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Remarks on positivity of α-determinants via SDP relaxation

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Abstract. After we survey our positivity problem for α-determinants of matrices, we reformulate it as polynomial optimization problems. By using SDP relaxation, we perform numerical computation for these optimization problems. We also give SOS representations for the α-determinants to obtain the optimal value when the matrix size $n = 3$, and give conjectures for $n = 4, 5$.

Keywords. α-determinant, determinant, permanent, point process, positivity, SDP relaxation, SOS representation

1. Introduction

We start by reviewing two 1-parameter interpolations between the determinant and the permanent of a matrix.

Definition 1.1. For a square matrix $A = (a_{ij})$ of size n,

$$\mathrm{det}_\alpha A = \sum_{\sigma\in\mathcal{S}_n} \alpha^{d(\sigma)} \prod_{i=1}^n a_{i\sigma(i)} \quad (\alpha\text{-determinant}),$$

$$q\text{-det}A = \sum_{\sigma\in\mathcal{S}_n} q^{\iota(\sigma)} \prod_{i=1}^n a_{i\sigma(i)} \quad (q\text{-determinant}),$$

where $\mathcal{S}_n$ is the symmetric group of order n, $d(\sigma)$ is the minimum number of transpositions whose product represents $\sigma \in \mathcal{S}_n$, i.e.,

$$d(\sigma) = n - \#\{\text{cycles of } \sigma\}$$

and $\iota(\sigma)$ is the inversion number defined by

$$\iota(\sigma) = \#\{1 \le i < j \le n \mid \sigma(i) > \sigma(j)\}.$$

The α-determinant is essentially the same as the α-permanent which was introduced by Vere-Jones [18], i.e., $\mathrm{per}_\alpha A = \alpha^n \mathrm{det}_{1/\alpha} A$. Here we prefer the α-determinants for later discussions.

The functions d and ι on $\mathcal{S}_n$ coincide when $n = 2$ and they do not when $n \ge 3$. The following table is for $n = 3$.

	e	(12)	(13)	(23)	(123)	(132)
$d(\sigma)$	0	1	1	1	2	2
$\iota(\sigma)$	0	1	3	1	2	2

Therefore, for a 3×3 matrix $A = (a_{ij})_{i,j=1}^3$, we have

$$\begin{aligned}\mathrm{det}_\alpha A &= a_{11}a_{22}a_{33} + (a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33})\alpha \\ &\quad + (a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32})\alpha^2, \\ q\text{-det}A &= a_{11}a_{22}a_{33} + (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})q \\ &\quad + (a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32})q^2 + a_{13}a_{22}a_{31}q^3.\end{aligned}$$

Their difference in this case is only for the coefficient of the transposition (13). For special values 1, 0, −1, it is clear that the q-determinant and the α-determinant coincide, and they are well-known quantities,

$$\mathrm{det}_{-1} A = \det A = \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \quad (\text{determinant}),$$

$$\mathrm{det}_0 A = \prod_{i=1}^n a_{ii} \quad (\text{product of diagonals}),$$

$$\mathrm{det}_1 A = \mathrm{per}A = \sum_{\sigma\in\mathcal{S}_n} \prod_{i=1}^n a_{i\sigma(i)} \quad (\text{permanent}).$$

It should be mentioned that when $A \ge O$, i.e., A is a positive semidefinite matrix, it holds that

$$\mathrm{per}A \ge \prod_{i=1}^n a_{ii} \ge \det A \ge 0, \tag{1.1}$$

or equivalently $\mathrm{det}_1 A \ge \mathrm{det}_0 A \ge \mathrm{det}_{-1} A \ge 0$. The first inequality is obtained by E. Lieb [4] and the second one is sometimes called Fischer's inequality. It is natural to ask whether similar inequalities hold for q-determinants and α-determinants. Although monotonicity in the parameter q or α cannot be expected in general, the following positivity result has been shown for q-determinants:

Theorem 1.2 (Bożejko and Speicher (1991))**.** For $-1 \le q \le 1$, it holds that $q\text{-det}A \ge 0$ for any $A \ge O$.

In [15], we raised the following positivity problem for α-determinants.

Problem 1. Find the range of $\alpha \in \mathbb{R}$ such that $\mathrm{det}_\alpha A \ge 0$ for any $A \ge O$.

For this problem, we obtained affirmative results for

$$\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup \{0\} \cup \{2/m \mid m \in \mathbb{N}\} \tag{1.2}$$

in the case of real-symmetric matrices and

$$\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup \{0\} \cup \{1/m \mid m \in \mathbb{N}\} \qquad (1.3)$$

in the case of Hermitian matrices [15, 16]. Also we can find real-symmetric matrices for which $\det_\alpha A$ is negative unless $\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup [0, 2]$ (resp. $\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup [0, 1]$) for real-symmetric (resp. Hermitian) matrices [11, 16]. From these results, we conjectured the following:

Conjecture 1.3 ([15, 16])**.** (1) When $\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup [0, 2]$,

$$\det_\alpha A \geq 0$$

for any *real-symmetric* positive semidefinite matrix A.
(2) When $\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup [0, 1]$,

$$\det_\alpha A \geq 0$$

for any *Hermitian* positive semidefinite matrix A.

In the present paper, concerning with this positivity conjecture, we discuss the infimum of α-determinants over a convex subset of the cone of positive semidefinite matrices. In Section 2 we explain why we need positivity for α-determinants by showing the probabilistic background of the problem. In Section 3 we review some known results towards Conjecture 1.3 above. In Section 4, we reformulate the positivity problem as polynomial optimization problems (POP) which depend on the size of matrices. In Section 5, we review semidefinite programming (SDP) relaxation problems associated with POP. In Section 6, we give the answer of the reformulated problem for positive semidefinite matrices of size 3 (Theorem 6.1), and give upper bounds (Propositions 6.7 and 6.9) and numerical lower bounds (Figure 3 to 5) for those of size 4 and 5, which lead us to Conjectures 6.8 and 6.10. In Section 7, we give a remark and some open questions.

2. α-DETERMINANTAL POINT PROCESSES

This positivity problem arises from the study of the existence of certain point processes. Here we briefly survey this point. Details can be found in [15] and [16].

Let R be a "nice" space such as $\mathbb{R}^d$ and λ a fixed reference measure on it. A (simple) point process on R is a random point configuration, i.e., a probability measure μ on the set of all locally finite subsets of R. It is uniquely determined by correlation functions (or joint intensities) $\rho_n \colon R^n \to [0, \infty)$ by the formula

$$E_\mu\left[\prod_{i=1}^{n} N(A_i)\right] = \int_{A_1 \times \cdots \times A_n} \rho_n(\mathbf{x}) \lambda^{\otimes n}(dx_1 \cdots dx_n)$$

for every $n \geq 1$ and disjoint Borel sets $A_1, \dots, A_n \subset R$, where $N(A)$ is the number of points inside a Borel set A. For example, the correlation functions of a stationary Poisson point process on $\mathbb{R}^d$ are given by $\rho_n(\mathbf{x}) = c^n$ ($\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$) for some $c > 0$.

