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The probability densities of the first hitting times of Bessel processes

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Abstract. We are concerned with the first hitting times of the Bessel processes. We give explicit expressions for the densities by means of the zeros of the Bessel functions and show their asymptotic behavior.

Keywords. Bessel process, first hitting time, Bessel functions

1. Introduction

The Bessel process is one of the fundamental stochastic processes. If the index is a half integer greater than or equal to $-1/2$, it is identical in law with the radial motion of a Euclidean Brownian motion. Moreover, in the black-Scholes model, a basic model in mathematical finance, a stock price processes is modelled by a geometric Brownian motion, which is a time change of some Bessel process. It is also known that the explicit computations on the Bessel process are useful in the study of so-called the CIR model of interest rates.

The first hitting time to a point or a domain of stochastic process is an important object in probability theory. When a geometric Brownian motion or a Bessel process is used to modell a stock price, the first hitting time plays an important role in the theory of exotic options. For applications of Bessel processes to mathematical finance, see e.g., Geman-Yor [3] and Yor [12].

In this article we are concerened with the first hitting time to a point of the Bessel process. The purpose is to show an explicit expression for the density and, by using it, to show asymptotic behavior of the density at infinity. The probability distributon has been recently studied in [4] and we use the results therein.

Let $\tau_{a,b}^{(\nu)}$ be the first hitting time to b of a Bessel process with index ν starting from a. When $a < b$, general theory on the eigenvalue expansions for the first hitting times of diffusion processes (cf. Kent [7]) may be applied since the boundary 0 is not natural, and an explicit simple expression via the zeros of the Bessel function for the density of $\tau_{a,b}^{(\nu)}$ is known. However, in the case of $a > b$, we cannot apply the general theory since the boundary ∞ is natural. Hence we concentrate on the latter case.

By the theory of one-dimensional diffusion processes, the Laplace transform of the density (the moment generating function) of $\tau_{a,b}^{(\nu)}$ can be computed and is expressed as a ratio of the modified Bessel functions. We will invert the Laplace transform along the same line as that in [4], where we concentrate on the distribution function. In this sence this article may be regarded as a companion of [4].

Recently Byczkowski et al [1, 2] have obtained other expression for the density and have shown the order of its decay at infinity. In some cases (see Remark 1) we need to use their results to show the exact asymptotics of the decay or to determine the constant. It should be also mentioned that Uchiyama [10] has studied the asymptotic behavior of the density in the case when $\nu = 0$.

2. Density of the first hitting time

For $\nu \in \mathbb{R}$ the diffusion process on $[0, \infty)$ with infinitesimal generator

$$\mathscr{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}$$

is called the Bessel process with index ν. If $2\nu + 2$ is a positive integer, the Bessel process is identical in law with the radial motion of a $(2\nu + 2)$-dimensional Brownian motion. Hence the number $2\nu + 2$ is called the dimension of the Bessel process. For all ν the boundary ∞ is natural. The classification of 0 depends on ν, but it is not natural for any ν. For details, see [5, 6, 7].

For $a, b > 0$, let $\tau_{a,b}^{(\nu)}$ denote the first hitting time to b of a Bessel process with index ν starting from a. Then it is a fundamental fact in the theory of one-dimensional diffusion processes (cf. [5]) that, for $\lambda > 0$, the function

$$x \mapsto E[e^{-\lambda \tau_{a,b}^{(\nu)}}] = u(x, \lambda)$$

is increasing (decreasing) on $[0, b)$ (resp. (b, ∞)) and satisfies $u(b, \lambda) = 1$ and

$$\mathscr{G}^{(\nu)} u = \lambda u, \tag{1}$$

where E denotes the expectation. Hence, solving this equation, we have an explicit expression for the Laplace transorm of the distribution of $\tau_{a,b}^{(\nu)}$ by means of the modified Bessel function.

