Bipartition of graphs based on the normalized cut and spectral methods, Part I : Minimum normalized cut

Perera, K. K. K. R.

Mizoguchi, Yoshihiro

http://hdl.handle.net/2324/1397719

バージョン：published
権利関係：
Bipartition of graphs based on the normalized cut and spectral methods, Part I: Minimum normalized cut

K. K. K. R. Perera and Yoshihiro Mizoguchi

Received on October 20, 2012 / Revised on April 7, 2013

Abstract. The main objective of this paper is to solve the problem of finding graphs on which the spectral clustering method and the normalized cut produce different partitions. To this end, we derive formulae for minimum normalized cut for graphs in some classes such as paths, cycles, complete graphs, double-trees, lollipop graphs $LP_{n,m}$, roach type graphs $R_{n,k}$ and weighted paths $P_{n,k}$.

Keywords. spectral clustering, normalized Laplacian matrices, difference Laplacian matrices, signless Laplacian matrices, normalized cut

1. Introduction

Clustering techniques are common in multivariate data analysis, data mining, machine learning, and so on. The goal of the clustering or partitioning problem is to find groups such that entities within a same group are similar and entities within different groups are dissimilar. In the graph-partitioning problem, much attention is given to find the precise criteria to obtain a good partition. Clustering methods that use eigenvalues and eigenvectors of matrices associated with graphs are called spectral clustering methods, and are widely used in graph-partitioning problems. In particular, eigenvalues and eigenvectors of Laplacian matrices play a vital role in graph-partitioning problems. In 1973, Fiedler defined the second smallest eigenvalue $\lambda_2$ of a difference Laplacian matrix as the algebraic connectivity of a graph [7]. In 1975, he showed that we can decompose a graph $G$ into two connected components by only using the sign structure of an eigenvector related to the second smallest eigenvalue $[8]$. In 2001, Fiedler’s investigation was extended by Davies using the discrete nodal domain theorem [5]. Eigenvectors of Laplacian, normalized Laplacian or adjacency matrices with negative off diagonal entries can be used for the nodal domain theorem. This theorem is useful to identify the number of connected sign graphs of a given graph on the basis of their eigenvectors and eigenvalues.

In 1984, Buser [3] investigated the graph invariant quantity $i(G) = \min_U \frac{|\partial U|}{|U|}$, which considers the relationship between the size of a cut and the size of a separate subset $U$ of vertices of $G$. He defined the isoperimetric number $i(G)$ and the optimal bisection given by the minimum $i(G)$. Guattery and Miller [9, 10] considered two spectral separation algorithms that partition vertices on the basis of the values of their corresponding entries in the second eigenvector and they provide some counter examples for which each of these algorithms produce poor separators. They used an eigenvector based on the second smallest eigenvalue of a difference Laplacian matrix as well as a specified number of eigenvectors corresponding to small eigenvalues. Finally, they extend their algorithm to a generalized version of spectral methods that allows for the use of more than a constant number of eigenvectors and showed that there are some graphs for which the performance of all the above spectral algorithms was poor. We follow their methods, especially in the cases of graph automorphism and even-odd eigenvector theorem, for the concrete classes of graphs such as roach graphs, double-trees, and double-tree cross paths. We prefer to use a normalized Laplacian matrix rather than a difference Laplacian matrix, and describe these properties in terms of formal graph notations.

In 1997, Fan Chung [4] discussed important theories and properties regarding eigenvalues of normalized Laplacian matrices and their applications to graph separator problems. She considered the partitioning problem using Cheeger constants and derived fundamental relations between the eigenvalues and Cheeger constants. In 2000, Shi and Malik [14] proposed a measure of disassociation, called normalized cut, for the image segmentations. This measure is defined by the cut cost as a fraction of total edge connections. The normalized cut is used to minimize the disassociation between groups and maximize the association within groups. However, minimization of normalized cut criteria is an non-deterministic polynomial-time hard (NP-hard) problem. Therefore, approximate solutions are required. A solution to the minimization problem of the normalized cut is given by the second smallest eigenvector of the generalized eigensystem, $(D - W)y = \lambda Dy$, where $D$ is the diagonal matrix with vertex degrees and $W$ is a
weighted adjacency matrix. Shi and Malik used a minimum
normalized cut value as a splitting point and found a
biason using a second smallest eigenvector. They rea-
alized that second smallest eigenvectors are well separated
and that this type of splitting point is very reliable. The
normalized cut introduced by Shi and Malik [14] is use-
ful in several areas. This measure is of interest, not only
for image segmentation, but also for network theories and
statistics [1, 13, 12, 6, 15].

In this paper, we review the known results regarding
the difference, normalized, and signless Laplacian matrices.
Then, we analyze the minimum normalized cut from the
view point of connectivity of graphs. We use the term
Mcut(G) to represent the minimum normalized cut. Find-
ing Mcut(G) for a graph is NP-hard. However, we derive
formulæ for Mcut(G) for some basic classes of graphs such
as paths, cycles, complete graphs, double-trees, and some
complex graphs like lollipop type graphs LP_{n,m}, roach
path graphs R_{n,k}, and weighted paths P_{n,k}.

This paper is organized as follows. In section 2, we
present basic terminologies and key results related to the
difference, normalized, and signless Laplacian matrices.
In particular, we summarize the upper and lower bounds of
the second smallest eigenvalues. We also define graphs that
are used in other sections using formal notation. In sec-
tion 3, we investigate properties of Mcut(G) of graphs
and derive formulæ for Mcut(G) of a graph G in some basic
classes of graphs including some complex graphs such as
R_{n,k}, P_{n,k} and LP_{n,m}.

2. Preliminaries

An undirected graph is an ordered pair \( G = (V(G), E(G)) \),
where \( V(G) = \{v_1, v_2, \ldots, v_n\} \) is a finite set, an ele-
ment of which is called vertex and \( E(G) \) is a set of two-element
subsets of \( V(G) \), called an edge. Conventionally, we denote
an edge \( \{v_i, v_j\} \) by \( (v_i, v_j) \) in this paper. Two vertices \( v_i \)
and \( v_j \) of \( G \) are called adjacent if \( (v_i, v_j) \in E(G) \).
For simplicity, we sometimes write \( V \) instead of \( V(G) \)
and \( E \) instead of \( E(G) \). The order of \( G \) is the number of vertices
in \( G \). A trivial graph is a graph of order 1 with no edges.
The size of \( G \) is the number of vertices. An empty graph is
a graph of size 0. For a given subset \( A \) of \( V, |A| \) represents
the size of the set \( A \). We denote the set of vertices not
belonging to \( A \) as \( V \setminus A = \{v \in V: v \notin A\} \).

Definition 1 (Adjacency matrix). Let \( G = (V(E)) \) be
a graph and \( |V| = n \). The adjacency matrix \( A(G) = (a_{ij}) \)
of an undirected graph \( G \) is an \( n \times n \) matrix whose entries are
given by
\[
a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise}. \end{cases}
\]

Definition 2 (Degree). The degree \( d_i \) of a vertex \( v_i \) of
a graph \( G \) is defined as \( d_i = \sum_{j=1}^{n} a_{ij} \). The minimum degree
and the maximum degree of a graph \( G \) are denoted by \( \delta(G) \)
and \( \Delta(G) \), respectively.

Definition 3 (Degree Matrix). The degree matrix of a
graph \( G \) is denoted by \( D(G) = \text{diag}(d_1, d_2, \ldots, d_n) \), where \( d_i \)
is the degree of the vertex \( v_i \).

Note: For simplicity, we sometimes write \( D \) instead of
\( D(G) \).

Definition 4 (Volume). The volume of a graph \( G = (V,E) \)
denoted by \( \text{vol}(G) = \sum_{i=1}^{n} d_i \), is the sum of degrees of
vertices in \( V \). The volume of a subset \( A \subset V \) is denoted
by \( \text{vol}(A) = \sum_{i \in A} d_i \).

Definition 5 (Weighted graph). A weighted graph is de-
noted by \( G = (V, E, w) \), where \( w: E \rightarrow \mathbb{R} \).

Definition 6 (Weighted adjacency matrix). The weighted
adjacency matrix \( W(G) = (w_{ij}) \) of a weighted graph \( G = (V,E,w) \)
is defined as
\[
w_{ij} = \begin{cases} w(i,j) & \text{if } (i,j) \in E, \\ 0 & \text{otherwise}. \end{cases}
\]
The degree \( d_i \) of a vertex \( v_i \) of a weighted graph is de-
noted by \( d_i = \sum_{j=1}^{n} w_{ij} \). Unweighted graphs are special cases,
where all edge weights are 0 or 1.

Definition 7 (Edge Connectivity). The edge connectiv-
ity of a graph \( G \), denoted by \( k'(G) \), is the minimum number of
edges needed to remove in order to disconnect the graph.
A graph is called \( k \)-edge connected if every disconnecting
set has at least \( k \) edges. A 1-edge connected graph is called
a connected graph.

Definition 8 (Cartesian product). The Carte-
sian product of graphs \( G \) and \( H \) is denoted by
\( G \Box H = (V(G \Box H), E(G \Box H)) \), where \( V(G \Box H) = V(G) \times V(H) \) and
\( E(G \Box H) = \{((u_1, v_1), (u_2, v_2)) \mid u_1 = u_2 \text{ and } v_1 = v_2 \text{ or } v_1 = v_2 \text{ and } u_1 = u_2 \in E(H)\} \).

We note that \( G_1 \boxtimes G_2 \cong G_2 \boxtimes G_1, \delta(G_1 \boxtimes G_2) = \delta(G_1) + \delta(G_2) \),
and \( k'(G_1 \Box H) = \min\{k'(G_1)|V(H)|, k'(H)|V(G)|, \delta(G) + \delta(H)\} \).

Definition 9 (Path). Let \( G = (V, E) \) be a graph, \( u \) and
\( v \) vertices. A path \( p \) from \( u \) to \( v \) in a graph \( G \) is a sequence
\( p = (v_0, v_1, \ldots, v_k) \), where \( v_0 = u, v_k = v \) and \( (v_i, v_{i+1}) \in E \)
for \( 0 \leq i \leq k - 1 \). The length of the path \( \ell(p) \) is the
number of edges encountered in \( p \), that is \( \ell(p) = k \).

Definition 10 (Shortest Path). Let \( G = (V,E,w) \) be a
weighted graph, \( p = (v_0, v_1, \ldots, v_k) \) a path from a vertex \( u \)
and \( v \) vertices. The length \( \ell(p) \) of the path \( p \) is defined by
\[
\ell(p) = \sum_{i=0}^{k-1} w((v_i, v_{i+1})).
\]

Let \( P \) be the set of all paths from a vertex \( u \) to a vertex
\( v \). We call a path \( p \in P \) a shortest path if \( \ell(p) = \min\{\ell(p') | p' \in P\} \).
Definition 12 (Diameter). The diameter of a graph $G = (V, E)$ is given by $\text{diam}(G) = \max\{\text{dist}(i, j) \mid i, j \in V\}$.

