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Abstract. We propose a fracture model of a vibrating wire modeled as a one dimensional spring-mass system. We introduce a phase field variable for damage of the springs and represent a fracture of the wire by a cutting of the spring. The model we propose consists of a system of ordinary differential equations and admits rigorous mathematical analysis, such as global existence of a unique solution and an energy decay estimate. The validity of our fracture model and its estimate is confirmed using numerical simulations.

Keywords. fracture model, vibration analysis, spring-mass system, phase filed model

1. Introduction

When an improperly constructed structure, such as bridge, tower and building, is oscillating in response to an external force with a certain frequency, such as that generated by a wind or an earthquake, a violent swaying motion can occur. This is referred to mechanical resonance and it can even lead to disastrous failure.

How to avoid such catastrophic failure has been a key activity in structural and vibration analysis for many years. On the other hand, over the last few decades, various numerical simulation techniques for crack propagation and structure failure have been developed and applied to various situations; e.g. DEM (discrete element method) [2, 3], RBSM (rigid body spring model) [4], and PDS-FEM (FEM-β) method [5, 9]. In most of the above studies, the fracture was modelled by the cutting of virtual springs. These simulation techniques are extremely powerful and can easily include various kind of effects besides fracture. On the other hand, one thing common to them is that only limited mathematical analysis has been undertaken because they include many articficial numerical steps to avoid numerical instabilities and to obtain desirable simulation results. The spring cutting criterion is one of such artificial choices in the crack simulation.

For more details about crack propagation simulation, we refer to [9]. It contains a very good literature survey about numerical techniques for crack propagation including the above methods as well as other methods such as X-FEM. For details about performing a vibration analysis with X-FEM in a cracked domain, we refer to [1].

The aim of this paper is the formulation of a rigorous mathematical vibration-fracture model. Here, we examine a one dimensional wave equation model of a straight vibrating wire (spring-mass system). The corresponding spring mass system is examined in Section 2. Following the approach in [7], we introduce in Section 3 a phase field variable for the springs to represent their damage. We propose a phase field model for the fracture of the vibrating wire and prove that there exists a unique global solution. In Section 4, we show that our model has a kind of gradient structure in terms of energies which are naturally defined. Using the gradient flow structure, we derive some uniform energy estimates which are independent of the number of the space division. The number of pieces of the broken wire is also uniformly estimated in Theorem 4.

In Section 5, some numerical results are derived using a time discretization scheme. In terms of the numerical examples, it is established that complex vibration-fracture phenomena can be modelled using our simple ODE model. In addition, our model can be analyzed mathematically to give results such as global existence of a unique solution and an energy decay estimate.

2. Spring-mass model for vibrating wire

We consider a spring-mass model for a straight vibrating wire of length l in the x-y plane. For $i = 0, \dots, N$, let P_i be a point mass $m_i > 0$, and let $(x_i, u_i(t))$ be the position of P_i at time t, where $0 = x_0 < x_1 < x_2 < \cdots < x_N = l$ are fixed. We call u_i the displacement of P_i in y-direction. We suppose that, for $i = 1, \dots, N$, the neighboring weights P_{i-1} and P_i are connected by a spring S_i and that P_{i-1} and P_i receive forces of $\kappa_i(u_i - u_{i-1})$ and $\kappa_i(u_{i-1} - u_i)$ respectively in the y-direction through S_i, where $\kappa_i > 0$ is the known spring constant of S_i. We assume that the boundary conditions at two the end points are given by

$$u_0(t) = a(t), \;\; u_N(t) = b(t), \tag{1}$$

and we denote the given external force acting on P_i in the y-direction by $F_i(t)$. Then, the equation of motion of P_i is given by

$$m_i \frac{d^2 u_i}{dt^2} = \kappa_i(u_{i-1} - u_i) + \kappa_{i+1}(u_{i+1} - u_i) + F_i(t) \quad (i = 1, \dots, N-1). \quad (2)$$

For the displacement $\boldsymbol{u} := (u_1, \dots, u_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1}$, taking (1) into account, we define an elastic energy and a kinetic energy by

$$E_1(\boldsymbol{u}) := \frac{1}{2} \sum_{i=1}^{N} \kappa_i |u_i - u_{i-1}|^2 - \boldsymbol{F} \cdot \boldsymbol{u}, \quad (3)$$

$$E_2(\dot{\boldsymbol{u}}) := \frac{1}{2} \sum_{i=1}^{N-1} m_i |\dot{u}_i|^2, \quad (4)$$

where $\dot{\boldsymbol{u}} := (\dot{u}_1, \dots, \dot{u}_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1}$, $\dot{u}_i := \frac{du_i}{dt}$, and $\boldsymbol{F} := (F_1, \dots, F_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1}$. Then, if $a(t)$, $b(t)$ and $F_i(t)$ do not depend on t, it is well-known that the following law of the energy conservation holds:

$$\frac{d}{dt}(E_1(\boldsymbol{u}(t)) + E_2(\dot{\boldsymbol{u}}(t))) = 0.$$

When the elastic wire is homogeneous, i.e. when its density ρ (mass per unit length) is constant, the mass of P_i is given by $m_i = \rho h_i$, where

$$h_i := \frac{x_{i+1} - x_{i-1}}{2} \qquad (i = 1, \dots, N-1). \quad (5)$$

Equation (2) is a finite difference scheme for a wave equation of a vibrating string

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2}(t,x) = \mu \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(t,x) \quad (t > 0,\ 0 < x < l) \\ u(t,0) = a(t), \quad u(t,l) = b(t) \quad (t > 0) \\ u(0,x) = u^0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v^0(x) \quad (0 < x < l) \end{cases} \quad (6)$$

where ρ and μ are given positive constants and f, a, b, u^0 and v^0 are given sufficiently smooth functions. For (6), we define

$$m_i = \rho h_i, \ \ \kappa_i = \frac{\mu}{x_i - x_{i-1}}, \ \ F_i(t) = h_i f(t, x_i). \quad (7)$$

