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A zero-range model for localized boundary stress on a tectonic plate with dissipative boundary conditions

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Abstract. We present spectral and scattering theories for a differential operator with a dissipation term that can be used to describe the dynamics of a tectonic plate with dissipative boundary conditions. The generating operator is non-selfadjoint causing some additional complexity. This difficulty has been overcome by developing the selfadjoint-dilation theory. We develop a functional model for the dissipative operator and the associated scattering theory of Lax-Phillips type.

Keywords. scattering theory, tectonic plate, dissipative boundary conditions, selfadjoint-dilation theory, spectrum

1. Structure & dynamics of tectonic plates.

By now the naive idea of the Earth as a hard body is viewed as far from reality. In fact the Earth is similar to an egg with a cracked shell. This shell – the *lithosphere* – is composed of numerous fragments – the most part covered by 14 large fragments, the *tectonic plates*. The scale of these large tectonic plates varies within a range of several thousand kilometers and bodies of neighboring plates contact directly along relatively small active zones with diameters of about 100 km. Along the remainder of the boundaries, neighboring tectonic plates are often separated by relatively narrow channels are filled with smaller fragments and disperse materials which cannot accumulate significant amounts of elastic energy, and so transform it into heat, causing the *dissipation* of stored *elastic energy* in the form of *seismo-gravitational oscillations* (SGO). These SGOs admit spectral interpretation, see [21, 44] and our discussion below. Tectonic plates are relatively thin (30–100 km thick) under the oceans, but are thicker (200–300 km) on the ancient continental platforms and along the oceanic ridges. The elastic properties of the plates are determined by Young's modulus $17.28 \times 10^{10}\,\mathrm{kg\,m^{-1}\,sec^{-2}}$, density $3380\,\mathrm{kg\,m^{-3}}$ and the Poisson coefficient 0.28. The velocity of longitudinal waves on the plates is about $8000\,\mathrm{m\ sec^{-1}}$, and the velocity of the flexural waves depends on the frequency and varies, with the type of the wave, on a wide range around $4500\,\mathrm{m\ sec^{-1}}$. The tectonic plates float on the *astenosphere*. The viscosity of the astenosphere is large for fast motions, but relatively small for slow ones.

The exterior dynamics of the tectonic plates is defined by their sliding along the astenosphere (with liquid friction) by convective flows in the Earth's mantle and also due to changes of the rotation speed of Earth caused by fluctuations in the moment of inertia. Both factors are caused by irregularities in the energy dissappation from the liquid upper core (see §4). Though forced oscillations may be present in the tectonic system most of the elastic dynamics are defined by the SGO of the plates caused by variation of the strain-stress conditions at the active zones. These elastic properties of the plates can be described in terms of thin plate models floating on a liquid with appropriate boundary conditions analogous to the model of floating ice, see [10]. Here we base our observations on the assumption of the spectral nature of SGO, see [21, 44]. This is supported by synchronous observations of SGO on several GEOSCOPE stations and confirmed by the direct calculation of flexural eigen-modes of a rectangular plate, see [27]. We note that the dissipation of elastic energy of these SGOs due to the liquid underlay and the presence of disperse materials on the boundary were not taken into account in that study. However the density of distribution of the eigenvalues of the model thin plate and their number below certain levels appears very much the same as the observed values of square frequencies of SGOs on an equivalent tectonic plate. Note that relatively minor details concerning the shape of the plate do not influence the lower eigenvalues corresponding to the long-wave elastic modes. An extended experimental materials science model supporting the hypothesis on the spectral nature of the flexural component of SGO on the tectonic plates can be found in numerous papers, see for instance [37, 38, 39, 40, 41, 42, 43, 45, 46].

The Synchronous observations of slow SGOs give a very rough estimation of the damping (or decay) of the elastic energy stored in the tectonic plates. These observations were made in Borovoe (Kazakhstan, 1992) and in Obninsk (Central Russia, 1987) simultaneously with corresponding observations by the seismogravimetric complex at St. Pe-

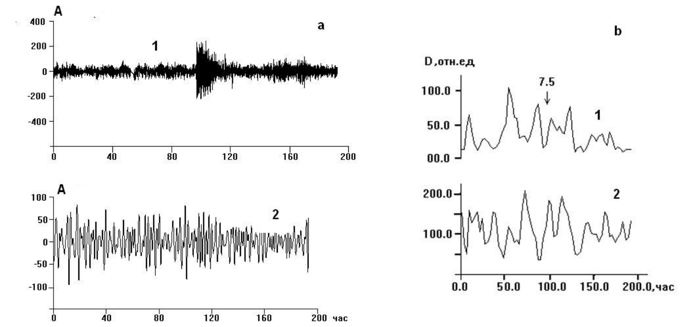

Figure 1: (a) Simultaneous observations by the gravimeter in Borovoe (1) and the seismogravimetric complex of St. Petersburg University (2), December 1992. (b) Pulsations intensity in the gliding window 6 hrs. The vertical arrow is the time of the earthquake in Indonesia (magnitude M = 7.5), 12 December 1992.

tersburg University.

Among various types of oscillations of the Earth, a special role is played by *pulsations* (observed first in [19]) and recognized as an important phenomenon [21, 19, 20]. Typical pulsations in December 1992 lasted up to 80 hrs and were followed by an earthquake of magnitude M = 7.5 in Indonesia, 12 December 1992, see Fig. 1. Spectra of the

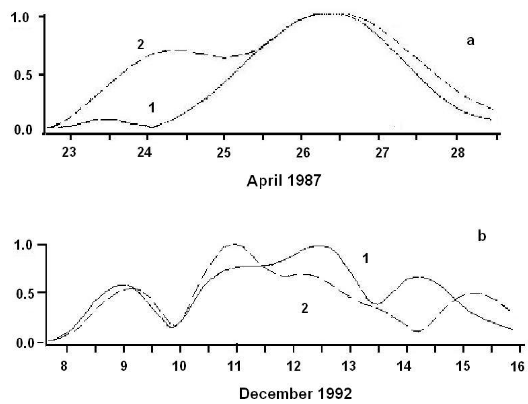

Figure 2: (a) The envelopes of SGO, 0.075–0.095 mHZ for Obninsk (1)and St. Petersgurg (2),1987. (b) The envelopes of SGO 0.153–0.167 mHZ for Borovoe (1) and St. Petersburg (2), 1992.

corresponding SGOs in synchronous observations in 1987 and 1992 had a similar structure and a stable maximum on the interval 0.110-0.130 mHZ. Beside each pair of synchronous spectra there is a common narrow interval with one peak in 1987 (0.075–0.095 mHz) or two peaks in 1992 (0.153–0.167 mHz). From these narrow frequency intervals the envelopes of the SGOs were calculated and the damping of the corresponding amplitudes was estimated. Despite the large distances between the stations involved

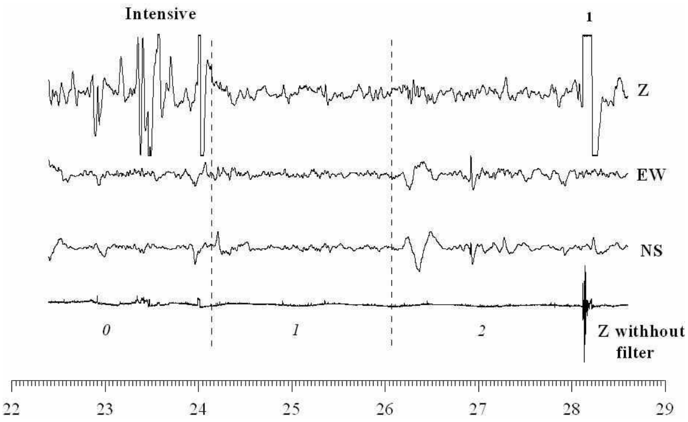

Figure 3: Dynamics of seismo-gravitational oscillations observed in St. Petersburg, 22–28 March 2000, before an earthquake in Japan (M = 7.6). Here Z, EW, NS mark components of the displacement. The magnitude of oscillation of the (vertical) Z-component was actually 2 times larger than the horizontal oscillations EW, NS. In the domain 0 one sees an intense pulsation, [37, 38]. In the domain 2 one sees a change in the SGO spectrum compared with domain 0 and 1. The last curve shows (without filtering) the record of all displacements of the base of the seismograph in a relative scale.

in the experiment, the envelopes look similar and contain the same number of maxima and minima, see Fig. 2, a. We calculated a decay rate τ (damping decrement), from the envelopes, see Fig. 2, based on the length of the time interval Δ_T characterized by the damping of the amplitude $A(\tau)$ in the ratio 2 : 1:

$$A(\Delta_T) = A(0)e^{-\tau\,\Delta_T},\ \tau = \ln 2/\Delta_T$$

The decrement found from the observations (with no pulsations taken into account) was 0.028 1/hrs (Obninsk) and 0.023 for St. Petersburg. For another interval of frequencies (with pulsations and two spectral peaks) the decrement was 0.040 for Borovoe and 0.046 for St. Petersburg. Note that the decay decrement, as a spectral parameter, is expected to be the same for a certain single mode of SGO (with selected frequency) independently of location.