Definition 13 (Permutation matrix). Let $G = (V, E)$ be a graph. The permutation $\phi$ defined on $V$ can be represented by a permutation matrix $P = (p_{ij})$, where

$$p_{ij} = \begin{cases} 
1 & \text{if } v_i = \phi(v_j), \\
0 & \text{otherwise}.
\end{cases}$$

Definition 14 (Automorphism). Let $G = (V, E)$ be a graph. Then a bijection $\phi : V \to V$ is an automorphism of $G$ if $(v_i, v_j) \in E$ then $(\phi(v_i), \phi(v_j)) \in E$. In other words, automorphisms of $G$ are the permutations of vertex set $V$ that maps edges onto edges.

Proposition 1 (Biggs [2]). Let $A(G)$ be the adjacency matrix of a graph $G = (V, E)$, and $P$ be the permutation matrix of permutation $\phi$ defined on $V$. Then $\phi$ is an automorphism of $G$ if and only if $PA = AP$. \hfill $\square$

Definition 15 (Graph cut). A subset of edges which disconnects the graph is called a graph cut. Let $G = (V, E, w)$ be a weighted graph and $W = (w_{ij})$ the weighted adjacency matrix. Then for $A, B \subset V$ and $A \cap B = \emptyset$, the graph cut is denoted by $\text{cut}(A, B) = \sum_{i \in A, j \in B} w_{ij}$.

Definition 16 (Isoperimetric number). The isoperimetric number $i(G)$ of a graph $G$ of order $n \geq 2$ is defined as

$$i(G) = \min \left\{ \frac{\text{cut}(S, V \setminus S)}{|S|} \mid S \subset V, 0 < |S| \leq \frac{n}{2} \right\}.$$ 

Definition 17 (Cheeger constant-edge expansion). Let $G = (V, E)$ be a graph. For a nonempty subset $S \subset V$, define

$$h_G(S) = \frac{\text{cut}(S, V \setminus S)}{\min(\text{vol}(S), \text{vol}(V \setminus S))}.$$ 

The Cheeger constant (edge expansion) $h_G$ is defined as $h_G = \min_S h_G(S)$.

Definition 18 (Cheeger constant-vertex expansion). Let $G = (V, E)$ be a graph. For a nonempty subset $S \subset V$, define

$$g_G(S) = \frac{\text{vol}(\delta S)}{\min(\text{vol}(S), \text{vol}(V \setminus S))},$$

where $\delta S = \{v \notin S \mid (u, v) \in E, u \in S\}$. Then the Cheeger constant (vertex expansion) $g_G$ is defined as $g_G = \min_S g_G(S)$.

Definition 19 (Weighted difference Laplacian). Let $G = (V, E, w)$ be a weighted graph and $W(G) = (w_{ij})$. The weighted difference Laplacian $L(G) = (l_{ij})$ is defined as

$$l_{ij} = \begin{cases} 
d_{ii} - w_{ii} & \text{if } i = j, \\
-w_{ij} & \text{if } (v_i, v_j) \in E \text{ and } i \neq j, \\
0 & \text{otherwise}.
\end{cases}$$

This can be written as $L(G) = D(G) - W(G)$.

Definition 20 (Weighted normalized Laplacian). Let $G = (V, E, w)$ be a weighted graph and $W(G) = (w_{ij})$. The weighted normalized Laplacian $L(G) = (\ell_{ij})$ is defined as

$$\ell_{ij} = \begin{cases} 
1 - \frac{w_{ij}}{d_{ii}} & \text{if } i = j, \\
-\frac{w_{ij}}{\sqrt{d_{ii}d_{jj}}} & \text{if } (v_i, v_j) \in E \text{ and } i \neq j, \\
0 & \text{otherwise}.
\end{cases}$$

We note that $L(G) = D^{-\frac{1}{2}}(D - W)D^{-\frac{1}{2}}$, where $D = D(G)$ and $W = W(G)$.

Lemma 1. Let $G$ be a graph, $n$ the size of the graph $G$, $W = (w_{ij})$ the weighted adjacency matrix of $G$, $\lambda$ an eigenvalue of $L(G)$ and $x = (x_i)$ an eigenvector corresponding to $\lambda$ such that $x^T x = 1$. Then,

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{x_i}{\sqrt{d_{ii}}} - \frac{x_j}{\sqrt{d_{jj}}} \right)^2 w_{ij}.$$ 

Proof. Let $D$ be the degree matrix of $G$. Let $y$ be a size $n$ vector and $x = D^{\frac{1}{2}} y$. Then

$$x^T L(G)x = (D^{\frac{1}{2}} y)^T (L(G)(D^{\frac{1}{2}} y) = y^T D^{\frac{1}{2}} L(G) D^{\frac{1}{2}} y = y^T (D - W)y = \sum_{i=1}^{n} y_i^2 d_{ii} - \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j w_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (y_i - y_j)^2 w_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{x_i}{\sqrt{d_{ii}}} - \frac{x_j}{\sqrt{d_{jj}}} \right)^2 w_{ij}.$$ 

Since $x$ is an eigenvector of $L(G)$ corresponding to $\lambda$ and $x^T x = 1$, we have

$$\lambda = \frac{x^T (\lambda x)}{x^T x} = \frac{x^T L(G)x}{x^T x} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{x_i}{\sqrt{d_{ii}}} - \frac{x_j}{\sqrt{d_{jj}}} \right)^2 w_{ij}. \hfill \square$$

There are several properties about bounds of the second eigenvalue $\lambda_2$.

Proposition 2 (Mohar [11]). Let $G = (V, E)$ be a graph and $\lambda_2$ be the second smallest eigenvalue of $L(G)$. Then,

$$\frac{\lambda_2}{2} \leq i(G) \leq \sqrt{2(\Delta(G) - \lambda_2)\lambda_2}. \hfill \square$$

Proposition 3 (Chung[4]). Let $G$ be a connected graph and $h_G$ the Cheeger constant of $G$. Then,

1. $\frac{2}{\text{vol}(G)} < h_G,$
2. $1 - \sqrt{1 - h_G^2} < \lambda_2,$ and
3. $\frac{h_G^2}{2} < \lambda_2 \leq 2h_G. \hfill \square$
This can be written as

\[ W = \text{a weighted graph and} \]

**Definition 22** (Path graph). A path graph \( P_n = (V, E) \) consists of a vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and an edge set \( E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\} \).

**Example 1.** The Table 1 shows the adjacency matrix and the three Laplacian matrices above for the path graph \( P_4 \).

**Lemma 2.** Let \( G = (V, E, w) \) be a weighted graph. Then the eigenvalues of \( L(G) \) and \( D(G)^{-1} L(G) \) are equal.

**Proof.** Let \( D = D(G) \) and \( W = W(G) \). We have

\[
D^{-1} L(G) = D^{-1/2} DD^{-1/2} - D^{-1/2} D^{-1/2} W
= D^{-1/2} (D - W)D^{1/2}.
\]

Therefore \( D^{-1} L(G) = L(G) \) and has the same spectrum.

**Definition 23** (Regular graph). A graph \( G = (V, E) \) is called \( r \)-regular, if \( d_i = r \) \((i = 1, \ldots, |V|)\).

**Lemma 3.** Let \( \mu_i \) \((i = 1, \ldots, n)\) be eigenvalues of difference Laplacian matrix \( L(G) = D(G) - A(G) \). Then for any regular graph of degree \( r \), normalized Laplacian eigenvalues are \( \lambda_i = \frac{\mu_i}{r} \) \((i = 1, \ldots, n)\).

**Proof.** \( L = (D - A) = r I - A \). Then

\[
L(G) = D^{-1/2} LD^{-1/2} = \frac{I}{r} I \frac{r I - A}{r} = I - A \frac{r}{r}.
\]

Then \( r L(G) = L(G) \). If \( \mu_i \) is an eigenvalue of \( L \), then it is an eigenvalue of \( r L(G) \). This shows that \( \lambda(L(G)) = \frac{\mu_i}{r} \) \((i = 1, \ldots, n)\).

**Proposition 4.** Let \( L(G) \) be the normalized Laplacian matrix of a graph \( G \) and \( P \) the permutation matrix corresponding to the automorphism \( \phi \) defined on \( V \). If \( U \) is an eigenvector of \( L(G) \) with an eigenvalue \( \lambda \), then \( PU \) is also an eigenvector of \( L(G) \) with the same eigenvalue.

**Proof.** From the definition of automorphism \( P^T L(G) P \) = \( L(G) \). Then \( L(G) U = \lambda U \) implies that \( (P^T L(G) P) U = \lambda U \). Since \( PP^T = I \), we get \( L(G) PU = \lambda PU \). If \( U \) is an eigenvector of \( L(G) \) with an eigenvalue \( \lambda \), then \( PU \) is also an eigenvector with the same eigenvalue.

**Remark.** This result holds for any matrix associated with a graph under the automorphism defined on a vertex set.

**Definition 24** (Odd-even vectors). Let \( G = (V, E) \) be a graph and \( \phi : V \to V \) an automorphism of order 2. A vector \( x \) is called an even vector if \( x_i = x_{\phi(i)} \) for all \( 1 \leq i \leq n \) and a vector \( y \) is called an odd vector if \( y_i = -y_{\phi(i)} \) for all \( 1 \leq i \leq n \), where \( n = |V| \).

**Proposition 5.** Let \( G \) be a graph, \( \phi \) an order 2 automorphism. If an eigenvalue of \( L(G) \) is simple, then the corresponding eigenvector is odd or even with respect to \( \phi \).

**Proof.** Let \( P \) be a permutation matrix of \( \phi \), \( \lambda \) an eigenvalue of \( L(G) \), \( U \) an eigenvector of \( L(G) \). If \( \lambda \) is simple, then \( PU \) and \( U \) are linearly dependent. Then there exists a constant \( c \) such that \( PU = c U \). Since \( P^2 = I \) for an automorphism of order 2, \( cU = cPU = c^2 U \) and \( c = \pm 1 \). Then \( PU = U \) or \( PU = -U \). Hence, an eigenvector \( U \) is odd or even with respect to \( \phi \).