If $0 < a < b$, we have (see [6])

$$E[e^{-\lambda\tau_{a,b}^{(\nu)}}] = \frac{a^{-\nu}I_\nu(a\sqrt{2\lambda})}{b^{-\nu}I_\nu(b\sqrt{2\lambda})}, \quad \nu > -1,$$

and

$$E[e^{-\lambda\tau_{a,b}^{(\nu)}}] = \frac{a^{-\nu}I_{-\nu}(a\sqrt{2\lambda})}{b^{-\nu}I_{-\nu}(b\sqrt{2\lambda})}, \quad \nu \leqq 1,$$

where I_ν is the modofied Bessel function. Since 0 is not a natural boundary, the solution $u(x, \lambda)$ for the equation (1) may be written as the canonical product (see [8, 9])

$$u(x, \lambda) = u(x, 0) \prod_{k=1}^{\infty} \Big(1 + \frac{\lambda}{\lambda_{k,x}}\Big),$$

where $\{\lambda_{k,x}\}_{k=1}^{\infty}$ is a sequence of the simple and positive zeros of $u(x, \bullet)$. Therefore the distribution of $\tau_{a,b}^{(\nu)}$ may be written as an infinite convolution of mixtures of exponential distribution. For the general theory, see Kent [7].

On the other hand, if $0 < b < a$, we have

$$E[e^{-\lambda\tau_{a,b}^{(\nu)}}] = \frac{a^{-\nu}K_\nu(a\sqrt{2\lambda})}{b^{-\nu}K_\nu(b\sqrt{2\lambda})}, \quad \nu > -1, \tag{2}$$

for every $\nu \in \mathbb{R}$, where K_ν is the other modified Bessel function called the Macdonald function. The boundary ∞ is natural and the general theory above are not applicable.

In the following we are only concerned with the latter case. We invert the Laplace transform (2) and we give an expression for the density of $\tau_{a,b}^{(\nu)}$ from Theorem 1 below, which is Theorem 3.3 in [4], on some ratio of the modified Bessel functions.

We denote by $z_{\nu,1}, \ldots, z_{\nu,N(\nu)}$ the zeros of K_ν. It is known (cf. [11]) that the real part of each zero is negative and that the number $N(\nu)$ is $|\nu| - 1/2$ if $\nu - 1/2 \in \mathbb{Z}$ and is the even number closest to $|\nu| - 1/2$ if otherwise.

Moreover, for $\mu \geqq 0$ and $c > 1$, we set

$$L_{\mu,c}(x) = \frac{\cos(\pi\mu)\{I_\mu(cx)K_\mu(x) - I_\mu(x)K_\mu(cx)\}}{\{K_\mu(x)\}^2 + \pi^2\{I_\mu(x)\}^2 + 2\pi\sin(\pi\mu)K_\mu(x)I_\mu(x)}.$$

By the estimates for the modified Bessel functions, we see that $L_{\mu,c}(x)$ decay exponentially as $x \to \infty$ and that, as $x \downarrow 0$,

$$L_{\mu,c}(x) = \begin{cases} \dfrac{\log c}{(\log x)^2}\{1 + o(1)\}, & \mu = 0, \\ \dfrac{\cos(\pi\mu)(c^\mu - c^{-\mu})x^{2\mu}}{2^{2\mu-1}\Gamma(\mu)\Gamma(\mu+1)}\{1 + o(1)\}, & \mu > 0. \end{cases} \tag{3}$$

Theorem 1. *Let $c > 1$, $\nu \in \mathbb{R}$ and $w \in \mathbb{C} \setminus \{0\}$ with $|\arg(w)| < \pi$ and $K_\nu(w) \neq 0$.*
(1) *If $\nu = \pm 1/2$, we have*

$$\frac{K_\nu(cw)}{K_\nu(w)} = \frac{e^{-(c-1)w}}{c^{|\nu|}}.$$

(2) *If $|\nu| < 3/2$ and $\nu \neq \pm 1/2$, we have*

$$\frac{K_\nu(cw)}{K_\nu(w)} = \frac{e^{-(c-1)w}}{c^{|\nu|}} - e^{(c-1)w}\int_0^\infty \frac{we^{-(c-1)x}L_{|\nu|,c}(x)}{x(x+w)}dx.$$

(3) *If $\nu - 1/2$ is an integer and $\nu \neq \pm 1/2$,*

$$\frac{K_\nu(cw)}{K_\nu(w)} = \frac{e^{-(c-1)w}}{c^{|\nu|}} - e^{(c-1)w}\sum_{j=1}^{N(\nu)} \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w - z_{\nu,j})}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}.$$

(4) *If $\nu - 1/2$ is not an integer and $|\nu| > 3/2$,*

$$\frac{K_\nu(cw)}{K_\nu(w)} = \frac{e^{-(c-1)w}}{c^{|\nu|}} - e^{(c-1)w}\sum_{j=1}^{N(\nu)} \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w - z_{\nu,j})}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})} - e^{(c-1)w}\int_0^\infty \frac{we^{-(c-1)x}L_{|\nu|,c}(x)}{x(x+w)}dx.$$

Combining the results in Theorem 1 with formula

$$\int_0^\infty e^{-\lambda t}q(t, a, b)dt = e^{-(a-b)\sqrt{2\lambda}},$$

where

$$q(t, a, b) = \frac{a-b}{\sqrt{2\pi t^3}}e^{-(a-b)^2/2t}, \quad a > b,\ t > 0,$$

we can invert the Laplace transform (2) and obtain an expression for the density of $\tau_{a,b}^{(\nu)}$.

