Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

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出版情報：Journal of Math-for-Industry (JMI). 1 (A), pp.7-15, 2009-04-08. 九州大学大学院数理学研究院

Journal of Math-for-industry, Vol.1(2009A-2), pp.7–15

Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

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Received on February 26, 2009

Abstract. Mack estimated the mean squared errors of the outstanding claims reserve of each accident year and of the overall claims reserve in order to obtain their confidence intervals within his distribution-free model. We generalise his formulae by allowing for arbitrary exponents in the variance assumption. Our formula is also capable of giving a confidence interval of the amount that the insurer is liable to pay each year.

Keywords. Mack's formula, claims reserving, chain-ladder method, mean squared error.

1. Introduction

The chain-ladder method is classical and yet probably the most widely used in stochastic claims reserving. Although formerly thought of simply as a deterministic algorithm, it has been justified so far by many stochastic models. Mack [1, 2] constructed one such model that is remarkable for being distribution-free, and obtained confidence intervals of the outstanding claims reserve of each accident year and of the overall claims reserve, via formulae estimating their mean squared errors. The aim of the present paper is to give a single formula that generalises Mack's in two senses. Firstly, our formula is general enough to yield a confidence interval not only of the outstanding claims reserve of each accident year and the overall claims reserve but also of the amount that the insurer is liable to pay each year. Secondly, we allow any real number to be the exponent in the assumption on the conditional variance of claims amounts, as opposed to Mack, who assumed that the conditional variance is proportional to the immediately preceding claims amount, i.e. the exponent is 1 (see Assumption 3 for further details).

We now introduce some notation. Let $(\Omega, \mathcal{F}, P)$ be a probability space, on which all random variables that appear below are defined. For a set $\mathcal{X}$ of random variables, we write $\sigma(\mathcal{X})$ for the sub-σ-algebra of $\mathcal{F}$ generated by the elements of $\mathcal{X}$. Equality between random variables is always understood to mean almost sure equality.

Denote by $C_{i,j}$ the cumulative claims amount of accident year i after development year j, where $i, j = 1, \ldots, n$. Mathematically speaking, we let $C_{i,j}$ be a positive-valued random variable, which is tacitly assumed to be square-integrable, so that its expectation and variance are well defined. We understand that the random variables $C_{i,j}$ have been observed if $i + j \leq n + 1$, and set

$$\mathcal{D} = \sigma\big(\{C_{i,j} \mid i + j \leq n + 1\}\big).$$

We further set

$$\mathcal{G}_{i,j} = \sigma\big(\{C_{i,1}, \ldots, C_{i,j}\}\big)$$

for $i, j = 1, \ldots, n$.

We shall make three assumptions on $C_{i,j}$. The first assumption is the independence of the accident years:

Assumption 1. The σ-algebras $\mathcal{G}_{1,n}, \ldots, \mathcal{G}_{n,n}$ are independent.

The second is the standard chain-ladder assumption:

Assumption 2. For each $j = 1, \ldots, n-1$, there exists a positive constant f_j such that

$$E[C_{i,j+1}|\mathcal{G}_{i,j}] = C_{i,j} f_j$$

for all $i = 1, \ldots, n$.

These two assumptions are also made by Mack. In addition, he assumed that for each $j = 1, \ldots, n-1$, there exists a positive constant v_j such that

$$V(C_{i,j+1}|\mathcal{G}_{i,j}) = C_{i,j} v_j$$

for all $i = 1, \ldots, n$. We shall generalise this variance assumption by replacing $C_{i,j}$ by $C_{i,j}^{\alpha}$, where α is an arbitrary real number:

Assumption 3. For each $j = 1, \ldots, n-1$, there exists a positive constant v_j such that

$$V(C_{i,j+1}|\mathcal{G}_{i,j}) = C_{i,j}^{\alpha} v_j$$

for all $i = 1, \ldots, n$, where α is any fixed real number.