**Definition 25.** Let \( G = (V, E) \) be a graph, \( V = \{v_i \mid 1 \leq i \leq n\} \) \((n = |V|)\) and \( U = \{u_1, u_2, \ldots, u_n\} \in \mathbb{R}^n \) a vector. We define three subsets of \( V \) as follows:

\[
U^+ = \{v_i \in V \mid u_i > 0\}, \quad U^- = \{v_i \in V \mid u_i < 0\}, \quad \text{and} \quad U^0 = \{v_i \in V \mid u_i = 0\}.
\]

**Lemma 4.** Let \( L(G) \) be the normalized Laplacian of graph \( G \) which has at least one edge and \( U = (u_i) \) \((1 \leq i \leq n)\) a second eigenvector. If \( U \neq 0 \), then \( V^+(U) \neq \emptyset \) and \( V^-(U) \neq \emptyset \).

**Proof.** The vector \( D^{1/2} U \) is an eigenvector corresponding to the zero eigenvalue. Since a second eigenvector \( U \) is orthogonal to \( D^{1/2} U \), \( (D^{1/2} U)^T U = 0 \) and \( \sum_i \sqrt{d_i} u_i = 0 \). Since \( d_i > 0 \) for some \( i \), \( \phi \neq 0 \), there exist at least two values such that \( u_i > 0 \) and \( u_j < 0 \) for \( i \neq j \). Hence \( V^+(U) \neq \emptyset \) and \( V^-(U) \neq \emptyset \).

**Lemma 5.** Let \( G \) be a graph with an automorphism \( \phi \) of order 2, \( U = (u_1, u_2, \ldots, u_n) \) an eigenvector and \( \phi(U) = (u_{\phi(1)}, u_{\phi(2)}, \ldots, u_{\phi(n)}) \). If \( U \neq 0 \) and \( \phi(U) = -U \), then \( V^+(U) \neq \emptyset \) and \( V^-(U) \neq \emptyset \).
Proof. Assume $V^+(U) = \emptyset$. If $u_i < 0$ ($i = 1, \ldots, n$), then $\phi(U) = -U$ implies that $u_{\phi(i)} > 0$. This contradicts $V^+(U) = \emptyset$. Similarly, if we assume that $V^-(U) = \emptyset$ and $u_i \geq 0$ for $i = 1, \ldots, n$, then $\phi(U) = -U$ implies that $u_{\phi(i)} < 0$. This contradicts $V^-(U) = \emptyset$. If $u_i = 0$ for all $i = 1, \ldots, n$, then $U = 0$ and this contradicts $U \neq 0$. □

**Proposition 6** (Guattery et al. [9]). Let $P_n$ be a weighted path graph and $\mathcal{L}(P_n)$ be its normalized Laplacian matrix. For any eigenvector $X = (x_1, x_2, \ldots, x_n)$,

1. $x_1 = 0$ implies $X = 0$.
2. $x_n = 0$ implies $X = 0$ and
3. $x_i = x_{i+1} = 0$ for some $i$ implies $X = 0$.

□

**Lemma 6** (Guattery et al. [9]). For a path graph $P_n$, $\mathcal{L}(P_n)$ has $n$ simple eigenvalues.

Proof. Let $U = (u_1, u_2, \ldots, u_n)$ and $\bar{U} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ be two eigenvectors of $\mathcal{L}(P_n)$ with common eigenvalue $\lambda$. From Proposition 6, we have $u_n \neq 0$ and $\bar{u}_n \neq 0$. Let $\alpha = \frac{\bar{u}_n}{u_n}$, where $\alpha \neq 0$. Consider $\mathcal{L}(P_n)(\alpha U - \bar{U}) = \lambda (\alpha U - \bar{U})$. The $n$-th element of $(\alpha U - \bar{U})$ is $\bar{u}_n u_n - \bar{u}_n u_n = 0$. Then $\alpha U = \bar{U}$. Thus $U$ and $\bar{U}$ are linearly dependent and hence $\lambda$ is simple. □

**Proposition 7.** Let $P_n$ be the path graph and $\phi$ be the automorphism of order 2 defined on $V(P_n)$. Then any second eigenvector $U_2$ of $\mathcal{L}(P_n)$ is an odd vector.

**Example 2.** Let

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $U_M$ is a second eigenvector of $M$, then by Proposition 4, $PU_M$ is also a second eigenvector. By Proposition 5, $PU_M = U_M$ or $PU_M = -U_M$. By Proposition 7, $U_M$ is an odd vector and $PU_M = -U_M$.

In the rest of this section, we describe concrete formal definitions of a weighted path $P_{n,k}$, a complete binary tree $T_n$, a double tree $DT_n$, a cycle $C_n$, a complete graph $K_n$, a roach type graph $R_{n,k}$, and a lollipop graph $LP_{n,m}$.

**Definition 26** (Weighted Path). For $n (n \geq 1)$ and $k (k \geq 1)$, the adjacency matrix $P = (p_{ij})$ of a weighted path $P_{n,k} = (V, E, w)$ is the $(n + k) \times (n + k)$ matrix defined by

$$p_{ij} = \begin{cases} 0 & (i = j \text{ and } i \leq n) \text{ or } (i \neq j + 1 \text{ and } j \neq i + 1), \\ 1 & (i = j \text{ and } n + 1 \leq i) \text{ or } (i = j + 1 \text{ and } j = i + 1). \end{cases}$$

That is $V = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}\}$, $E = \{(x_i, x_j) | p_{ij} = 1, 1 \leq i, j \leq n + k\}$ and $w((x_i, x_j)) = p_{ij}$ for $(x_i, x_j) \in E$.

Let $\Sigma$ be an alphabet and $\Sigma^*$ a set of strings over $\Sigma$ including the empty string $\epsilon$. We denote the length of $w \in \Sigma^*$ by $|w|$. Let $\Sigma^n = \{w \in \Sigma^* | |w| < n\}$ and $\Sigma^n_1 = \{w \in \Sigma_1^* | 1 \leq |w| < n\}$. Throughout the paper, we assume $\Sigma = \{0, 1\}$.

**Definition 27** (Complete binary tree). A complete binary tree $T_n = (V, E)$ of depth $n$ is defined as follows:

$$V = \Sigma^{<n}, \text{ and}$$

$$E = \{(w, wu) | w \in \Sigma^{<(n-1)}, u \in \Sigma\}.$$

**Definition 28** (Double tree). A double tree $DT_n = (V, E)$, where $n$ is the depth of the tree, consists of two complete binary trees with a connection of their roots. We define a double tree as follows:

$$V = \{x(w) | w \in \Sigma^{<n} \cup \{y(w) | w \in \Sigma^{<n}\},$$

$$E_1 = \{(x(w), x(wu)) | w \in \Sigma^{<(n-1)}, u \in \Sigma\},$$

$$E_2 = \{(y(w), y(wu)) | w \in \Sigma^{<(n-1)}, u \in \Sigma\},$$

and $E = E_1 \cup E_2 \cup \{(x(e), y(e))\}$.

**Definition 29** (Cycle). A cycle $C_n = (V_n, E_n)$ consists of a vertex set $V_n = \{v_1 | i \in \mathbb{Z}^+, 1 \leq |v| \leq n\}$ and an edge set $E_n = \{(v_i, v_{i+1}) | 1 \leq i < n \cup \{(v_1, v_n)\}.$

**Definition 30** (Complete graph). A complete graph $K_n = (V_n, E_n)$ consists of a vertex set $V_n = \{v_1 | i \leq n \}$ and an edge set $E_n = \{(v_i, v_j) | i \neq j \text{ and } 1 \leq i, j \leq n \}$.

**Definition 31** (Graph $R_{n,k}$). A graph $R_{n,k} = (V, E)$ $(n > 1, k \geq 2)$ consists of a vertex set $V = V_1 \cup V_2$ and an edge set $E = E_1 \cup E_2 \cup E_3$, where

$$V_1 = \{x_i | 1 \leq i \leq n + k\},$$

$$V_2 = \{y_i | 1 \leq i \leq n + k\},$$

$$E_1 = \{(x_i, x_{i+1}) | 1 \leq i \leq n + k - 1\},$$

$$E_2 = \{(y_i, y_{i+1}) | n + k + 1 \leq i \leq 2(n + k) - 1\},$$

and $E_3 = \{(x_i, y_i) | n + 1 \leq i \leq n + k\}$.

**Definition 32** (Lollipop graph $LP_{n,m}$). A lollipop graph $LP_{n,m}$ $(n \geq 3, m \geq 1)$ is obtained by connecting a vertex of $K_n = (V_K, E_K)$ to the end vertex of $P_m = (V_P, E_P)$. Define $LP_{n,m} = (V, E)$ as follows:

$$V = \{x_1, x_2, \ldots, x_m, y_1, \ldots, y_n\},$$

$$E = \{(x_i, x_j) | 1 \leq i \leq m - 1 \cup \{(y_i, y_j) | i \neq j, 1 \leq i \leq n, 1 \leq j \leq n \cup \{(x_m, y_1)\}.$$}

We note that $V_P = \{x_1, x_2, \ldots, x_m\}$, $V_K = \{y_1, y_2, \ldots, y_n\}$ and $E = E_P \cup E_K \cup \{(x_m, y_1)\}$.

**Example 3.** The double tree $DT_3$ shown in the Figure 1(a) has a vertex set $V = \{x(e), x(0), x(1), y(e), y(0), y(1), x(00), x(01), x(10), x(11), y(00), y(01), y(10), y(11)\}$ and an edge set $E = \{(x(e), y(e)), (x(e), x(0)), (x(e), x(1)),$
(y(e), y(0)), (y(e), y(1)), (x(0), x(00)), (x(0), x(01)),
(x(1), x(10)), (x(1), x(11)), y(0), y(00)), ((y(0), y(01)),
(y(1), y(10)), (y(1), y(11)). The graph $R_{5,5}$ shown in the
Figure 1(b) has a vertex set $V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7,$
$x_8, x_9, x_{10}, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$ and an edge
set $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \{(x_i, x_{i+1}) \mid 1 \leq i \leq 9\}$,
$E_2 = \{(y_i, y_{i+1}) \mid 1 \leq i \leq 9\}$ and $E_3 = \{(x_6, y_8), (x_7, y_7),$
$(x_8, y_8), (x_9, y_9), (x_{10}, y_{10})\}$. A lollipop graph $LP_{10,2}$ is
shown in the Figure 2.

Figure 1: Double tree $DT_3$ and graph $R_{n,k}$ ($n = 5, k = 5$).

Figure 2: Graph $LP_{n,m}$ ($n = 10, m = 2$)

3. MINIMUM NORMALIZED CUT OF GRAPHS

We use the term $\text{Mcut}(G)$ to represent the minimum normalized
Cut of a graph $G$. In this section, we review the basic properties of $\text{Mcut}(G)$ and its relation to the connectivity and a second smallest eigenvalue of the normalized
Laplacian. We derive $\text{Mcut}(G)$ of a graph in basic classes of
graphs such as paths, cycles, double trees, complete graphs and other graphs like $R_{n,k}$, $P_{n,k}$ and $LP_{n,m}$.