Estimates of the decay rate obtained for both pairs of observations are close despite the naive method of calculation. Small differences in decay rate calculated from envelopes observed in remoted locations for the same interval of frequencies enable decay rates to be viwed as a spectral characteristic of the mode, similar to frequency. Mathematically both observables arise as the imaginary and the real parts of the corresponding complex “frequency” – an eigenvalue of a relevant *dissipative* generator of the wave dynamics of the thin plate floating on the liquid underlay, see section 2.

The monitoring of the SGO background provides information on the amount of elastic energy accumulated, due to stress, at the active zone of a tectonic plate. Evidence of the growth of stored energy is seen from the growing frequencies of SGO before the earthquakes. Indeed, the effect of growing of frequencies of SGO under a localized boundary stress was observed in [21] from analysis of spectral-time (ST) cards (St. Petersburg 22–28 March 2000) just before an earthquake in Japan, and noted in [44], from which Fig. 4. is taken.

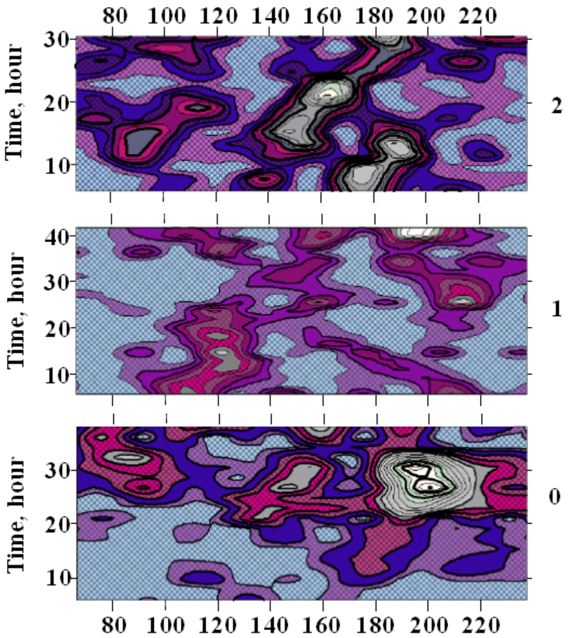

Figure 4: The time- frequency cards showing the variation of the frequency of the corresponding SGO

Space-time card analysis reveals an important connection between the stress magnitude and the frequency of the SGOs observed. In spectral terms, stress magnitude and frequency are obtained from the eigenvalues of the generator of the wave process. According to the celebrated variational principle, see [7], the elastic deformation caused by the stress enhances the corresponding Hamiltonian, and hence the growth of the corresponding conditional minima, coincident with the eigenvalues of the biharmonic Hamiltonian of the tectonic plate – the squares of the eigenfrequencies of the corresponding generator, §2. A similar well known effect is used in musical instruments attenuated via regulation of tension of strings.

In what follows we quote from [44] concerning the dynamics of the frequencies revealed from analysis of the spectral-time cards (ST-cards) of the **vertical** component of SGOs, based on comparison of data represented in Fig. 4. Frequency growth of SGOs is seen on ST-card, Fig. 3. The spectral amplitudes are represented in Fig. 4 by variation of the shade. Maximal amplitudes are white. The amplitudes on the shaded domains are 3.5 times greater than the average amplitude (frequency $\sim$ 200 mcHz). One can see from Fig. 4 (part 1) a few domains, where the frequencies of the modes increase or decrease. This is interpreted as evidence of an accumulation (or discharge, by forming local cracks) of the elastic energy due to stress. Growth/decrease of the frequency is characterized by the ratio $\Delta v/t$, where Δ is the increment of the frequency on the interval $[0, t]$. We have obtained from [44] the relevant data from observations of real processes, including an essential damping of amplitudes of the SGO. That paper [44] is dedicated to an analysis of the elastic dynamics of a thin plate with no dissipation.

In this paper we attempt to develop an appropriate analysis for a more realistic model, with dissipation of elastic energy and the damping of SGO amplitudes is taken into account. The analysis of the dynamics with no damping can be developed, with use of an appropriate fitted zero-range model, based on spectral theory and an operator extensions procedure for *symmetric operators*, the corresponding analysis with SGO amplitudes damping requires spectral analysis and the operator extensions for non-symmetric – *dissipative* – operators.

Key features of dissipative spectral theory were developed in [18, 24, 28], unfortunately the corresponding operator extension techniques was absent. This is a major obstacle in using appropriate zero-range models of the boundary stress as a tool of the corresponding analytic perturbation procedure, c.f. similar techniques for selfadjoint operators, [29].

Experimental results show that the damping decrements of the SGO amplitudes are important spectral variables, and can be used, together with the frequencies and amplitudes of the SGO, and the shapes of the corresponding modes for fitting of a model of a tectonic plate under stress. Monitoring of SGOs would then allow a fit of the corresponding model of the plate under stress. Mathematical modeling would then enable calculation of the eigenvalues and eigenfunctions of the fitted solvable model, this in turn would allow estimation of the elastic energy accumulated in the active zone of the plate and, eventually, predicting the power of expected earthquake.

Here we aim to attract the attention of mathematicians and geologists to an interesting area of seismology providing an alternative approach to prediction of earthquakes and/or tsunami and ultimately a preliminary estimation of their power. A further part of the paper suggests new mathematical methods in the dissipative operator extension theory based on Lax-Phillips analysis of the generator of the corresponding contracting dynamics. We need Lax-Phillips analysis as a background for construction of a fitted soluble model of the stressed tectonic plate using the model as a first step in the relevant analytic perturbation procedure for calculation of the perturbed dynamics under stress.

2. Plates in a dissipative environment

There are two ways in which dissipation of the elastic energy accumulated on a thin tectonic plate due to boundary stress may occur. First, through the disperse materials filling the channel between the neighboring plates beyond the active zones of direct contact. Second, by the contact of the plate with the liquid underlay.

The disperse material cannot accumulate an essential amount of elastic energy, but simply transforms it into heat, [17, 48]. We do not discuss here the physics of this process, but assume that the dissipation is described by a phenomenological dissipative boundary condition, defined by a real operator matrix (see §2.1). We neglect the presence of neighboring plates and hence assume that a single thin plate is embedded into the domain filled with disperse materials, emulated by a dissipative boundary condition on the boundary of the plate.

To take into account the dissipation defined by the fluid underlay, we should consider contact of the plate with a viscous liquid, described by a modified Navier-Stokes equation and derive the dissipation, assuming a small, but non-zero liquid viscosity with respect to slow movements. We will not do this here, but assume, see §2.2, that the dynamics of SGOs on the thin plate floating on the fluid is described by the bi-harmonic wave equation (5), see [10], with a positive damping parameter β on the velocity, assuming that this phenomenological parameter defines the dissipation due to small viscosity and does not vanish, contrary to [10].

We aim to construct a fitted zero-range solvable model of a floating plate under boundary stress in a dissipative environment. Contrary to the conservative case, where the corresponding unperturbed operator is Hermitian or selfadjoint, we need a version of the operator-extension theory for dissipative operators $\mathcal{L}$. We will achieve this based on construction of the selfadjoint dilation $\hat{\mathcal{L}}$ of the dissipative generator of the wave dynamics in an extended space via attachment of incoming and outgoing channels on the plate Ω and it's boundary γ. Then the zero-range perturbation of the dilation $\hat{\mathcal{L}}$ is introduced via special boundary condition on the selected deficiency subspace N orthogonal to incoming and outgoing channels which remain unaffected. Subsequent splitting of these gives a dissipative operator $\mathcal{L}$, – a zero-range perturbation of the original dissipative operator L. This program includes several steps:

1. Description of dissipative boundary conditions for the thin-plate model (bi-harmonic operator on a bounded domain).

2. Construction of a selfadjoint dilation, see [24], $\hat{\mathcal{L}} : \mathcal{H} \to \mathcal{H}$ of the dissipative thin-plate model $\mathcal{L} : \mathcal{K} \to \mathcal{K} \subset \mathcal{H}$.