Then, (2) and (1) can be viewed a Shortley-Weller type finite difference scheme [8, 10]. A standard and elementary error estimate is given by:

Proposition 1. *For a smooth solution $u(t,x)$ $(0 \le t \le T)$ to* (6), *if $\boldsymbol{u}(t)$ $(0 \le t \le T)$ is a solution of* (2) *and* (1) *with the initial conditions:*

$$u_i(0) = u^0(x_i), \quad \dot{u}_i(0) = v^0(x_i) \quad (i = 1, \dots, N-1),$$

there exists a constant $C_T > 0$ such that

$$|u(t, x_i) - u_i(t)| \le C_T(h^2 + \triangle h) \quad (0 \le t \le T,\ i = 1, \dots, N-1)$$

holds, where

$$h := \max_{1 \le i \le N} (x_i - x_{i-1}),$$
$$\triangle h := \max_{1 \le i \le N-1} |x_{i+1} - 2x_i + x_{i-1}|.$$

A proof of this proposition is given using standard stability and truncation error estimates. Since it is an elemental calculation, it is omitted.

In the simulation of this spring-mass model, the choice of values of the spring constants is a most delicate task. Proposition 1 suggests a reasonable choice for the spring constant as given in (7).

3. Fracture model for the vibrating wire

Under a suitable periodic external force, the wave equation (6) or the corresponding spring-mass system (2) can exhibit a resonance phenomena. Then, physically, the strength of the wire cannot resist large vibrations and a failure can be expected to occur at a certain moment. In this section, we propose a phase field model for the damage of the spring, by which we can simulate the motion of the wire from resonance to fracture. We suppose that each spring S_i is damaged if the stress of S_i exceeds a given threshold and that its spring constant κ_i is weakened and changes to $\tilde{\kappa}_i < \kappa_i$. We introduce a phase field variable $z_i \in [0,1]$ for each spring S_i and suppose that $(1 - z_i)^2$ is the relative damage of the spring constant, i.e., we suppose that

$$\tilde{\kappa}_i = (1 - z_i)^2 \kappa_i \qquad (i = 1, 2, \dots, N).$$

When $z_i = 0$, this corresponds to the case that the spring S_i has no damage, whereas $z_i = 1$ corresponds to the case that the spring is totally broken. Let $\boldsymbol{z}(t) = (z_1(t), \dots, z_N(t))^{\mathrm{T}} \in \mathbb{R}^N$.

We also assume that a weakened spring cannot recover and that the damage of the spring accumulates and weakens its spring constant, and leads to failure of the wire. We propose the following phase field model for fracture of the straight vibrating wire

$$\begin{cases} m_i \dfrac{d^2 u_i}{dt^2} = \tilde{\kappa}_i(u_{i-1} - u_i) + \tilde{\kappa}_{i+1}(u_{i+1} - u_i) + F_i(t) \\ \qquad\qquad\qquad\qquad (i = 1, \dots, N-1) \\ u_0(t) = a(t), \ \ u_N(t) = b(t) \\ \alpha \dfrac{dz_i}{dt} = [Q_i - \gamma_i]_+ (1 - z_i) \ \ (i = 1, \dots, N) \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \ \ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0, \ \ \boldsymbol{z}(0) = \boldsymbol{z}^0, \end{cases} \quad (8)$$

where $[c]_+ := \max(c, 0)$ and $\alpha > 0$ is a suitable time constant, and $\boldsymbol{u}^0 \in \mathbb{R}^{N-1}$, $\boldsymbol{v}^0 \in \mathbb{R}^{N-1}$, $\boldsymbol{z}^0 \in \mathbb{R}^N$ are given. The term Q_i is defined by

$$Q_i := \kappa_i (u_i - u_{i-1})^2 \qquad (i = 1, \dots, N),$$

and represents the magnitude of the stress on the spring S_i. We suppose that the strength of the spring S_i is represented by the given positive constant $\gamma_i > 0$. The larger γ_i

is, the harder it is to break the spring S_i. Here, we call γ_i fracture toughness of S_i using the analogy from the fracture mechanics. The first equation of (8) is an equation of motion similar to (2). The third equation is an evolution equation of the i-th damage variable z_i and it represents that z_i can increase and approach to 1 if and only if Q_i exceeds the threshold γ_i. We define a convex set $K \subset \mathbb{R}^N$ by

$$K := \{\boldsymbol{z} = (z_1, \dots, z_N)^{\mathrm{T}} \in \mathbb{R}^N;\ z_i \in [0,1] \quad (i = 1, \dots, N)\},$$

and we suppose that

$$\boldsymbol{z}^0 \in K. \tag{9}$$

Suppose that $N \geq 3$. In the following, we denote the maximum norm in $\mathbb{R}^{N-1}$ or $\mathbb{R}^N$ by $\|\cdot\|_\infty$. The ODE system (8) can be rewritten as the following first order system

$$\begin{cases} \dot{\boldsymbol{u}}(t) = \boldsymbol{v}(t) \\ \dot{\boldsymbol{v}}(t) = \boldsymbol{f}(t, \boldsymbol{u}(t), \boldsymbol{z}(t)) \\ \dot{\boldsymbol{z}}(t) = \boldsymbol{g}(t, \boldsymbol{u}(t), \boldsymbol{z}(t)) \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \boldsymbol{v}(0) = \boldsymbol{v}^0, \quad \boldsymbol{z}(0) = \boldsymbol{z}^0, \end{cases} \tag{10}$$