3.1. PROPERTIES OF MINIMUM NORMALIZED CUT $\text{Mcut}(G)$

Definition 33 (Normalized cut). Let $G = (V, E)$ be a
connected graph. Let $A, B \subset V, A \neq \emptyset, B \neq \emptyset$ and $A \cap B = \emptyset$. The normalized cut $\text{Net}(A, B)$ of $G$ is defined by

$$\text{Net}(A, B) = \text{cut}(A, B) \left( \frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(B)} \right),$$

Definition 34 ($\text{Mcut}(G)$). Let $G = (V, E)$ be a
cconnected graph. The minimum normalized cut $\text{Mcut}(G)$ is defined by

$$\text{Mcut}(G) = \min \{\text{Net}(j \{G \mid j = 1, 2, \ldots \} \},$$

where

$$\text{Net}_j(G) = \min \{|\text{cut}\left(A, V \setminus A \right) \mid A \subseteq V,$$

$$\text{cut}(A, V \setminus A) = j, \ A \text{ and } V \setminus A \text{ are connected.}\}$$

We call a subset $A$ of $V$ connected if there exists a path
from $u$ to $v$ for any two vertices $u$ and $v$ in $A$.

In [14], they concerned about $\text{Ncut}(A, V \setminus A)$, but there
is no mention about connectivity about $A$ and $V \setminus A$. To
give a precise discussion about the minimum of this value, we define $\text{Mcut}$ for the minimum of connected subsets. A
partition defined by a second eigenvector of the Laplacian
matrix of a given graph is always connected [5]. So this is
a kind of general limitation for precise discussions. Even
though we can prove the minimum value of $\text{Net}$ of two
separated subsets without limitation is always same as our
limited $\text{Mcut}$, most of our proofs of properties will be used
as they are.

Example 4. The graph $G = (V, E)$ shown in the Figure 3
has the vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$ and the edge set
$E = \{(1, 2), (2, 3), (3, 1), (3, 4), (1, 4), (1, 5), (3, 6), (6, 5),$
$(7, 5), (7, 6)\}$. The volume of the graph is 20. We compute
normalized cuts for the following cases.

Case (1): $A = \{1, 2, 3, 4\}, B = \{5, 6, 7\}, \text{vol}(A) = 12,$
$\text{vol}(B) = 8, \text{cut}(A, B) = 2$ and $\text{Ncut}(A, B) = 0.417$.

Case (2): $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}, \text{vol}(A) = 10,$
$\text{vol}(B) = 10, \text{cut}(A, B) = 4$ and $\text{Ncut}(A, B) = 0.8$.

Case (3): $A = \{1, 3, 4, 5, 6, 7\}, B = \{2\}, \text{vol}(A) = 2,$
$\text{vol}(B) = 18, \text{cut}(A, B) = 2$ and $\text{Ncut}(A, B) = 1.1111$.

We note that the set $B = \{4, 5, 6, 7\}$ in Case (2) is not a
connected subset and we see $\text{Mcut}(G)$ in Case (1).

Figure 3: Normalized cut examples
2. \( \text{Mcut}(G) \geq \frac{4\kappa'(G)}{\Delta(G)|V|} \), and

3. \( \text{cut}(A, V \setminus A) = \kappa'(G) \) and \( 2\text{vol}(A) = \text{vol}(G) \) implies \( \text{Mcut}(G) = \frac{4\kappa'(G)}{\text{vol}(G)} \).

**Proof.**

1. Since \( \kappa'(G) \) is the edge connectivity, we have \( \text{cut}(A, V \setminus A) \geq \kappa'(G) \) for any \( A \subseteq V \).

2. \( \left( \frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(V \setminus A)} \right) \) is minimum when \( \text{vol}(A) = \text{vol}(V \setminus A) \). That is
   \[
   \left( \frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(V \setminus A)} \right) \geq \frac{2}{\text{vol}(A)} = \frac{4}{\text{vol}(G)}.
   
   Since \( \text{vol}(G) = \sum_{i=1}^{|V|} d_i \leq |V|\Delta(G) \), we have
   \[
   \text{Ncut}(A, V \setminus A) = \text{cut}(A, V \setminus A) \times \left( \frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(V \setminus A)} \right)
   \geq \frac{4\kappa'(G)}{\Delta(G)|V|}.
   
   3. If \( \text{cut}(A, V \setminus A) = \kappa'(G) \) and \( 2\text{vol}(A) = \text{vol}(G) \), then it is clear that, \( \text{Mcut}(G) = \frac{4\kappa'(G)}{\text{vol}(G)} \). \( \square \)

**Proposition 9** (Luxburg [16]). **Let** \( G = (V, E) \) **be a connected graph**, \( D = D(G) \), \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) **be eigenvalues of** \( \mathcal{L}(G) \). **Then** \( \text{Mcut}(G) \geq \lambda_2(\mathcal{L}(G)) \).

**Proof.** **Let** \( V = \{1, 2, \ldots, n\} \), \( d_i \) **the degree of vertex** \( i \in V \), \( W(G) = (w_{ij}) \), and \( A \) **a subset of** \( V \) **such that** \( A \) **and** \( V \setminus A \) **are connected. Define** \( f = (f_1, \ldots, f_n) \) **as**

\[
    f_i = \begin{cases} 
    a & \text{if } i \in A, \\
    -b & \text{if } i \notin A,
    \end{cases}
\]

where \( a = \text{vol}(V \setminus A) = \sum_{i \notin A} d_i \) **and** \( b = \text{vol}(A) = \sum_{i \in A} d_i \).

Then we have

\[
\frac{\sum_{i=1}^n \sum_{j=1}^n (f_i - f_j)^2 w_{ij}}{2 \sum_{i=1}^n f_i d_i} = \frac{2\text{cut}(A, (V \setminus A)) (a + b)^2}{2(a^2 \text{vol}(A) + b^2 \text{vol}(V \setminus A))} = \frac{\text{cut}(A, (V \setminus A)) (\text{vol}(G))^2}{\text{vol}(V \setminus A)^2 \text{vol}(A) + \text{vol}(A)^2 \text{vol}(V \setminus A)} = \frac{\text{cut}(A, (V \setminus A)) (\text{vol}(G))}{\text{vol}(V \setminus A) \text{vol}(A)} = \text{cut}(A, (V \setminus A)) \left( \frac{1}{\text{vol}(V \setminus A)} + \frac{1}{\text{vol}(A)} \right) = \text{Ncut}(A, V \setminus A).
\]

Let \( D \) **be the degree matrix of** \( G \), \( y \) **a size** \( n \) **vector and** \( x = D^T y \). **As** **we noticed in** **the proof of** **Lemma 1**, \( x^T \mathcal{L}(G)x = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (y_i - y_j)^2 w_{ij} \) **and** \( x^T x = \sum_{i=1}^n x_i^2 = \sum_{i=1}^n d_i y_i^2 \).

Since \( x \perp D^T \Rightarrow y \perp D \), **we have**

\[
\lambda_2(\mathcal{L}(G)) = \inf_{x \perp D^T} \frac{x^T \mathcal{L}(G)x}{x^T x} = \inf_{y \perp D} \frac{\sum_{i=1}^n \sum_{j=1}^n (y_i - y_j)^2 w_{ij}}{2 \sum_{i=1}^n y_i^2 d_i}.
\]

Since \( (D^T)^T f = \sum_{i=1}^n d_i f_i = \sum_{i \in A} d_i a - \sum_{i \notin A} d_i b = 0 \), **we have** \( f \perp D \) **and**

\[
\lambda_2(\mathcal{L}(G)) \leq \frac{\sum_{i=1}^n \sum_{j=1}^n (f_i - f_j)^2 w_{ij}}{2 \sum_{i=1}^n f_i^2 d_i} = \text{Ncut}(A, (V \setminus A)).
\]

We can take any arbitrary subset \( A \), **so** **we have**

\[
\lambda_2(\mathcal{L}(G)) \leq \min_{A \subseteq V} \text{Ncut}(A, V \setminus A) \leq \text{Mcut}(G). \quad \square
\]

**Lemma 7.** **Let** \( G = (V, E) \) **be a connected graph**, \( A \) **a nonempty subset of** \( V \). **Then**

(i) \( |\text{vol}(A) - \text{vol}(V \setminus A)| = 2 \left| \text{vol}(A) - \frac{\text{vol}(V)}{2} \right| \),

(ii) \( \text{Ncut}(A, V \setminus A) = \frac{4 \text{cut}(A, V \setminus A) \cdot \text{vol}(V)}{(\text{vol}(V))^2 - (\text{vol}(A) - \text{vol}(V \setminus A))^2} \), and

(iii) \( \text{Mcut}_j(G) = \frac{4 j \text{vol}(V)}{\text{vol}(V)^2 - X_j^2} \), **where**

\[
X_j = \min \{|\text{vol}(A) - \text{vol}(V \setminus A)| \mid \text{cut}(A, V \setminus A) = j, A \subseteq V, A \cap V \setminus A \text{ are connected.}\}.
\]

(iv) **In particular**, \( \text{Mcut}_j(G) \geq \frac{4 j \text{vol}(V)}{\text{vol}(V)^2} \).

**Proof.** **Let** \( s = \text{vol}(V) \), \( s_A = \text{vol}(A) \) **and** \( s_{\bar{A}} = \text{vol}(V \setminus A) \).

Since \( s = s_A + s_{\bar{A}} \), **we have** \( s_A - s_{\bar{A}} = 2s_A - s \).

(i) **If** \( s_A \geq s_{\bar{A}} \) **then** \( 2s_A \geq s \) **and**

\[
|s_A - s_{\bar{A}}| = s_A - s_{\bar{A}} = 2s_A - s = |2s_A - s| = 2 \left| s_A - \frac{s}{2} \right|.
\]

Similarly, if \( s_A < s_{\bar{A}} \) **then** **we have** \( |s_A - s_{\bar{A}}| = 2 \left| s_A - \frac{s}{2} \right| \).