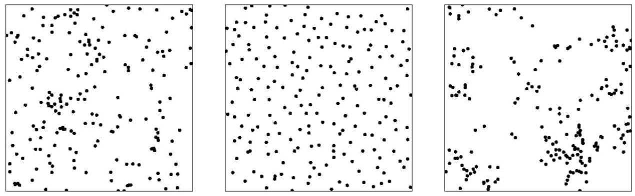

Figure 1: From the left, Poisson, determinantal and permanental point processes on $\mathbb{R}^2$. The figure is borrowed from [3].

A point process is called *determinantal* or *fermionic* if its correlation functions are given by determinants, i.e.,

$$\rho_n(\mathbf{x}) = \det(K(x_i, x_j))_{i,j=1}^n$$

with a positive semidefinite kernel $K \colon R \times R \to \mathbb{C}$ satisfying certain conditions. This is originally considered as fermionic particles in quantum physics; however, there found many interesting examples of determinantal point processes, such as eigenvalues of some random matrices, e.g., Gaussian Unitary Ensemble, the Ginibre point process, zeros of a certain Gaussian analytic function, non-intersecting Brownian motions in one-dimension, uniform spanning trees on a graph, random domino tiling (dimer model).

From the physical point of view, boson is also important as well as fermion. We can also define point processes corresponding to boson. A point process is called *permanental* or *bosonic* if its correlation functions are given by permanents, i.e.,

$$\rho_n(\mathbf{x}) = \mathrm{per}(K(x_i, x_j))_{i,j=1}^n.$$

For both determinantal and permanental point processes, it follows from the inequality (1.1) that correlation functions are non-negative if the kernel K is positive semidefinite. See Figure 1 for typical samples of Poisson, determinantal and permanental point processes.

A point process μ is uniquely determined by its Laplace transform

$$L_\mu(f) = E_\mu[\exp(-\langle \xi, f \rangle)],$$

where $\langle \xi, f \rangle = \sum_i f(x_i)$ for a point configuration $\xi = \{x_i\}_i \subset R$ and a nonnegative function f of compact support on R. The Laplace transform of determinantal point process $\mu_{K,-1}$ and permanental point process $\mu_{K,+1}$ are given by

$$L_{\mu_{K,\pm 1}}(f) = \det(I \mp K_\phi)^{\pm 1}, \qquad (2.1)$$

where K_ϕ is an integral operator with kernel $K_\phi(x, y) = \phi^{1/2}(x) K(x, y) \phi^{1/2}(y)$ with $\phi = 1 - e^{-f}$. We can naturally interpolate it by introducing 1-parameter α to the above Laplace transforms as

$$L_{\mu_{K,\alpha}}(f) = \det(I + \alpha K_\phi)^{-1/\alpha}. \qquad (2.2)$$

The equality (2.2) defines a signed measure $\mu_{K,\alpha}$ in general and by expanding the right-hand side we obtain

$\det_\alpha(K(x_i, x_j))_{i,j=1}^n$ as "signed" correlation functions for $\mu_{K,\alpha}$. However, if $\mu_{K,\alpha}$ is indeed a probability measure, or equivalently there exists a point process, which we call α-deteminatal point process, the correlation functions $\det_\alpha(K(x_i, x_j))_{i,j=1}^n$ are nonnegative for any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in R$. Hence, the positivity problem of α-determinants is equivalent to the existence problem of point processes. We remark that the limiting case $\alpha = 0$ corresponds to Poisson point processes.

3. Known results towards positivity conjecture

Here we list some known results for the positivity. More details can be found in [11, 15, 16]

1. For the all 1 matrix $\mathbf{1}$ of size n, we have

$$\begin{aligned}\det_\alpha \mathbf{1} &= (1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)\\ &=: c_n(\alpha).\end{aligned}$$

   More generally, if A is an n by n matrix of rank 1, or equivalently $A = \mathbf{u}^*\mathbf{u}$ for a row vector $\mathbf{u} \in \mathbb{C}^n$,

$$\det_\alpha A = c_n(\alpha)\prod_{i=1}^n |u_i|^2.$$

   Unless $\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup [0, \infty)$, the above can be negative if n is big enough.

2. When A is of rank p (≥ 2), it holds that $\det_\alpha A \geq 0$ whenever $\alpha \in [0, \frac{1}{p-1}]$. In particular, when A is of rank 2, $\det_\alpha A \geq 0$ whenever $\alpha \in [0, 1]$ (cf. (4.12), [16]).

3. Real-symmetric case [16]: define a $(2N+2)\times(2N+2)$ matrix of rank two by $A_N = \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v}$ with

$$\begin{aligned}\mathbf{u} &= (1, 1, \underbrace{1, \dots, 1}_N, \underbrace{0, \dots, 0}_N),\\ \mathbf{v} &= (1, -1, \underbrace{0, \dots, 0}_N, \underbrace{1, \dots, 1}_N).\end{aligned}$$

   When $\alpha > 2$, $\det_\alpha A_N < 0$ for sufficiently large N.

4. Hermitian case [11]: define a $(2N+K)\times(2N+K)$ matrix of rank two by $A_{K,N} = \mathbf{u}^*\mathbf{u} + \mathbf{v}^*\mathbf{v}$ with

$$\begin{aligned}\mathbf{u} &= (\underbrace{1, \dots, 1}_K, \underbrace{1, \dots, 1}_N, \underbrace{0, \dots, 0}_N),\\ \mathbf{v} &= (\underbrace{1, e^{2\pi i/K}, \dots, e^{2\pi i(K-1)/K}}_K, \underbrace{0, \dots, 0}_N, \underbrace{1, \dots, 1}_N).\end{aligned}$$

   When $\alpha > 1$, one can show that $\det_\alpha A_{K,N} < 0$ for sufficiently large K and N.

5. For real-symmetric case, $\det_2 A$ is nonnegative whenever $A \geq O$. This follows from the fact that $\det_2 A$ is equal to the correlation function of a Cox process, which is a Poisson point process with random intensity being *squared real Gaussian field* with covariance A.

6. The correlation function of the superposition of m-independent copies of Cox process the above is equal to $\det_{2/m} A$. Hence, $\det_{2/m} A$ is nonnegative for any real-symmetric positive semidefinite matrices. Similarly, the correlation function of the superposition of m-independent copies of determinantal point process the above is equal to $\det_{-1/m} A$, which implies that $\det_{-1/m} A$ is nonnegative for any real-symmetric positive semidefinite matrices. Therefore, when

$$\alpha \in \{-1/m \mid m \in \mathbb{N}\} \cup \{0\} \cup \{2/m \mid m \in \mathbb{N}\}, \quad (3.1)$$

   the α-determinant is nonnegative for any real-symmetric matrices.