3. Restriction of the dilation $\hat{\mathcal{L}} \to \hat{\mathcal{L}}_0$, with deficiency subspaces $N_{\pm i}$ from the original co-invariant subspace $\mathcal{K}$, and subsequent extension of $\hat{\mathcal{L}}_0 \to \hat{\mathcal{L}}_B$ with a Hermitian matrix extension parameter $B : N \equiv N_i \dot{+} N_i \to N$ and corresponding selfadjoint generator $\hat{\mathcal{L}}_B$ of the relevant unitary group $U_B(t) = \exp i\hat{\mathcal{L}}_B t$.

4. Description of spectral properties of the dissipative generator of the relevant Lax-Phillips semigroup, see [18], $P_K U_B(t)\big|_{\mathcal{K}}$, $t \geq 0$, obtained as a zero-range perturbation $\mathcal{L}_B$ of the original dissipative generator $\mathcal{L}$ of the wave dynamics of the tectonic plate emulating the stressed dynamics of the tectonic plate in a dissipative environment.

Steps 3 & 4 are standard in dissipative operator theory, so we concentrate here on 1 & 2 and provide, in the Appendix a brief review of the spectral analysis of dissipative operators in terms of the functional model. There are considerable technical details which are necessary for the proofs of a number of the results we claim here which underpin the mathematical phenomena we discuss. We shall present those in a forthcoming manuscript. Indeed, similar techniques to those presented here can be used for the stress applied on the body of the plate, caused by violent convective flows in the asthenosphere, or the stresses on the oceanic plated, which are responsible for tsunami, see §4 and an extended discussion in [14].

2.1. Dissipative boundary conditions.

The bi-harmonic operator Δ^2 acts on $L_2(\Omega)$, $\Omega \subset \mathbb{R}^2$ a relatively compact domain with smooth boundary $\partial\Omega \equiv \gamma$. Integration by parts yields the boundary form :

$$\begin{aligned} J_{int}(u,v) &= \int_{\tilde{\Omega}} \left[\Delta^2 \bar{u}\, v - \bar{u}\Delta^2 v\right] = \langle \Xi_+^u, \Xi_-^v\rangle - \langle \Xi_-^u, \Xi_+^v\rangle \\ &= \int_\gamma d\gamma \left[\langle \begin{pmatrix} \frac{\partial \Delta u}{\partial n} \\ -\Delta u \end{pmatrix}, \begin{pmatrix} v \\ \frac{\partial v}{\partial n} \end{pmatrix}\rangle \right. \\ &\qquad \left. -\langle \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix}, \begin{pmatrix} \frac{\partial \Delta v}{\partial n} \\ -\Delta v \end{pmatrix}\rangle \right] \end{aligned} \tag{1}$$

Or, alternatively the form

$$\begin{aligned} J_N(u,v) &\equiv \int_\gamma d\gamma \left[\langle \begin{pmatrix} \frac{\partial \Delta u}{\partial n} \\ \frac{\partial u}{\partial n} \end{pmatrix}, \begin{pmatrix} v \\ \Delta v \end{pmatrix}\rangle \right. \\ &\qquad \left. -\langle \begin{pmatrix} u \\ \Delta u \end{pmatrix}, \begin{pmatrix} \frac{\partial \Delta v}{\partial n} \\ \frac{\partial u}{\partial n} \end{pmatrix}\rangle \right]. \end{aligned} \tag{2}$$

We select standard selfadjoint boundary conditions for the bi-harmonic operator so that the Lagrangian form (1) vanishes. The Lagrangian plane (defined by boundary conditions) is an Hermitian matrix $\mathcal{B}$ in the appropriate Sobolev class of the boundary data, for instance

$$\begin{pmatrix} \frac{\partial \Delta u}{\partial n} \\ -\Delta u \end{pmatrix} + B_t \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \mathcal{B} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix}, \tag{3}$$

where

$$B_t = (\sigma - 1) \begin{pmatrix} 0 & \frac{\partial^2}{\partial t^2} \\ \frac{\partial^2}{\partial t^2} & 0 \end{pmatrix},$$

with the Poisson coefficient $\sigma, 0 \leq \sigma \leq 1/2$ and $\frac{\partial^2}{\partial t^2}$ taken along the boundary. Equation (3) with $\mathcal{B} = 0$ gives natural boundary conditions for the thin plate model. Similarly for the dissipative boundary condition we substitute $\mathcal{B}$ for a dissipative matrix operator. Thus for a dissipative model and in the simplest case $\mathcal{B} = m\frac{i}{2}\,\Gamma_\gamma^+\Gamma_\gamma$ with a constant 2×2 nonsingular matrix operator Γ_γ acting from $L_2(\gamma, E)$ to an auxiliary space $K\big|_\gamma$:

$$\begin{pmatrix} \frac{\partial \Delta u}{\partial n} \\ -\Delta u \end{pmatrix} + (\sigma - 1)\begin{pmatrix} 0 & \frac{\partial^2}{\partial t^2} \\ \frac{\partial^2}{\partial t^2} & 0 \end{pmatrix}\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \frac{i}{2}\Gamma_\gamma^+\,\Gamma_\gamma \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix}. \qquad (4)$$

The bi-harmonic operator with this boundary condition is dissipative in $L_2(\Omega)$ and is connected to the generator $\mathcal{L}$ of the wave dynamics of the tectonic plate with dissipation.

2.2. Damped wave process on the tectonic plate.

Flexural oscillations of the thin plate Ω floating on a liquid with small viscosity are described in non-dimensional coordinates by the wave equation for the vertical displacements η [10] on the horizontal surface $\Omega \subset \partial\tilde{\Omega}$ of the liquid filling the 3D domain $\tilde{\Omega}$ with $\Delta_\Omega = \Delta$:

$$\Delta^2_\Omega\,\eta + m\eta_{tt} + \beta\eta_t + \eta + \phi_t = p_a, \qquad (5)$$

Here m is the relative 2D density of the 2D plate, ϕ is the hydrodynamic potential in $\tilde{\Omega}$, β – a positive damping parameter, and p_a is an additional pressure applied on the surface Ω in the upward direction. The hydrodynamic potential satisfies 3D-Laplace equation $\Delta\phi = 0$ in $\tilde{\Omega}$ and Neumann boundary conditions on $\partial\tilde{\Omega} \setminus \Omega$, $\frac{\partial\phi}{\partial n}\big|_{\partial\tilde{\Omega}\setminus\Omega} = 0$. Kinematic matching conditions are imposed on the plate: $\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial t}\big|_\Omega = 0$. The hydrodynamic potential satisfies the volume conservation condition $\int_\Omega \frac{\partial\phi}{\partial z} d\Omega = 0$, which allows us to disregard the zero eigenvalue of the Laplacian in $\tilde{\Omega}$ and define the partial Neumann-to-Dirichlet map $\mathcal{ND}_0$ on the orthogonal complement of constants in an appropriate Sobolev class from $L_2(\Omega)$. The relative ND-map is represented by the formal spectral series over the system $\varphi_l^\Omega \equiv \varphi_l\big|_\Omega$ of the eigenfunctions of the Neumann Laplacian in $\tilde{\Omega}$, restricted to Ω

$$\mathcal{ND}_0 \equiv \sum_{l>0} \frac{\varphi_l^\Omega\rangle\,\langle\varphi_l^\Omega}{\lambda_l}. \qquad (6)$$

This series is divergent, but can be regularized based on the resolvent equation, [33], and implies the positivity of the relative ND-map. Then we eliminate the hydrodynamic potential from (5) based on the kinematic matching condition $\frac{\partial\phi}{\partial z} = \mathcal{ND}\frac{\partial^2\eta}{\partial t^2}$. This yields a "wave equation" for the vertical displacement η on Ω:

$$(m + \mathcal{ND}_0)\,\eta_{tt} + \beta\eta_t + \Delta^2\eta + \eta = p_a. \qquad (7)$$

With negligeable upward pressure p_a (i.e. thin plate) the equation (5) is homogeneous:

$$(m + \mathcal{ND}_0)\,\eta_{tt} + \beta\eta_t + \Delta^2\eta + \eta \equiv \mathcal{D}\eta_{tt} + \beta\eta_t + L\eta = 0, \quad (8)$$

The homogeneous wave equation (8) can be presented as a first order equation in Lax-Phillips form, [18], with operator coefficient $(m + \mathcal{ND}_0) \equiv \mathcal{D}$ and $\mathcal{D}^{-1}L \equiv L_{\mathcal{D}}$, as follows:

$$\frac{1}{i}\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_t \end{pmatrix} = i\begin{pmatrix} 0 & -1 \\ L_{\mathcal{D}} & \mathcal{D}^{-1}\beta \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} \equiv \mathcal{L}_{\mathcal{D}}\begin{pmatrix} u \\ u_t \end{pmatrix} \equiv \mathcal{L}_{\mathcal{D}}\vec{u}. \qquad (9)$$