where $\boldsymbol{u}(t) \in \mathbb{R}^{N-1}$, $\boldsymbol{v}(t) \in \mathbb{R}^{N-1}$, and $\boldsymbol{z}(t) \in \mathbb{R}^N$. The functions $\boldsymbol{f} = (f_1, \dots, f_{N-1})^{\mathrm{T}} : \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^N \to \mathbb{R}^{N-1}$ and $\boldsymbol{g} = (g_1, \dots, g_N)^{\mathrm{T}} : \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^N \to \mathbb{R}^N$ are defined as follows. For $\boldsymbol{u} = (u_1, \dots u_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1}$ and $\boldsymbol{z} = (z_1, \dots z_N)^{\mathrm{T}} \in \mathbb{R}^N$, we define

$$f_i(t, \boldsymbol{u}, \boldsymbol{z}) :=$$

$$\begin{cases} \dfrac{1}{m_1}\{(1 - z_1)^2 \kappa_1 (a(t) - u_1) \\ \qquad + (1 - z_2)^2 \kappa_2 (u_2 - u_1) + F_1(t)\} \qquad (i = 1), \\ \dfrac{1}{m_i}\{(1 - z_i)^2 \kappa_i (u_{i-1} - u_i) \\ \qquad + (1 - z_{i+1})^2 \kappa_{i+1} (u_{i+1} - u_i) + F_i(t)\} \\ \qquad\qquad (i = 2, \dots, N-2), \\ \dfrac{1}{m_{N-1}}\{(1 - z_{N-1})^2 \kappa_{N-1} (u_{N-2} - u_{N-1}) \\ \qquad + (1 - z_N)^2 \kappa_N (b(t) - u_{N-1}) + F_{N-1}(t)\} \\ \qquad\qquad (i = N-1), \end{cases}$$

and

$$g_i(t, \boldsymbol{u}, \boldsymbol{z}) := \frac{1}{\alpha}[Q_i(t, \boldsymbol{u}) - \gamma_i]_+ (1 - z_i) \quad (i = 1, \dots, N),$$

$$Q_i(t, \boldsymbol{u}) := \begin{cases} \kappa_1 (a(t) - u_1)^2 & (i = 1), \\ \kappa_i (u_{i-1} - u_i)^2 & (i = 2, \dots, N-1), \\ \kappa_N (b(t) - u_{N-1})^2 & (i = N)\ . \end{cases}$$

We have the following lemmas.

Lemma 1. *Let $T > 0$. We suppose that $a \in C^0([0,T])$, $b \in C^0([0,T])$ and $\boldsymbol{F} \in C^0([0,T], \mathbb{R}^{N-1})$. Then $\boldsymbol{f} \in C^0(\mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^N,\ \mathbb{R}^{N-1})$ and $\boldsymbol{g} \in C^0(\mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^N,\ \mathbb{R}^N)$ hold. Moreover, for an arbitrary bounded subset $G \subset \mathbb{R}^{N-1} \times \mathbb{R}^N$, there exists $M > 0$ such that the following inequalities hold.*

$$\|\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{z}) - \boldsymbol{f}(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{z}})\|_\infty \leq M(\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_\infty + \|\boldsymbol{z} - \tilde{\boldsymbol{z}}\|_\infty),$$
$$\|\boldsymbol{g}(t, \boldsymbol{u}, \boldsymbol{z}) - \boldsymbol{g}(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{z}})\|_\infty \leq M(\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_\infty + \|\boldsymbol{z} - \tilde{\boldsymbol{z}}\|_\infty),$$

for all $t \in [0,T]$, $(\boldsymbol{u}, \boldsymbol{z}) \in G$, $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{z}}) \in G$. Moreover, we have

$$\|\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{z})\|_\infty \leq \frac{4\kappa}{m}\|\boldsymbol{u}\|_\infty + A(t) \qquad (t \in [0,T],\ \boldsymbol{u} \in \mathbb{R}^{N-1}, \boldsymbol{z} \in K), \tag{11}$$

where

$$m := \min_{1 \leq i \leq N-1} m_i, \qquad \kappa := \max_{1 \leq i \leq N} \kappa_i,$$

$$A(t) := \frac{1}{m}\left(\kappa \max(|a(t)|,\ |b(t)|) + \|\boldsymbol{F}(t)\|_\infty\right). \tag{12}$$

Proof. Since each f_i is a polynomial of $a(t)$, $b(t)$, $F_j(t)$, $u_j(t)$ and $z_j(t)$, the Lipschitz condition for $\boldsymbol{f}$ is clear. We can also easily show the Lipschitz condition for $\boldsymbol{g}$ from the facts that Q_i is a polynomial of $a(t)$, $b(t)$ and $u_j(t)$ and that

$$\Big|[c_1]_+ - [c_2]_+\Big| \leq |c_1 - c_2| \qquad (c_1,\ c_2 \in \mathbb{R}).$$

The estimate (11) directly follows from the definition of f_i. □

Lemma 2. *Under the conditions of Lemma 1 and (9), we suppose that $(\boldsymbol{u}, \boldsymbol{z}) \in C^2([0,T], \mathbb{R}^{N-1}) \times C^1([0,T], \mathbb{R}^N)$ is a solution to (8). Then the following inequalities hold:*

$$\dot{z}_i(t) \geq 0, \quad z_i(t) \in [0,1] \quad (i = 1, \dots, N,\ t \in [0,T]), \tag{13}$$

$$\max\left(\|\boldsymbol{u}(t)\|_\infty, \|\dot{\boldsymbol{u}}(t)\|_\infty\right) \leq e^{c_0 t} \max\left(\|\boldsymbol{u}^0\|_\infty, \|\boldsymbol{v}^0\|_\infty\right) + \int_0^t e^{c_0(t-s)} A(s)\, ds \quad (t \in [0,T]),$$

where $A(s)$ is defined by (12) and $c_0 := \max(1,\ 4\kappa/m)$.