7. Roughly speaking, $\det_\alpha A$ can be written in terms of a Poisson point process with intensity being diagonals of Wishart random matrix (cf. [16]). From this fact, for $n = 2, 3, \dots$, $\det_\alpha A \geq 0$ for real-symmetric, n by n positive semidefinite matrices when

$$\alpha \in \left[0, \frac{2}{n-1}\right] \cup \left\{\frac{2}{n-1}, \frac{2}{n-2}, \dots, \frac{2}{2} = 1, 2\right\}.$$

   Remark that the numerator 2 is replaced by 1 for the Hermitian case.

4. Reformulation of the positivity problem

In what follows, we only consider the real-symmetric matrices. Hermitian cases can also be considered.

Since the conjugation by a regular diagonal matrix does not change the sign of α-determinant, for our positivity problem, without loss of generality, we may assume that the diagonals of matrices are all 1, e.g., when $n = 3$

$$A = A_3[\mathbf{x}] = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2, x_3)$. Let $\mathcal{P}_n$ be the totality of such *positive semidefinite* real-symmetric matrices of size n. The set $\mathcal{P}_n$ is a compact convex subset of $[-1, 1]^{d_n}$ with $d_n = n(n-1)/2$, the number of variables. We consider a minimization problem for fixed matrix size, and for each n we define

$$\lambda_n(\alpha) := \min_A \det_\alpha A \quad \text{s.t. } A \in \mathcal{P}_n.$$

It is easy to see that (i) for any α and n, $\lambda_n(\alpha) \leq 1$ since $\det_\alpha I_n = 1$ for the identity matrix $I_n \in \mathcal{P}_n$, and (ii) for fixed α, the quantity $\lambda_n(\alpha)$ is non-increasing in n since $\mathcal{P}_n$ can be naturally embedded in $\mathcal{P}_{n+1}$. We now define

$$\lambda_\infty(\alpha) := \lim_{n\to\infty} \lambda_n(\alpha) \in [-\infty, 1].$$

We note that $\lambda_\infty(\alpha)$ is upper semi-continuous since $\det_\alpha A$ is a continuous function of α for each n and A.

Now Problem 1 in the introduction is rewritten as follows:

Problem 2. Find the range of $\alpha \in \mathbb{R}$ such that $\lambda_\infty(\alpha) \geq 0$.

From 1. in Section 3, we can see that $\lambda_\infty(\alpha) = -\infty$ when $\alpha \in (-\infty, 0) \setminus \{-1/n, n \in \mathbb{N}\}$. In this setting, when $\alpha \geq 0$, Conjecture 1.3 is equivalent to

$$\lambda_\infty(\alpha) \geq 0 \text{ if and only if } \alpha \in [0,2] \text{ (resp. } \alpha \in [0,1]) \quad (4.1)$$

for real-symmetric case (resp. Hermitian case). We restate known sufficient conditions for positivity mentioned at 6. in Section 3. For $n = 2, 3, \ldots,$ it holds that $\lambda_n(\alpha) \geq 0$ when

$$\alpha \in \left[0, \frac{2}{n-1}\right] \cup \left\{\frac{2}{n-1}, \frac{2}{n-2}, \ldots, 1, 2\right\}$$

and thus $\lambda_\infty(\alpha) \geq 0$, if $\alpha \in \left\{\frac{2}{k} \;\middle|\; k = 1, 2, \ldots\right\}$.

5. POP and SDP relaxation

Semidefinite programming (SDP) is a convex optimization over the cone of positive semidefinite matrices. It has been developed for last decade from the theoretical and practical points of view, and the development is still ongoing. Here we formulate our problem as a polynomial optimization problem (POP) and estimate it from below by SDP relaxation.

Although the relaxation procedure below is rather standard, here we explain, for simplicity, when $n = 3$. Our objective function is written with the variable $\mathbf{x} = (x_1, x_2, x_3)$ as

$$f_\alpha(\mathbf{x}) = \det_\alpha \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} = 1 + \alpha(x_1^2 + x_2^2 + x_3^2) + 2\alpha^2 x_1 x_2 x_3.$$

Then, our problem is the following:
POP$_3$: Find

$$\lambda_3(\alpha) := \min f_\alpha(\mathbf{x}) \text{ s.t. } A = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \geq O,$$

or equivalently

$$\min f_\alpha(\mathbf{x}) \text{ s.t. } \begin{cases} 1 - x_1^2 \geq 0 \\ 1 - x_2^2 \geq 0 \\ 1 - x_3^2 \geq 0 \\ 1 - (x_1^2 + x_2^2 + x_3^2) + 2x_1x_2x_3 \geq 0. \end{cases}$$

By introducing the vector consisting of all monomials of degree $\leq d$

$$u_d[\mathbf{x}] = (1, x_1, x_2, x_3, x_1^2, x_1x_2, \ldots, x_3^d)$$

and a positive semidefinite matrix for which the degree of each element $\leq 2d$

$$M_d[\mathbf{x}] = u_d[\mathbf{x}]^T u_d[\mathbf{x}],$$

we rewrite the constraint in another equivalent form as

$$\begin{cases} (1 - x_1^2) M_{r-1}(\mathbf{x}) \geq O \\ (1 - x_2^2) M_{r-1}(\mathbf{x}) \geq O \\ (1 - x_3^2) M_{r-1}(\mathbf{x}) \geq O \\ (1 - (x_1^2 + x_2^2 + x_3^2) + 2x_1x_2x_3) M_{r-2}(\mathbf{x}) \geq O \end{cases}$$

for $r \geq 2$. Here r is called the relaxation order. By linearization of variables as $x_1^a x_2^b x_3^c \Rightarrow m_{abc}$

$$\text{e.g. } 1 \Rightarrow m_{000}, \; x_1 \Rightarrow m_{100}, \; x_1^2 x_3 \Rightarrow m_{201} \text{ etc.}$$

and omitting the constraints $x_1^a x_2^b x_3^c = m_{abc}$ for each a, b, c, we have the SDP relaxation problems SDP$_{3,r}$ ($r \geq 2$) as follows:
SDP$_{3,r}$. Find $\mu_3^{(r)}(\alpha) := \min\{1 + \alpha(m_{200} + m_{020} + m_{002}) + 2\alpha^2 m_{111}\}$

$$\text{s.t. } \begin{cases} M_r(\mathbf{m}) \geq O \\ (1 - \sigma_{200}) M_{r-1}(\mathbf{m}) \geq O \\ (1 - \sigma_{020}) M_{r-1}(\mathbf{m}) \geq O \\ (1 - \sigma_{002}) M_{r-1}(\mathbf{m}) \geq O \\ \{1 - (\sigma_{200} + \sigma_{020} + \sigma_{002}) + 2\sigma_{111}\} M_{r-2}(\mathbf{m}) \geq 0, \end{cases}$$

where $\mathbf{m} = (m_{000}, m_{100}, m_{010}, \ldots, m_{2r,2r,2r})$ and σ_{ijk} is acting on each variable m_{abc} as shift of indices, i.e., $\sigma_{ijk} m_{abc} = m_{a+i,b+j,c+k}$. For example,

$$\sigma_{201} \begin{pmatrix} m_{000} & m_{100} & m_{010} & m_{001} \\ m_{100} & m_{200} & m_{110} & m_{101} \\ m_{010} & m_{110} & m_{020} & m_{011} \\ m_{001} & m_{101} & m_{011} & m_{002} \end{pmatrix} = \begin{pmatrix} m_{201} & m_{301} & m_{211} & m_{202} \\ m_{301} & m_{401} & m_{311} & m_{302} \\ m_{211} & m_{311} & m_{221} & m_{212} \\ m_{202} & m_{302} & m_{212} & m_{203} \end{pmatrix}.$$