We put $c = a/b > 1$ and define the following functions.

$$\Phi_{a,b}^{(\nu),1}(t) = \sum_{j=1}^{N(\nu)} \frac{e^{(c-1)z_{\nu,j}}}{z_{\nu,j}}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})} \cdot q(t, a, b),$$

$$\Phi_{a,b}^{(\nu),2}(t) = \frac{1}{b}\sum_{j=1}^{N(\nu)} \frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}\frac{1}{\sqrt{2\pi t^3}}\int_{a-b}^\infty \xi e^{-\frac{\xi^2}{2t} + \frac{z_{\nu,j}\xi}{b}}d\xi,$$

$$\Psi_{a,b}^{(\nu),1}(t) = \int_0^\infty \frac{1}{x}e^{-(c-1)x}L_{|\nu|,c}(x)dx \cdot q(t, a, b),$$

$$\Psi_{a,b}^{(\nu),2}(t) = \frac{1}{b}\int_0^\infty L_{|\nu|,c}(x)dx\frac{1}{\sqrt{2\pi t^3}}\int_{a-b}^\infty \xi e^{-\frac{\xi^2}{2t} - \frac{x\xi}{b}}d\xi.$$

Then we obtain the following from Theorem 1.

Theorem 2. *The first hitting time $\tau_{a,b}^{(\nu)}$ has the density $f_{a,b}^{(\nu)}$ which is given by the following.*
(1) *If $\nu = \pm 1/2$, we have*

$$f_{a,b}^{(\nu)}(t) = c^{-\nu-|\nu|}q(t, a, b).$$

(2) *If $|\nu| < 3/2$ and $\nu \neq \pm 1/2$, we have*

$$f_{a,b}^{(\nu)}(t) = c^{-\nu}\big(e^{-|\nu|}q(t, a, b) - \Psi_{a,b}^{(\nu),1}(t) + \Psi_{a,b}^{(\nu),2}(t)\big).$$

(3) *If $\nu - 1/2$ is an integer and $\nu \neq \pm 1/2$,*

$$f_{a,b}^{(\nu)}(t) = c^{-\nu}\big(e^{-|\nu|}q(t,a,b) - \Phi_{a,b}^{(\nu),1}(t) - \Phi_{a,b}^{(\nu),2}(t)\big).$$

(4) *If $\nu - 1/2$ is not an integer and $|\nu| > 3/2$,*

$$f_{a,b}^{(\nu)}(t) = c^{-\nu}\big(e^{-|\nu|}q(t,a,b) - \Phi_{a,b}^{(\nu),1}(t) - \Phi_{a,b}^{(\nu),2}(t) - \Psi_{a,b}^{(\nu),1}(t) + \Psi_{a,b}^{(\nu),2}(t)\big).$$

3. Asymptotic behavior of the densities

In this section we study the asymptotic behavior of the density $f_{a,b}^{(\nu)}$ of $\tau_{a,b}^{(\nu)}$ as $t \to \infty$ by using the expression given in Theorem 2. It is shown in [2] that $f_{a,b}^{(\nu)}(t) = c_0 t^{-1}(\log t)^{-2}(1+o(1))$ if $\nu = 0$ and $f_{a,b}^{(\nu)}(t) = c_\nu t^{-1-|\nu|}(1+o(1))$ if $\nu \neq 0$.

At first we note that, for every $\nu \in \mathbb{R}$, the functions $q(t,a,b), \Phi_{a,b}^{(\nu),1}(t), \Phi_{a,b}^{(\nu),2}(t)$ and $\Psi_{a,b}^{(\nu),1}(t)$ are $O(t^{-3/2})$ as $t \to \infty$. About $\Psi_{a,b}^{(\nu),2}(t)$, since

$$\int_{a-b}^{\infty} \xi e^{-x\xi/b} d\xi = O(x^{-2}) \quad \text{as} \quad x \downarrow 0,$$

we obtain

$$\int_0^\infty L_{|\nu|,c}(x)\frac{1}{x^2}dx = \infty$$

from (3). This means $t^{3/2}\Psi_{a,b}^{(\nu),2}(t) \to \infty$, $t \to \infty$ when $0 < |\nu| < 1/2$. Hence we need to consider separately for the four cases, $\nu = 0$, $0 < |\nu| < 1/2$, $|\nu| = 1/2$ and $|\nu| > 1/2$.