Proof. Let $\boldsymbol{v}(t) := \dot{\boldsymbol{u}}(t)$. Then $(\boldsymbol{u}(t), \boldsymbol{v}(t), \boldsymbol{z}(t))$ is a solution to (10) for $t \in [0,T]$. For $i = 1, \dots, N$, from (10), we have

$$\dot{z}_i(t) = h_i(t)(1 - z_i(t))\ (t \in [0,T]),\ z_i(0) = z_i^0 \in [0,1], \tag{14}$$

where

$$h_i(t) := \frac{1}{\alpha}[Q_i(t, \boldsymbol{u}(t)) - \gamma_i]_+\ .$$

We remark that $h_i \in C^0([0,T])$ and $h_i(t) \geq 0$. Solving the ODE (14), we have

$$z_i(t) = 1 - (1 - z_i^0) e^{-\int_0^t h_i(s) ds}.$$

and from this expression, we obtain (13). Since $\boldsymbol{z}(t) \in K$, from Lemma 1, we obtain

$$\|\dot{\boldsymbol{v}}(t)\|_\infty = \|\boldsymbol{f}(t, \boldsymbol{u}(t), \boldsymbol{z}(t))\|_\infty \leq \frac{4\kappa}{m}\|\boldsymbol{u}(t)\|_\infty + A(t), \tag{15}$$

for $t \in [0,T]$. We define

$$\varphi(t) := \max\left(\|\boldsymbol{u}(t)\|_\infty, \|\dot{\boldsymbol{u}}(t)\|_\infty\right), \ \psi(t) := \int_0^t \varphi(s)\,ds.$$

From (10), we have

$$\begin{aligned}\|\boldsymbol{u}(t)\|_\infty &= \left\|\boldsymbol{u}^0 + \int_0^t \dot{\boldsymbol{u}}(s)\,ds\right\|_\infty \\ &\le \|\boldsymbol{u}^0\|_\infty + \int_0^t \|\dot{\boldsymbol{u}}(s)\|_\infty\,ds \le \varphi(0) + \psi(t),\end{aligned}$$

and from (15), we have

$$\begin{aligned}\|\dot{\boldsymbol{u}}(t)\|_\infty &= \left\|\boldsymbol{v}^0 + \int_0^t \dot{\boldsymbol{v}}(s)\,ds\right\|_\infty \\ &\le \|\boldsymbol{v}^0\|_\infty + \int_0^t \|\dot{\boldsymbol{v}}(s)\|_\infty\,ds \\ &\le \|\boldsymbol{v}^0\|_\infty + \frac{4\kappa}{m}\int_0^t \|\boldsymbol{u}(s)\|_\infty\,ds + \int_0^t A(s)\,ds \\ &\le \varphi(0) + \frac{4\kappa}{m}\psi(t) + \int_0^t A(s)\,ds.\end{aligned}$$

Hence we obtain the following Gronwall type inequality:

$$\psi'(t) = \varphi(t) \le \varphi(0) + c_0\psi(t) + \int_0^t A(s)\,ds.$$

Solving this differential inequality, we have

$$\varphi(t) \le e^{c_0 t}\varphi(0) + \int_0^t e^{c_0(t-s)}A(s)\,ds. \qquad \square$$

We prove the unique existence of the global solution.

Theorem 1. *We suppose that* $a \in C^0([0,\infty))$, $b \in C^0([0,\infty))$, $\boldsymbol{F} \in C^0([0,\infty),\mathbb{R}^{N-1})$, *and* (9). *Then there exists a unique solution* $(\boldsymbol{u},\boldsymbol{z}) \in C^2([0,\infty),\mathbb{R}^{N-1}) \times C^1([0,\infty),\mathbb{R}^N)$ *to* (8). *Moreover,* $\boldsymbol{z}(t) \in K$ *holds for* $t \in [0,\infty)$.

Proof. Instead of (8), we consider the equivalent first order system (10). From Lemma 1, there exists a unique local solution to the initial value problem (10). Let $T^* > 0$ be the maximal time of the existence of the solution to (10). If $T^* < \infty$, from a general theory of the ODE system, the solution $(\boldsymbol{u}(t),\boldsymbol{v}(t),\boldsymbol{z}(t))$ has to blow up as $t \to T^* - 0$, i.e.,

$$\lim_{t\to T^*-0} \max\left(\|\boldsymbol{u}(t)\|_\infty,\ \|\boldsymbol{v}(t)\|_\infty,\ \|\boldsymbol{z}(t)\|_\infty\right) = \infty.$$

But it contradicts the estimates in Lemma 2. Hence, it follows that $T^* = \infty$. $\square$

4. Uniform energy estimates

We consider the following energies. Similarly to (3), the elastic energy E_1 is defined by

$$E_1(\boldsymbol{u},\boldsymbol{z}) := \frac{1}{2}\sum_{i=1}^N \tilde{\kappa}_i|u_i - u_{i-1}|^2 - \boldsymbol{F}\cdot\boldsymbol{u},$$

and the kinetic energy $E_2(\dot{\boldsymbol{u}})$ is defined by (4). We additionally consider the energy:

$$E_3(\boldsymbol{z}) := \frac{1}{2}\sum_{i=1}^N \gamma_i(1-(1-z_i)^2), \tag{16}$$

which represents the energy consumed by the damage of the springs. The following theorem shows that our phase field model possesses a kind of gradient flow structure.