By SDP relaxation, we have obvious inequalities

$$\mu_3^{(r)}(\alpha) \leq \mu_3^{(r+1)}(\alpha) \leq \lambda_3(\alpha).$$

This means that $\mu_3^{(r)}(\alpha)$ gives a lower bound for $\lambda_3(\alpha)$, and hence positivity of $\lambda_3(\alpha)$ follows from that of $\mu_3^{(r)}(\alpha)$ for some r.

In the same manner as above, we can also define SDP relaxation problems for any $n \in \mathbb{N}$ and $\mu_n^{(r)}(\alpha)$ which gives a lower bound of $\lambda_n(\alpha)$.

6. The case $n = 3, 4, 5$

6.1. POP and SOS relaxations

Let $\mathbb{R}[\mathbf{x}]$ be the ring of real polynomials of the variables $\mathbf{x} = (x_1, \ldots, x_n)$. We say that $f \in \mathbb{R}[\mathbf{x}]$ is a positive polynomial on $\mathbb{R}^n$ if $f(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \mathbb{R}^n$. We denote the set of all positive polynomials by Π. A positive polynomial f is said to be a sum of squares polynomial (for short, SOS polynomial) if there exist some polynomials $p_1, \ldots, p_L \in$

$\mathbb{R}[\mathbf{x}]$ so that $f = \sum_{i=1}^{L} p_i^2$. We denote the set of all SOS polynomials by Σ. Obviously, $\Sigma \subset \Pi$. Conversely, every positive polynomial of one variable is an SOS polynomial, however it is not always the case for positive polynomials of several variables. For example, the Motzkin polynomial $f(x, y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2$ is a positive polynomial on $\mathbb{R}^2$ but not an SOS polynomial.

It is easy to see that $g \in \mathbb{R}[\mathbf{x}]$ is an SOS polynomial of degree $2d$ if and only if there exists a real-symmetric, positive semidefinite matrix Q of size $\binom{n+d}{d}$ such that

$$g(\mathbf{x}) = u_d[\mathbf{x}]Qu_d[\mathbf{x}]^T.$$

The problem of finding an SOS representation for a given positive polynomial is often reduced to the semidefinite optimization problem, i.e., the problem of finding such a real-symmetric positive semidefinite matrix Q.

In Section 5, we discussed the following constrained POP for $\mathbf{x} \in \mathbb{R}^n$: for polynomials $f, g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$,

$$\min f(\mathbf{x}) \text{ s.t. } g_j(\mathbf{x}) \geq 0 \ (\forall j = 1, 2, \ldots, m). \tag{6.1}$$

Here we consider its dual problem:

$$\rho^* := \max \rho \ \text{ s.t. } f(\mathbf{x}) - \rho \geq 0 \ (\forall \mathbf{x} \in \mathbb{K}), \tag{6.2}$$

where $\mathbb{K}$ is a basic semi-algebraic set defined by

$$\mathbb{K} = \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \geq 0 \ (\forall j = 1, 2, \ldots, m)\}.$$

We say that a positive polynomial p on $\mathbb{K}$ admits an SOS representation if there exist SOS polynomials $\sigma_0, \ldots, \sigma_m$ such that

$$p(\mathbf{x}) = \sigma_0(\mathbf{x}) + \sum_{j=1}^{m} g_j(\mathbf{x})\sigma_j(\mathbf{x}). \tag{6.3}$$

The set of all such $p(\mathbf{x})$ is denoted by $Q(g_1, \ldots, g_m)$, which is called the quadratic module generated by $g_1, \ldots, g_m$. If the level set $\{\mathbf{x} \in \mathbb{R}^n \mid u(\mathbf{x}) \geq 0\}$ is compact for some $u \in Q(g_1, \ldots, g_m)$, by Putinar's Positivstellensatz (cf. Theorem 2.14 in [6]), one can find such $\sigma_j \in \Sigma$, $j = 1, 2, \ldots, m$ when p is strictly positive on $\mathbb{K}$.

If we replace the condition "$f(\mathbf{x}) - \rho \geq 0$ on $\mathbb{K}$" by "$f(\mathbf{x}) - \rho$ on $\mathbb{K}$ admits an SOS representation" in (6.2), we have an SOS relaxation problem as

$$\rho_{\text{SOS}} := \max \rho \ \text{ s.t. } f - \rho \text{ admits an SOS representation.}$$

It is obvious that $\rho^* \geq \rho_{\text{SOS}}$. SDP solvers can find positive semidefinite matrices $Q_i, i = 0, 1, \ldots, m$ such that

$$f(\mathbf{x}) - \rho_{\text{SOS}}^{(r)} = \sum_{j=0}^{m} g_j(\mathbf{x})u_{r-p_j}(\mathbf{x})Q_j u_{r-p_j}(\mathbf{x}) \geq 0 \quad (\forall \mathbf{x} \in \mathbb{K}),$$

where we put $g_0(\mathbf{x}) \equiv 1$ and $p_j = \lceil \deg g_j/2 \rceil$. Here $r \geq \max(\lceil \deg f/2 \rceil, p_j, j = 0, 1, \ldots, m)$ corresponds to the relaxation order. Therefore, we obtain $f(\mathbf{x}) \geq \rho_{\text{SOS}}^{(r)}$ for all $\mathbf{x} \in \mathbb{K}$.

Note that $\mathbb{K}$ has a non-empty interior in our problem. Hence by the strong duality theorem of SDP (cf. [6]), we have $\mu_n^{(r)}(\alpha) = \rho_{\text{SOS}}^{(r)}$, where $\mu_n^{(r)}(\alpha)$ is the minimum of the SDP relaxation problem for (6.1) with relaxation order r as in Section 5.

6.2. For $n = 3$ (i)

For $n = 3$, we can compute $\lambda_3(\alpha)$ explicitly by direct computation as follows. See Figure 2.