(ii) **Let** \( j = \text{cut}(A, V \setminus A) \). **Since** \( s^2 - (s_A - s_{\bar{A}})^2 = 4s_A s_{\bar{A}} \), **we have**

\[
\text{Ncut}(A, V \setminus A) = j \cdot \left( \frac{1}{s_A} + \frac{1}{s_{\bar{A}}} \right) = \frac{js(s_A + s_{\bar{A}})}{s_A s_{\bar{A}}} = \frac{js}{s_A s_{\bar{A}}} = \frac{4js}{s^2 - (s_A - s_{\bar{A}})^2}.
\]

(iii) **Follows from the definition of** \( \text{Mcut}_j(G) \) **and (i).**

(iv) **Since** \( X_j \geq 0 \), **it follows from (ii). \( \square \)
Lemma 8. Let $G = (V, E)$ be a graph. If there exists a nonempty subset $A \subset V$ such that $A$ and $V \setminus A$ are connected, and

$$|\text{vol}(A) - \text{vol}(V \setminus A)| \leq \frac{\text{vol}(V)}{\sqrt{\text{cut}(A, V \setminus A) + 1}},$$

then

$$\text{Mcut}(G) = \min\{\text{Mcut}_j(G) \mid j = 1, 2, \ldots, \text{cut}(A, V \setminus A)\}.$$ 

Proof. Let $j = \text{cut}(A, V \setminus A), a = |\text{vol}(A) - \text{vol}(V \setminus A)|, s = \text{vol}(V), s_A = \text{vol}(A)$ and $s_B = \text{vol}(V \setminus A)$. Since $a^2 \leq \frac{2}{j+1}$ and $\text{Ncut}(A, V \setminus A) = \frac{4js}{s^2 - a^2}$ by Lemma 7 (ii), we have $s^2 - (j+1)a^2 \geq 0$ and

$$\frac{4(j+1)}{s} = \frac{4(j+1) - 4js}{s^2 - a^2} = \frac{4(j+1)(s^2 - a^2) - 4js^2}{s(s^2 - a^2)} = \frac{4(s^2 - (j+1)a^2)}{s(s^2 - a^2)} \geq 0.$$

Hence $\text{Mcut}_j(G) \leq \text{Mcut}(A, V \setminus A) \leq \frac{4(j+1)}{s}$.

For any $j' > j$, we have the following using Lemma 7 (iv).

$$\text{Mcut}_{j'}(G) \geq \frac{4j'}{s} \geq \frac{4(j+1)}{s} \geq \text{Mcut}_j(G).$$

Lemma 9. Let $G = (V, E)$ be a graph and $\text{vol}(G) \geq 9$. If there exists a subset $A$ of $V$ such that $\text{cut}(A, V \setminus A) = 1$, $|\text{vol}(A) - \text{vol}(G)| \leq 3$, and $A$ and $V \setminus A$ are connected, then

$$\text{Mcut}(G) = \text{Mcut}_1(G).$$

Proof. Let $s = \text{vol}(G), s_A = \text{vol}(A)$ and $s_B = \text{vol}(V \setminus A)$. Since $|s_A - s| \leq 3$ and $s = s_A + s_B$, we have $|s_A - s_B| \leq 6$. Since $\sqrt{1 + \frac{1}{s}} |s_A - s_B| \leq 6\sqrt{\frac{1}{s}} < 9 \leq s$, we have $\text{Mcut}(G) = \text{Mcut}_1(G)$ by Lemma 8.

Lemma 10. Let $G = (V, E)$ be a graph and $\text{vol}(G) \geq 11$. If there exists a set $A \subset V$ such that $\text{cut}(A, V \setminus A) = 2$, $|\text{vol}(A) - \text{vol}(G)| \leq 3$, and $A$ and $V \setminus A$ are connected, then

$$\text{Mcut}(G) = \min\{\text{Mcut}_1(G), \text{Mcut}_2(G)\}.$$ 

Proof. Let $s = \text{vol}(G), s_A = \text{vol}(A)$ and $s_B = \text{vol}(V \setminus A)$. Since $|s_A - s| \leq 3$ and $s = s_A + s_B$, we have $|s_A - s_B| \leq 6$ and $\sqrt{s} |s_A - s_B| \leq \frac{6\sqrt{s}}{s} < 11$. So we have $\text{Mcut}(G) = \min\{\text{Mcut}_1(G), \text{Mcut}_2(G)\}$ by Lemma 8.

Lemma 11. Let $G = (V, E)$ be a graph with $\text{vol}(G) \geq 11$. Suppose there exists a subset $A \subset V$ such that $\text{cut}(A, V \setminus A) = 2, |\text{vol}(A) - \text{vol}(G)| \leq 3$, and $A$ and $V \setminus A$ are connected. If there exists no subset $B \subset V$ such that $\text{cut}(B, V \setminus B) = 1, |\text{vol}(B) - \text{vol}(G)| \leq \frac{\sqrt{36 + (\text{vol}(G))^2}}{2\sqrt{2}}$, and $B$ and $V \setminus B$ are connected, then

$$\text{Mcut}(G) = \text{Mcut}_1(G).$$

Proof. Let $s = \text{vol}(G), s_A = \text{vol}(A)$ and $s_B = \text{vol}(V \setminus A)$. Since $|s_A - s| \leq 3$ and $s = s_A + s_B$, we have $|s_A - s_B| \leq 6$ and $\sqrt{s} |s_A - s_B| \leq \frac{6\sqrt{s}}{s} < 11$. So we have $\text{Mcut}(G) = \text{Mcut}_1(G)$ by Lemma 8.
Proof. 1. For a regular graph of degree $d$, $\kappa'(G) = \Delta(G) = \delta(G) = d$. For $A \subset V$,

$$\text{Ncut}(A, V \setminus A) \geq \kappa'(G) \left( \frac{1}{d|A|} + \frac{1}{d|V \setminus A|} \right) = \frac{|V|}{|A||V \setminus A|}.$$ 

If $\text{cut}(A, V \setminus A) = \kappa'(G)$, then we have $\text{Ncut}(A, V \setminus A) = \frac{|V|}{|A||V \setminus A|}$. $\text{Ncut}(A, V \setminus A)$ is minimum, when $|A| = |V \setminus A|$. If $V$ is even, then $\text{Mcut}(G) \geq \frac{4}{n} \frac{|V|}{|V| + 1}$.

If $|V|$ is odd, then we can write $|V| = \frac{|V| - 1}{2} + \frac{|V| + 1}{2}$, where $-1 \leq |A| - |V \setminus A| \leq 1$. Then

$$\text{Mcut}(A, V \setminus A) \geq \kappa'(G) \left( \frac{2}{d(|V| - 1)} + \frac{2}{d(|V| + 1)} \right) \frac{4|V|}{(|V| + 1)(|V| - 1)} = \frac{4n}{n^2 - 1}. \quad \Box$$

Hence, $\text{Mcut}(G) \geq \frac{4|V|}{(|V| + 1)(|V| - 1)} = \frac{4n}{n^2 - 1}$.

2. Let $A_k = \{ x_i | i \leq k \} (k = 1, \ldots, n - 1)$. We note that

$$\text{vol}(C_n) = 2n, \quad \text{vol}(A_k) = 2k, \quad \text{vol}(V \setminus A_k) = 2n - 2k$$

and $\text{vol}(V \setminus A_k) - \text{vol} \{ x_i | i \leq k \} = n - 2k$ and $\text{Ncut}(A_k, V \setminus A_k) = \frac{4n}{n^2 - (2k - n)^2}.$

If $n$ is even, then $\text{Ncut}(A_k, V \setminus A_k) = \frac{k}{n}$ is the minimum of $\text{Ncut}(A_k, V \setminus A_k)$.

If $n$ is odd, then

$$\text{Ncut}(A_{n+1}, V \setminus A_{n+1}) = \text{Ncut}(A_{n+1}, V \setminus A_{n+2}) = \frac{4n}{n^2 - 1}$$

is the minimum of $\text{Ncut}(A_k, V \setminus A_k)$. Since $\text{vol}(A_{n+1}) - \text{vol}(V \setminus A_{n+1}) = 0$, $\text{vol}(A_{n+2}) - \text{vol}(V \setminus A_{n+2}) = -2$ and

$$\frac{\text{vol}(V)}{\sqrt{\text{cut}(A_k, V \setminus A_k) + 1}} \geq \frac{6}{\sqrt{3}} \text{ we have} \text{Mcut}(C_n) = \text{Mcut}(C_{n+1}) \text{ by Lemma 8.}$$

We note that for any nonempty subset $A \subset V$ such that $\text{cut}(A, V \setminus A) = 2$, there exists $k$ such that $\text{Ncut}(A, V \setminus A) = \text{Ncut}(A_k, V \setminus A_k)$ and $\kappa'(C_n) = 2$.

For even $n, n = \left( \frac{n}{2} \right)$ and for odd $n, n = \left( \frac{n}{2} \right)$ and $n = \left( \frac{n}{2} \right)$. Combining the odd and even cases together, we can write $\text{Mcut}(C_n) = \frac{4n}{4\left( \frac{n}{2} \right)^2} \cdot \frac{1}{2}$. \Box

3. For a complete graph $K_n$, $|V| = n$, $\kappa'(K_n) = n - 1$ and $\text{vol}(K_n) = n(n - 1)$. For any subset $A \subset V$, we have $\text{vol}(A) = |A|(n - 1)$ and $\text{cut}(A, V \setminus A) = |A|(n - |A|)$. Then $\text{Mcut}(K_n) = |A|(n - 1) = \frac{n}{n - 1}$.

4. Let $A_k = \{ x_i | i \leq k \} (k = 1, \ldots, n - 1)$. We note that

$$\text{vol}(P_n) = 2n - 2, \quad \text{vol}(A_k) = 2k - 1, \quad \text{vol}(V \setminus A_k) = 2n - 2k - 1, \quad \text{vol}(V \setminus A_k) = 4k - 2n$$

and

$$\text{Ncut}(A_k, V \setminus A_k) = \frac{2(n - 1)}{(n - 1)^2 - (2k - n)^2}.$$ 

If $n$ is even, then $\text{Ncut}(A_{2k}, V \setminus A_{2k}) = \frac{2}{n}$ is the minimum of $\text{Ncut}(A_k, V \setminus A_k)$. If $n$ is odd, then

$$\text{Ncut}(A_{2k+1}, V \setminus A_{2k+1}) = \text{Ncut}(A_{2k+1}, V \setminus A_{2k+1}) = \frac{2(n - 1)}{(n - 1)^2 - 1} = \frac{2(n - 1)}{(n - n - 2)}$$

is the minimum of $\text{Ncut}(A_k, V \setminus A_k)$. Since $\text{vol}(A_{2k}) - \text{vol}(V \setminus A_{2k}) = 0$, $\text{vol}(A_{2k+1}) - \text{vol}(V \setminus A_{2k+1}) = -2$ and

$$\frac{\sqrt{\text{cut}(A_k, V \setminus A_k) + 1}}{2\text{vol}(V)} \geq \frac{2n - 2}{\sqrt{2}} \geq \frac{2}{\sqrt{2}} \text{ we have} \text{Mcut}(C_n) = \text{Mcut}(P_n) \text{ by Lemma 8.} \quad \Box$$

5. The size of a tree is $|T_n| = 1 + 2 + \cdots + 2^n = 2^n - 1$ and the size of a double tree is $|DT_n| = 2|T_n| = 2^n + 2 - 2$. The volume of a tree is $\text{vol}(T_n) = 2\text{vol}(T_n) + 4$, which can be written as $\text{vol}(T_n) + 4 = 2\text{vol}(T_n) + 4 = 2^2(\text{vol}(T_n - 4) + 4) = \cdots = 2^{n-1}\text{vol}(T_1) + 4$.