$\mathcal{L}_{\mathcal{D}}$ is dissipative in $\mathcal{E}$ (Cauchy data $\vec{u} = (u, u_t) \equiv (u_0, u_1)$), with energy norm

$$\langle\vec{u},\,\vec{v}\rangle_{\mathcal{E}} = \frac{1}{2}\int_\Omega \left[\Delta\bar{u}_0\Delta v_0 + \bar{u}_0 v_0 + \mathcal{D}\bar{u}_1\,v_1\right] d\Omega, \qquad (10)$$

Lemma 1. *The boundary form of the Lax-Phillips generator $\mathcal{L}_{\mathcal{D}}$ in the energy-normed space of the Cauchy data $\vec{\eta}$ is represented as*

$$\langle\mathcal{L}_{\mathcal{D}}\vec{u},\vec{v}\rangle_{\mathcal{E}} - \langle\vec{u},\mathcal{L}_{\mathcal{D}}\vec{v}\rangle_{\mathcal{E}} = -\frac{i}{2}\left[\langle\Xi_+^{u_0},\Xi_-^{v_1}\rangle_{L_2(\gamma)} - \langle\Xi_-^{u_1},\Xi_+^{v_0}\rangle_{L_2(\gamma)}\right] - i\langle\beta u_1, v_1\rangle_{L_2}.$$

The boundary form is often interpreted as a "current" associated with the "wave function" of the problem. For selfadjoint operators - Hamiltonians of conservative physical systems - boundary forms vanish due to conservative boundary conditions imposed on the symplectic data $\Xi_\pm\big|_\gamma,\,u_{0,1}\big|_\Omega$. For multi-channel systems, conservative boundary conditions are interpreted as balancing the currents on the contact of the channels.

Hereafter we consider the dissipative boundary conditions for the generator of the wave dynamics coming from the boundary conditions and data $\Xi_+ = i\Gamma^+\Gamma\Xi_-$, as in (16) below, if $\Gamma^+\Gamma = \mathcal{B} > 0$.

2.3. The selfadjoint dilation of a dissipative generator.

Unfortunately rigorously formulated multi-channel problems are usually too complicated for explicit analysis because of the problem of multiple returning waves. Lax-Phillips systems are characterized by the condition that all such waves (exiting from $\mathcal{K}$ into an outgoing channel) never return, [18]. Thus the structure of the outgoing channel is not important for damped evolution and only the balance of currents is essential. Therefore, for Lax-Phillips systems, we are able to replace the outgoing channels with the *simplest* outgoing channels providing the required balance. Then the damped evolution $e^{i\mathcal{L}t}$ in $\mathcal{K}$ can be embedded into the larger space $\mathcal{H} = \mathcal{K} \oplus \mathcal{D}_{in} \oplus \mathcal{D}_{out}$ via the attachment of trivial artificial incoming and outgoing channels $\mathcal{D}_{in,out} = L_2(R_\mp)$ so that the system obtained is Lax-Phillips with a selfadjoint generator $\hat{\mathcal{L}}_{\mathcal{D}}$. We use the analogue of the boundary conditions (3,4) disregarding the term with the Poisson coefficient. For instance, the generator $\hat{\mathcal{L}}_{\mathcal{D}}$ is obtained from the original dissipative operator by attachment of truncated momenta $i\frac{d}{dx}$ on $L_2(R_\mp)$. Then the positive cone of the relevant unitary evolution group $e^{i\hat{\mathcal{L}}_{\mathcal{D}}t}$, $t \geq 0$ restricted onto the co-invariant subspace $\mathcal{K}$ coincides with the original damped evolution, see [18, 24]:

$$P_{\mathcal{K}} e^{i\hat{\mathcal{L}}_{\mathcal{D}}t}\big|_{\mathcal{K}} = e^{i\mathcal{L}_{\mathcal{D}}t},\ t > 0. \tag{11}$$

This remarkable connection between the dissipative operator $\mathcal{L}_{\mathcal{D}}$ and its nearest selfadjoint relative $\hat{\mathcal{L}}_{\mathcal{D}}$ - the *self-adjoint dilation* - is the most important feature of the corresponding spectral theory, see [24]. Essential details of the spectral picture of the dissipative operator $\mathcal{L}_{\mathcal{D}}$ can be interpreted in terms of the corresponding scattering matrix/characteristic function, see [24, 28, 31]. Here we use the dilation as a tool for explicit construction of the zero-range model of a weakly perturbed dissipative operator.

Construct the selfadjoint dilation of $\mathcal{L}_{\mathcal{D}}$ by attachment of incoming/outgoing channels to the boundary of Ω and to the support *supp* $\beta \subset \Omega$ of $\beta > 0$. Set

$$\mathcal{D}^{\gamma}_{in,out} = L_2(R_{\mp}, E_{\gamma}),\ \mathcal{D}^{\Omega}_{in,out} = L_2(R_{\mp}, E_{\Omega}),$$

The truncated momenta are defined on $\mathcal{D}^{\gamma}_{in,out}$, $\mathcal{D}^{\Omega}_{in,out}$ as

$$i\frac{du^{\gamma}_{+}}{ds},\ i\frac{du^{\gamma}_{-}}{ds},\ i\frac{du^{\Omega}_{+}}{ds},\ i\frac{du^{\Omega}_{-}}{ds}. \tag{12}$$

The boundary forms of the truncated momenta on $u^{\gamma}_{\pm} \in \mathcal{H}^{\gamma}$ and $u^{\Omega}_{\pm} \in \mathcal{H}^{\Omega}$ are represented in terms of boundary data $u^{\gamma}_{\pm}(0_{\pm},\gamma)$, $u^{\Omega}_{\pm}(0_{\pm},\Omega)$ or in terms of the corresponding symplectic variables

$$\Xi^{\gamma}_{+} = \frac{u^{\gamma}_{+} + u^{\gamma}_{-}}{2}, \qquad \Xi^{\gamma}_{-} = \frac{u^{\gamma}_{+} - u^{\gamma}_{-}}{i},$$
$$\Xi^{\Omega}_{+} = \frac{u^{\Omega}_{+} + u^{\Omega}_{-}}{2}, \qquad \Xi^{\Omega}_{-} = \frac{u^{\Omega}_{+} - u^{\Omega}_{-}}{i}$$

$$\begin{aligned} J_{\gamma}(u^{\gamma}, v^{\gamma}) &= i\langle u^{\gamma}_{+}(0_{+},\gamma), v^{\gamma}_{+}(0,\gamma)\rangle_{E_{\gamma}} \\ &\quad -i\langle u^{\gamma}_{-}(0_{-},\gamma), v^{\gamma}_{-}(0_{-},\gamma)\rangle_{E_{\gamma}} \\ &= \langle \Xi^{u^{\gamma}}_{+}, \Xi^{v^{\gamma}}_{-}\rangle - \langle \Xi^{u^{\gamma}}_{-}, \Xi^{v^{\gamma}}_{+}\rangle, \\ J_{\Omega}(u^{\Omega}, v^{\Omega}) &= \langle iu^{\Omega}_{+}(0_{+},\Omega), v^{\Omega}_{+}(0,\Omega)\rangle_{E_{\Omega}} \\ &\quad -\langle iu^{\Omega}_{-}(0_{-},\Omega), v^{\Omega}_{-}(0_{-},\Omega)\rangle_{E_{\Omega}} \\ &= \langle \Xi^{u^{\Omega}}_{+}, \Xi^{v^{\Omega}}_{-}\rangle - \langle \Xi^{u^{\Omega}}_{-}, \Xi^{v^{\Omega}}_{+}\rangle. \end{aligned} \tag{13}$$

We factorize the operators $\beta = \Gamma^{+}_{\Omega}\Gamma_{\Omega}$, $\mathcal{B} = \Gamma^{+}_{\gamma}\Gamma_{\gamma}$ so as to have nonsingular factors acting on the boundary spaces of $\vec{u}, u^{\Omega}_{\pm}(0_{\pm}), u^{\gamma}_{\pm}(0_{\pm})$. Define $\mathcal{L}_{\mathcal{D}}$, associated with the biharmonic operator with dissipative boundary conditions $\Xi^{u}_{+}\big|_{\gamma} = i\mathcal{B}\,\Xi^{u}_{+}\big|_{\gamma}$. We verify the following elsewhere.