Theorem 6.1. For $\alpha \geq 0$,

$$\lambda_3(\alpha) = \mu_3^{(r)}(\alpha) = \min\left\{\frac{1}{4}(1+\alpha)(4-\alpha), 1\right\}$$

for any $r \geq 3$.

We omit a direct proof as it is elementary but tedious. Instead we prove this formula by giving SOS representations for $\det_\alpha A$.

From SDP relaxation problem $\text{SDP}_{3,r}$, we can easily compute $\mu_3^{(r=2)}(\alpha)$ and $\mu_3^{(r=3)}(\alpha)$ numerically for given α (see Figure 2) by using SDPA 7.3.5 [21] on a Mac. From this solution, as mentioned in the last subsection, we also find an SOS representation of $\det_\alpha A$ as in (6.3). We can then construct an expression for general α, using the procedure in [12], as follows.

Proposition 6.2. Let $V_\alpha = \begin{pmatrix} \alpha/2 & \alpha^2/3 \\ \alpha^2/3 & \alpha/2 \end{pmatrix}$. Then, the α-determinant for $n = 3$ admits an SOS representation

$$f_\alpha(\mathbf{x}) = 1 + \sum_{ijk:\text{cyclic}} \left\{\mathbf{v}_{ijk}V_\alpha\mathbf{v}_{ijk}^T + \frac{\alpha}{4}(1 - x_i^2)(x_j^2 + x_k^2)\right\},$$

where $\mathbf{v}_{ijk} = (x_i, x_jx_k)$ and

$$\sum_{ijk:\text{cyclic}} p_{ijk} = p_{123} + p_{231} + p_{312}.$$

Since V_α is positive semidefinite when $\alpha \in [0, 3/2]$, we see that $f_\alpha(\mathbf{x}) \geq 1$ for any $A_3[\mathbf{x}] \in \mathcal{P}_3$. Hence, we can conclude that $\lambda_3(\alpha) = 1$ at least for $\alpha \in [0, 3/2]$. This type of representation is not unique. In the next subsection, we will find another representation for $\lambda_3(\alpha)$.

6.3. For $n = 3$ (ii)

Here we discuss the problem in the extended framework of SDP and SOS relaxations for POP developed in [7, 9].

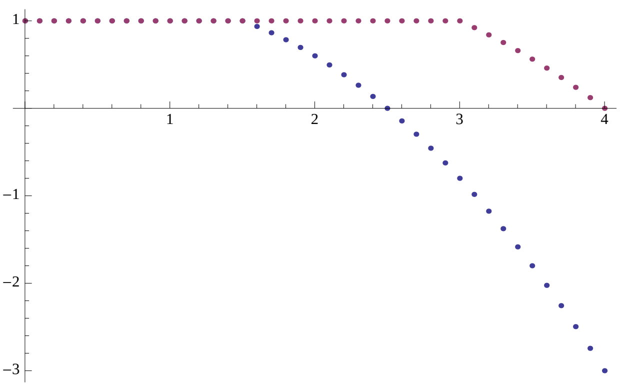

Figure 2: $\mu_3^{(r=2)}(\alpha)$ (blue) and $\mu_3^{(r=3)}(\alpha)$ (purple) for $0 \leq \alpha \leq 4$.

An $m \times m$ symmetric matrix $F(\mathbf{x})$ is said to be *a polynomial matrix* if there exists a finite number of $m \times m$ symmetric constant matrices H_b such that

$$F(\mathbf{x}) = \sum_b H_b \mathbf{x}^b,$$

where $\mathbf{x}^b = x_1^{b_1} x_2^{b_2} \cdots x_d^{b_d}$ for given $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and the multi-index $b \in \mathbb{N}^d$. We call the inequality $F(\mathbf{x}) \in \mathbb{S}_+^m$ *polynomial matrix inequality (PMI)*, where $\mathbb{S}_+^m$ is the closed cone of the positive semidefinite matrices of size m. Note that $F(\mathbf{x})$ is just a polynomial if the size of H_b is 1×1. An $m \times m$ polynomial matrix $F(\mathbf{x})$ is said to be *an $m \times m$ SOS polynomial matrices* if there exist $m \times m$ polynomial matrices $G_1(\mathbf{x}), \dots, G_q(\mathbf{x})$ such that $F(\mathbf{x}) = \sum_{i=1}^q G_i(\mathbf{x})^T G_i(\mathbf{x})$. We denote the set of $m \times m$ SOS polynomial matrices by $\Sigma^{m\times m}$ and that with degrees at most $2r$ by $\Sigma_r^{m\times m}$. The following lemma is presented in [7]. This plays an essential role in SDP relaxation for polynomial optimization problems that contain PMIs.

Lemma 6.3. Let $W(\mathbf{x})$ be an $m \times m$ polynomial matrix with degree $2r$. Then $W(\mathbf{x}) \in \Sigma_r^{m\times m}$ if and only if there exists a positive semidefinite matrix $V \in \mathbb{S}_+^{mn_0}$ such that $W(\mathbf{x}) = \sum_{a\in\mathbb{N}_r^n} \sum_{b\in\mathbb{N}_r^n} V_{ab}\mathbf{x}^{a+b}$, where $n_0 := \binom{n+r}{r}$, $\mathbb{N}_r^n = \{a \in \mathbb{N}^n \mid \sum_{i=1}^n a_i \le r\}$, and $V_{ab} \in \mathbb{S}^m$ is the (a, b)-th block of $V \in \mathbb{S}_+^{mn_0}$.

We recall our positivity problem:

$$\lambda_n(\alpha) = \min\{\det_\alpha A_n[\mathbf{x}] \mid A_n[\mathbf{x}] \in \mathbb{S}_+^n\}. \qquad (6.4)$$

This can also be written as

$$\lambda_n(\alpha) = \min\{\det_\alpha A_n[\mathbf{x}] \mid M_{r-1}[\mathbf{x}] \otimes A_n[\mathbf{x}] \in \mathbb{S}_+^{n_1}\}$$

since $A_n[\mathbf{x}] \in \mathbb{S}_+^n$ is equivalent to $M_{r-1}[\mathbf{x}] \otimes A_n[\mathbf{x}] \in \mathbb{S}_+^{n_1}$ where $n_1 = \binom{d_n+r-1}{r-1} n$ with $d_n = n(n-1)/2$. We can rewrite it as

$$\lambda_n(\alpha) = \min\left\{ \sum_b f_b(\alpha)\mathbf{x}^b \;\middle|\; \sum_b E_{r,b}\mathbf{x}^b \in \mathbb{S}_+^{n_1} \right\}$$

for some $f_b(\alpha) \in \mathbb{R}$ and $E_{r,b} \in \mathbb{S}_+^{n_1}$. By replacing all monomials $\mathbf{x}^b$ with new variables y_b (linearlization) as in Section 5, we obtain the following SDP relaxation problem