Proposition 1. *Assume $0 < b < a$ and put $c = b/a$.*
(1) *If $\nu = 0$, we have*

$$f_{a,b}^{(\nu)}(t) = 2\log c \cdot t^{-1}(\log t)^{-2}\cdot(1+o(1)). \tag{4}$$

(2) *If $0 < |\nu| < 1/2$, we have*

$$f_{a,b}^{(\nu)}(t) = \frac{b^{2|\nu|}(c^{|\nu|} - c^{-|\nu|})}{c^\nu 2^{|\nu|}\Gamma(|\nu|)} t^{-|\nu|-1}(1+o(1)). \tag{5}$$

Remark 1. When $|\nu| > 1/2$, we can prove that (5) holds if $\nu - 1/2 \notin \mathbb{Z}$. When $\nu - 1/2 \in \mathbb{Z}$ and $\nu \neq \pm 1/2$, we can prove that

$$f_{a,b}^{(\nu)}(t) = C(\nu)t^{-[|\nu|-1/2]-3/2}\cdot(1+o(1))$$

holds for some constant in a similar way to that in Section 4 of [4], where we have studied the tail probability for $\tau_{a,b}^{(\nu)}$. We can give an expression for the constant $C(\nu)$, but it is so complicated that we omit it. We believe that $C(\nu)$ coincides with the constant on the right hand side of (5). In order to prove the above mentioned results when $|\nu| > 1/2$, we need to use the result in [2] on the order of decay of the density.

In the rest of this article we directly deduce (4) and (5) from the expression for $f_{a,b}^{(\nu)}(t)$ given in Theorem 2.

Proof of (4). We have

$$f_{a,b}^{(0)}(t) = \Big\{1 - \int_0^\infty \frac{1}{x}e^{-(c-1)x}L_{0,c}(x)dx\Big\}q(t,a,b) + \Psi_{a,b}^{(0),2}(t),$$

where $c = a/b$ and $\Psi_{a,b}^{(0),2}(t)$ is given in the previous section. Setting

$$S(t) = \frac{1}{\sqrt{2\pi t}}\int_0^\infty \Big(u + \frac{a-b}{\sqrt{t}}\Big)e^{-\frac{1}{2}(u+\frac{a-b}{\sqrt{t}})^2}du \times \int_0^\infty L_{0,c}(x)e^{-(c-1)x}e^{-\frac{x\sqrt{t}u}{b}}dx,$$

we have $\Psi_{a,b}^{(0),2}(t) = b^{-1}S(t)$.

Fix arbitrary $\varepsilon > 0$. Then, by (3), there exists $\delta \in (0,1)$ such that

$$\Big|L_{0,c}(x) - \frac{\log c}{(\log x)^2}\Big| < \frac{\varepsilon}{(\log x)^2}$$

holds for every $x \in (0,\delta)$. We also fix $\eta > 1$ and assume that t satisfies

$$\frac{(\log t)^3}{\sqrt{t}} < \delta, \quad \eta < \min\{\sqrt{t}, (\log t)^3\}, \quad \frac{\log \eta}{\log\sqrt{t}} < \frac{1}{2}.$$

We devide the integral defining $S(t)$ into four parts, that is, we set

$$S_1(t) = \frac{1}{\sqrt{2\pi t}}\int_0^\infty \Big(u + \frac{a-b}{\sqrt{t}}\Big)e^{-\frac{1}{2}(u+\frac{a-b}{\sqrt{t}})^2}du \times \int_\delta^\infty L_{0,c}(x)e^{-(c-1)x}e^{-\frac{x\sqrt{t}u}{b}}dx,$$

$$S_2(t) = \frac{1}{\sqrt{2\pi t}}\int_0^\infty \Big(u + \frac{a-b}{\sqrt{t}}\Big)e^{-\frac{1}{2}(u+\frac{a-b}{\sqrt{t}})^2}du \times \int_{(\log t)^3/\sqrt{t}}^\delta L_{0,c}(x)e^{-(c-1)x}e^{-\frac{x\sqrt{t}u}{b}}dx,$$