Theorem 1. *The matrix operator defined in*

$$\mathcal{H} = \mathcal{D}^{\Omega}_{out} \oplus \mathcal{D}^{\gamma}_{out} \oplus \mathcal{K} \oplus \mathcal{D}^{\gamma}_{in} \oplus \mathcal{D}^{\Omega}_{out} \tag{14}$$

by the formula

$$\hat{\mathcal{L}}_{D}\begin{pmatrix} u^{\Omega}_{+} \\ u^{\gamma}_{+} \\ \vec{u} \\ u^{\gamma}_{-} \\ u^{\Omega}_{-} \end{pmatrix} = \begin{pmatrix} i\frac{du^{\Omega}_{+}}{ds} \\ i\frac{du^{\gamma}_{+}}{dr} \\ i\begin{pmatrix} 0 & -1 \\ L_{\mathcal{D}} & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \Gamma^{+}_{\Omega}\Xi^{u^{\Omega}_{1}}_{+} \end{pmatrix} \\ i\frac{du^{\gamma}_{-}}{dr} \\ i\frac{du^{\Omega}_{+}}{ds} \end{pmatrix} \tag{15}$$

with the boundary conditions imposed on $\Xi^{\Omega}_{\pm}, \Xi^{\gamma}_{\pm}, \Xi^{u_0}_{\pm}, \Xi^{u_1}_{\pm}$*:*

$$\Xi^{\Omega}_{-} = \Gamma_{\Omega}u^{1},\ \Xi^{u_0}_{+} = \Gamma^{+}_{\gamma}\Xi^{\gamma}_{+},\ \Xi^{\gamma}_{-} = \Gamma_{\gamma}\Xi^{u_1}_{-} \tag{16}$$

is a selfadjoint dilation of the operator $\mathcal{L}_{D}$*.*

A similar statement is true for the dissipative wave generator with the boundary condition (4) again for the biharmonic operator. In fact description of the dilation can be simplified by introducing total incoming and outgoing spaces $\mathcal{D}_{in,out} = \mathcal{D}^{\gamma}_{in,out} \oplus \mathcal{D}^{\Omega}_{in,out}$ and total vectors $\vec{u}_{\pm} = u^{\gamma}_{\pm} \oplus u^{\Omega}_{\pm}$. Then, instead of Γ_{γ}, Γ_{Ω} use $\Gamma : E \to E_{\gamma} \oplus E_{\Omega}$ and the adjoint Γ^{+}, the formula (15) for the dilation reduces to

$$\hat{\mathcal{L}}\begin{pmatrix} u_{+} \\ u \\ u_{-} \end{pmatrix} = \begin{pmatrix} i\frac{du_{+}}{ds} \\ \mathcal{A}_{sa}u + \Gamma^{+}\frac{u_{+}+u_{-}}{2} \\ i\frac{du_{-}}{ds} \end{pmatrix} \tag{17}$$

with boundary condition $\frac{u_{+}-u_{-}}{i} = \Gamma u$. Here $\mathcal{A}_{sa}$ is the self-adjoint part, with zero contribution to the boundary form of $\hat{\mathcal{L}}$. In this form the dilation formula was presented in [28].

3. Restriction-extension scheme for dilation

Originally von Neumann's operator extension theory was developed as a tool to construct the functional calculus, in particular to construct the appropriate dynamic group. Though the zero-range model arose in [9] independently, the connection between them was noted in [5] and von Neumann's theory is considered foundational to zero-range models of quantum and acoustic systems, see [8, 3, 4, 6, 22] and references therein. Now zero-range models are used in analytic perturbation procedures to improve convergence of the perturbation series in resonance spectral problems with unperturbed eigenvalues embedded in continuous spectrum, see [34, 29, 35].

Since we need to mark the operators by their extension parameter α, we will write $\mathcal{L}_{\alpha}$ instead of $\mathcal{L}_{\mathcal{D},\alpha}$ and $\hat{\mathcal{L}}_{\alpha}$ instead of $\hat{\mathcal{L}}_{\mathcal{D},\alpha}$ for the dilations.

To construct a zero-range model of the dissipative operator $\mathcal{L}$, we develop an appropriate restriction-extension procedure $\hat{\mathcal{L}} \to \hat{\mathcal{L}}_{\alpha}$ for the corresponding dilation, and then consider the Lax-Phillips semigroup generated by $\hat{\mathcal{L}}_{\alpha}$. Due to the special selection of the deficiency subspace of the dilation $N_{\pm i} \subset \mathcal{K}$, the incoming and outgoing subspaces remain $L_2(R_{\pm}, E)$ – the same as incoming and outgoing subspaces of the dilation $\hat{\mathcal{L}}$. Then the restriction $e^{i\mathcal{L}_{\alpha}t}$ of the extended unitary group $e^{i\hat{\mathcal{L}}_\alpha t}$ to the co-invariant subspace $\mathcal{K}$ yields a contracting semigroup $P_{\mathcal{K}} e^{i\hat{\mathcal{L}}_\alpha t}\big|_{\mathcal{K}} \equiv e^{i\mathcal{L}_\alpha t}$, $t \geq 0$, whose generator serves as a zero-range perturbation of the original dissipative operator $\mathcal{L}_{\mathcal{D}} \equiv \mathcal{L}$.

The deficiency elements at the complex point $-i$ of a local restriction of a self-adjoint differential operator L to the domain of all smooth elements, vanishing near given point a coincide with square integrable derivatives of the corresponding Green function $G_i^L(x,a)$. Thus the deficiency index is $(6,6)$ of the 2D bi-harmonic operator at $a \in \Omega$, see [15] and the index is $(3,3)$ for the local restriction of the bi-harmonic operator with the Neumann homogeneous boundary condition. Further, restriction of the bi-harmonic operator with natural boundary conditions for a thin plate, (3) with $\mathcal{B} = 0$, has deficiency index $(5,5)$ (an important remark of C. Fox and S. Nazarov shows the deficiency index of the local restriction of the bi-harmonic operator may be greater than the index of the similar restriction of the operator with the Neumann homogeneous boundary condition). Generally the index of the locally restricted unperturbed bi-harmonic operator does not exceed $(6,6)$, but may vary depending on the original boundary condition. The restriction of the selfadjoint operator $\hat{\mathcal{L}} \equiv A$ is equivalent to selection of the deficiency subspace for given value $i\delta$ of the spectral parameter. The restriction is local if the deficiency elements are generalized solutions of the corresponding homogeneous equation and belong to the Hilbert space $\mathcal{H}$ in question. In particular, for the local restriction of the dilation $\hat{\mathcal{L}}$ one can use the Green function or the derivatives of the Green function of the dissipative bi-harmonic operator. We assume that the deficiency subspace is a generating subspace $N_{_i}$ (or develop the restriction-extension procedure in the invariant subspace generated by the defect $N_i + N_i$, and assume that the deficiency subspaces do not overlap:$N_{-i} \cap N_i = \frac{A+iI}{A-iI} N_i \cap N_i = 0$, $\dim\, N_i = d$). Then set $D_0^{^A} = (A - iI)^{-1}\,(E_{_A} \ominus N_i)$ and define the restriction as $A \to A_0 = A\big|_{D_0^A}$.

The procedure of restriction suggested above does not guarantee density of the domain of the restricted operator. Nevertheless, the deficiency subspaces $N_{\pm i}$ do not overlap, the extension procedure for the orthogonal sum $l_0 \oplus A_0$ can be developed based on von Neumann's formula. We follow [32]. The "formal adjoint" operator for $A_{_0}$ is initially defined on the defect $N_{_i} + N_{_{-i}} := \mathcal{N}$ by $A_{_0}^{^+} e \pm i\, e = 0$ for $e \in N_{\pm i}$, which makes sense as N_i, N_{-i} do not overlap. When the formal adjoint on the defect is constructed, we restrict it to a certain plane in the defect where the boundary form vanishes. These Lagrangian planes are parametrized by isometries $V : N_i \to N_{-i}$ of the form $\mathcal{T}_{_V} = (I - V)\, N_{_i}$. When the deficiency subspaces do not overlap, the corresponding isometry is admissible, and from [16] there is a self-adjoint extension A_V of the restricted operator A_0.

Lemma 2. *The lagrangian plane $\mathcal{T}_V$ in the defect forms a non-zero angle with the domain D_0^A of the restricted operator A_0.*

It follows that once the extension is constructed on the Lagrangian plane the extended operator can be found in the form of a direct sum of the closure of the restricted operator and the extended operator on the Lagrangian plane.