$$\nu_n^{(r)}(\alpha) = \min\left\{ \sum_b f_b(\alpha)y_b \;\middle|\; \sum_b E_{r,b}y_b \in \mathbb{S}_+^{n_1} \right\}$$

and $\lambda_n(\alpha) \ge \nu_n^{(r)}(\alpha)$. Its dual problem is obtained from the following SOS problem:

$$\rho_{\mathrm{SOS}}^{(r)}(\alpha) = \sup\left\{ \rho \;\middle|\; \begin{array}{l} \sum_b f_b(\alpha)\mathbf{x}^b - \rho = W(\mathbf{x}) \cdot A_n[\mathbf{x}] \\ \hfill (\mathbf{x} \in \mathbb{R}^{d_n}), \\ \rho \in \mathbb{R},\ W(\mathbf{x}) \in \Sigma_{r-1}^{n\times n} \end{array} \right\},$$

where $A \cdot B = \mathrm{Tr}(AB)$. From Lemma 6.3, there exists $V \in \mathbb{S}_+^{n_1}$ such that $W(\mathbf{x}) \cdot A_n[\mathbf{x}] = V \cdot (M_{r-1}[\mathbf{x}] \otimes A_n[\mathbf{x}])$. Then, we have an equivalent form:

$$\rho_{\mathrm{SOS}}^{(r)}(\alpha) = \sup\left\{ \rho \;\middle|\; \begin{array}{l} \sum_b f_b(\alpha)\mathbf{x}^b - \rho = V \cdot (M_{r-1}[\mathbf{x}] \otimes A_n[\mathbf{x}]) \\ \hfill (\mathbf{x} \in \mathbb{R}^{d_n}) \\ \rho \in \mathbb{R},\ V \in \mathbb{S}_+^{n_1}. \end{array} \right\}.$$

Since the feasible region of (6.4) has a non-empty interior and the SOS relaxation problem is feasible, by the strong duality for SDP (cf. [6]), we have $\lambda_n(\alpha) \ge \nu_n^{(r)}(\alpha) = \rho_{\mathrm{SOS}}^{(r)}(\alpha)$. SDP solvers can find $V \in \mathbb{S}_+^{n_1}$, from which we found the following SOS representation.

Proposition 6.4. For $\alpha \ge 0$,

$$\begin{aligned} &\det_\alpha A_3[\mathbf{x}] - \min\left\{\frac{1}{4}(1+\alpha)(4-\alpha),\ 1\right\} \\ &= p_1(\alpha) \begin{pmatrix} x_3^2 & x_1x_2x_3 & x_1x_2x_3 \\ x_1x_2x_3 & x_2^2 & x_1x_2x_3 \\ x_1x_2x_3 & x_1x_2x_3 & x_1^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &+ p_2(\alpha) \sum_{ijk:\text{cyclic}} \begin{pmatrix} 1 & x_i^2 & x_j^2 \\ x_i^2 & x_i^2 & x_ix_jx_k \\ x_j^2 & x_ix_jx_k & x_j^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}, \end{aligned}$$

where

$$p_1(\alpha) = \begin{cases} \frac{1}{3}\alpha(2\alpha-3) & 3/2 \le \alpha \le 3, \\ \alpha & \alpha \ge 3, \end{cases}$$

and

$$p_2(\alpha) = \begin{cases} \frac{1}{3}\alpha(3-\alpha) & 3/2 \le \alpha \le 3, \\ \frac{1}{12}\alpha(\alpha-3) & \alpha \ge 3. \end{cases}$$

In particular, when $\alpha \ge 3/2$, we have the inequality

$$\det_\alpha A_3[\mathbf{x}] \ge \min\left\{\frac{1}{4}(1+\alpha)(4-\alpha),\ 1\right\}$$

whenever $A_3[\mathbf{x}] \in \mathbb{S}_+^3$.

Proof. We can directly verify the above equality. Since any principal minor of the tensor product of two positive semidefinite matrices is also positive semidefinite, we see that

$$\begin{aligned} &(\check{M}_1[\mathbf{x}] \otimes A_3[\mathbf{x}])_{(3,5,7),(3,5,7)} \\ &= \left(\begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \right)_{(3,5,7),(3,5,7)} \\ &= \begin{pmatrix} x_3^2 & x_1x_2x_3 & x_1x_2x_3 \\ x_1x_2x_3 & x_2^2 & x_1x_2x_3 \\ x_1x_2x_3 & x_1x_2x_3 & x_1^2 \end{pmatrix} \ge O, \end{aligned}$$

where $\check{M}_1[\mathbf{x}] = (x_1, x_2, x_3)^T(x_1, x_2, x_3) \ge O$. □

Remark that for $0 \le \alpha \le 3$, we also have the following more neat SOS representation.

Proposition 6.5. For $0 \le \alpha \le 3$,

$$\det_\alpha A_3[\mathbf{x}] - 1 = \begin{pmatrix} x_3^2 & x_1x_2x_3 & x_1x_2x_3 \\ x_1x_2x_3 & x_2^2 & x_1x_2x_3 \\ x_1x_2x_3 & x_1x_2x_3 & x_1^2 \end{pmatrix} \cdot \begin{pmatrix} \alpha & \alpha^2/3 & \alpha^2/3 \\ \alpha^2/3 & \alpha & \alpha^2/3 \\ \alpha^2/3 & \alpha^2/3 & \alpha \end{pmatrix} \ge 0.$$

Proof of Theorem 6.1. Combining Propositions 6.4 and 6.5, we obtain the lower bound of $\lambda_3(\alpha)$. Also when $\mathbf{x} = (1/2, 1/2, -1/2)$, the optimal values are attained for every $\alpha \ge 0$. We obtain the assertion. □

6.4. For $n = 4$

For $\mathbf{x} = (x_1, x_2, \dots, x_6) \in \mathbb{R}^6$ and

$$A_4[\mathbf{x}] = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & 1 & x_4 & x_5 \\ x_2 & x_4 & 1 & x_6 \\ x_3 & x_5 & x_6 & 1 \end{pmatrix}$$

we see that the α-determinant is given by

$$\begin{aligned} g_\alpha(\mathbf{x}) &:= \det_\alpha(A_4[\mathbf{x}]) \\ &= 1 + \alpha \sum_{i=1}^{6} x_i^2 \\ &+ 2\alpha^2(x_1x_2x_4 + x_1x_3x_5 + x_2x_3x_6 + x_4x_5x_6) \\ &+ \alpha^2(x_1^2x_6^2 + x_2^2x_5^2 + x_3^2x_4^2) \\ &+ 2\alpha^3(x_2x_3x_4x_5 + x_1x_3x_4x_6 + x_1x_2x_5x_6). \end{aligned}$$

We consider the minimization problem as follows:

$$\lambda_4(\alpha) = \min g_\alpha(\mathbf{x}) \text{ s.t. } A_4[\mathbf{x}] \in \mathcal{P}_4.$$

Lemma 6.6. When $0 \le \alpha \le 1$, it holds that $\lambda_4(\alpha) = 1$.

