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Rossman, Wayne

Yasumoto, Masashi

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Weierstrass representation for semi-discrete minimal surfaces, and comparison of various discretized catenoids

Wayne Rossman and Masashi Yasumoto

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Abstract. We give a Weierstrass type representation for semi-discrete minimal surfaces in Euclidean 3-space. We then give explicit parametrizations of various smooth, semi-discrete and fully-discrete catenoids, determined from either variational or integrable systems principles. Finally, we state the shared properties that those various catenoids have.

Keywords. discrete differential geometry, isothermic surfaces, discrete surfaces, semi-discrete surfaces, minimal surfaces, catenoids, Weierstrass representation

1. Introduction

The well known minimal surface of revolution in $\mathbb{R}^3 = \{(x_1, x_2, x_3)^t \mid x_j \in \mathbb{R}\}$ called the catenoid, which we refer to as the *smooth catenoid* here and which can be parametrized by

$$x(u,v) = \begin{pmatrix} \cosh u \cos v \\ \cosh u \sin v \\ u \end{pmatrix}, \; v \in [0, 2\pi), \; u \in \mathbb{R}, \qquad (1)$$

has a number of discretized versions. A fully discretized version can be found in [11] by Polthier and the first author, which is defined using a variational approach, that is, those surfaces are triangulated meshes that are critical for area with respect to smooth variations of the vertex set. A different approach for defining fully discrete catenoids, using quadrilateral faces and based on integrable systems methods, was found by Bobenko and Pinkall [1, 2]. Both approaches apply to much wider classes of surfaces.

One can also consider semi-discrete catenoids, that is, catenoids that are discretized in only one of the two parameter directions corresponding to u and v in (1). There are now four choices for how to proceed with this, by choosing either the u direction or v direction to discretize, and by choosing to use either variational principles or integrable systems principles to determine the discretizations. Again, these approaches apply to much wider classes of surfaces.

Here we compare these various smooth, semi-discrete and fully-discrete catenoids to see in what ways they do or do not coincide. For the smooth and fully-discrete catenoids, the parametrizations have already been determined, making comparisons between them elementary. However, for some of the semi-discrete cases, we will need to first establish those parametrizations here. In particular, we will provide a Weierstrass representation for determining semi-discrete minimal surfaces as defined by Mueller and Wallner [9, 13]. Construction of the semi-discrete catenoids in particular, via an integrable systems approach, can be done either with this Weierstrass representation, or without it (instead using the results by Mueller and Wallner). However, the usefulness of the Weierstrass representation comes when one wishes to consider the full class of semi-discrete minimal surfaces based on an integrable systems approach, as this representation gives a classification of such surfaces in terms of semi-discrete holomorphic functions. This Weierstrass representation can be regarded as a restatement of the definition of such surfaces (Definition 4), but in a more explicit form that tells us how the surface is constructed from the given dual surface inscribed in a sphere.

Once we have established this representation for semi-discrete minimal surfaces (Theorem 2), we compare the various types of catenoids (Theorem 1).

To make semi-discrete catenoids based on variational principles, Machigashira [7] chose to discretize them in the u direction. He then classified these surfaces and studied their stability properties. The surfaces obtained by Machigashira will be seen (Proposition 2) to be limiting cases of the discrete catenoids found in [11].

From the point of view of architectural structures in the shape of a semi-discrete catenoid, Machigashira's catenoids would involve producing circular-shaped flat pieces that cannot be so efficiently made as cut-outs from planar sheets, since there would be a large amount of waste material. So from the architectural point of view, a more efficient use of materials would be to discretize in the v direction instead. Such semi-discrete catenoids are considered here as well.

To distinguish between various catenoids, we write the superscript va (resp. in) when the catenoid is constructed by a variational (resp. integrable systems) approach, and write the subscript pd (resp. ps) when the catenoid has a discrete profile curve (resp. smooth profile curve) and the

	associated authors
smooth catenoid	(classically known surface)
$BP^{in}_{pd,rd}$-catenoid	Bobenko and Pinkall
$PR^{va}_{pd,rd}$-catenoid	Polthier and Rossman
$M^{va}_{pd,rs}$-catenoid	Machigashira
$MW^{in}_{pd,rs}$-catenoid	Mueller and Wallner
$MW^{in}_{ps,rd}$-catenoid	Mueller and Wallner
$M^{va}_{ps,rd}$-catenoid	(Machigashira analogue)

Table 1: Names of seven types of catenoids

subscript rd (resp. rs) when the catenoid is discrete (resp. smooth) in the rotational direction. Thus, in total, we consider the seven types of catenoids in Table 1.

For catenoids with discrete profile curves, we will assume them to have a "neck vertex". In other words, we assume there exists a plane of reflective symmetry of the catenoids that is perpendicular to the axis of rotation symmetry and also contains one vertex of each profile curve. We note that there do exist discrete catenoids that do not have this neck-vertex symmetry.

Theorem 1. *After appropriate normalizations, we have the following:*

1. *$PR^{va}_{pd,rd}$-catenoid profile curves and $M^{va}_{pd,rs}$-catenoid profile curves are never the same, but $PR^{va}_{pd,rd}$-catenoid profile curves converge to $M^{va}_{pd,rs}$-catenoid profile curves as the angle of rotation symmetry approaches* 0.

2. *$BP^{in}_{pd,rd}$-catenoids and $MW^{in}_{pd,rs}$-catenoids have the same profile curves.*

3. *$BP^{in}_{pd,rd}$-catenoid ($MW^{in}_{pd,rs}$-catenoid) profile curves and $PR^{va}_{pd,rd}$-catenoid profile curves are never the same, and $BP^{in}_{pd,rd}$-catenoid ($MW^{in}_{pd,rs}$-catenoid) profile curves and $M^{va}_{pd,rs}$-catenoid profile curves are never the same.*

4. *The smooth catenoid and $MW^{in}_{ps,rd}$-catenoid have the same profile curve.*

5. *$M^{va}_{ps,rd}$-catenoid profile curves and the smooth catenoid's profile curve are never the same. $M^{va}_{ps,rd}$-catenoid profile curves converge to the smooth catenoid ($MW^{in}_{ps,rd}$-catenoid) profile curve as the angle of rotation symmetry approaches* 0.

6. *For all types of catenoids, the profile curves have vertices lying on affinely scaled graphs of the hyperbolic cosine function.*

2. Notation for semi-discrete surfaces

To consider semi-discrete minimal surfaces from an integrable systems approach, we set some notations in this section.