The extension of the dissipative operator on the defect $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_{-1}$ can be constructed by a symplectic procedure. Indeed if we choose an ortho-normal basis in N_i : $\{f_{_s}\}$, $s = 1, 2, \ldots, d$, as a set of deficiency vectors of the restricted operator A_0. The vectors $\hat{f}_s = \frac{A+iI}{A-iI} f_s$ form an orthonormal basis in the dual deficiency subspace N_{-i}. Under the non-overlapping condition one can use the formal adjoint operator A_0^+ defined on the defect $N_i + N_{-i} = \mathcal{N}$, [2, 1]

$$u \;=\; \sum_{s=1}^{d} [x_s\, f_s + \hat{x}_s\, \hat{f}_s] \in \mathcal{N}, \tag{18}$$

$$A_0^+ u \;=\; \sum_{s=1}^{d} [-i\; x_s\, f_s + i\; \hat{x}_s\, \hat{f}_s]. \tag{19}$$

To use the symplectic version of the operator-extension techniques we introduce in the defect $\mathcal{N}$ a new basis $W_s^{\pm}$, on which the formal adjoint A_0^+ is correctly defined

$$W_s^+ = \frac{f_s + \hat{f}_s}{2} = \frac{A}{A - iI} f_s,\; W_s^- = \frac{f_s - \hat{f}_s}{2i} = -\frac{I}{A - iI} f_s,$$

$A_0^+ W_s^+ = W_s^-$, $A_0^+ W_s^- = -W_s^+$.. Then represent elements $u \in \mathcal{N}$ via the new basis as

$$u = \sum_{s=1}^{d} [\xi_s^{^+}\, W_s^+ + \xi_s^{^-}\, W_s^-]. \tag{20}$$

Then, with $\sum_{s=1}^{d} \xi_{s,\pm}\, e_{_s} := \vec{\xi}_{\pm}$ we re-write the above as

$$\begin{aligned} u &= \frac{A}{A - iI} \vec{\xi}_+^{\,u} - \frac{1}{A - iI} \vec{\xi}_-^{\,u}, \\ A_{_0}^{^+} u &= -\frac{1}{A - iI} \vec{\xi}_+^{\,u} - \frac{A}{A - iI} \vec{\xi}_-^{\,u} \end{aligned} \tag{21}$$

The formula of integration by parts for abstract operators found in [32]

Lemma 3. *Consider the elements u, v from the domain of the (formal) adjoint operator A_0^+:*

$$u = \frac{A}{A - iI} \vec{\xi}_+^{\,u} - \frac{1}{A - iI} \vec{\xi}_-^{\,u},\; v = \frac{A}{A - iI} \vec{\xi}_+^{\,v} - \frac{1}{A - iI} \vec{\xi}_-^{\,v}$$

with coordinates $\vec{\xi}_{\pm}^{u}$, $\vec{\xi}_{\pm}^{v}$:

$$\vec{\xi}_{\pm}^{\,u} = \sum_{s=1}^{d} \xi_{_{s,\pm}}^{^u} f_{_{s,i}} \in N_{_i},\; \vec{\xi}_{\pm}^{\,v} = \sum_{s=1}^{d} \xi_{_{s,\pm}}^{^v} f_{_s} \in N_{_i}.$$

Then, the boundary form of the formal adjoint operator is equal to

$$\mathcal{J}_{_A}(u,v) = \langle A_0^+ u, v\rangle - \langle u, A_0^+ v\rangle = \langle \vec{\xi}_+^{\,u}, \vec{\xi}_-^{\,v}\rangle_{_N} - \langle \vec{\xi}_-^{\,u}, \vec{\xi}_+^{\,v}\rangle_{_N}.$$

The symplectic coordinates $\vec{\xi}_{\pm}^{u}, \vec{\xi}_{\pm}^{v}$ of the elements u, v play the role of the boundary values

$\{U'(0), U(0), V'(0), V(0)\}$ for the 1D Schrödinger equation. The boundary form vanishes on the Lagrangian plane $\mathcal{T}_\alpha$ defined in $D(A_0^+)$ defined by the "boundary condition" with an Hermitian operator $\mathbf{B}_\alpha : N_i \to N_i$: $\vec{\xi}_+ = \mathbf{B}_\alpha \vec{\xi}_-$. This boundary condition defines a self-adjoint operator A_α as a restriction of A_0^+ onto the Lagrangian plane $\mathcal{T}_\alpha \in D(A_0^+)$. The resolvent of A_α so defined is represented at regular points of A_α by the following Krein formula, see [2],

$$(A_\alpha - \lambda I)^{-1} = \frac{I}{A-\lambda I} - \frac{A+iI}{A-\lambda I} P\Gamma_\alpha \frac{I}{I + P\frac{I+\lambda A}{A-\lambda I} P\Gamma} P \frac{A-iI}{A-\lambda I}. \quad (22)$$

Next, from [32], for generalized resolvents of symmetric operators.

Lemma 4. *The vector-valued function of the spectral parameter*

$$u(\lambda) = \frac{A+iI}{A-\lambda I}\ \vec{\xi}^{\,u}_+ := u_0 + \frac{A}{A-iI}\vec{\xi}^{\,u}_+ - \frac{1}{A-iI}\vec{\xi}^{\,u}_-, \quad (23)$$

satisfies the adjoint equation $[A_0^+ - \lambda I]u = 0$, and the symplectic coordinates $\vec{\xi}^{\,u}_\pm \in N_i$ of it are connected by the formula

$$\vec{\xi}^{\,u}_- = -P_{N_i}\frac{I+\lambda A}{A-\lambda}\vec{\xi}^{\,u}_+ \quad (24)$$

Introduce the map

$$-P_{N_i}\frac{I+\lambda A}{A-\lambda I}P_{N_i} =: \mathcal{M}(\lambda) : N_i \to N_i.$$

The matrix-function $\mathcal{M}(\lambda) = P_{N_i} A P_{N_i} - P_{N_i}\frac{I+A^2}{A-\lambda I}P_{N_i}$ has negative imaginary part in the upper half-plane $\Im\lambda > 0$ and serves an analog of the celebrated Weyl-Titchmarsh function. The operator-function $\mathcal{M}$ exists on the real axis λ, on the complement of the spectrum of the restriction of the operator A onto the invariant subspace generated by the defect. In particular it has simple poles at the eigenvalues a_r^2 of A. Then (22) gives an explicit equation for the perturbed resolvent of the extension, so the resolvent of the finite-dimensional perturbation $\hat{\mathcal{L}}_\alpha$ of the dilation $\hat{\mathcal{L}}$ of $\mathcal{L}$. In [44] the parameters of the extension were interpreted as Saint-Venant parameters. The finite-dimensional formula (22) allows us, in principle, to calculate the evolution operator $e^{i\hat{\mathcal{L}}_\alpha t}$, and the restriction of it onto co-invariant subspace $\mathcal{K}$ – the corresponding Lax-Phillips semigroup,

$$P_{\mathcal{K}} e^{i\hat{\mathcal{L}}_\alpha t}\big|_{\mathcal{K}} \equiv e^{i\mathcal{L}_\alpha t} \quad (25)$$

generated by the finite-dimensional dissipative perturbation $\mathcal{L}_\alpha$ of $\mathcal{L}$ of the Lax-Phillips semigroup.

4. Computation

The dynamical picture exposed above is united by the common theme of the spectral nature of SGOs. One of most important questions in mathematical modeling is on the nature of the dissipation of elastic energy stored on the tectonic plates due to the fluid underlay. We avoid this question by introducing the positive damping parameter β and so the basic physical question on the nature of the damping remains unsolved. Next we suggest a program of study of SGO- related phenomena based on synchronous observations of frequencies, decay decrements and the shape of amplitudes of SGOs.