Proof. Since $\lambda_4(\alpha) \le 1$, it suffices to show that $\lambda_4(\alpha) \ge 1$. Indeed, we have the following representation

$$\begin{aligned} &g_\alpha(\mathbf{x}) - 1 \\ &= \frac{\alpha}{2} \sum_{(i,j,k)\in\Lambda} \begin{pmatrix} x_i^2 & x_ix_jx_k & x_ix_jx_k \\ x_ix_jx_k & x_j^2 & x_ix_jx_k \\ x_ix_jx_k & x_ix_jx_k & x_k^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{2\alpha}{3} & \frac{2\alpha}{3} \\ \frac{2\alpha}{3} & 1 & \frac{2\alpha}{3} \\ \frac{2\alpha}{3} & \frac{2\alpha}{3} & 1 \end{pmatrix} \\ &+ \alpha^2 \begin{pmatrix} x_1^2x_6^2 & x_1x_2x_5x_6 & x_1x_3x_4x_6 \\ x_1x_2x_5x_6 & x_2^2x_5^2 & x_2x_3x_4x_5 \\ x_1x_3x_4x_6 & x_2x_3x_4x_5 & x_3^2x_4^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix} \\ &\ge 0 \end{aligned}$$

when $\alpha \in [0, 1]$, where

$$\Lambda = \{(1,2,4), (1,3,5), (2,3,6), (4,5,6)\}.$$

Positivity follows from the fact that all matrices appeared in the above equality are positive semidefinite whenever $A_4[\mathbf{x}] \in \mathbb{S}^4_+$ and $\alpha \in [0, 1]$. Therefore, we obtain the desired inequality. □

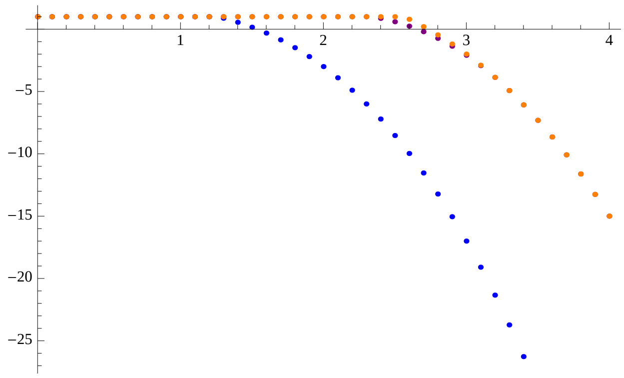

Figure 3: $\mu_4^{(r=2)}(\alpha)$ (blue), $\mu_4^{(r=3)}(\alpha)$ (purple) and $\mu_4^{(r=4)}(\alpha)$ (orange) for $0 \le \alpha \le 4$.

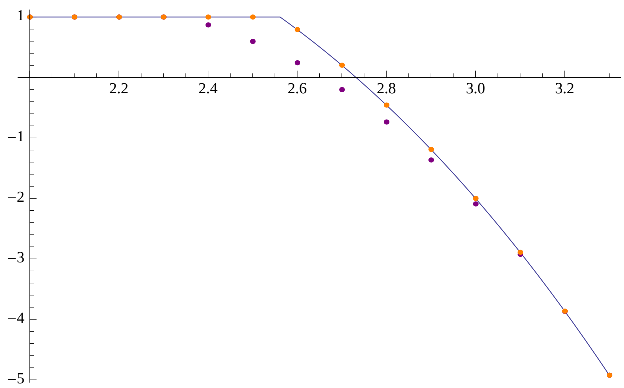

Figure 4: $\mu_4^{(r=3)}(\alpha)$ (purple) and $\mu_4^{(r=4)}(\alpha)$ (orange) for $2 \le \alpha \le 3.3$. The solid line is the upper bound given in Proposition 6.7.

Proposition 6.7. For $\alpha \ge 0$,

$$\lambda_4(\alpha) \le \min\left\{\frac{1}{2}(1+\alpha)(2+2\alpha-\alpha^2),\ 1\right\}.$$

Proof. Let $\alpha_* = \frac{1+\sqrt{17}}{2} = 2.56155\dots$. Then, the right-hand side attains at the identity matrix I_4 for $0 \le \alpha \le \alpha_*$ and at $A = \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v}$ with

$$\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, 0\right), \quad \mathbf{v} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 1\right)$$

for $\alpha \ge \alpha_*$. Therefore, the upper bound for $\lambda_4(\alpha)$ follows. □

The numerical lower bound matches the upper bound (see Figure 4). We conjecture the following:

Conjecture 6.8. For $\alpha \ge 0$,

$$\lambda_4(\alpha) = \mu_4^{(r)}(\alpha) = \min\left\{\frac{1}{2}(1+\alpha)(2+2\alpha-\alpha^2),\ 1\right\}$$

for any $r \ge 4$.

In Figures 3 and 4, we used a MacBook Air to solve SDP relaxation problems for $(n, r) = (3,2), (3,3), (4,2), (4,3)$ in the framework of Section 5. The specification is as follows: OS is Mac OS X 10.6, CPU is Intel Core 2 Duo with 1.4 GHz and the memory is 4 GByte.

For $(n, r) = (4, 4)$, we used SDPA online solver at Fujisawa lab., Chuo University [20].

6.5. For $n = 5$

For $\mathbf{x} = (x_1, x_2, \ldots, x_{10}) \in \mathbb{R}^{10}$ and

$$A_5[\mathbf{x}] = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 \\ x_1 & 1 & x_5 & x_6 & x_7 \\ x_2 & x_5 & 1 & x_8 & x_9 \\ x_3 & x_6 & x_8 & 1 & x_{10} \\ x_4 & x_7 & x_9 & x_{10} & 1 \end{pmatrix},$$