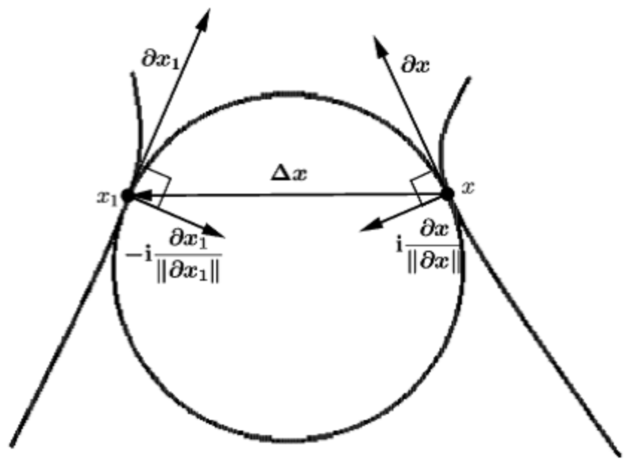

Figure 1: Sketch of the proof of Remark 2

Let $x = x(k,t)$ be a map from a domain in $\mathbb{Z}\times\mathbb{R}$ to $\mathbb{R}^3$ ($k\in\mathbb{Z}$, $t\in\mathbb{R}$). We call x a *semi-discrete surface.* Set

$$\partial x = \frac{\partial x}{\partial t},\ \Delta x = x_1 - x,\ \partial\Delta x = \partial x_1 - \partial x,$$

where $x_1 = x(k+1,t)$. The following definitions can be found in [9], and are all naturally motivated by geometric properties found in previous works, such as [1, 2, 3, 4, 5, 6, 8, 9, 13].

Definition 1. Let x be a semi-discrete surface.

- x is a *semi-discrete conjugate net* if ∂x, Δx and $\partial\Delta x$ are linearly dependent.
- x is a *semi-discrete circular net* if there exists a circle $\mathscr{C}$ passing through x and x_1 that is tangent to ∂x, ∂x_1 there (for all k, t).

Remark 1. If x lies in $\mathbb{R}^2\cong\mathbb{C}$, circularity is equivalent to the following condition: there exists a non-zero-valued function s such that

$$\Delta x = \mathrm{i}s\left(\frac{\partial x}{\|\partial x\|}+\frac{\partial x_1}{\|\partial x_1\|}\right), \quad (2)$$

which follows from

$$\Delta x = \mathrm{i}s\frac{\partial x}{\|\partial x\|}-\left(-\mathrm{i}s\frac{\partial x_1}{\|\partial x_1\|}\right)$$

for some $s\in\mathbb{R}$, in the setting shown in Figure 1.

Definition 2. Suppose x, x^* are semi-discrete conjugate surfaces. Then x and x^* are *dual* to each other if there exists a function $\nu\colon\mathbb{Z}\times\mathbb{R}\to\mathbb{R}^+$ so that

$$\partial x^* = -\frac{1}{\nu^2}\partial x,\ \Delta x^* = \frac{1}{\nu\nu_1}\Delta x.$$

Definition 3. A semi-discrete circular surface x is *isothermic* if there exist positive functions $\nu,\ \sigma,\ \tau$ such that

$$\|\Delta x\|^2 = \sigma\nu\nu_1,\ \|\partial x\|^2 = \tau\nu^2,\ \text{with } \partial\sigma = \Delta\tau = 0.$$

Remark 2. For circular semi-discrete surfaces, dualizability and isothermicity are equivalent, by Theorem 11 in [9]. In particular, the ν, ν_1 in Definitions 2 and 3 are the same.

Definition 4. A semi-discrete isothermic surface x is *minimal* if its dual x^* is inscribed in a sphere.

3. Semi-discrete catenoids with discrete profile curve

Take the following parametrization for $MW^{in}_{pd,rs}$-catenoids:

$$x(k,t) = \begin{pmatrix} f(k)\cos t \\ f(k)\sin t \\ h\cdot k \end{pmatrix}$$

where $f = f(k)$ and h are positive. Then, with $f_1 = f(k+1)$,

$$\|\Delta x\|^2 = (f_1 - f)^2 + h^2,$$
$$\|\partial x\|^2 = f^2.$$

One can check that x is isothermic by taking

$$\nu = f, \quad \tau = 1 \quad \text{and} \quad \sigma = \frac{(\Delta f)^2 + h^2}{f \cdot f_1}.$$

We compute x^* by solving

$$x^* = -\frac{1}{\nu^2}\int \partial x dt = -\frac{1}{f}\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + \vec{c}_k,$$

where $\vec{c}_k$ depends on k but not t. We now have

$$\Delta x^* = \frac{1}{f\cdot f_1}\begin{pmatrix} \Delta f \cdot \text{cos}t \\ \Delta f \cdot \text{sin}t \\ 0 \end{pmatrix} + \vec{c}_{k+1} - \vec{c}_k.$$

Therefore, x^* is dual to x if

$$\vec{c}_{k+1} - \vec{c}_k = \frac{h}{f\cdot f_1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Without loss of generality, we can take $\vec{c}_k$ as $(0,0,c(k))^t$ with

$$c(k+1) = c(k) + \frac{h}{f(k)f(k+1)}. \tag{3}$$

For x to be minimal, we wish to have

$$\|x^*\| \equiv \text{constant}$$

for some choice of $c(0)$. Substituting Equation (3) into the equation $\|x_1^*\| = \|x^*\|$, we obtain

$$f(k+1) = hc(k)f(k) + \sqrt{(hc(k)f(k))^2 + f(k)^2 + h^2}. \tag{4}$$

Then we can recursively solve the system of difference equations (3)-(4).

In order to compare the other catenoids with $MW^{in}_{pd,rs}$-catenoids, we wish to reduce the above system of difference equations to one equation.