$$S_3(t) = \frac{1}{\sqrt{2\pi t}}\int_0^\infty \Big(u + \frac{a-b}{\sqrt{t}}\Big)e^{-\frac{1}{2}(u+\frac{a-b}{\sqrt{t}})^2}du \times \int_{\eta/\sqrt{t}}^{(\log t)^3/\sqrt{t}} L_{0,c}(x)e^{-(c-1)x}e^{-\frac{x\sqrt{t}u}{b}}dx,$$

$$S_4(t) = \frac{1}{\sqrt{2\pi t}}\int_0^\infty \Big(u + \frac{a-b}{\sqrt{t}}\Big)e^{-\frac{1}{2}(u+\frac{a-b}{\sqrt{t}})^2}du \times \int_0^{\eta/\sqrt{t}} L_{0,c}(x)e^{-(c-1)x}e^{-\frac{x\sqrt{t}u}{b}}dx.$$

For an estimate for $S_1(t)$ we recall $I_0(z) = O(z^{-1/2}e^z)$ and $K_0(z) = O(z^{-1/2}e^{-z})$ as $z \to \infty$. Then we see that there exists $C_1 > 0$ such that

$$|L_{0,c}(x)e^{-(c-1)x}| \leqq C_1 e^{-2x}, \quad x > \delta.$$

Hence we obtain

$$|S_1(t)| \leqq \frac{C_1}{\sqrt{2\pi t}}\int_0^\infty \Big(u + \frac{a-b}{\sqrt{t}}\Big)du \int_\delta^\infty e^{-2x - \frac{x\sqrt{t}u}{b}}du,$$

which is $O(t^{-3/2})$. This implies $t(\log t)^2 S_1(t) \to 0,\ t \to \infty$.

By the choice of the constant δ, there exists $C_2 > 0$ such that

$$|L_{0,c}(x)| \leqq \frac{C_2}{(\log x)^2}, \quad 0 < x < \delta. \tag{6}$$

Hence we get

$$\begin{aligned}|S_2(t)| &\leqq \frac{1}{\sqrt{2\pi t}}\int_0^\infty \left(u + \frac{a-b}{\sqrt{t}}\right)du \\ &\qquad \times \int_{(\log t)^3/\sqrt{t}}^{\delta} \frac{C_2}{(\log x)^2} e^{-\frac{\sqrt{t}ux}{b}}dx \\ &\leqq \frac{C_3}{t(\log t)^3}\int_{(\log t)^3/\sqrt{t}}^{\delta}\frac{dx}{x(\log x)^2}.\end{aligned}$$

From this we easily conclude that $S_2(t) = O(t^{-1}(\log t)^{-3})$.

For an estimate for $S_3(t)$ we write

$$\begin{aligned}S_3(t) = &\frac{1}{\sqrt{2\pi t}}\int_{(a-b)/\sqrt{t}}^\infty u e^{-u^2/2}du \\ &\times \int_{\eta/\sqrt{t}}^{(\log t)^3/\sqrt{t}} L_{0,c}(x) e^{-\frac{\sqrt{t}ux}{b}}dx.\end{aligned}$$

Then, by (6), we get

$$\begin{aligned}|S_3(t)| &\leqq \frac{C_4}{\sqrt{t}}\int_0^\infty u du \int_{\eta/\sqrt{t}}^{(\log t)^3/\sqrt{t}} \frac{1}{(\log x)^2} e^{-\frac{\sqrt{t}xu}{b}}dx \\ &\leqq \frac{C_5}{t(\log t)^2 \eta}\end{aligned}$$

for some positive constants C_4 and C_5.

We next set

$$\begin{aligned}\overline{S}_4(t) = &\frac{1}{\sqrt{2\pi t}}\int_0^\infty \left(u + \frac{a-b}{\sqrt{t}}\right) e^{-\frac{1}{2}(u+\frac{a-b}{\sqrt{t}})^2}du \\ &\times \int_0^{\eta/\sqrt{t}} \frac{\log c}{(\log x)^2} e^{-(c-1)x} e^{-\frac{\sqrt{t}ux}{b}}dx.\end{aligned}$$