we see that the α-determinant is given by

$$\begin{aligned} h_\alpha(\mathbf{x}) &:= \det_\alpha(A_5[\mathbf{x}]) \\ &= 1 + \alpha \sum_{i=1}^{10} x_i^2 \\ &\quad + 2\alpha^2(x_1x_2x_5 + x_1x_3x_6 + x_1x_4x_7 + x_2x_3x_8 + x_2x_4x_9 \\ &\qquad + x_3x_4x_{10} + x_5x_6x_8 + x_5x_7x_9 + x_6x_7x_{10} + x_8x_9x_{10}) \\ &\quad + \alpha^2(x_1^2x_8^2 + x_1^2x_9^2 + x_1^2x_{10}^2 + x_2^2x_6^2 + x_2^2x_7^2 + x_2^2x_{10}^2 \\ &\qquad + x_3^2x_5^2 + x_3^2x_7^2 + x_3^2x_9^2 + x_4^2x_5^2 + x_4^2x_6^2 + x_4^2x_8^2 \\ &\qquad + x_5^2x_{10}^2 + x_6^2x_9^2 + x_7^2x_8^2) \\ &\quad + 2\alpha^3(x_2x_3x_5x_6 + x_2x_4x_5x_7 + x_3x_4x_6x_7 + x_1x_3x_5x_8 \\ &\qquad + x_1x_2x_6x_8 + x_1x_4x_5x_9 + x_1x_2x_7x_9 + x_3x_4x_8x_9 \\ &\qquad + x_6x_7x_8x_9 + x_1x_4x_6x_{10} + x_1x_3x_7x_{10} + x_2x_4x_8x_{10} \\ &\qquad + x_5x_7x_8x_{10} + x_2x_3x_9x_{10} + x_5x_6x_9x_{10}) \\ &\quad + 2\alpha^3(x_1^2x_8x_9x_{10} + x_2^2x_6x_7x_{10} + x_3^2x_5x_7x_9 + x_4^2x_5x_6x_8 \\ &\qquad + x_3x_4x_5^2x_{10} + x_2x_4x_6^2x_9 + x_2x_3x_7^2x_8 + x_1x_4x_7x_8^2 \\ &\qquad + x_1x_3x_6x_9^2 + x_1x_2x_5x_{10}^2) \\ &\quad + 2\alpha^4(x_1x_2x_6x_9x_{10} + x_1x_3x_5x_9x_{10} + x_1x_2x_7x_8x_{10} \\ &\qquad + x_1x_4x_5x_8x_{10} + x_2x_3x_5x_7x_{10} + x_2x_4x_5x_6x_{10} \\ &\qquad + x_1x_3x_7x_8x_9 + x_1x_4x_6x_8x_9 + x_2x_3x_6x_7x_9 \\ &\qquad + x_3x_4x_5x_6x_9 + x_2x_4x_6x_7x_8 + x_3x_4x_5x_7x_8). \end{aligned}$$

We consider the minimization problem as follows:

$$\lambda_5(\alpha) = \min h_\alpha(\mathbf{x}) \text{ s.t. } A_5[\mathbf{x}] \in \mathcal{P}_5.$$

Here we give an upper bound by computing a matrix of rank two.

Proposition 6.9. For $\alpha \geq 0$,

$$\lambda_5(\alpha) \leq \min\left\{\frac{1}{4}(1+\alpha)(1+2\alpha)(4+3\alpha-2\alpha^2),\ 1\right\}.$$

Proof. It is obvious that $\lambda_5(\alpha) \leq 1$. Set

$$A = \begin{pmatrix} 1 & s & -t & s & -t \\ s & 1 & -s & t & t \\ -t & -s & 1 & t & -s \\ s & t & t & 1 & -s \\ -t & t & -s & -s & 1 \end{pmatrix}$$

with

$$(s, t) = \left(\frac{\sqrt{5}+1}{4}, \frac{\sqrt{5}-1}{4}\right),\ \left(-\frac{\sqrt{5}-1}{4}, -\frac{\sqrt{5}+1}{4}\right).$$

Then, A is of rank 2 and we obtain the assertion by direct computation of $\det_\alpha A$. □

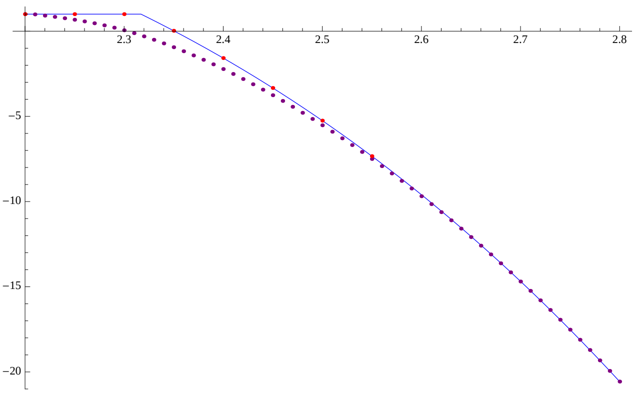

Figure 5: $\nu_5^{(3)}(\alpha)$ (purple) for $2.2 \leq \alpha \leq 2.8$ and $\nu_5^{(4)}(\alpha)$ (red) for $2.2 \leq \alpha \leq 2.55$. The solid line is the upper bound given in Proposition 6.9. The two functions 1 and $1/4(1+\alpha)(1+2\alpha)(4+3\alpha-2\alpha^2)$ meet at $\alpha = 1/2(5^{1/3}+5^{2/3}) = 2.31699\ldots$.

In Figure 5, we used a Linux machine to solve SDP relaxation problems in the framework of Subsection 6.3. The specification is as follows: OS is Ubuntu 12.10, the model name of cpu is Intel(R) Xeon(R) CPU E5530 with 2.40 GHz, the number of physical cpu is two and the memory is 24 GByte. We used eight CPU cores of this computer. In particular, to solve SDP relaxation for $n = 5$ with relaxation order $r = 4$, we used SDPA 7.3.5 [21] linked with GotoBLAS2 1.13 [22]. In other cases, we used SeDuMi 1.3 [23] with MATLAB R2012b at the same Linux computer.

In Figure 5, for small α, there is a gap between the upper bound given in Proposition 6.9 and the numerical lower bound when the relaxation order $r = 3$. However, when $r = 4$, they seem to match. Proposition 6.9 and Figure 5 support the following conjecture.

Conjecture 6.10. For $\alpha \geq 0$,

$$\lambda_5(\alpha) = \nu_5^{(r)}(\alpha) = \min\left\{\frac{1}{4}(1+\alpha)(1+2\alpha)(4+3\alpha-2\alpha^2),\ 1\right\}$$

for $r \geq 4$.

7. Concluding remarks and open questions

During the preparation of this paper, we found a paper [1] which states that positivity holds only for parameters given in (1.2) for real-symmetric case and (1.3) for Hermitian case. The proof relies on a result due to A. D. Scott-A. D. Sokal for complete monotonicity of inverse powers of certain polynomials that are defined through determinants [17]. It is still unknown which matrices break the positivity. Finding such matrices would also be an interesting problem.

Here are some open questions.

- Compute $\lambda_n(\alpha)$ for $n \geq 4$. Find corresponding SOS representations.
- Is $\lambda_n(\alpha)$ monotone decreasing in $\alpha \in [0, \infty]$?
- Compute $\lambda_\infty(\alpha)$.
- Compute $\lambda_n(\alpha)$, $\lambda_\infty(\alpha)$ for Hermitian positive semidefinite matrices.
- For the q-determinant, we can also define the similar quantities, say $\lambda_n(q)$ and $\lambda_\infty(q)$, just by replacing the α-determinant with the q-determinant. By Theorem 1.2, we see that $\lambda_\infty(q) \geq 0$ for $-1 \leq q \leq 1$. Study these quantities.

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