Therefore the volume of a tree is $\text{vol}(T_n) = 2^{n-1} - 4$ and the volume of a double tree is $\text{vol}(DT_n) = 2\text{vol}(T_n) + 2 = 2^n + 6 - 2n$.

Let $A_1 = \{ x_i | w \in \Sigma^n \}$ and $V \setminus A_1 = \{ y_i | w \in \Sigma^n \}$. Then we have $\text{vol}(A_1) = \text{vol}(A_1) + 1 = 2^n - 3, \text{vol}(V \setminus A_1) = 2^n + 2$ and $\text{cut}(A_1, V \setminus A_1) = 1$. Therefore $\text{Ncut}(A_1, V \setminus A_1) = \frac{2}{2\text{vol}(T_n) + 1} = \frac{2}{2^n + 3} = \frac{4}{2\text{vol}(DT_n)}$. Here $\kappa'(DT_n) = 1$ and $\text{vol}(A_1) = \text{vol}(DT_n)$. Then we have $\text{Mcut}(DT_n) = \frac{2}{2^n + 3} \cdot \frac{1}{2}$.

3.3. Mcut of roach type graphs $R_{n,k}$

Next, we consider the graph $R_{n,k}$ and derive a formula for $\text{Mcut}(R_{n,k})$ in terms of $n, k$.

**Theorem 2.** Let $n \geq 1$ and $k \geq \frac{2}{3}n + 2$. For a graph $R_{n,k}$,

$$\text{Mcut}(R_{n,k}) = \begin{cases} \frac{3}{2} \frac{4(3k + 2n - 2)}{4(3k + 2n - 2)^2 - (3k + 2n - 2)^2} & (n, k) = (2, 1), \\ \frac{3}{2} \frac{2(3k + 2n - 2)^2 - (3k + 2n - 2)^2}{2(3k + 2n - 2)^2 - (3k + 2n - 2)^2} & (n, k) \in D_1, \\ \frac{2(3k + 2n - 2)^2 - (3k + 2n - 2)^2}{2(3k + 2n - 2)^2 - (3k + 2n - 2)^2} & (n, k) \in D_2, \end{cases}$$

where

$$p_0 = \min_{1 \leq \alpha < k} \{ 6\alpha - (3k - 2n) \},$$

$$D_1 = \{ (n, k) | k \geq 4 \wedge 9k^2 - 12nk - 4n^2 + 8n - 2 > p_0 \} \cup \{ (1, 3) \}, \text{ and }$$

$$D_2 = \{ (n, k) | k \geq 4 \wedge 9k^2 - 12nk - 4n^2 + 8n - 2 \leq p_0 \}.$$ 

We note that $p_0$ takes the values as follows:

$$p_0 = \begin{cases} 0 & (2 | k) \wedge (3 | n), \\ 1 & (2 | k) \wedge (3 | n), \\ 2 & (2 | k) \wedge (3 | n), \\ 3 & (2 | k) \wedge (3 | n). \end{cases}$$
Proof. Let $V(R_{n,k}) = \{x_i \mid 1 \leq i \leq n + k\} \cup \{y_i \mid 1 \leq i \leq n + k\}$. The volume of $R_{n,k}$ is $\text{vol}(R_{n,k}) = 2(2n - 1 + 3k - 1) = 6k + 4n - 4$.

We consider the following cases in order to find $\text{Mcut}(R_{n,k})$.

**Case (i):** Let $A_1 \subseteq V(R_{n,k})$, where $A_1 = \{x_i \mid 1 \leq i \leq n\}$ and $V \setminus A_1 = \{y_i \mid 1 \leq i \leq n + k\}$. Then the volume $\text{vol}(A_1) = \frac{3k + 2n - 2}{2}$ and cut$(A_1, V \setminus A_1) = 1$.

So we have

$$\text{Mcut}(A_1, V \setminus A_1) = k\left(\frac{1}{3k + 2n - 2} + \frac{1}{3k + 2n - 2}\right) = \frac{2k}{3k + 2n - 2}.$$

Let $c_1 = \text{Mcut}(A_1, V \setminus A_1)$.

**Case (ii):** Let $A_2 \subseteq V(R_{n,k})$ such that $A_2 = \{x_i \mid 1 \leq i \leq n\}$ and $V \setminus A_2 = \{x_i \mid n + 1 \leq i \leq n + k\} \cup \{y_i \mid 1 \leq i \leq n + k\}$. Then the volume $\text{vol}(A_2) = 2n - 1$, $\text{vol}(V \setminus A_2) = \text{vol}(R_{n,k}) - \text{vol}(A_2) = 2n + 6k - 3$ and cut$(A_2, V \setminus A_2) = 1$.

So we have

$$\text{Mcut}(A_2, V \setminus A_2) = \frac{(6k + 4n - 4)}{(2n - 1)(6k + 2n - 3)}.$$

Let $c_2 = \text{Mcut}(A_2, V \setminus A_2)$.

**Case (iii):** Let $A_3(\alpha) = \{x_i \mid 1 \leq i \leq \alpha\}$ where $1 \leq \alpha < n$ and $V \setminus A_3(\alpha) = \{x_i \mid \alpha + 1 \leq i \leq n + k\} \cup \{y_i \mid 1 \leq i \leq n + k\}$. Then $\text{vol}(A_3(\alpha)) = 2n - 1$, $\text{vol}(V \setminus A_3(\alpha)) = 4n + 6k - 2a - 3$ and cut$(A_3(\alpha), V \setminus A_3(\alpha)) = 1$.

Since

$$\left|\frac{\text{vol}(A_3(\alpha)) - \text{vol}(V \setminus A_3(\alpha))}{2}\right| = \left|\frac{-3k - 2(n - \alpha) + 1}{2}\right| \geq \left|\frac{-3k + 1}{2}\right| = \left|\frac{\text{vol}(A_2) - \text{vol}(V \setminus A_2)}{2}\right|,$$

we have $\text{Mcut}(A_3(\alpha), V \setminus A_3(\alpha)) \geq \text{Mcut}(A_2, V \setminus A_2)$ for any $\alpha (1 \leq \alpha < n)$ by Lemma 7 (i) and (ii). So we can ignore this case.

**Case (iv):** Let $A_4(\alpha) = \{x_i \mid 1 \leq i \leq n + \alpha\} \cup \{y_i \mid 1 \leq i \leq n + \alpha\}$, where $1 \leq \alpha < k$ and $V \setminus A_4(\alpha) = \{x_i \mid n + \alpha + 1 \leq i \leq n + k\} \cup \{y_i \mid n + \alpha + 1 \leq i \leq n + k\}$. Then $\text{vol}(A_4(\alpha)) = 2(2n - 1 + 3\alpha) = 4n + 6\alpha - 2$, $\text{vol}(V \setminus A_4(\alpha)) = 6k - 2 - 6\alpha$ and cut$(A_4(\alpha), V \setminus A_4(\alpha)) = 2$. Then we have

$$\text{Mcut}(A_4(\alpha), V \setminus A_4(\alpha)) = \frac{4 \cdot 2 \cdot \text{vol}(V)}{\text{vol}(V)^2 - (\text{vol}(A_4) - \text{vol}(V \setminus A_4))^2} = \frac{4 \cdot 2 \cdot (6k + 4n - 4)}{(6k + 4n - 4)^2 - (\text{vol}(A_4) - \text{vol}(V \setminus A_4))^2} = \frac{4 \cdot 2 \cdot (6k + 4n - 4)}{4(3k + 2n - 2)^2 - (6\alpha - (3k - 2n))^2} = \frac{4(3k + 2n - 2)^2 - p_4(\alpha)^2}{4(3k + 2n - 2)^2}$,$$

where $p_4(\alpha) = 6a - (3k - 2n) = \text{vol}(A_4(\alpha)) - \frac{\text{vol}(V)}{2}$.

Let $c_4 = \text{Mcut}(A_4(\alpha), V \setminus A_4(\alpha))$. The minimum of $c_4$ can be obtained when $|p_4(\alpha)|$ is the minimum with respect to $\alpha$. That is if $\alpha_0 = \frac{3k - 2n}{6}$ is an integer and $1 \leq \alpha_0 < k$ then $p_4(\alpha_0) = 0$ and $c_4(\alpha_0)$ is the minimum.

But $\alpha_0$ is not an integer for some $n$, $k$. Since $k \geq \frac{2}{3}$, we have

$$\frac{3k - 2n}{6} \geq 1.$$ If $\frac{3k - 2n}{6} \in \mathbb{Z}$, then $p_4\left(\frac{3k - 2n}{6}\right) = 0$ and the minimum value of $c_4(\alpha)$ is

$$c_4(\alpha) = \frac{4}{3k + 2n - 2}.$$

If $2 \nmid k$ and $3 \nmid n$, the minimum value of $p_4(\alpha) = 3$ and the minimum value is

$$c_4(\alpha) = \frac{4(3k + 2n - 2)}{(3k + 2n - 2)^2 - 9}.$$

If $2 \mid k$ and $3 \nmid n$, the minimum value of $p_4(\alpha) = 2$ and the minimum value is

$$c_4(\alpha) = \frac{4(3k + 2n - 2)}{(3k + 2n - 2)^2 - 4}.$$

If $2 \nmid k$ and $3 \nmid n$, the minimum value of $p_4(\alpha) = 1$ and the minimum value is

$$c_4(\alpha) = \frac{4(3k + 2n - 2)}{(3k + 2n - 2)^2 - 1}.$$

**Case (v):** Let $A_5 = \{x_i \mid 1 \leq i \leq n + 1\}$ and $V \setminus A_5 = \{x_i \mid n + 2 \leq i \leq n + k\} \cup \{y_i \mid 1 \leq i \leq n + k\}$. Then $\text{vol}(A_5) = 2n + 2$ and $\text{vol}(V \setminus A_5) = 2n + 6k - 6$. Then we have

$$\text{Mcut}(A_5, V \setminus A_5) = 2\left(\frac{1}{2n + 2} + \frac{1}{2n + 6k - 6}\right) = \frac{2n + 3k - 2}{(n + 1)(n + 3k - 3)}.$$

Before comparing all the cases, we want to show

$$\text{Mcut}(R_{n,k}) = \min(\text{Mcut}_1(R_{n,k}), \text{Mcut}_2(R_{n,k})).$$

This means that we do not need to investigate other kind of subsets except for subsets considered in case (i) to (v). We note that every connected subset $A$ with $\text{cut}(A, V \setminus A) = 1$ is $A_2$ or $A_3$ and every connected subset $A$ with $\text{cut}(A, V \setminus A) = 2$ are $A_1$, $A_5$, or $A_4$. We consider all cases with $\text{cut}(A, V \setminus A) = 1$ and the minimum occurs at $A_2$ in case (iii). Since $n \geq 1$ and $k \geq 2$, we have $\text{vol}(R_{n,k}) \geq 12$. As we noticed in case (iv), there exists an $\alpha$ such that $p_4(\alpha) = \left|\frac{\text{vol}(A_4(\alpha)) - \frac{\text{vol}(V)}{2}}{2}\right| \leq 3$. So we have $\text{Mcut}(R_{n,k}) = \min(\text{Mcut}_1(R_{n,k}), \text{Mcut}_2(R_{n,k}))$ by the Lemma 10.