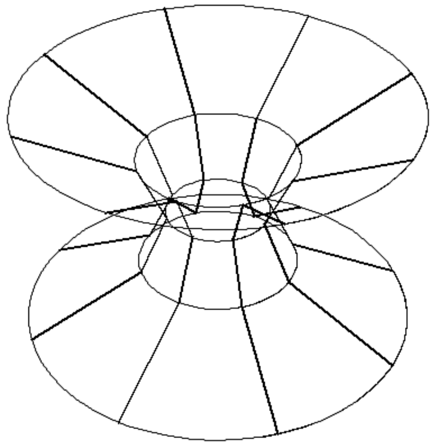

Figure 2: a semi-discrete $MW^{in}_{pd,rs}$-catenoid with discretized profile curve

Lemma 1. *We have*

$$f(k+2) = \frac{f(k+1)^2 + h^2}{f(k)}. \tag{5}$$

Proof. By Equation (4),

$$c(k) = \frac{f(k+1)^2 - f(k)^2 - h^2}{2hf(k)f(k+1)}. \tag{6}$$

Inserting (6) into (3), we have

$$(f(k+2) + f(k))(f(k+2)f(k) - f(k+1)^2 - h^2) = 0,$$

which implies (5). □

Lemma 1 implies

$$f(1)f(-1) = f(0)^2 + h^2,$$

and then neck-vertex symmetry (in particular, $f(1) = f(-1)$) gives

$$f(1)^2 = f(0)^2 + h^2.$$

Then Equation (6) implies $c(0) = 0$. Without loss of generality, we can take $f(0) = 1$, and then the solution to Equation (5) is

$$f(k) = \cosh(\text{arcsinh}(h)\cdot k).$$

4. Semi-discrete catenoids foliated by discrete circles

Take the following parametrization for $MW^{in}_{ps,rd}$-catenoids:

$$x(k,t) = \begin{pmatrix} f(t)\cos\alpha k \\ f(t)\sin\alpha k \\ t \end{pmatrix},$$

where $f(t)$ and α are positive. We assume

$$f(0) = 1 \text{ and } f'(0) = 0. \tag{7}$$

Then

$$\|\Delta x\|^2 = 4f(t)^2 \sin^2\frac{\alpha}{2},$$
$$\|\partial x\|^2 = (f'(t))^2 + 1.$$

One can confirm that x is isothermic by taking

$$\nu = f(t), \quad \tau = \frac{(f'(t))^2+1}{(f(t))^2} \quad \text{and} \quad \sigma = 4\sin^2\frac{\alpha}{2}.$$

Now,

$$x^* = -\int \frac{1}{\nu^2}\partial x dt = \begin{pmatrix} -\cos\alpha k \int \frac{f'}{f^2}dt \\ -\sin\alpha k \int \frac{f'}{f^2}dt \\ -\int \frac{1}{f^2}dt \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\cos\alpha k}{f} \\ \frac{\sin\alpha k}{f} \\ \ell(t) \end{pmatrix} + \vec{c}_k, \tag{8}$$

where $\vec{c}_k$ depends on k but not t, and $\ell(t) = -\int_0^t f(t)^{-2}dt$ depends on t but not k. We compute that

$$\Delta x^* = \frac{1}{f}\begin{pmatrix} \Delta\cos\alpha k \\ \Delta\sin\alpha k \\ 0 \end{pmatrix} + \vec{c}_{k+1} - \vec{c}_k.$$

Therefore, x^* is dual to x if $\vec{c}_k$ is a constant vector. For x to be minimal, x^* must be inscribed in a sphere and therefore we can choose $\vec{c}_0 = (0,0,c_0)^t$ so that $\|x^*\|$ is constant. From (8) we have

$$\frac{1}{f^2} + \left(\int_0^t \frac{1}{(f(t))^2}dt - c_0\right)^2 = \text{constant}.$$

Differentiation gives that

$$\int_0^t \frac{1}{(f(t))^2}dt = \frac{f'}{f} + c_0$$

and

$$f''f - (f')^2 = 1.$$

We find from (7) that

$$f(t) = \cosh t.$$

Thus semi-discrete catenoids with smooth profile curves and fixed α are unique up to homotheties. The picture in Figure 3 is such a semi-discrete catenoid. In fact, we have proven:

Proposition 1. *The profile curve of $MW^{in}_{ps,rd}$-catenoids is independent of choice of α.*

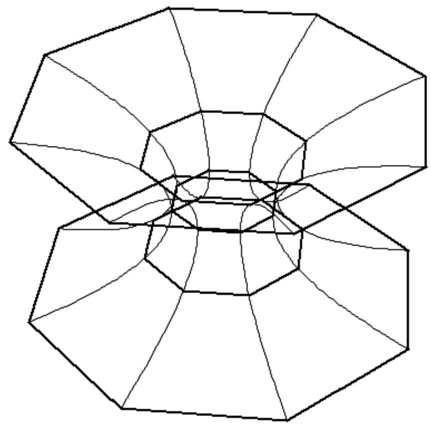

Figure 3: a semi-discrete $MW^{in}_{ps,rd}$-catenoid discretized in the direction of rotation

5. Weierstrass representation for semi-discrete minimal surfaces

We now give a Weierstrass representation for semi-discrete minimal surfaces. First we define semi-discrete holomorphic functions.

Definition 5. A semi-discrete isothermic surface g is a *semi-discrete holomorphic function* if the image of g lies in a plane.

Remark 3. Semi-discrete holomorphic maps have the following property: With σ and τ as in Definition 3 (with x replaced by g),

$$\frac{\|\Delta g\|^2}{\|g'\|\,\|g_1'\|} = \frac{\sigma}{\tau}, \tag{9}$$

where $g' = \partial g$. So we can think of τ and σ in the semi-discrete case as an analogy to the (absolute values of the) cross-ratio factorizing functions in the fully discrete case, see [1, 2, 4, 5, 6, 12].

We introduce the following recipe for constructing semi-discrete minimal surfaces.

Theorem 2 (Weierstrass representation)**.** *Let g be a semi-discrete holomorphic function with data τ, σ, and ν as in Definition 3. Then we can construct a semi-discrete minimal surface by solving*

$$\partial x = -\frac{\tau}{2}\mathrm{Re}\left(\frac{1}{g'}\begin{pmatrix} 1-g^2 \\ \mathrm{i}(1+g^2) \\ 2g \end{pmatrix}\right),$$
$$\Delta x = \frac{\sigma}{2}\mathrm{Re}\left(\frac{1}{\Delta g}\begin{pmatrix} 1-gg_1 \\ \mathrm{i}(1+gg_1) \\ g+g_1 \end{pmatrix}\right). \tag{10}$$

Conversely, any semi-discrete minimal surface is described in this way by some semi-discrete holomorphic function g.