The paper is organised as follows. We first provide point estimators of f_j, v_j, and $C_{i,j}$ in Section 2, and justify them in Section 3. In Section 4, we give estimators of the mean squared errors of what actuaries, rather than mathematicians, are interested in. Section 5 is devoted to stating our main formula, which will be justified in Section 6, and to showing that it does indeed lead to the estimators given in Section 4. Practising actuaries who are not keen on knowing our formula in full generality or on understanding its proof are advised to read Sections 2 and 4 only.

2. Point estimators

Estimate 1. *We estimate f_j by*

$$\hat{f}_j = \frac{\sum_{i=1}^{n-j} C_{i,j}^{1-\alpha} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}^{2-\alpha}}$$

for $j = 1, \dots, n-1$.

Remark 1. If $\alpha = 1$, then

$$\hat{f}_j = \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}}$$

is the chain-ladder estimator. This is why Mack adopted the variance assumption with $\alpha = 1$.

If $\alpha = 2$, then

$$\hat{f}_j = \frac{1}{n-j} \sum_{i=1}^{n-j} \frac{C_{i,j+1}}{C_{i,j}}$$

is the arithmetic mean of the age-to-age factors $C_{i,j+1}/C_{i,j}$. Note that $\alpha = 2$ is necessarily the case in some natural models stronger than ours. As an example, let us adopt Assumption 1 and assume, instead of Assumptions 2 and 3, that $C_{i,j+1}/C_{i,j}$ is independent of $\mathcal{G}_{i,j}$ for any $i = 1, \dots, n$ and $j = 1, \dots, n-1$, and that the random variables $C_{i,j+1}/C_{i,j}$ for $i = 1, \dots, n$ are identically distributed for each $j = 1, \dots, n-1$. Then it is easy to derive Assumption 2 and Assumption 3 with $\alpha = 2$.

Estimate 2. *We estimate v_j by*

$$\hat{v}_j = \frac{1}{n-j-1} \sum_{i=1}^{n-j} C_{i,j}^{2-\alpha} \left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j \right)^2$$

for $j = 1, \dots, n-2$, and v_{n-1} by

$$\hat{v}_{n-1} = \min \left\{ \frac{\hat{v}_{n-2}^2}{\hat{v}_{n-3}}, \hat{v}_{n-2}, \hat{v}_{n-3} \right\}.$$

Remark 2. Since $C_{1,n}/C_{1,n-1}$ is the only age-to-age factor observed from $n-1$ to n, it is impossible to obtain an estimator of v_{n-1} in the same way as other v_j; here we use the estimator $\hat{v}_{n-1}$ in accordance with Mack.

The estimator $\hat{v}_{n-1}$ is defined only if $n \geq 4$. Although we normally have large enough n in practice, we would have to construct a different estimator if we should have n less than 4. If $n = 3$, setting $\hat{v}_2 = \hat{v}_1$ would be legitimate. If $n = 2$, constructing $\hat{v}_1$ would be very difficult if not impossible. The case $n = 1$ is meaningless.

Estimate 3. *We estimate $C_{i,j}$ by*

$$\hat{C}_{i,j} = C_{i,n+1-i} \hat{f}_{n+1-i} \cdots \hat{f}_{j-1}$$

whenever $i + j \geq n + 2$.

3. Justification for the point estimators

The following σ-algebras are of great use in our model:

Definition 1. We set

$$\mathcal{B}_j = \sigma\bigl(\{C_{i,k} \mid i + k \leq n+1,\ k \leq j\}\bigr) \subset \mathcal{D}$$

for $j = 1, \dots, n$.

Proposition 1. *The estimator $\hat{f}_j$ is $\mathcal{B}_{j+1}$-measurable for $j = 1, \dots, n-1$. It follows that the estimator $\hat{C}_{i,j}$ is $\mathcal{B}_j$-measurable whenever $i + j \geq n + 2$.*

Proof. Obvious. □

Remark 3. If $i+j \leq n+1$, then Assumption 1 shows that

$$E[C_{i,j+1}|\mathcal{B}_j] = E[C_{i,j+1}|\mathcal{G}_{i,j}] = C_{i,j} f_j,$$
$$V(C_{i,j+1}|\mathcal{B}_j) = V(C_{i,j+1}|\mathcal{G}_{i,j}) = C_{i,j}^{\alpha} v_j$$

together with Assumptions 2 and 3.