Now we can compare all the cases (i), (ii), (iii), (iv) and (v).

If $k = 3$ and $n = 1$, then $c_4(\frac{3k - 2n}{6} - \frac{1}{6})$ is the minimum. If $k \geq 4$ and $n = 1$, then $c_4$ is the minimum. Next,
we assume that \( k \geq 4 \) and \( n \geq 2 \). It is easy to check that \( c_2 \) is smaller than \( c_1 \), \( c_3 \) and \( c_5 \). So we compare \( c_2 \) with \( c_4 \) for \( k \geq 4 \). Let \( s = \frac{vol(V)}{2}, \ p_2 = vol(A_2) - \frac{vol(V)}{2}, \) and \( p_4(\alpha) = vol(A_4(\alpha)) - \frac{vol(V)}{2} \) where \( 1 \leq \alpha < k \). Since 
\[
\text{cut}(A_2, V \setminus A_2) = 1 \quad \text{and} \quad \text{cut}(A_4, V \setminus A_4) = 2
\]
we can denote 
\[
c_2 = \frac{4(2s)^2 - 2p_2^2}{8(2s)} \quad \text{and} \quad c_4(\alpha) = \frac{(2s)^2 - (2p_4(\alpha))^2}{8(2s)}
\]
by Lemma 7 (ii). So we have 
\[
c_4(\alpha) < c_2 \iff \frac{1}{c_2} < \frac{1}{c_4(\alpha)} \iff 2(2s)^2 - 2(2p_2)^2 < (2s)^2 - (2p_4(\alpha))^2 \iff 2p_2^2 - s^2 > p_4(\alpha)^2.
\]
Since \( p_2 = -3k+1 \) and \( s = 2n+3k-2 \), we have a condition
\[
9k^2 - 12k - 4n^2 + 8n - 2 > p_4(\alpha)^2
\]
for \( c_4(\alpha) < c_2 \). Let \( p_0 = \min_{1 \leq \alpha < n+k} |p_4(\alpha)|. \) If 
\[(n, k) \in \{(n, k) \mid k \geq 4 \land 9k^2 - 12k - 4n^2 + 8n - 2 > p_0\}
\]
then there exists an \( \alpha \) such that \( \text{Neut}(A_4(\alpha), V \setminus A_4(\alpha)) < c_2 \). This means
\[
\text{Mcut}(P_{n,k}) = \frac{8(2s)^2 - 2(2p_2)^2}{8(2s)} = \frac{4(3k+2n-2)}{(3k+2n-2)^2}.
\]

### 3.4. Mct of weighted paths \( P_{n,k} \)

In this section, we consider a weighted path graph \( P_{n,k} \) and find a formula for \( \text{Mcut}(P_{n,k}) \) based on \( n, k \). We consider subsets of \( V(P_{n,k}) \) defined by \( A(\alpha) = \{x_i \mid 1 \leq i \leq \alpha\} \) for \( 1 \leq \alpha \leq n+k-1 \). We note that every subset \( A \subset V(P_{n,k}) \) with \( \text{cut}(A, V \setminus A) = 1 \) is \( A = A(\alpha) \) for some \( \alpha \).

**Lemma 12.** Let \( G = P_{n,k} \). There exists a subset \( A \subset V(P_{n,k}) \) such that \( \text{cut}(A, V \setminus A) = 1 \) and \( \text{Mcut}(G) = \text{Neut}(A, V \setminus A) \).

**Proof.** Since \( vol(P_{n,k}) = 3k+2n-2 \), if \( k \geq \frac{1}{3}(11-2n) \), then \( vol(P_{n,k}) \geq 9 \). By Lemma 9, we have \( \text{Mcut}(G) = \text{Mcut}_1(G) \).

If \( k < \frac{1}{4}(11-2n) \), then we have only five cases \( (n, k) = (1,1), (2,1), (3,1), (1,2) \) and \( (2,2) \). For each case 
\[
\text{Mcut}(P_{1,1}) = \text{Neut}(A(1), V \setminus A(1)), \text{Mcut}(P_{2,1}) = \text{Neut}(A(2), V \setminus A(2)), \text{Mcut}(P_{1,2}) = \text{Neut}(A(1), V \setminus A(1)), \text{Mcut}(P_{2,2}) = \text{Neut}(A(2), V \setminus A(2)), \)
\]

Let \( P_{n,k} \) \( (k \geq 1, n \geq 1) \) be a weighted path graph and \( \alpha \) an integer \( (1 \leq \alpha < n+k) \). We first note that 
\[
vol(P_{n,k}) = 3k+2n-2, \quad vol(A(\alpha)) = \begin{cases} 2\alpha - 1 & (\alpha \leq n) \\ 3\alpha - n - 1 & (n+1 \leq \alpha) \end{cases},
\]
\[
\text{Neut}(A(\alpha), V \setminus A(\alpha)) = c(\alpha),
\]
where functions \( c(\alpha) \) and \( p(\alpha) \) \( (1 \leq \alpha < n+k) \) are defined by
\[
c(\alpha) = \frac{(3k+2n-2)}{(3k+2n-2)^2} \quad \text{and} \quad p(\alpha) = \begin{cases} 4a - 3k - 2n & (1 \leq \alpha \leq n) \\ 6a - 3k - 4n & (n+1 \leq \alpha < n+k). \end{cases}
\]

We note that \( p(\alpha) = vol(A(\alpha)) - \frac{vol(P_{n,k})}{2} \) and
\[
\text{Mcut}(P_{n,k}) = \min_{1 \leq \alpha \leq n+k} c(\alpha) = \frac{4(3k+2n-2)}{(3k+2n-2)^2} - \frac{p_0}{2},
\]
where \( p_0 = \min_{1 \leq \alpha \leq n+k} |p(\alpha)|. \) To find the \( \min_{1 \leq \alpha \leq n+k} |p(\alpha)| \), we consider the following four cases:

**Case (i):** \( \frac{1}{2} vol(P_{n,k}) \leq vol(A(n)) \).

**Case (ii):** \( vol(A(n)) < \frac{1}{4} vol(P_{n,k}) < \frac{1}{2} (vol(A(n)) + vol(A(n+1))) \).

**Case (iii):** \( vol(A(n)) + vol(A(n+1)) < \frac{1}{2} vol(P_{n,k}) < vol(A(n+1)) \).

**Case (iv):** \( vol(A(n+1)) \leq \frac{1}{2} vol(P_{n,k}). \)

**Case (i):** Assume \( \frac{1}{2} vol(P_{n,k}) \leq vol(A(n)). \) That is \( k \leq \frac{2n}{3} \). In this case \( p_0 = \min_{1 \leq \alpha \leq n} |p(\alpha)| = \min_{1 \leq \alpha \leq n} |p(\alpha)|. \) We find \( \alpha \) \( (1 \leq \alpha \leq n) \) minimizing \( |p(\alpha)| = \left| \frac{4a - 3k - 2n}{2} \right| \).

For such \( \alpha \), we have
\[
\alpha - 1 < \frac{2n+3k}{4} \leq \alpha + 1/2.
\]

This means \( \alpha \) is the nearest integer of \( \frac{2n+3k}{4} \) which attains the minimum of \( \{ |p(\alpha)| \mid 1 \leq \alpha \leq n \} \). The value \( \min_{1 \leq \alpha \leq n} |p(\alpha)| \) is
\[
\min_{1 \leq \alpha \leq n} |p(\alpha)| = \begin{cases} 0 & (\{4 \mid k \land (2 \mid n)\}), \\ \frac{1}{2} & (2 \mid k), \\ 1 & (\{4 \mid k \land (2 \mid n)\}) \lor (k \text{ mod } 4 = 2 \land (2 \mid n)). \end{cases}
\]

**Case (ii):** Assume \( vol(A(n)) < \frac{1}{2} vol(P_{n,k}) < \frac{1}{2} (vol(A(n)) + vol(A(n+1))). \) 
In this case \( \text{Mcut}(P_{n,k}) = c(n) \). By assumptions, we have \( 2n < 3k < 2n+3 \) and so \( 2n < 3k \leq 2n+2. \) Then we obtain
\[
n < \frac{2n+3k}{4} < n + 1/2.
\]
This means \( n \) is the nearest integer of \( \frac{2n+3k}{4} \). So we can merge this case into case (i).

**Case (iv):** For technical reasons, we consider case (iv) before case (iii). Assume \( vol(A(n+1)) \leq \frac{1}{2} vol(P_{n,k}) \).

That is \( \frac{2}{3} n + 2 \leq k \). In this case \( p_0 = \min_{1 \leq \alpha \leq n+k} |p(\alpha)| \)
Corollary 1. For $P_{2k}(k \geq 1)$,

\[
\text{Mcut}(P_{2k},k) = \begin{cases}
\frac{4k}{7k-2} & (4 \mid k), \\
\frac{4k}{7k-4(k+1)} & (2 \mid k), \\
\frac{4k}{7k-2} & (4 \mid k) \land (2 \mid k).
\end{cases}
\]

Proof. By substituting $n = 2k$ in the formula given for \text{Mcut}(P_{n,k})\), we can directly obtain the result from Theorem 3.

\[
\text{Mcut}(P_{2k},k) = \frac{4(7k-2)}{(7k-2)^2 - (2p_0)^2} = \frac{4(7k-2)}{(7k-2 + 2p_0)(7k-2 - 2p_0)}.
\]

Since $k = \frac{1}{2}n \leq \frac{1}{2}n$, the result follows from case (i). \qed

3.5. Mecut of Graph $LP_{m,m}$

Here, we consider the lollipop graph $LP_{m,m}$ and derive a formula for \text{Mcut}(LP_{m,m}). A lollipop graph $LP_{m,m}$ defined in Definition 32 is constructed by joining an end vertex of a path graph $P_m$ to a vertex of a complete graph $K_n$.