Proof. We start proving the first half of Theorem 2. Let g be a semi-discrete holomorphic function such that $|\Delta g|^2 = \sigma\nu\nu_1$, $|g'| = \tau\nu^2$ for some positive functions ν, σ, τ. Then

$$x^* := \frac{1}{1+\|g\|^2}\begin{pmatrix} 2g \\ -1+\|g\|^2 \end{pmatrix} \in \mathbb{S}^2 \subset \mathbb{C}\times\mathbb{R} = \mathbb{R}^3$$

is semi-discrete isothermic, because x^* is the image of g under the inverse of stereographic projection. Then

$$\partial x^* = \frac{2}{(1+\|g\|^2)^2}\begin{pmatrix} g' - \overline{g'}g^2 \\ g'\bar{g} + \overline{g'}g \end{pmatrix},$$
$$\Delta x^* = \frac{2}{(1+\|g\|^2)(1+\|g_1\|^2)}\begin{pmatrix} \Delta g - \overline{\Delta g}gg_1 \\ \Delta g\bar{g_1} + \overline{\Delta g}g \end{pmatrix}.$$

It follows that

$$\|\partial x^*\|^2 = \frac{4\|g'\|^2}{(1+\|g\|^2)^2} = \frac{4\tau\nu^2}{(1+\|g\|^2)^2},$$
$$\|\Delta x^*\|^2 = \frac{4\|\Delta g\|^2}{(1+\|g\|^2)(1+\|g_1\|^2)} = \frac{4\sigma\nu\nu_1}{(1+\|g\|^2)(1+\|g_1\|^2)},$$

so we can take the data τ^*, σ^*, ν^* for the isothermic surface x^* to be

$$\tau^* = \tau,\ \sigma^* = \sigma,\ \nu^* = \frac{2\nu}{1+\|g\|^2}.$$

Here σ^* depends only on k, and τ^* depends only on t. Then

$$\frac{-1}{(\nu^*)^2}\partial x^* = \frac{-1}{2\nu^2}\begin{pmatrix} g' - \bar{g}'g^2 \\ g'\bar{g} + \bar{g}'g \end{pmatrix} = \frac{-\tau}{2\|g'\|^2}\begin{pmatrix} g' - \bar{g}'g^2 \\ g'\bar{g} + \bar{g}'g \end{pmatrix}$$
$$= -\frac{\tau}{2}\begin{pmatrix} \mathrm{Re}\left(\frac{1-g^2}{g'}\right) + \mathrm{i}\mathrm{Re}\left(\frac{\mathrm{i}(1+g^2)}{g'}\right) \\ \mathrm{Re}\left(\frac{2g}{g'}\right) \end{pmatrix} = \partial x,$$

where we have identified $\mathbb{C}\times\mathbb{R}$ and $\mathbb{R}^3$ in the final equality. Similarly,

$$\frac{1}{\nu^*\nu_1^*}\Delta x^* = \frac{\sigma}{2\|\Delta g\|^2}\begin{pmatrix} \Delta g - gg_1\overline{\Delta g} \\ \overline{g_1}\Delta g + g\overline{\Delta g} \end{pmatrix}$$
$$= \frac{\sigma}{2}\begin{pmatrix} \mathrm{Re}\left(\frac{1-gg_1}{\Delta g}\right) + \mathrm{i}\mathrm{Re}\left(\frac{\mathrm{i}(1+gg_1)}{\Delta g}\right) \\ \mathrm{Re}\left(\frac{g+g_1}{\Delta g}\right) \end{pmatrix} = \Delta x.$$

Thus if x solving (10) exists, x and x^* are dual to each other. A direct computation shows

$$\|\Delta x\|^2 = \sigma\left(\frac{1+\|g\|^2}{2\nu}\right)\left(\frac{1+\|g_1\|^2}{2\nu_1}\right),$$
$$\|\partial x\|^2 = \tau\left(\frac{1+\|g\|^2}{2\nu}\right)^2,$$

so x will be isothermic if it is circular. Since x^* is inscribed in a sphere, x must then be a semi-discrete minimal surface. Thus it remains to check existence and circularity of x.

To show existence of x, we need to show compatibility of the two equations in (10), and this amounts to showing that the two operators Δ and ∂ in (10) commute, that is,

$$\partial\left(\frac{\sigma}{2}\mathrm{Re}\begin{pmatrix} \frac{1-gg_1}{\Delta g} \\ \frac{\mathrm{i}(1+gg_1)}{\Delta g} \\ \frac{g+g_1}{\Delta g} \end{pmatrix}\right) = \Delta\left(-\frac{\tau}{2}\mathrm{Re}\begin{pmatrix} \frac{1-g^2}{g'} \\ \frac{\mathrm{i}(1+g^2)}{g'} \\ \frac{2g}{g'} \end{pmatrix}\right). \quad (11)$$

One can compute

Left-hand side of (11)

$$= \frac{\sigma}{2}\mathrm{Re}\left(\frac{1}{(\Delta g)^2}\begin{pmatrix} g^2g' - g_1' - g'g_1^2 + g' \\ \mathrm{i}(g'g_1^2 + g' - g^2g_1' - g_1') \\ 2g'g_1 - 2gg_1' \end{pmatrix}\right)$$
$$= \frac{\tau\|\Delta g\|^2}{2\|g'\|\|g_1'\|}\mathrm{Re}\left(\frac{1}{(\Delta g)^2}\begin{pmatrix} g^2g' - g_1' - g'g_1^2 + g' \\ \mathrm{i}(g'g_1^2 + g' - g^2g_1' - g_1') \\ 2g'g_1 - 2gg_1' \end{pmatrix}\right)$$
$$= \frac{\tau}{2}\mathrm{Re}\left(\frac{\overline{\Delta g}}{\|g'\|\|g_1'\|\Delta g}\begin{pmatrix} g^2g' - g_1' - g'g_1^2 + g' \\ \mathrm{i}(g'g_1^2 + g' - g^2g_1' - g_1') \\ 2g'g_1 - 2gg_1' \end{pmatrix}\right)$$
$$= -\frac{\tau}{2}\mathrm{Re}\left(\frac{1}{g'g_1'}\begin{pmatrix} g^2g' - g_1' - g'g_1^2 + g' \\ \mathrm{i}(g'g_1^2 + g' - g^2g_1' - g_1') \\ 2g'g_1 - 2gg_1' \end{pmatrix}\right).$$

= Right-hand side of (11).