Theorem 2. *Let* $(\boldsymbol{u}(t),\boldsymbol{z}(t))$ *be a solution to* (8). *If* $a(t)$, $b(t)$ *and* $\boldsymbol{F}(t)$ *do not depend on* t, *then the following equality holds.*

$$\begin{aligned}\frac{d}{dt}\big(E_1(\boldsymbol{u}(t),\boldsymbol{z}(t)) + E_2(\dot{\boldsymbol{u}}(t)) + E_3(\boldsymbol{z}(t))\big) \\ = -\alpha\sum_{i=1}^N \left|\frac{dz_i}{dt}\right|^2 \le 0.\end{aligned} \tag{17}$$

Proof. Under the assumptions, since $\dot{u}_0 = \dot{u}_N = 0$ and $\dot{F}_i = 0$, we obtain

$$\begin{aligned}&\frac{d}{dt}\big(E_1(\boldsymbol{u}(t),\boldsymbol{z}(t)) + E_2(\dot{\boldsymbol{u}}(t)) + E_3(\boldsymbol{z}(t))\big) \\ &= \sum_{i=1}^N \tilde{\kappa}_i(u_i - u_{i-1})(\dot{u}_i - \dot{u}_{i-1}) - \sum_{i=1}^{N-1} F_i\dot{u}_i \\ &\quad - \sum_{i=1}^N (1-z_i)\dot{z}_i\kappa_i|u_i - u_{i-1}|^2 + \sum_{i=0}^N m_i\dot{u}_i\ddot{u}_i \\ &\quad + \sum_{i=1}^N \gamma_i(1-z_i)\dot{z}_i \\ &= \sum_{i=1}^{N-1}\{\tilde{\kappa}_i(u_i - u_{i-1}) - \tilde{\kappa}_{i+1}(u_{i+1}-u_i) - F_i + m_i\ddot{u}_i\}\dot{u}_i \\ &\quad + \sum_{i=1}^N\{\gamma_i(1-z_i) - (1-z_i)\kappa_i|u_i - u_{i-1}|^2\}\dot{z}_i \\ &= \sum_{i=1}^N (\gamma_i - Q_i)(1-z_i)\dot{z}_i\,.\end{aligned}$$

If $Q_i > \gamma_i$, since $\alpha\dot{z}_i = (Q_i - \gamma_i)(1-z_i)$, we have

$$(\gamma_i - Q_i)(1-z_i)\dot{z}_i = -\alpha|\dot{z}_i|^2 \tag{18}$$

On the other hand, if $Q_i \le \gamma$, we have $\dot{z}_i = 0$. Hence, in both cases, we obtain (18). The estimate (17) follows from the above equalities. $\square$

Let $l > 0$ and $T > 0$ be fixed. We consider a space division $0 = x_0 < x_1 < \cdots < x_N = l$ with $N \in \mathbb{N}$, $N \ge 2$, and set the following assumptions for the parameters and the initial and boundary conditions of the problem (8).

(H1) We suppose that $u^0 \in C^1([0,l])$, $v^0 \in C^0([0,l])$, $a \in C^1([0,T])$, $b \in C^1([0,T])$, and $f \in C^0([0,T]\times[0,l])$ are given independently of the space division, and set

$$u_i^0 := u(x_i),\ v_i^0 := v(x_i),\ F_i(t) := h_i f(t,x_i) \qquad (i = 1,\ldots,N-1).$$

(H2) $\rho > 0$, $m_i = \rho h_i$ $(i = 1, \dots, N-1)$

(H3) $\mu > 0$, $\kappa_i = \dfrac{\mu}{x_i - x_{i-1}}$ $(i = 1, \dots, N)$

(H4) $\alpha > 0$,

(H5) $\gamma_i > 0$, $z_i^0 \in [0,1]$ $(i = 1, \dots, N)$ and there exists $C_0 > 0$ which is independent of the space division such that $\displaystyle\sum_{i=1}^{N} \gamma_i z_i^0 \le C_0$.

We define the following discrete energies

$$
\begin{aligned}
U_0(t) &:= \frac{1}{2}\sum_{i=1}^{N-1} m_i |u_i(t)|^2, \\
U_1(t) &:= \frac{1}{2}\sum_{i=1}^{N} \tilde{\kappa}_i(t) |u_i(t) - u_{i-1}(t)|^2, \\
U_2(t) &:= \frac{1}{2}\sum_{i=1}^{N-1} m_i |\dot{u}_i(t)|^2, \\
U_3(t) &:= \frac{1}{2}\sum_{i=1}^{N} \gamma_i \big(1 - (1 - z_i(t))^2\big), \\
U_4(t) &:= \alpha \sum_{i=1}^{N-1} \int_0^t |\dot{z}_i(s)|^2 \, ds.
\end{aligned}
$$

Theorem 3. *Under the conditions (H1)–(H5), there exists $M_k(t) > 0$ $(0 \le t \le T,\ k = 0, \dots, 4)$ which do not depends on α, $\{\gamma_i\}_{i=1}^N$ and the space division such that, if the conditions*

$$
\begin{cases}
(u_1(t) - a(t))\dot{a}(t) \ge 0 \\
(u_{N-1}(t) - b(t))\dot{b}(t) \ge 0
\end{cases}
\qquad (0 \le t \le T), \tag{19}
$$

hold, then a solution of (8) *satisfies the following uniform estimates:*

$$
U_k(t) \le M_k(t) \quad (0 \le t \le T,\ k = 0, \dots, 4).
$$

Proof. We remark that $U_i(t) \ge 0$ for $i = 0, \dots, 4$ and that the following estimates hold:

$$
\begin{aligned}
&U_0(0) \le \frac{\rho l}{2}\|u_0\|_\infty^2, \quad U_1(0) \le \frac{\mu l}{2}\|u_0'\|_\infty^2, \\
&U_2(0) \le \frac{\rho l}{2}\|v_0\|_\infty^2, \quad U_3(0) \le C_0, \quad U_4(0) = 0,
\end{aligned}
$$