We consider three kinds of subsets of $V(LP_{m,m})$ defined by $A_1(\alpha) = \{x_i \mid 1 \leq i \leq \alpha\}$ for $1 \leq \alpha \leq m$, $A_2(\beta) = \{x_i \mid 1 \leq i \leq \beta\}$ for $1 \leq \beta < n$, and, $B(\alpha, \beta) = \{x_i \mid 1 \leq i \leq \alpha\} \cup \{x_m\} \cup \{y_i \mid 1 \leq i \leq \beta\}$ for $1 \leq \alpha < m - 1, 1 \leq \beta < n$.

Lemma 13. Let $A$ be a subset of the vertex set $V(LP_{m,n})$ of a lollipop graph $LP_{m,n}$.

1. If $y_i \notin A$ and $y_{i+1} \notin A$ for some $i (2 \leq i \leq n - 1)$, then $\text{Ncut}(A', V \setminus A') = \text{Ncut}(A, V \setminus A)$, where $A' = (A \setminus \{y_i\}) \cup \{y_{i+1}\}$.

2. If $x_i \in A$, $x_{i+1}, \ldots, x_j \notin A$, and $x_{j+1} \in A$ for some $i, j (1 \leq i \leq j \leq m - 1)$, then $\text{Ncut}(A, V \setminus A') \leq \text{Ncut}(A, V \setminus A)$, where $A' = (A \setminus \{x_{i+1}\}) \cup \{x_{j+1}\}$.

3. There exists a subset $A_1(\alpha), A_2(\beta)$ or $B(\alpha, \beta)$ such that $\text{Mcut}(LP_{m,m}) = \text{Mcut}(A_1(\alpha), V \setminus A_1(\alpha))$, $\text{Mcut}(LP_{m,m}) = \text{Mcut}(A_2(\beta), V \setminus A_2(\beta))$, or $\text{Mcut}(LP_{m,m}) = \text{Mcut}(B(\alpha, \beta), V \setminus B(\alpha, \beta))$.

Proof. 1. It is easy to check that $\text{vol}(A) = \text{vol}(A')$ and $\text{cut}(A, V \setminus A) = \text{cut}(A', V \setminus A')$.

2. It is easy to check that $\text{vol}(A) = \text{vol}(A')$ and $\text{cut}(A', V \setminus A') \leq \text{cut}(A, V \setminus A)$.

3. Let $A$ be a subset of $V(LP_{m,m})$ such that $\text{Mcut}(LP_{m,m}) = \text{Ncut}(A, V \setminus A)$. Using the above results 1. and 2., we have a subset $A'$ which is one of $A_1(\alpha), A_2(\alpha)$ or $B(\alpha, \beta)$ such that $\text{Ncut}(A, V \setminus A') = \text{Mcut}(LP_{m,m})$. \qed

Let $n \geq 3$ and $m \geq 1$. We first note the followings for a lollipop graph $LP_{m,m}$,

\[
\text{vol}(LP_{m,m}) = 2m + n(n - 1),
\]

\[
\text{vol}(A_1(\alpha)) = 2\alpha - 1,
\]

\[
\text{cut}(A_1(\alpha), V \setminus A_1(\alpha)) = 1,
\]

\[
\text{vol}(A_2(\beta)) = 2m + \beta(n - 1),
\]

\[
\text{cut}(A_2(\beta), V \setminus A_2(\beta)) = \beta(n - \beta),
\]

\[
\text{vol}(B(\alpha, \beta)) = 2\alpha + 2 + \beta(n - 1),
\]

\[
\text{cut}(B(\alpha, \beta), V \setminus B(\alpha, \beta)) = \beta(n - \beta) + 2, \text{ and}
\]

\[
\text{Ncut}(A_1(\alpha), V \setminus A_1(\alpha)) = c(\alpha),
\]
where a function $c(t) (1 \leq t \leq m)$ is defined by

$$c(t) = \frac{2m + n(n - 1)}{(1 + 2(m - t) + n(n - 1))(2t - 1)}.$$

It is also shown that

$$\text{Ncut}(A_2(\beta), V \setminus A_2(\beta)) = \frac{\beta(2m + n(n - 1))}{(-1 + n)(2m + \beta(n - 1))},$$

and

$$\text{Ncut}(B(\alpha), V \setminus B(\alpha)) = \frac{(2m + n(n - 1))(2 + (n - 1)\beta)}{(2\alpha + 1) + (n - 1)\beta(2m + n(n - 1) - 2(\alpha + 1) - (n - 1)\beta)}.$$

**Lemma 14.** Let $n \geq 3$ and $m \geq 2$. We have the following for a lollipop graph $L_{P,m}$.

1. $c(\alpha - 1) < c(\alpha)$ iff $m > \frac{1}{2}(n^2 - n + 4)$ ($2 \leq \alpha \leq m$).
2. $c(m) \leq \frac{1}{2}\text{vol}(LP_{n,m})$ iff $m \leq \frac{1}{2}(n^2 - n + 4)$.
3. $c(m) \leq \text{Ncut}(A_2(\beta), V \setminus A_2(\beta))$ (1 $\leq \beta < n$).
4. If $m \leq \frac{1}{2}(n^2 - n + 2)$, then
   $$c(m) \leq \text{Ncut}(B(\alpha), V \setminus B(\alpha)),$$
   $$(1 \leq \alpha \leq m - 2, 1 \leq \beta < n).$$

**Proof.** Each item is given by straightforward computations.

Since $\text{cut}(A_1(\alpha), V \setminus A_1(\alpha)) = 1$, if $\text{vol}(A_1(m)) \geq \frac{1}{2}\text{vol}(LP_{n,m})$, then there exists some $\alpha$ such that

$$\text{Mcut}(LP_{n,m}) = \text{Ncut}(A_1(\alpha), V \setminus A_1(\alpha)).$$

To find $\alpha$, we solve

$$\text{vol}(A_1(\alpha)) - 1 < \frac{1}{2}\text{vol}(LP_{n,m}) \leq \text{vol}(A_1(\alpha)) + 1.$$

That is

$$\alpha - 1 < \frac{n^2 - n + 2m + 2}{4} \leq \alpha + \frac{1}{2}.$$

This means $\alpha$ is the nearest integer of $\frac{n^2 - n + 2m + 2}{4}$. We consider two cases ($K \in \mathbb{Z}$) and ($K \not\in \mathbb{Z}$), where $K = \frac{n^2 - n + 2m + 2}{2}$. If $K \in \mathbb{Z}$, then $\alpha = K$. If $K \not\in \mathbb{Z}$, then $K + \frac{1}{2}$ is an integer and $\alpha = K + \frac{1}{2}$ or $\alpha = K - \frac{1}{2}$. Since $c(K + \frac{1}{2}) = c(K - \frac{1}{2})$, $\text{Mcut}(LP_{n,m})$ will be

$$c(K) = \frac{4}{n^2 - n + 2m},$$

or

$$c(K + \frac{1}{2}) = \frac{4(n^2 - n + 2m)}{(n(n - 1) + 2(m - 1))(n(n - 1) + 2(m + 1))}.$$

By Lemma 14, if $m \leq \frac{1}{2}(n^2 - n + 4)$, then $\text{Mcut}(LP_{n,m}) = \text{Ncut}(A_1(m), V \setminus A_1(m))$. That is

$$\text{Ncut}(A_1(m), V \setminus A_1(m)) = \frac{n^2 - n + 2m}{(2m - 1)(n^2 - n + 1)}.$$

If $m = 1$, then it is easy to verify $\text{Mcut}(LP_{1,1}) = \text{Ncut}(A_2(1), V \setminus A_2(1)) = \frac{n^2 - n + 2}{(n + 1)(n - 1)}$.

**Theorem 4.** Let $n \geq 3$ and $m \geq 1$. For a lollipop graph $LP_{n,m}$,

$$\text{Mcut}(LP_{n,m}) = \begin{cases} 
\frac{n^2 - n + 2m}{(2m - 1)(n^2 - n + 1)} & (2 \leq m \leq \frac{n^2 - n + 4}{2}), \\
\frac{n^2 - n + 2m}{(n + 1)(n - 1)(n(n - 1) + 2(m + 1))} & (m = 1), \\
\frac{n^2 - n + 2m}{(n^2 - n + 2m)} & (o_1 \land m > \frac{n^2 - n + 4}{2}), \\
\frac{n^2 - n + 2m}{(n + 1)(n - 1)(n^2 - n + 1) + 2(m + 1)} & (o_2 \land m > \frac{n^2 - n + 4}{2}), \\
\end{cases}$$

where

$$o_1 = \left(\frac{n^2 - n + 2m + 2}{4} \in \mathbb{Z}\right),$$

and

$$o_2 = \left(\frac{n^2 - n + 2m + 2}{4} \not\in \mathbb{Z}\right).$$

4. Conclusion

We have presented a survey of some known results associated with difference, normalized, and signless Laplacian matrices. We also stated upper and lower bounds for the difference and normalized Laplacian matrices using isoperimetric numbers and the Cheeger constant. We derived concrete formulae for $\text{Mcut}(G)$ for some classes of graphs such as paths, cycles, complete graphs, double-trees, lollipop graphs $LP_{n,m}$, roach type graphs $R_{n,k}$ and weighted paths $P_{n,k}$.

**Acknowledgments**

We would like to especially thank Professor Hiroyuki Ochiai for his ideas pertaining to computations and comparisons of the second eigenvalues of $L(P_{n,k})$, which gave us useful hints to finish this study. We also thank to the anonymous referee for his very careful reviews of the paper that include many important points and will improve significantly the clarity of our proofs in this paper. We are also grateful to Dr. Tetsuji Taniguchi for his helpful comments and encouragement during the course of this study. This research was partially supported by the Global COE Program “Educational-and-Research Hub for Mathematics-for-Industry” at Kyushu University.

**References**


K. K. K. R. Perera  
Department of Mathematics, University of Kelaniya, Sri Lanka  
E-mail: kkkrperera(at)kln.ac.lk  

Yoshihiro Mizoguchi  
Institute of Mathematics for Industry, Kyushu University, Japan  
E-mail: ym(at)imi.kyushu-u.ac.jp