The last task is to check that x is circular. By a rotation and translation, we can assume that $\mathrm{span}\{\partial x, \partial x_1, \Delta x\} = \mathbb{C}\times\{0\}$ for one edge $\overline{xx_1}$. We fix $(k,t) = (k_0,t_0)$ arbitrarily, and write $x(k_0,t_0)$ simply as x. It suffices to show the existence of $s \in \mathbb{R}$ such that (2) holds. Now,

$$\frac{g}{g'},\ \frac{g_1}{g_1'},\ \frac{g+g_1}{\Delta g} \in \mathrm{i}\mathbb{R}.$$

Expressing g as $g = re^{\mathrm{i}\theta}$ in polar form, we have

$$r' = r_1' = 0,\ r_1e^{\mathrm{i}\theta_1} + re^{\mathrm{i}\theta} = \mathrm{i}\rho(r_1e^{\mathrm{i}\theta_1} - re^{\mathrm{i}\theta})$$

for some $\rho \in \mathbb{R}$. Taking the absolute value of

$$r_1(\mathrm{i}\rho - 1)e^{\mathrm{i}\theta_1} = r(\mathrm{i}\rho + 1)e^{\mathrm{i}\theta},$$

we find that $r = r_1$. The left and right hand sides of Equation (2) are real scalar multiples of

$$\Delta g - gg_1\overline{\Delta g} = r(1+r^2)(e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta}),$$
$$\mathrm{i}\left(\frac{g' - \overline{g'}g^2}{|g' - \overline{g'}g^2|} + \frac{g_1' - \overline{g_1'}g_1^2}{|g_1' - \overline{g_1'}g_1^2|}\right)$$
$$= \mathrm{i}\left(\frac{\mathrm{i}\theta' r(1+r^2)e^{\mathrm{i}\theta}}{|\mathrm{i}\theta' r(1+r^2)e^{\mathrm{i}\theta}|} + \frac{\mathrm{i}\theta_1' r_1(1+r_1^2)e^{\mathrm{i}\theta_1}}{|\mathrm{i}\theta_1' r_1(1+r_1^2)e^{\mathrm{i}\theta_1}|}\right)$$
$$= \pm(e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_1}),$$

respectively, where we used the following lemma in the final equality above. This lemma follows from Lemma 6 and Theorem 11 in [9], because g is isothermic.

Lemma 2. *We have the following property:*

$$\theta' \cdot \theta_1' < 0.$$

Therefore, we have

$$\arg\left(\Delta g - gg_1\overline{\Delta g}\right) = \arg\left(\pm \mathrm{i}\left(\frac{g' - \overline{g'}g^2}{|g' - \overline{g'}g^2|} + \frac{g'_1 - \overline{g'_1}g_1^2}{|g'_1 - \overline{g'_1}g_1^2|}\right)\right),$$

which implies (2). Now we prove the final sentence of Theorem 2. Let x be a semi-discrete minimal surface and ψ be stereographic projection $\psi\colon \mathbb{S}^2 \to \mathbb{C}$. Then by definition, there exists a dual x^* that is semi-discrete isothermic and inscribed in $\mathbb{S}^2$. Take

$$g := \psi \circ x^*,$$

then g is a semi-discrete holomorphic function (see Example 1 of [9]). Setting

$$x^* = (X_1, X_2, X_3)^t,\ x_1^* = (X_{1,1}, X_{2,1}, X_{3,1})^t,$$

we have

$$g = \frac{X_1 + \mathrm{i}X_2}{1 - X_3},\ X_1^2 + X_2^2 + X_3^2 = X_{1,1}^2 + X_{2,1}^2 + X_{3,1}^2 = 1,$$

$$(X_1')^2 + (X_2')^2 + (X_3')^2 = \frac{\tau}{\nu^2},$$

$$1 - (X_1X_{1,1} + X_2X_{2,1} + X_3X_{3,1}) = \frac{\sigma}{2\nu\nu_1}.$$

Using the above equations and Definition 2, computations give

$$-\frac{\tau}{2}\mathrm{Re}\begin{pmatrix}\frac{1-g^2}{g'}\\ \frac{\mathrm{i}(1+g^2)}{g'}\\ \frac{2g}{g'}\end{pmatrix} = -\nu^2\begin{pmatrix}X_1'\\ X_2'\\ X_3'\end{pmatrix} = \partial x,$$

$$\frac{\sigma}{2}\mathrm{Re}\begin{pmatrix}\frac{1-gg_1}{\Delta g}\\ \frac{\mathrm{i}(1+gg_1)}{\Delta g}\\ \frac{g+g_1}{\Delta g}\end{pmatrix} = \nu\nu_1\begin{pmatrix}X_{1,1} - X_1\\ X_{2,1} - X_2\\ X_{3,1} - X_3\end{pmatrix} = \Delta x. \tag{12}$$

Thus g produces x via Equation (10), which completes the proof.

Because the computation of (12) in particular is rather laborious, we outline one part of that computation here: Since

$$\frac{\sigma}{\Delta g} = \nu\nu_1[X_{1,1}(1 - X_3) - X_1(1 - X_{3,1}) - \mathrm{i}\{X_{2,1}(1 - X_3) - X_2(1 - X_{3,1})\}],$$

we have

$$\begin{aligned}
&\frac{\sigma}{2}\mathrm{Re}\frac{1 - gg_1}{\Delta g}\\
&= \frac{\nu\nu_1}{2(1-X_3)(1-X_{3,1})}\mathrm{Re}([X_{1,1}(1 - X_3) - X_1(1 - X_{3,1})\\
&\qquad - \mathrm{i}\{X_{2,1}(1 - X_3) - X_2(1 - X_{3,1})\}]\\
&\qquad \cdot [(1 - X_3)(1 - X_{3,1}) - (X_1 + \mathrm{i}X_2)(X_{1,1} + \mathrm{i}X_{2,1})])\\
&= \frac{\nu\nu_1}{2(1-X_3)(1-X_{3,1})}[\{X_{1,1}(1 - X_3) - X_1(1 - X_{3,1})\}\\
&\qquad \cdot \{(1 - X_3)(1 - X_{3,1}) - X_1X_{1,1} + X_2X_{2,1}\}\\
&\quad - \{X_{2,1}(1 - X_3) - X_2(1 - X_{3,1})\}\\
&\qquad \cdot (X_1X_{2,1} + X_{1,1}X_2)]\\
&= \frac{\nu\nu_1}{2(1-X_3)(1-X_{3,1})}\{(1 - X_3)(1 - X_{3,1})\\
&\qquad \cdot (X_{1,1} - X_1 - X_{1,1}X_3 + X_1X_{3,1})\\
&\quad - X_1(1 - X_3)(X_{1,1}^2 + X_{2,1}^2)\\
&\quad + X_{1,1}(1 - X_{3,1})(X_1^2 + X_2^2)\}\\
&= \frac{\nu\nu_1}{2(1-X_3)(1-X_{3,1})}\{(1 - X_3)(1 - X_{3,1})\\
&\qquad \cdot (X_{1,1} - X_1 - X_{1,1}X_3 + X_1X_{3,1})\\
&\quad - (1 - X_3)(1 - X_{3,1})(1 + X_{3,1})X_1\\
&\quad + (1 - X_3)(1 - X_{3,1})(1 + X_3)X_{1,1}\}\\
&= \nu\nu_1(X_{1,1} - X_1).
\end{aligned}$$