from the conditions (H1)–(H5), which are uniformly bounded independently of α, $\{\gamma_i\}_{i=1}^N$ and the space division. We define

$$
U(t) := U_1(t) + U_2(t) + U_3(t) + U_4(t).
$$

Then, similarly to Theorem 2, we obtain

$$
\begin{aligned}
\frac{d}{dt}U(t) = &-\sum_{i=1}^{N} \kappa_i (1 - z_i(t))\dot{z}_i(t)|u_i(t) - u_{i-1}(t)|^2 \\
&+ \sum_{i=1}^{N} \tilde{\kappa}_i(t)(u_i(t) - u_{i-1}(t))(\dot{u}_i(t) - \dot{u}_{i-1}(t)) \\
&+ \sum_{i=1}^{N-1} m_i \dot{u}_i(t)\ddot{u}_i(t) + \sum_{i=1}^{N} \gamma_i (1 - z_i(t))\dot{z}_i(t) \\
&+ \alpha \sum_{i=1}^{N-1} |\dot{z}_i(t)|^2 \\
= &\sum_{i=1}^{N-1} F_i(t)\dot{u}_i(t) + \tilde{\kappa}_1(t)(a(t) - u_1(t))\dot{a}(t) \\
&+ \tilde{\kappa}_N(t)(b(t) - u_{N-1}(t))\dot{b}(t) \\
\le &\sum_{i=1}^{N-1} F_i(t)\dot{u}_i(t),
\end{aligned}
$$

where we used the conditions (19). Using $|F_i(t)| \le h_i \|f\|_\infty$ and the Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
\frac{d}{dt}U(t) &\le \sum_{i=1}^{N-1} F_i(t)\dot{u}_i(t) \le \|f\|_\infty \sqrt{\frac{2l}{\rho}}\sqrt{U_2(t)} \\
&\le \|f\|_\infty \sqrt{\frac{2l}{\rho}}\sqrt{U(t)}.
\end{aligned}
$$

Solving this differential inequality, we obtain that

$$
U(t) \le \left(\sqrt{U(0)} + \sqrt{\frac{l}{2\rho}}\|f\|_\infty t\right)^2 \quad (0 \le t \le T).
$$

Similarly, from the inequality:

$$
\frac{d}{dt}U_0(t) \le \sum_{i=1}^{N-1} m_i u_i(t)\dot{u}_i(t) \le 2\sqrt{U_2(t)}\sqrt{U_0(t)},
$$

we obtain that

$$
U_0(t) \le \left(\sqrt{U_0(0)} + \int_0^t \sqrt{M_2(s)}\, ds\right)^2 \quad (0 \le t \le T). \ \square
$$

As shown in numerical examples in the next section, the wire breaks at the spring S_i if the damage variable $z_i(t)$ is close to 1. For fixed $0 \le \varepsilon \ll 1$, we define $I(t)$ which is the number of broken springs as follows:

$$
I(t) := \#\{S_i;\ z_i(t) \ge 1 - \varepsilon\}.
$$

In other words, the wire is broken into $I(t)+1$ pieces. The number of broken pieces of the wire is uniformly estimated as follows.

Theorem 4. *Under the conditions of Theorem 3,*

$$
I(t) \le \frac{2}{\left(\min\limits_{i=1,\cdots,N} \gamma_i\right)(1 - \varepsilon)} M_3(t)
$$

holds. In particular, if there exists a constant $\gamma > 0$ independent of the space dividion such that $\gamma_i \geq \gamma > 0$ for all $i = 1, \ldots, N$, then $I(t) \leq 2\gamma^{-1}(1-\varepsilon)^{-1}M_3(t)$ holds, namely the number of broken pieces of the wire is uniformly bounded.

Proof. The assertion follows from the following inequalities:

$$\begin{aligned} M_3(t) \geq U_3(t) &= \frac{1}{2}\sum_{i=1}^{N} \gamma_i(2 - z_i(t))z_i(t) \\ &\geq \frac{1}{2}\Big(\min_{i=1,\ldots,N} \gamma_i\Big)\sum_{i=1}^{N} z_i(t) \\ &\geq \frac{1}{2}\Big(\min_{i=1,\ldots,N} \gamma_i\Big)(1-\varepsilon)I(t). \qquad \square \end{aligned}$$

5. Numerical results

We consider the following time discretization scheme for the phase field model of (8). Let $\tau > 0$ be a time increment. For $k = 0, 1, 2, \ldots$, we denote numerical approximations of $\boldsymbol{u}$ and $\boldsymbol{z}$ at time $t = k\tau$ by $\boldsymbol{u}^k = (u_0^k, \ldots, u_N^k)^{\mathrm{T}}$ and $\boldsymbol{z}^k = (z_1^k, \ldots, z_N^k)^{\mathrm{T}}$, respectively. We adopt the three point central difference for $\ddot{\boldsymbol{u}}$ and the backward Euler scheme for $\dot{\boldsymbol{z}}$. Then we have the following finite difference scheme:

$$\begin{cases} m_i \dfrac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{\tau^2} \\ \quad = \tilde{\kappa}_i^k(u_{i-1}^k - u_i^k) + \tilde{\kappa}_{i+1}^k(u_{i+1}^k - u_i^k) + F_i(k\tau) \\ \qquad\qquad (i = 1, \ldots, N-1,\ k = 0, 1, \ldots) \\ u_0^k = a(k\tau),\ u_N^k = b(k\tau) \qquad (k = 0, 1, 2, \ldots) \\ \alpha \dfrac{z_i^{k+1} - z_i^k}{\tau} = [Q_i^k - \gamma_i]_+ (1 - z_i^{k+1}) \quad (i = 1, \ldots, N) \\ \boldsymbol{u}^0, \boldsymbol{u}^{-1}, \boldsymbol{z}^0\text{: given,} \end{cases} \tag{20}$$

where

$$\begin{cases} \tilde{\kappa}_i^k := (1 - z_i^k)^2 \kappa_i \\ Q_i^k := \kappa_i(u_i^k - u_{i-1}^k)^2 \end{cases} \quad (i = 1, \ldots, N,\ k = 0, 1, 2, \ldots).$$