3.1. Justification for $\hat{f}_j$

Proposition 2. *Let $j = 1, \dots, n-1$. Then the estimator $\hat{f}_j$ is unbiased. More generally, whenever $\lambda = (\lambda_1, \dots, \lambda_{n-j})$ is a $\mathcal{B}_j$-measurable $\mathbb{R}^{n-j}$-valued random variable with non-negative components that add up to 1, the estimator*

$$\hat{f}_j^{\lambda} = \sum_{i=1}^{n-j} \lambda_i \frac{C_{i,j+1}}{C_{i,j}}$$

satisfies $E[\hat{f}_j^{\lambda}|\mathcal{B}_j] = f_j$ and therefore is an unbiased estimator of f_j.

Moreover, $\hat{f}_j$ is the best unbiased estimator in the sense that it minimises $V(\hat{f}_j^{\lambda}|\mathcal{B}_j)$ and $V(\hat{f}_j^{\lambda})$ amongst all such random variables λ.

Proof. We should first bear in mind that since

$$\hat{f}_j = \frac{\sum_{i=1}^{n-j} C_{i,j}^{1-\alpha} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}^{2-\alpha}} = \sum_{i=1}^{n-j} \left(\frac{C_{i,j}^{2-\alpha}}{\sum_{i'=1}^{n-j} C_{i',j}^{2-\alpha}} \cdot \frac{C_{i,j+1}}{C_{i,j}} \right),$$

we have $\hat{f}_j^{\lambda} = \hat{f}_j$ if $\lambda_i = C_{i,j}^{2-\alpha} / \sum_{i'=1}^{n-j} C_{i',j}^{2-\alpha}$ for all $i = 1, \dots, n-j$.

The unbiasedness can be checked as follows:

$$E[\hat{f}_j^{\lambda}|\mathcal{B}_j] = \sum_{i=1}^{n-j} \lambda_i \frac{E[C_{i,j+1}|\mathcal{B}_j]}{C_{i,j}} = \sum_{i=1}^{n-j} \lambda_i f_j = f_j.$$

For the bestness of $\hat{f}_j$, since

$$V(\hat{f}_j^{\lambda}|\mathcal{B}_j) = \sum_{i=1}^{n-j} \lambda_i^2 \frac{V(C_{i,j+1}|\mathcal{B}_j)}{C_{i,j}^2} = v_j \sum_{i=1}^{n-j} \frac{\lambda_i^2}{C_{i,j}^{2-\alpha}},$$

the Cauchy-Schwarz inequality implies that

$$V(\hat{f}_j^\lambda|\mathcal{B}_j)\sum_{i=1}^{n-j}C_{i,j}^{2-\alpha} \geq v_j\left(\sum_{i=1}^{n-j}\lambda_i\right)^2 = v_j,$$

i.e. $V(\hat{f}_j^\lambda|\mathcal{B}_j) \geq v_j/\sum_{i=1}^{n-j}C_{i,j}^{2-\alpha}$, with equality if and only if $\hat{f}_j^\lambda = \hat{f}_j$. The unconditional variance satisfies

$$V(\hat{f}_j^\lambda) = V\bigl(E[\hat{f}_j^\lambda|\mathcal{B}_j]\bigr) + E\bigl[V(\hat{f}_j^\lambda|\mathcal{B}_j)\bigr] = E\bigl[V(\hat{f}_j^\lambda|\mathcal{B}_j)\bigr]$$

because $E[\hat{f}_j^\lambda|\mathcal{B}_j] = f_j$ is a constant, and so the inequality above shows that $V(\hat{f}_j^\lambda) \geq E[v_j/\sum_{i=1}^{n-j}C_{i,j}^{2-\alpha}]$, with equality if and only if $\hat{f}_j^\lambda = \hat{f}_j$. This completes the proof. □