Then we have $|S_4(t) - \overline{S}_4(t)| \leqq \varepsilon \overline{S}_4(t)$ and

$$\begin{aligned}t(\log\sqrt{t})^2 \overline{S}_4(t) = &\frac{\log c}{\sqrt{2\pi}}\int_{(a-b)/\sqrt{t}}^\infty u e^{-u^2/2}du \\ &\times \int_0^\eta \left(\frac{\log\sqrt{t}}{\log(\sqrt{t}/y)}\right)^2 e^{-\frac{uy}{b}}dy.\end{aligned}$$

By the choice of η and t, we have for $0 < y < \eta$

$$0 < \frac{\log\sqrt{t}}{\log(\sqrt{t}/y)} \leqq \frac{\log\sqrt{t}}{\log\sqrt{t} - \log\eta} \leqq 2.$$

Hence the dominated convergence theorem implies

$$\lim_{t\to\infty} t(\log\sqrt{t})^2 \overline{S}_4(t) = \frac{b\log c}{2} - \frac{b\log c}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}-\frac{u\eta}{b}}du.$$

Combining the above mentioned estimates and letting $\eta \to \infty$, we obtain

$$\limsup_{t\to\infty}\left|t(\log\sqrt{t})^2 S(t) - \frac{b\log c}{2}\right| \leqq \frac{b\log c}{2}\varepsilon$$

and the desired result. □

Remark 2. By using the formula (see [11, p.80])

$$K_0(z) = -\log(z/2)I_0(z) + \sum_{m=0}^\infty \frac{(z/2)^{2m}}{(m!)^2}\psi(m+1),$$

where $\psi(m+1) = \sum_{k=1}^m k^{-1} - \gamma$ for the Euler constant γ, we have asymptotic expansion for $L_{0,c}(x)$ as $x \downarrow 0$ and can give the asymptotic expansion for $f_{a,b}^{(0)}(t)$ in the form

$$f_{a,b}^{(0)}(t) = \frac{2\log c}{t(\log t)^2}\left\{1 + \frac{\alpha_1}{\log t} + \frac{\alpha_2}{(\log t)^2} + \cdots\right\}.$$

After some computations, we can show $\alpha_1 = 2(\gamma - \log 2 + 2\log b)$. In general the explicit expressions for the constants $\alpha_2, \alpha_3, \ldots$ are complicated and we omit them. For a related result, see [10].

Proof of (5). We start from

$$\begin{aligned}f_{a,b}^{(\nu)}(t) = &c^{-\nu}q(t,a,b) \\ &\times \left\{c^{-|\nu|} - \int_0^\infty \frac{1}{x}L_{|\nu|,c}(x)e^{-(c-1)x}dx\right\} \\ &+ c^{-\nu}\Psi_{a,b}^{(\nu),2}(t).\end{aligned}$$

Then, by a simple change of variables in the defining integral for $\Psi_{a,b}^{(\nu),2}(t)$, we obtain

$$\begin{aligned}\Psi_{a,b}^{(\nu),2}(t) = &\frac{1}{b\sqrt{2\pi}t^{1+|\nu|}}\int_0^\infty\int_0^\infty \mathbf{1}_{[(a-b)/\sqrt{t},\infty)}(u) \\ &\times u e^{-u^2/2}\frac{L_{|\nu|,c}(y/\sqrt{t})}{(y/\sqrt{t})^{2|\nu|}} y^{2|\nu|} e^{-\frac{uy}{b}}dudy.\end{aligned}$$

By (3) the function $L_{|\nu|,c}(x)/x^{2|\nu|}$ is bounded near 0 and, on $(0,\infty)$ since $L_{|\nu|,c}(x)$ decays exponentially at ∞. Therefore, by the dominated convergence theorem, we obtain after some manupulations

$$\begin{aligned}&\lim_{t\to\infty} t^{1+|\nu|}\Psi_{a,b}^{(\nu),2}(t) \\ &= \frac{1}{\sqrt{2\pi}}C_\nu\Gamma(1+2|\nu|)b^{2|\nu|}2^{-|\nu|-1/2}\Gamma\left(\frac{1-2|\nu|}{2}\right),\end{aligned} \tag{7}$$

where the constant C_ν is given by

$$C_\nu = \lim_{x\downarrow 0}\frac{L_{|\nu|,c}(x)}{x^{2|\nu|}} = \cos(\nu\pi)\frac{c^{|\nu|} - c^{-|\nu|}}{2^{2|\nu|-1}\Gamma(|\nu|)\Gamma(|\nu|+1)}.$$

Finally, by simplifying the right hand side of (7) via the functional equality and duplication formula for the Gamma function, we arrive at the desired result. □

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