4.1. Computing for the Weyl function.

To apply the results we have determined mathematically we need to compute the Neumann-to-Dirichlet map or the corresponding Weyl function. We consider the relative ND-map for the 3D Laplacian with hydrodynamic potential ϕ on the astenosphere and with given inhomogeneous boundary conditions on the tectonic plate Ω situated on the surface of the Earth Σ, the homogeneous Neumann boundary condition on the complement $\Sigma\backslash\Omega$, Robin boundary conditions in the depth on the mutual boundaries and perhaps the homogeneous Neumann boundary condition on the inner core. The relative Weyl function M is defined for the bi-harmonic equation with the spectral parameter (26) as a map connecting the boundary data $\Xi^u_\pm$ on $\partial\Omega \equiv \gamma$ of the solution u of the adjoint bi-harmonic boundary problem with Ξ^u_+ fixed on γ and the spectral parameter λ in the weighted space of the square-integrable functions:

$$\Delta^2 u + u = \lambda \mathcal{D} u. \quad (26)$$

Selecting an orthogonal basis $\psi_{\vec{l}}$, $\vec{l} = (l_1, l_2)$, $l_s = 1, 2, \dots$ in the space of the two-component vectors Ξ_-, and solving the corresponding boundary problem

$$\Delta^2\Psi_{\vec{l}} + \Psi_{\vec{l}} = \lambda\mathcal{D}\Psi_{\vec{l}},\ \Xi_+^{\Psi_{\vec{l}}}\big|_\gamma = \psi_{\vec{l}}, \quad (27)$$

we extend the bases from the boundary onto the plate Ω and find the matrix elements $M_{\vec{l}\vec{m}}$ of the Weyl function from the bilinear form:

$$\int_\Omega \left[\Delta\bar{\Psi}_{\vec{l}}\,\Delta\Psi_{\vec{m}} + \bar{\Psi}_{\vec{l}}\,\Psi_{\vec{m}} - \lambda\bar{\Psi}_{\vec{l}}\Psi_{\vec{m}}\right] d\Omega + \int_\gamma \langle \Xi_+^{\bar{\Psi}_{\vec{l}}}\,\Xi_-^{\bar{\Psi}_{\vec{m}}}\rangle d\gamma \qquad = 0.$$

If λ is not an eigenvalue of the spectral problem for the bi-harmonic equation $\Delta^2 u + u = \mathcal{D}\,\lambda u$ with the boundary condition $\Xi^u_+\big|_\gamma = 0$, then the corresponding solutions $\Psi_{\vec{l}}$ are found uniquely and the Weyl function is defined. One can construct a rational approximation to the Weyl function near the regular point and near the pole of M at an eigenvalue of the above spectral problem.

4.2. Rational approximation

Rational approximation of the Weyl function serves a basis for approximate calculation of the spectral properties of the dissipative bi-harmonic operator and the relevant Lax-Phillips generator with weak dissipative perturbations $\mathcal{B}, \beta$

near a fixed spectral point. For the dissipative biharmonic equation

$$\Delta^2 u + u = \lambda \mathcal{D} u, \quad \Xi_+\big|_\gamma = \mathcal{B}\Xi_-\big|_\gamma \tag{28}$$

the eigenvalues are found from the homogeneous equation $[M - \mathcal{B}]\,\Xi_-\big|_\gamma = 0$. This rational approximation of $M \to M^N$ reduces the equation to a finite-dimensional one and we calculate the shift of the eigenvalues of the spectral problem while the boundary condition $\Xi_+^u\big|_\gamma = 0$ becomes $\Xi_+^u\big|_\gamma = \mathcal{B}\Xi_-^u\big|_\gamma$, with $\mathcal{B} \approx 0$. The rational approximation can be chosen in various ways, in particular so that poles of the rational approximation of the corresponding scattering matrix coincide with the poles of the original scattering matrix, see for instance [22].

4.3. Zero-range perturbation.

The explicit finite-dimensional (not more than 6) perturbation of the dissipative generator of the wave evolution at (22), does not give an explicit form for the quantities which may be observed. This makes the process of fitting the solvable model difficult. Indeed, both terms on the right-hand side of the Krein formula contain the resolvent of the "unperturbed" operator, which has singularities on the spectrum of the corresponding problem. We expect that these singularities compensate each other so only the singularities in the spectrum of the perturbed operator $\mathcal{L}_\alpha$ remain. The question of compensation singularities in the Krein formula was considered in [23] [35, 22]. It is possible that these ideas give an explicit formula for the polar term of the resolvent of $\mathcal{L}_\alpha$ near it's eigenvalue. This gives a finite-dimensional expression for the perturbed eigenvalues (eigenfrequencies and the corresponding damping decrements) and eigenfunctions, which can be monitored as SGO -modes. This would help fit the solvable model and then estimate, based on the fitted model, the amount of elastic energy, stored on the modes which may give an alternative approach to the important problem of prediction of earthquakes and tsunami, and the estimation of their expected power.

5. Observations and conclusions.

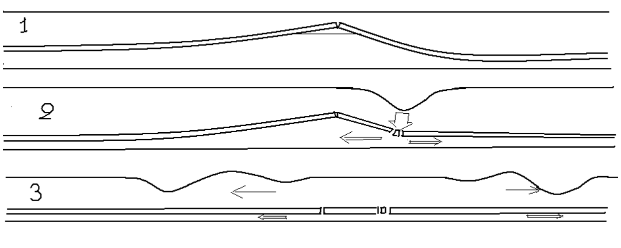

Figure 5: Tsunami collision senario

The basic observation of instability of frequencies of SGO motivates numerous questions concerning both the reasons for the instability and the influence of these instabilities upon various processes in the troposphere.

These questions and the associated hypotheses around the models deserve careful analysis.

The variation of the frequencies of SGO may be considered not only as a precursor of an earthquake, but also as a precursor of a tsunami caused by the collisions of tectonic plates in active zones. However deformation of the plates in an underwater active zone may be greater than usual because the oceanic plates are relatively thin leading to the possible formation of a decompression area under the active zone. The specific character of the destruction of these stressed plates may allow formation of deep splits and cracks through which oceanic water enters the decompressed area and mixes with the liquid component of the astenosphere. The amount of water entering the under-plates area in active zone may be quite significant - perhaps up to 10 cubic kilometers, as for instance during the recent Christmas tsunami in Malasia, 2009. This hot mixture may spread along an under-plate channel for large distances, thus transferring the energy and potentially triggering new geological events, see [14].

Therefore monitoring of the SGO frequencies and the damping parameters and use of a fitted model for the active zone with correctly chosen Saint-Venant parameters should allow us to calculate, in an explicit form, the deformation of the eigenmodes of the stressed plate. Then, comparing the results of monitoring frequencies, amplitudes and the decay parameters of SGO with the corresponding data recovered from the *already fitted* mathematical model of the tectonic plate under the dissipative boundary conditions and the point-wise stress at the active zones, it might be possible in practice to define the active zone from which the perturbation is coming. This leaves the problem of estimating, based on the fitted solvable model, the elastic energy, accumulated due to stress. The technique of using solvable models for localized excitations of tectonic plates allows us to estimate the elastic energy stored on the corresponding SGO. In particular, returning to our original problem we expect that the rational approximation of the characteristic function of the dissipative operator $\mathcal{L}$ and its local zero-range perturbation $\mathcal{L}_\mathcal{B}$ at the active zone may allow us to solve the inverse problem and recover, in explicit form, the dependence of the frequencies of SGO and decay decrements from the parameters $\mathcal{B}$ of of the zero-range model. We expect that under the special conditions on the parameters of the dissipative operator β, Γ_γ and its singular perturbation, some of perturbed eigenvalues may endure anomalously small decay decrements. In this case the corresponding SGO modes are characterized by extremely slow dissipation of the elastic energy stored in them and allow observation of a corresponding pumping effect on the eigenmode, while the corresponding eigenvalue is near a stable eigenvalue for the tectonic plate.

We hope that the necessary data for the relevant analysis could therefore be collected by the appropriate monitoring of SGO.

Appendix: Scattering and the spectral analysis of the dissipative operator.

Here we omit the lower index $\mathcal{D}$ for $\mathcal{L}_{\mathcal{D}}$ and $\hat{\mathcal{L}}_{\mathcal{D}}$. We consider the simplified version, of Theorem 1, with $\Gamma : \mathcal{K} \to \mathcal{E}_\gamma \oplus \mathcal{E}_\Omega$. Denote by $\mathcal{A}_{sa}$ the self-adjoint part of the original dissipative operator $\mathcal{L}$.