3.2. Justification for $\hat{v}_j$

Proposition 3. *For $j = 1, \ldots, n-2$, the estimator $\hat{v}_j$ satisfies $E[\hat{v}_j|\mathcal{B}_j] = v_j$ and therefore is unbiased.*

Proof. Write $U = \sum_{i=1}^{n-j}C_{i,j}^{1-\alpha}C_{i,j+1}$ and $V = \sum_{i=1}^{n-j}C_{i,j}^{2-\alpha}$, so that $\hat{f}_j = U/V$. Then

$$\begin{aligned}
&(n-j-1)\hat{v}_j\\
&\quad= \sum_{i=1}^{n-j}C_{i,j}^{-\alpha}C_{i,j+1}^2 - 2\hat{f}_j\sum_{i=1}^{n-j}C_{i,j}^{1-\alpha}C_{i,j+1} + \hat{f}_j^2\sum_{i=1}^{n-j}C_{i,j}^{2-\alpha}\\
&\quad= \sum_{i=1}^{n-j}C_{i,j}^{-\alpha}C_{i,j+1}^2 - 2\cdot\frac{U}{V}\cdot U + \left(\frac{U}{V}\right)^2 V\\
&\quad= \sum_{i=1}^{n-j}C_{i,j}^{-\alpha}C_{i,j+1}^2 - \frac{U^2}{V},
\end{aligned}$$

and so

$$(n-j-1)E[\hat{v}_j|\mathcal{B}_j] = \sum_{i=1}^{n-j}C_{i,j}^{-\alpha}E[C_{i,j+1}^2|\mathcal{B}_j] - \frac{E[U^2|\mathcal{B}_j]}{V}.$$

Here

$$\begin{aligned}
E[C_{i,j+1}^2|\mathcal{B}_j] &= V(C_{i,j+1}|\mathcal{B}_j) + E[C_{i,j+1}|\mathcal{B}_j]^2\\
&= C_{i,j}^\alpha v_j + C_{i,j}^2 f_j^2
\end{aligned}$$

for $i = 1, \ldots, n-j$, and

$$\begin{aligned}
E[U^2|\mathcal{B}_j] &= V(U|\mathcal{B}_j) + E[U|\mathcal{B}_j]^2\\
&= \sum_{i=1}^{n-j}C_{i,j}^{2-2\alpha}V(C_{i,j+1}|\mathcal{B}_j)\\
&\qquad + \left(\sum_{i=1}^{n-j}C_{i,j}^{1-\alpha}E[C_{i,j+1}|\mathcal{B}_j]\right)^2\\
&= \sum_{i=1}^{n-j}C_{i,j}^{2-\alpha}v_j + \left(\sum_{i=1}^{n-j}C_{i,j}^{2-\alpha}f_j\right)^2\\
&= Vv_j + V^2 f_j^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(n-j-1)E[\hat{v}_j|\mathcal{B}_j] &= \sum_{i=1}^{n-j}C_{i,j}^{-\alpha}(C_{i,j}^\alpha v_j + C_{i,j}^2 f_j^2)\\
&\qquad - \frac{Vv_j + V^2 f_j^2}{V}\\
&= (n-j-1)v_j,
\end{aligned}$$

which completes the proof. □

Remark 4. Since $\hat{v}_{n-1}$ was defined artificially, we cannot hope for its unbiasedness.

3.3. Justification for $\hat{C}_{i,j}$

Proposition 4. *We have*

$$E[C_{i,j}|\mathcal{D}] = C_{i,n+1-i}f_{n+1-i}\cdots f_{j-1}$$

whenever $i + j \geq n+2$.