When m_i and κ_i are defined as (7), since $\tilde{\kappa}_i < \kappa_i$, the Courant-Friedrichs-Lewy condition becomes

$$\tau \leq \sqrt{\rho} \min_{1 \leq i \leq N} (x_i - x_{i-1}).$$

In the third equation of (20), we define $A = [Q_i^k - \gamma_i]_+ \geq 0$. Then z_i^{k+1} is explicitly solved as

$$z_i^{k+1} = \frac{\tau A + \alpha z_i^k}{\tau A + \alpha} = z_i^k + \frac{\tau A(1 - z_i^k)}{\tau A + \alpha} \quad .$$

From this expression, we can compute the solution of (20) successively. We remark that $z_i^k \leq z_i^{k+1} \leq 1$ if $0 \leq z_i^k \leq 1$, and that $z_i^{k+1} = z_i^k$ if $Q_i^k \leq \gamma_i$.

In the following numerical examples, if $z_i^{k+1} \geq 0.99$ then z_i^{k+1} is replaced by 1 and we disconnect the wire at the

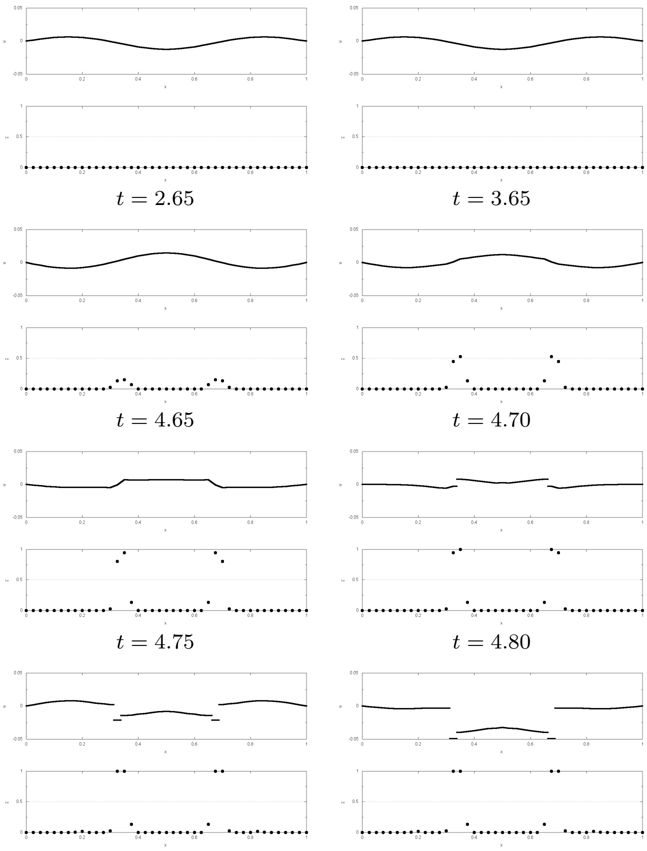

Figure 1: A fracture phenomena of a resonating wire simulated by (20) with $f(x,t) = \sin 3\pi t$ and $N = 40$.

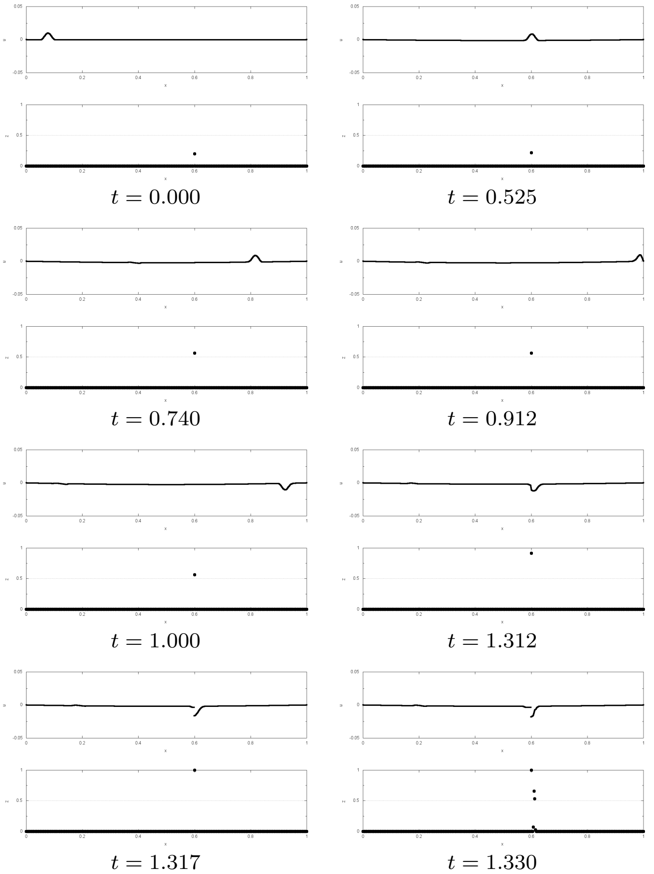

Figure 2: A fracture phenomena of a damaged wire with a solitary wave simulated by (20) with $f(x,t) = -0.01$ and $N = 400$.