□

Example 1. The semi-discrete minimal Enneper surface, has been given in [9]. We can also obtain that surface by taking $g(k,t) = k + \mathrm{i}t$ in Theorem 2.

Example 2. The $MW^{in}_{pd,rs}$ (resp. $MW^{in}_{ps,rd}$) catenoid can be constructed via Theorem 2 with

$$g(k,t) = ce^{\alpha k + \mathrm{i}\beta t} \quad (\text{resp. } g(k,t) = ce^{\alpha t + \mathrm{i}\beta k}),$$

for the right choices of $c,\ \alpha,\ \beta \in \mathbb{R}\setminus\{0\}$.

6. Fully-discrete catenoids of Bobenko-Pinkall

The fully discrete catenoids of Bobenko and Pinkall [1] can be given explicitly by using the Weierstrass representation for discrete minimal surfaces (in the integrable systems sense), that is, we can use

$$x(q) - x(p) = \mathrm{Re}\left(\frac{a_{pq}}{g_q - g_p}\begin{pmatrix}1 - g_qg_p\\ \mathrm{i} + \mathrm{i}g_qg_p\\ g_q + g_p\end{pmatrix}\right) \tag{13}$$

with the choice of g as $g_p = g_{n,m} = ce^{c_1n + ic_2m}$, where c, c_1, c_2 are nonzero real constants, and $p = (n,m)$ and $q = (n+1,m)$ or $q = (n,m+1)$, and a_{pq} is a cross ratio factorizing function for g. This formulation can be found in [1, 2, 4, 6, 12].

This choice of g has cross ratios

$$\mathrm{cr}(g_{n,m}, g_{n+1,m}, g_{n+1,m+1}, g_{n,m+1}) = \frac{-\sinh^2\frac{c_1}{2}}{\sin^2\frac{c_2}{2}}.$$

So we can take $a_{pq} = -\alpha\sinh^2\frac{c_1}{2}$ (resp. $a_{pq} = \alpha\sin^2\frac{c_2}{2}$), when $q = (n+1,m)$ (resp. $q = (n,m+1)$). The value $\alpha \in \mathbb{R}\setminus\{0\}$ can be chosen as we like.

Taking the axis of the surface to be

$$\left\{\begin{pmatrix}0\\0\\t\end{pmatrix}\middle|\ t \in \mathbb{R}\right\},$$

and taking the vertex in the profile curve at the neck to be $f(0,0) = (1,0,0)^t$, we can propagate to find the discrete profile curve in the x_1x_3-plane. For this purpose, $\alpha = -2$

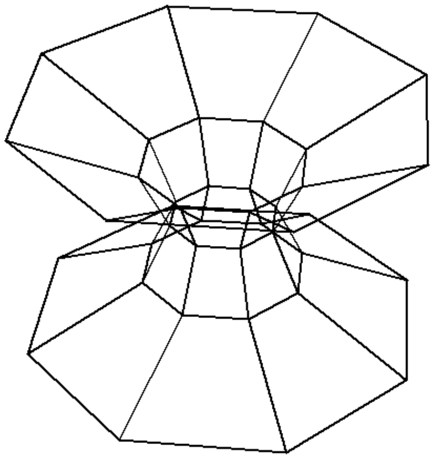

Figure 4: a (fully-discrete) $BP^{in}_{pd,rd}$-catenoid

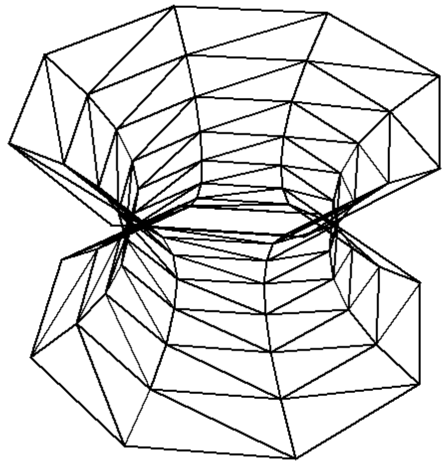

Figure 5: a (fully-discrete) $PR^{va}_{pd,rd}$-catenoid

and $c = -1$ are suitable values. One can check that, for all $m \in \mathbb{Z}$,

$$x(0,m) = \begin{pmatrix} \cos(c_2 m) \\ \sin(c_2 m) \\ 0 \end{pmatrix}.$$

By (13), the discrete profile curve in the x_1x_3-plane is, for all $n \in \mathbb{Z}$,

$$x(n,0) = \begin{pmatrix} \cosh(c_1 n) \\ 0 \\ n \cdot \sinh c_1 \end{pmatrix}.$$

Again by (13), we obtain

$$x(n,m) = \begin{pmatrix} \cosh(c_1 n)\cos(c_2 m) \\ \cosh(c_1 n)\sin(c_2 m) \\ n \cdot \sinh c_1 \end{pmatrix}.$$

Setting $\ell = \sinh c_1$, one profile curve of the $BP^{in}_{pd,rd}$-catenoid is as written in the upcoming Section 9. Note that the profile curves do not depend on c_2.