Proof. We fix $i = 2, \ldots, n$ and proceed by induction on j. If $j = n+2-i$, then

$$\begin{aligned}
E[C_{i,n+2-i}|\mathcal{D}] &= E[C_{i,n+2-i}|\mathcal{G}_{i,n+1-i}]\\
&= C_{i,n+1-i}f_{n+1-i}.
\end{aligned}$$

Suppose that the equality holds for j. Then

$$\begin{aligned}
E[C_{i,j+1}|\mathcal{D}] &= E[C_{i,j+1}|\mathcal{G}_{i,n+1-i}]\\
&= E\bigl[E[C_{i,j+1}|\mathcal{G}_{i,j}]\big|\mathcal{G}_{i,n+1-i}\bigr]\\
&= E[C_{i,j}f_j|\mathcal{G}_{i,n+1-i}]\\
&= E[C_{i,j}|\mathcal{G}_{i,n+1-i}]f_j\\
&= C_{i,n+1-i}f_{n+1-i}\cdots f_{j-1}f_j,
\end{aligned}$$

the last equality following from the inductive hypothesis. This establishes the equality for $j+1$. □

Proposition 5. *We have*

$$E[\hat{C}_{i,j}|\mathcal{B}_{n+1-i}] = C_{i,n+1-i}f_{n+1-i}\cdots f_{j-1}$$

whenever $i + j \geq n+2$.

Proof. We fix $i = 2, \ldots, n$ and proceed by induction on j. If $j = n+2-i$, then we have

$$\begin{aligned}
E[\hat{C}_{i,n+2-i}|\mathcal{B}_{n+1-i}] &= E[C_{i,n+1-i}\hat{f}_{n+1-i}|\mathcal{B}_{n+1-i}]\\
&= C_{i,n+1-i}E[\hat{f}_{n+1-i}|\mathcal{B}_{n+1-i}]\\
&= C_{i,n+1-i}f_{n+1-i}
\end{aligned}$$

by Proposition 2. Suppose that the equality holds for j. Then, using Propositions 1 and 2 and the inductive hypothesis, we have

$$\begin{aligned}
E[\hat{C}_{i,j+1}|\mathcal{B}_{n+1-i}] &= E\bigl[E[\hat{C}_{i,j+1}|\mathcal{B}_j]\big|\mathcal{B}_{n+1-i}\bigr]\\
&= E\bigl[E[\hat{C}_{i,j}\hat{f}_j|\mathcal{B}_j]\big|\mathcal{B}_{n+1-i}\bigr]\\
&= E\bigl[\hat{C}_{i,j}E[\hat{f}_j|\mathcal{B}_j]\big|\mathcal{B}_{n+1-i}\bigr]\\
&= E[\hat{C}_{i,j}f_j|\mathcal{B}_{n+1-i}]\\
&= E[\hat{C}_{i,j}|\mathcal{B}_{n+1-i}]f_j\\
&= C_{i,n+1-i}f_{n+1-i}\cdots f_{j-1}f_j,
\end{aligned}$$

establishing the equality for $j+1$. □

The following corollary means that $\hat{C}_{i,j}$ is an unbiased estimator of $C_{i,j}$ in some sense:

Corollary 1. *Whenever $i+j \geq n+2$, we have*

$$E[\hat{C}_{i,j}|\mathcal{B}_{n+1-i}] = E[C_{i,j}|\mathcal{B}_{n+1-i}]$$

and so

$$E[\hat{C}_{i,j}] = E[C_{i,j}].$$

Proof. Propositions 4 and 5 show that

$$E[\hat{C}_{i,j}|\mathcal{B}_{n+1-i}] = E[C_{i,j}|\mathcal{D}],$$

from which the corollary easily follows. □

4. Estimators of mean squared errors

Linear combinations of the random variables $C_{i,j}$ include many practically important values; for example, the overall claims reserve can be written as

$$\sum_{i=2}^{n}(C_{i,n} - C_{i,n+1-i}).$$

If S is a linear combination of the random variables $C_{i,j}$, then its natural estimator $\hat{S}$ can be constructed from S by replacing $C_{i,j}$ with $\hat{C}_{i,j}$ whenever $i+j \geq n+2$. For instance, the estimator of the overall claims reserve is

$$\sum_{i=2}^{n}(\hat{C}_{i,n} - C_{i,n+1-i}).$$

We shall always use this estimator for linear combinations of $C_{i,j}$. Note that the estimator $\hat{S}$ is a $\mathcal{D}$-measurable random variable and satisfies $E[\hat{S}] = E[S]$ because of Corollary 1.

