtain

$$\frac{1}{\mu_1} = \frac{-2}{b\nu_3^3} = \frac{2(\Delta^2 + 1)^{3/2}}{b\Delta^3}, \quad (67)$$

$$\frac{1}{\mu_2} = \frac{-2}{b\nu_3} = \frac{2(\Delta^2 + 1)^{1/2}}{b\Delta}. \quad (68)$$

By Lemma 4 with (67) and (68), we see that $\Lambda = 0$ if and only if

$$\begin{aligned} (rf'' - 2r'f') \cos \theta + (rg'' - 2r'g') \sin \theta \\ + (r''r - (r')^2 - (f')^2 - (g')^2) = 0 \end{aligned}$$

holds. This gives the following system of ordinary differential equations:

$$rf'' - 2r'f' = 0, \quad (69)$$

$$rg'' - 2r'g' = 0, \quad (70)$$

$$r''r - (r')^2 - (f')^2 - (g')^2 = 0. \quad (71)$$

From (69) and (70), we have

$$f' = c_1 r^2, \quad g' = c_2 r^2. \quad (72)$$

When $f' = g' = 0$ holds, (71) is equivalent to

$$r''r - (r')^2 = 0. \quad (73)$$

The general solution of (73) is

$$r = e^{c_1 t + c_2},$$

which gives the formula (33).

When $f' \neq 0$ or $g' \neq 0$ holds, by rotating the surface around the x_3 -axis if necessary, we may assume that

$$f' = c_1 r^2 \quad (c_1 \neq 0), \quad g(t) \equiv 0 \quad (74)$$

holds. Then, (71) is equivalent to

$$(r')^2 - r''r + c_1^2 r^4 = 0. \quad (75)$$

From this, by a standard argument, we obtain

$$\frac{dr}{dt} = \pm \sqrt{c_1^2 r^4 + c_2 r^2}. \quad (76)$$

Hence,

$$t = \pm \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2}}. \quad (77)$$

By using $f' = c_1 r^2$ and (76), we obtain

$$f = \pm \int \frac{c_1 r dr}{\sqrt{c_1^2 r^2 + c_2}}, \quad c_1 \neq 0. \quad (78)$$

(77) with (78) gives the formula (34).

Next, we will check whether the surface is compatible with γ or not. Note that the surface X is compatible with γ if and only if its Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies $\nu_3 \neq 0$ for all θ .

For the surface X given by (33), the Gauss map ν is

$$\begin{aligned} \nu &:= \frac{X_\theta \times X_t}{|X_\theta \times X_t|} \\ &= \left(1 + c^2 e^{2ct}\right)^{-1/2} \left(\cos \theta, \sin \theta, -ce^{ct}\right). \end{aligned}$$

This shows that, $\nu_3 \neq 0$ for all θ if and only if $c \neq 0$.

For the surface X given by (34), the Gauss map ν is

$$\begin{aligned} \nu &:= \frac{X_\theta \times X_r}{|X_\theta \times X_r|}, \\ X_\theta \times X_r &= \left(\frac{\cos \theta}{(c_1^2 r^2 + c_2)^{1/2}}, \frac{\sin \theta}{(c_1^2 r^2 + c_2)^{1/2}}, \right. \\ &\quad \left. -r \left(1 + \frac{c_1 r \cos \theta}{(c_1^2 r^2 + c_2)^{1/2}}\right) \right). \end{aligned}$$

This shows that, $\nu_3 \neq 0$ for all θ if and only if $c_2 > 0$ holds. \square

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