7. Fully-discrete catenoids of Polthier-Rossman

The catenoids described in [11] are fully discrete and have discrete rotational symmetry, thus the symmetry group is a dihedral group. Taking the dihedral angle to be $\theta = 2\pi K^{-1}$ for a constant $K \in \mathbb{N}$ and $K \geq 3$, the vertices of a profile curve (when the x_3-axis is the central axis of symmetry) in the x_1x_3-plane can be taken to be points that are vertically equally spaced apart with height difference between adjacent vertices denoted as ℓ, and the x_1 coordinates of the vertices can be taken as $x(n) = r \cdot \cosh(r^{-1}a\ell n)$, where $a = r\ell^{-1}\text{arccosh}(1+r^{-2}\ell^2(1+\cos\theta)^{-1})$. Here r is the waist radius of the interpolated hyperbolic cosine curve. Taking $r = 1$ without loss of generality, we can take one profile curve to be

$$\begin{pmatrix} \cosh(n \cdot \text{arccosh}(1 + \ell^2(1+\cos\theta)^{-1})) \\ 0 \\ n \cdot \ell \end{pmatrix}, \quad n \in \mathbb{Z}, \qquad (14)$$

so when we take the limit as $\theta \to 0$, we have

$$\begin{pmatrix} \cosh(n \cdot \text{arccosh}(1 + \frac{1}{2}\ell^2)) \\ 0 \\ n \cdot \ell \end{pmatrix}.$$

A direct computation, as in the proof of the next proposition, shows that this is exactly what was obtained by Machigashira [7], although it was not described in terms of the hyperbolic cosine function there, but rather by using Chebyshev polynomials and Gauss hypergeometric functions.

Proposition 2. *The $M^{va}_{pd,rs}$-catenoid equals the limiting case of the $PR^{va}_{pd,rd}$-catenoids as $\theta \to 0$, and no $PR^{va}_{pd,rd}$-catenoid (with positive θ) will ever have the same profile curve as the $M^{va}_{pd,rs}$-catenoid.*

Proof. The vertices of an $M^{va}_{pd,rs}$-catenoid profile curve can be written as

$$\begin{pmatrix} T_n(1 + \frac{1}{2}\Lambda^2) \\ 0 \\ n \cdot \Lambda \end{pmatrix}, \qquad (15)$$

where T_k can be defined by the recursion

$$T_0(z) = 1,\ T_1(z) = z,\ T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)$$

for $n \geq 2$. The T_n are called Chebyshev polynomials of the first kind, and are described in [7]. Suppose, for the vertex on the profile curve where $n = 1$, we equate (14) and (15), i.e.

$$\begin{pmatrix} 1+\frac{1}{2}\Lambda^2 \\ 0 \\ \Lambda \end{pmatrix} = \begin{pmatrix} 1+\ell^2(1+\cos\theta)^{-1} \\ 0 \\ \ell \end{pmatrix}. \tag{16}$$

The third coordinate in (16) implies $\Lambda = \ell$, and then the first coordinate implies $\theta = 0$. Then we would need to check that all other corresponding vertices in (14) and (15) also become equal.

Letting x denote the first coordinate of the profile curve, the $M^{va}_{pd,rs}$-catenoid satisfies

$$\begin{aligned} x(n) &= T_n(1+\tfrac{1}{2}\Lambda^2) \\ &= 2(1+\tfrac{1}{2}\Lambda^2)\cdot T_{n-1}(1+\tfrac{1}{2}\Lambda^2) - T_{n-2}(1+\tfrac{1}{2}\Lambda^2). \end{aligned}$$

For the limiting $PR^{va}_{pd,rd}$-catenoid ($\theta = 0$), we would like to see the same recursion for the first coordinate of the profile curve. That is, we wish to have

$$\begin{aligned} &\cosh(n\cdot \operatorname{arccosh}(1+\tfrac{1}{2}\ell^2)) \\ &= 2(1+\tfrac{1}{2}\ell^2)\cosh((n-1)\cdot\operatorname{arccosh}(1+\tfrac{1}{2}\ell^2)) \\ &\quad - \cosh((n-2)\cdot\operatorname{arccosh}(1+\tfrac{1}{2}\ell^2)), \end{aligned}$$

and this is indeed true, which proves the proposition. □

8. Another type of semi-discrete catenoid

Consider the two discrete loops, for a constant $K \in \mathbb{N}$, $K \geq 3$,

$$\begin{pmatrix} \cos(2\pi K^{-1}) \\ \sin(2\pi K^{-1}) \\ \pm r \end{pmatrix}$$

in the horizontal planes at height $\pm r$. We consider a semi-discrete catenoid (i.e. a surface with rotational symmetry by angle $2\pi K^{-1}$ about the x_3-axis) with those two loops as boundary. This catenoid is comprised of K congruent pieces, and each piece is foliated by horizontal line segments. One such piece would have two boundary curves parametrized by

$$c_1(t) = \begin{pmatrix} x(t) \\ 0 \\ t \end{pmatrix} \text{ and } c_2(t) = \begin{pmatrix} x(t)\cos(2\pi K^{-1}) \\ x(t)\sin(2\pi K^{-1}) \\ t \end{pmatrix} \tag{17}$$

in vertical planes, with $t \in [-r, r]$ and with

$$x(r) = x(-r) = 1.$$

The area of this piece is

$$A = \int_{-r}^{r} x\cdot\sqrt{2(1-\cos(2\pi K^{-1})) + (\sin(2\pi K^{-1}))^2(x')^2}dt.$$

Then consider a variation $x(t) \to x(t,\lambda)$ with $x(t,0) = x(t)$ and $x(\pm r, \lambda) = 1$, so λ is the variation parameter. Note that we are only considering rotationally invariant variations here, as was done by Machigashira [7]. An interesting question is whether we are also in effect considering variations that are not rotationally invariant as well, by some semi-discrete version of the symmetric criticality principle, see [10]. Set

$$x' := \frac{\partial x}{\partial t},\ x_\lambda := \frac{\partial x}{\partial \lambda},\ (x')_\lambda := \frac{\partial^2 x}{\partial\lambda\partial t}.$$