Although the point estimator $\hat{S}$ is easy to find, a confidence interval of S is much more difficult, partly because our model does not specify a distribution of $C_{i,j}$. For this purpose, Mack looked at the mean squared error of the point estimator $\hat{S}$:

Definition 2. Let S be a linear combination of the random variables $C_{i,j}$. Then the *mean squared error* $\operatorname{mse}\hat{S}$ of its estimator $\hat{S}$ is defined by

$$\operatorname{mse}\hat{S} = E\big[(S-\hat{S})^2\big|\mathcal{D}\big].$$

There are several approaches to a confidence interval of S via the mean squared error $\operatorname{mse}\hat{S}$. It is reasonable to estimate the 95% confidence interval of S by

$$\big(\hat{S} - 2(\operatorname{mse}\hat{S})^{1/2}, \hat{S} + 2(\operatorname{mse}\hat{S})^{1/2}\big)$$

or by

$$\big(\hat{S} - 3(\operatorname{mse}\hat{S})^{1/2}, \hat{S} + 3(\operatorname{mse}\hat{S})^{1/2}\big).$$

Chebyshev's inequality ensures that

$$\big(\hat{S} - 2\sqrt{5}(\operatorname{mse}\hat{S})^{1/2}, \hat{S} + 2\sqrt{5}(\operatorname{mse}\hat{S})^{1/2}\big)$$

is at least 95% confidence interval because

$$P\big(|S-\hat{S}| \geq 2\sqrt{5}(\operatorname{mse}\hat{S})^{1/2}\big|\mathcal{D}\big) \leq \frac{E\big[(S-\hat{S})^2\big|\mathcal{D}\big]}{20\operatorname{mse}\hat{S}} = 0.05;$$

but the interval is usually too large to be of practical use.

The aim of this paper is to estimate the mean squared errors for several important linear combinations of $C_{i,j}$ within our model. The present section will focus on giving estimators, and they will be justified in subsequent sections.

For notational convenience, we set $\hat{C}_{i,j} = C_{i,j}$ whenever $i+j \leq n+1$, and make the following definition:

Definition 3. We define

$$\hat{A}_{i,l} = \frac{\hat{v}_l}{\hat{f}_l^2}\left(\frac{1}{\hat{C}_{i,l}^{2-\alpha}} + \frac{1}{\sum_{m=1}^{n-l} C_{m,l}^{2-\alpha}}\right),$$
$$\hat{B}_l = \frac{\hat{v}_l}{\hat{f}_l^2 \sum_{m=1}^{n-l} C_{m,l}^{2-\alpha}}$$

for $i, l = 1, \ldots, n$.

Estimate 4. *Suppose that $i+j \geq n+2$. Then we estimate $\operatorname{mse}\hat{C}_{i,j}$ by*

$$\hat{C}_{i,j}^2 \sum_{l=n+1-i}^{j-1} \hat{A}_{i,l}.$$

Remark 5. If $\alpha = 1$ and $j = n$, then this estimator was given in [1, Theorem 3] and [2, Equation (7)].