Theorem 2. *The matrix operator defined in*

$$\mathcal{H} = \mathcal{D}_{out} \oplus \mathcal{K} \oplus \mathcal{D}_{in} \tag{29}$$

by the formula

$$\hat{\mathcal{L}} \begin{pmatrix} u_+ \\ u \\ u_- \end{pmatrix} = \begin{pmatrix} i\frac{du_+}{ds} \\ \mathcal{A}_{sa} u + \Gamma^+ (u_+ + u_-)/2 \\ i\frac{du_-}{ds} \end{pmatrix} \tag{30}$$

with the corresponding boundary condition $\frac{u_+ - u_-}{i} = \Gamma u$ *is a selfadjoint dilation of the dissipative operator* $\mathcal{L}$*: the compression of the resolvent* $\mathcal{R}_\lambda$, $\Im\lambda < 0$ *of the operator* $\hat{\mathcal{L}}$ *onto the subspace* $\mathcal{K}$ *coincides with the resolvent of the operator* L*:*

$$P_{_K} [\mathcal{L} - pI]^{-1} P_{_K} = (L - pI)^{^{-1}}, \ \Im p < 0. \tag{31}$$

The spectrum of the operator $\hat{\mathcal{L}}$ is absolutely continuous and consists of two branches $\hat{\sigma}_{\mathcal{L}} = R \cup \sigma_c$. The *incoming* eigenfunctions of the first branch are scattered waves

$$\Psi^{in}_\nu = \begin{pmatrix} e^{-ips} S^+ \nu \\ u^{in}_\nu \\ e^{-ips} \nu, \end{pmatrix} \tag{32}$$

with the *transmission coefficient* $S^+ = \frac{\mathcal{L}^+ - pI}{\mathcal{L} - pI}$ found as the limit on the real axis of the spectral parameter p from the lower half-plane. The term u^{in}_ν is a limit from the lower half-plane of the solution of

$$[\mathcal{A}_{sa} - Ip]\, u^{in}_\nu + \Gamma^+ (S^+ + I)/2\nu = 0. \tag{33}$$

These form a basis in the invariant subspace $\mathcal{H}_{in}$ of the dilation obtained as a closure $\bigvee_{t>0} e^{i\hat{\mathcal{L}}t} \mathcal{D}_{in}$. Similarly the outgoing scattered waves are defined.

Theorem 3. *The incoming and outgoing eigen-functions of the dilation* $\mathcal{L}$ *are generalized solutions of the corresponding homogeneous equation with exponential behavior in* $L_2(R_\pm, E)$ *:*

$$\Psi^{in}_\nu = \begin{cases} e^{-ikx}\nu & \nu \in \mathcal{E} \quad x \in R_- \\ u^{in}_\nu & in \quad \mathcal{K} \\ e^{-ikx}\mathbf{S}^+\nu & \nu \in \mathcal{E} \quad x \in R_+ \end{cases}$$

$$\Psi^{out}_\nu = \begin{cases} e^{-ikx}\nu & \nu \in \mathcal{E} \quad x \in R_+ \\ u^{out}_\nu & in \quad \mathcal{K} \\ e^{-ikx}\mathbf{S}\nu & \nu \in \mathcal{E} \quad x \in R_- \end{cases}$$

These eigenfunctions are labeled by direction vectors $\nu \in E$. *The components* $u^{in,out}$ *are generalized solutions of the inhomogeneous equation (33) with complex spectral parameter* $p = k \mp i0$ *and are uniquely defined by the direction vectors, see (3) below, as images of strong limits of properly framed resolvent of the self-adjoint operator* A_{sa} *or strong limits of the resolvent of* L, L^+ *on the real axis from the lower (upper) half-planes. The transmission coefficients* $\mathbf{S}$, $\mathbf{S}^+$ *are also uniquely defined from the homogeneous equation. In particular,* $\mathbf{S}$, $\mathbf{S}^+$ *are analytic matrix-function in upper and lower half-planes* $\Im p > 0$, $\Im p < 0$

$$\mathbf{S}^+(k - i0) = I - i \lim_{p \to k - i0} \Gamma \frac{I}{\mathcal{L} - pI} \Gamma^+$$

$$\mathbf{S}(k + i0) = I + i \lim_{p \to k + i0} \Gamma \frac{I}{\mathcal{L}^+ - \lambda I} \Gamma^+. \tag{34}$$

$$\begin{aligned} u^{in}_\nu &= -\frac{1}{2} \frac{1}{A_{sa} - (k - i0)} \left(I + \mathbf{S}^+(k - i0)\right) \nu \\ &= \left[\nu + \frac{i\Gamma}{2} \frac{I}{\mathcal{L}^+ - (k + i0)} \Gamma^+ \nu\right], \ \nu \in E. \\ u^{out}_\nu &= -\frac{1}{2} \frac{I}{A_{sa} - (k + i0)} \left(I + \mathbf{S}(k + i0)\right) \nu \\ &= \left[\nu - \frac{i\Gamma}{2} \frac{I}{\mathcal{L} - (k - i0)} \Gamma^+ \nu\right], \ \nu \in E.. \end{aligned} \tag{35}$$

The eigenfunctions $\psi^>$, $\psi^<$ *of the complementary components of the dilation in* $\mathcal{E} \ominus \mathcal{E}_- = \mathcal{E}^<$ *and* $\mathcal{E} \ominus \mathcal{E}_+ = \mathcal{E}^>$ *have the form:*

$$\Psi^< = \begin{pmatrix} 0 \\ u^< \\ e^{-ikx}\nu^< \end{pmatrix}, \ \Psi^> = \begin{pmatrix} e^{-ikx}\nu^> \\ u^> \\ 0 \end{pmatrix}, \tag{36}$$

and are obtained as normalized linear combinations of the incoming and outgoing waves. Choosing vectors $\nu^>$, $\nu^<$ *as eigenvectors of operators* $\Delta^> = I - \mathbf{S}^+\mathbf{S}$, $\Delta^< = I - \mathbf{S}\mathbf{S}^+$ *with non-zero eigenvalues* $\delta^>$, $\delta^<$ *respectively, we obtain:*

$$u^>_{\nu^>} = \frac{1}{\delta^>} \left[\Psi^{in}_{\nu^>} - \mathbf{S}^+\nu^> \Psi^{out}_{\nu^>}\right],$$

$$u^<_{\nu^<} = \frac{1}{\delta^<} \left[\Psi^{out}(\nu^<) - \mathbf{S}\nu^< \Psi^{in}\right].$$

The systems $\left\{\Psi^{in}_\nu, \Psi^<_{\nu^<}\right\}$ *and* $\left\{\Psi^{out}_\nu, \Psi^>_{\nu^>}\right\}$ *are orthogonal, with spectral density matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & \Delta_> \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & \Delta_< \end{pmatrix}.$$

The system $\Psi^{in,out}_\nu$ *is not orthogonal, but with respect to the data* $(f^{in}, f^{out}) \equiv \left(\langle \Psi^{in}, f\rangle, \langle \Psi^{out}, f\rangle\right)$ *has the spectral density matrix*

$$\begin{pmatrix} I_{in,in} & S \\ S^+ & I_{out,out} \end{pmatrix}$$

with blocks

$$\langle \Psi^{\in}_\nu, \Psi^{in}_\mu \rangle = \delta(\nu - \mu), \ \equiv I_{in,in} \langle \Psi^{\in}_\nu,$$

$$\Psi^{out}_\mu \rangle = S_{in,out}(\nu, \mu) = \delta(\nu - \mu) S, \ \langle \Psi^{out}_\mu, \Psi^{in}_\nu \rangle = \delta(\nu - \mu) S^+.$$

Then a functional model of the dissipative operator is obtained, see [24] by rewriting the dissipative operator in terms of this system of eigenfunctions $\{\Psi^{out}, \Psi^{>}\}$ of the dilation. This system $\{\Psi^{in}, \Psi^{out}\}$ gives the most simple formulae for the spectral quantities, see [28, 26, 31] in terms of the so called "symmetric functional model".

The functional model allows us to calculate angles between invariant subspaces of the absolutely continuous, discrete and singular spectrum of the dissipative operator in terms of its characteristic function (playing the role of the relevant scattering matrix). In particular the eigenfunctions of the absolutely continuous "complementary component" of the absolutely continuous part of the dissipative operator are identified up to the parametrization by the direction vectors. The mid-components $u^{<}$, $u^{>}$ $\mathcal{E}^{<}$, $\mathcal{E}^{>}$ serve as a canonical system of eigenfunctions for the absolutely continuous spectrum of the original dissipative operator and adjoint operator.

We have the corresponding spectral expansion, see [28, 31]:

$$u = \frac{1}{2\pi} \int_{\sigma_a} \frac{|\mathbf{S}(k)|^2 - 1}{\mathbf{S}^{+}(k)} u^{<}(k) \langle u, u^{>}(k) \rangle dk, \tag{37}$$

converging for elements u represented as orthogonal projections of elements of the complementary subspace $\mathcal{E}^{<}$ onto K. This set is dense in the absolutely-continuous subspace of the operator L, [24] and [30, 47]. Thus the incoming-outgoing eigenfunctions of the dilation and eigenfunctions in the complementary subspaces $\mathcal{E}^{<}$, $\mathcal{E}^{>}$ play essentially different roles in the spectral problem for the dissipative operator.

For discussion of choice of the canonical system of eigenfunctions of the absolutely-continuous spectrum in case of spectral multiplicity one for a unitary operator and a canonical system of eigenfunction of it's contracting perturbation. see [36].

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