spring S_i in the figures if $z_i^{k+1} = 1$. We also set $l = 1$, $\rho = 1$, $\alpha = 10^{-5}$ and $a(t) = b(t) \equiv 0$, and we choose $\tau = h = 1/N$ and $x_i = ih$ $(i = 0, \dots, N)$. The fracture toughness γ_i is defined as follows:

$$\gamma_i := \gamma\, \varphi\left(\frac{x_{i-1} + x_i}{2}\right) \qquad (i = 1, \dots, N), \tag{21}$$

$$\varphi(x) := \begin{cases} -180x + 10 & (0 \le x < 0.05) \\ 1 & (0.05 \le x \le 0.95) \\ 180x - 170 & (0.95 < x \le 1), \end{cases}$$

with $\gamma > 0$. Since we consider the Dirichlet boundary condition, the two end points $x = 0,\ 1$ have relatively large stress. Actually, if we choose uniform fracture toughness $\gamma_i = \gamma$ for all spring S_i, the springs S_1 and S_N at the end points are often first to be broken. In order to avoid such end point fracture, we reinforce the fracture toughness by (21).

A numerical example of the fracture of a resonating wire is shown in Fig. 1, where we set $f(t,x) = 0.1\sin(3\pi t)$, $\boldsymbol{u}^0 = \boldsymbol{u}^{-1} = \boldsymbol{0}$, $\boldsymbol{z}^0 = \boldsymbol{0}$, $N = 40$, $T = 6$, and $\gamma = 2.25 \times 10^{-4}$. In the first stage in the period $0 \le t \le 4.5$, under the periodic forcing term $\sin 3\pi t$, the wire resonates and a large vibration of the mode of $\sin 3\pi x$ is induced. The resonance causes large stress around $x = 0,\ 1/3,\ 2/3,\ 1$ and the stress

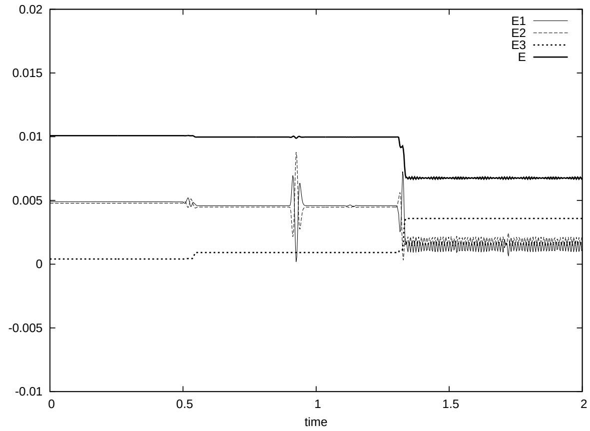

Figure 3: This graph shows the energy profiles in time of E_1, E_2, E_3 and $E := E_1 + E_2 + E_3$ for the simulation shown in Fig. 2. The horizontal axis is time t.

gives damage to some springs around $x = 1/3$ and $2/3$ in $4.5 \le t \le 4.75$. Around $t = 4.75 \sim 4.8$, two phase field variables reach to 1 and the wire breaks at those points. Four z_i variables reach to 1 in the end and the wire breaks into three long pieces and two short ones $(4.8 \le t \le 6)$.

Another example of the fracture caused by a traveling solitary wave is shown in Fig. 2, where we set $N = 400$, $T = 2$, $\gamma = 2.25 \times 10^{-3}$, $f(t,x) \equiv -0.01$, and $u_i^k = \bar{u}(x_i - k\tau - 0.075)$ $(i = 0, \dots, N,\ k = 0, -1)$ with

$$\bar{u}(x) := \begin{cases} 0 & (|x| \ge 0.025) \\ 0.01\cos^2(20\pi x) & (|x| < 0.025)\ . \end{cases}$$

We suppose that the initial spring S_{240} at $x = 0.6$ has a damage of twenty percent, i.e., the initial phase field variable for the relative damage is set as

$$z_i^0 := \begin{cases} 0.2 & (i = 240) \\ 0 & (\text{else})\ . \end{cases}$$

A solitary wave starts from $x = 0.075$ and moves to the right. It pass the damaged spring S_{240} at $t = 0.525$. At that time, S_{240} is more weakened by the wave and damaged more than fifty percent but not yet broken (figure of $t = 0.740$). After that, the solitary wave becomes upside down by the reflection at the right boundary and moves to the left (figure of $t = 1.000$). Around $t = 1.312 \sim 1.317$, it again arrives at S_{240}, which has been damaged more than fifty percent by the first arrival, and it finally breaks the wire.

The energy profiles for the simulation of Fig. 2 in time are shown in Fig. 3, where E_1, E_2, E_3 and $E := E_1 + E_2 + E_3$ are plotted with the horizontal axis t. When the solitary wave passes the damaged spring at $t = 0.525$, E_3 increases and the total energy E decreases. At the second passing $(t = 1.312 \sim 1.330)$, E_3 increases and E decreases again. Globally, the total energy E is monotonically decreasing as proved in Theorem 2 except for some tiny oscillations due to the time discretization.

6. Concluding remarks

Various aspects of a vibrating wire and its fracture were observed in terms of the numerical simulations of our phase field model. In addition to its ability to represent complex phenomena, our phase field model is very simple and was derived in a rigorous mathematical manner. We were able to prove the global existence of a unique solution in Section 3 and even some uniform energy estimates in Section 4.

Although we have confined our attention to the one dimensional wave equation model in this research, it is relatively starightforward to extend our model to the two dimensional wave equation includiong our mathematical estimates. It is more important and interesting to extend our approach to vibrating beam or shell and to two or three dimensional linear elasticity problem. We remark that a basic idea in this direction in static linear elasticity problem is described in [7].

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