We wish to have that the following derivative with respect to λ is zero, where $c := \cos(2\pi K^{-1})$ and $s := \sin(2\pi K^{-1})$ and $D := 2(1-c) + s^2(x')^2$:

$$\begin{aligned} \frac{d}{d\lambda}A(\lambda)\Big|_{\lambda=0} &= \int_{-r}^{r} \frac{x_\lambda D + xx'(x')_\lambda s^2}{\sqrt{D}}dt\Big|_{\lambda=0} \\ &= \int_{-r}^{r}\left(\hat{x}\sqrt{D} + s^2\hat{x}'\frac{1}{2}\frac{((x(t))^2)'}{\sqrt{D}}\right)dt, \end{aligned}$$

when $x(t,\lambda) = x(t) + \lambda\cdot\hat{x}(t) + \mathcal{O}(\lambda^2)$. Then, using integration by parts, we wish to have, with $x = x(t)$,

$$\begin{aligned} 0 &= \int_{-r}^{r}\hat{x}\left(\sqrt{D} - s^2\frac{2(1-c)((x')^2+xx'') + s^2(x')^4}{\sqrt{D}^3}\right)dt \\ &= 2\int_{-r}^{r}\hat{x}\left((1-c)^2\frac{2-(1+c)(xx''-(x')^2)}{\sqrt{D}^3}\right)dt \end{aligned}$$

for all variations. This implies

$$xx'' - (x')^2 = \frac{2}{1+c},$$

and hence we obtain that

$$x = c_1 e^{-c_3 t} + c_2 e^{c_3 t}\ ,\quad c_1 = \frac{1}{2(1+c)c_2 c_3^2},$$

where c_2, c_3 are free constants. The conditions $x(\pm r) = 1$ imply $c_1 = c_2 = (2\cosh(c_3 r))^{-1}$, so we obtain

$$x(t) = 2c_1\cosh(c_3 t) = \frac{\cosh(c_3 t)}{\cosh(c_3 r)}. \tag{18}$$

Then automatically $x'(0) = 0$. From the above relations amongst the c_j, we obtain that

$$\cosh^2(c_3 r) = \frac{c_3^2(1+c)}{2}. \tag{19}$$

Thus c_3 is determined by r. These catenoids have been determined by using a variational property, like the $M^{va}_{pd,rs}$-catenoids were, so we call them $M^{va}_{ps,rd}$-catenoids.

For r that allow for solutions c_3 to (19), a profile curve of an $M^{va}_{ps,rd}$-catenoid is $c_1(t)$ as in (17) with $x(t)$ as in (18). Rescaling this c_1 by $\cosh(c_3 r)$ and appropriately rescaling the parameter t, we find that this catenoid's profile curve can be parametrized as

$$\begin{pmatrix} \cosh\left(\frac{\sqrt{2}t}{\sqrt{1+\cos\frac{2\pi}{K}}}\right) \\ 0 \\ t \end{pmatrix}.$$

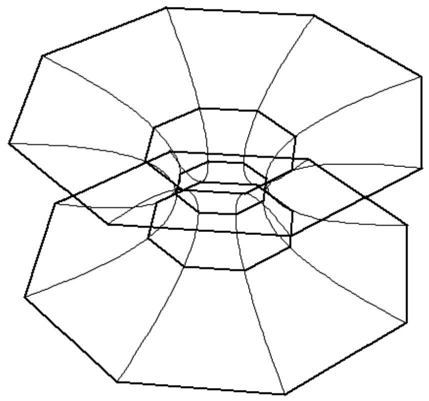

Figure 6: a $M^{va}_{ps,rd}$-catenoid

9. Proof of Theorem 1

We list parametrizations of the profile curves of the various catenoids again here in Table 2.

Comparing all profile curves, we obtain the following proof of Theorem 1:

Proof. The statements in items 1, 2, 4, 5 and 6 of Theorem 1 are obvious, so we prove only item 3 here. By way of contradiction, suppose $BP^{in}_{pd,rd}$-catenoid profile curves and $M^{va}_{pd,rs}$-catenoid profile curve can be the same. From the parametrizations in Table 2,

$$\cosh(\mathrm{arcsinh}\ell) = 1 + \frac{1}{2}\ell^2. \tag{20}$$

Since $\cosh(\mathrm{arcsinh}\ell) = \sqrt{1+\ell^2}$, (20) implies $\ell = 0$, which does not occur. Similarly, suppose $BP^{in}_{pd,rd}$-catenoid profile curves and $PR^{va}_{pd,rd}$-catenoid profile curve can be the same. Then we have

$$\cosh(\mathrm{arcsinh}\ell) = 1 + \frac{1}{1+\cos\frac{2\pi}{K}}\ell^2,$$

namely,

$$\Leftrightarrow \left(\frac{-2}{1+\cos\frac{2\pi}{K}} + 1\right)\ell^2 = \left(\frac{1}{1+\cos\frac{2\pi}{K}}\right)^2 \ell^4. \tag{21}$$

The left-hand-side of (21) is negative and the right-hand-side of (21) is positive, which is impossible. □

	parametrizations of profile curves
smooth catenoid	$\begin{pmatrix}\cosh t\\ 0\\ t\end{pmatrix}$ $(t \in \mathbb{R})$
$PR^{va}_{pd,rd}$-catenoid	$\begin{pmatrix}\cosh(n\cdot \mathrm{arccosh}(1+\frac{\ell^2}{1+\cos\frac{2\pi}{K}}))\\ 0\\ n\cdot\ell\end{pmatrix}$
$M^{va}_{pd,rs}$-catenoid	$\begin{pmatrix}\cosh(n\cdot \mathrm{arccosh}(1+\frac{1}{2}\ell^2))\\ 0\\ n\cdot\ell\end{pmatrix}$
$BP^{in}_{pd,rd}$-catenoid	$\begin{pmatrix}\cosh(n\cdot \mathrm{arcsinh}\ell)\\ 0\\ n\cdot\ell\end{pmatrix}$
$MW^{in}_{pd,rs}$-catenoid	$\begin{pmatrix}\cosh(n\cdot \mathrm{arcsinh}\ell)\\ 0\\ n\cdot\ell\end{pmatrix}$
$MW^{in}_{ps,rd}$-catenoid	$\begin{pmatrix}\cosh t\\ 0\\ t\end{pmatrix}$
$M^{va}_{ps,rd}$-catenoid	$\begin{pmatrix}\cosh\left(\frac{\sqrt{2}t}{\sqrt{1+\cos\frac{2\pi}{K}}}\right)\\ 0\\ t\end{pmatrix}$

Table 2: Parametrizations of seven types of catenoids

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Wayne Rossman and Masashi Yasumoto
Faculty of Science, Department of Mathematics, Kobe University, 1-1, Rokkodai, Nada-ku, Kobe, 657-8501, Japan
E-mail: wayne(at)math.kobe-u.ac.jp
masashi.yasumoto(at)gmail.com