Estimate 5. *Let*

$$S = \sum_{i=2}^{n}(C_{i,n} - C_{i,n+1-i})$$

be the overall claims reserve. Then we estimate $\operatorname{mse}\hat{S}$ by

$$\sum_{i=2}^{n}\left(\hat{C}_{i,n}^2 \sum_{l=n+1-i}^{n-1} \hat{A}_{i,l}\right.$$
$$\left. + 2\sum_{i=2}^{n}\left(\hat{C}_{i,n}\left(\sum_{i'=i+1}^{n} \hat{C}_{i',n}\right)\left(\sum_{l=n+1-i}^{n-1} \hat{B}_l\right)\right)\right).$$

Remark 6. If $\alpha = 1$, then this estimator was given in [1, Corollary] and [2, Equation (11)].

Estimate 6. *Let $t = 1, \ldots, n-1$ and let*

$$S = \sum_{i=t+1}^{n}(C_{i,n+1-i+t} - C_{i,n-i+t})$$

be the amount that the insurer is liable to pay in t years' time for the claims between accident years $t+1$ and n.

Then we estimate $\operatorname{mse}\hat{S}$ *by*

$$\begin{aligned}
&\sum_{i=t+1}^{n}\sum_{l=n+1-i}^{n-i+t-1}\hat{X}_{i,n+1-i+t}^2\hat{A}_{i,l}\\
&\quad+\sum_{i=t+1}^{n}\hat{C}_{i,n+1-i+t}^2\hat{A}_{i,n-i+t}\\
&\quad+2\sum_{i=t+1}^{n-1}\sum_{i'=i+1}^{\min\{i+t-1,n\}}\sum_{l=n+1-i}^{n-i'+t-1}\hat{X}_{i,n+1-i+t}\hat{X}_{i',n+1-i'+t}\hat{B}_l\\
&\quad+2\sum_{i=t+1}^{n-1}\sum_{i'=i+1}^{\min\{i+t-1,n\}}\hat{X}_{i,n+1-i+t}\hat{C}_{i',n+1-i'+t}\hat{B}_{n-i'+t},
\end{aligned}$$

where $\hat{X}_{i,j}$ *is the estimator of the incremental claims amount of accident year* i *after development year* j*, defined by*

$$\hat{X}_{i,j}=\begin{cases}\hat{C}_{i,j}-\hat{C}_{i,j-1} & \text{if } 2\le j\le n;\\ \hat{C}_{i,1} & \text{if } j=1.\end{cases}$$

In particular, setting $t=1$*, we estimate* $\operatorname{mse}\hat{S}$ *by*

$$\sum_{i=2}^{n}\hat{C}_{i,n+2-i}^2\hat{A}_{i,n+1-i}.$$

5. Statement of the main formula

Estimate 7 (Main formula)**.** *For each* $i=1,\ldots,n$*, let* $j_i,k_i\in\mathbb{Z}$ *be given so that*

$$n+1-i\le j_i\le k_i\le n.$$

Define

$$S=\sum_{i=1}^{n}(C_{i,k_i}-C_{i,j_i}).$$

Then we estimate $\operatorname{mse}\hat{S}$ *by*

$$\sum_{i,l=1}^{n}\hat{\varphi}_{i,l}^2\hat{A}_{i,l}+2\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat{\varphi}_{i,l}\hat{\varphi}_{i',l}\hat{B}_l,$$

where we set

$$\hat{\varphi}_{i,l}=\begin{cases}\hat{C}_{i,k_i}-\hat{C}_{i,j_i} & \text{if } n+1-i\le l<j_i;\\ \hat{C}_{i,k_i} & \text{if } j_i\le l<k_i;\\ 0 & \text{otherwise}\end{cases}$$

for $i,l=1,\ldots,n$.

Postponing justifying Estimate 7 until Section 6, we first show that Estimate 7 does indeed lead to the estimators given in Section 4.

Example 1 (Estimate 5)**.** Set $j_i=n+1-i$ and $k_i=n$ for all $i=1,\ldots,n$. Then

$$S=\sum_{i=1}^{n}(C_{i,n}-C_{i,n+1-i})=\sum_{i=2}^{n}(C_{i,n}-C_{i,n+1-i}).$$

Since

$$\hat{\varphi}_{i,l}=\begin{cases}\hat{C}_{i,n} & \text{if } n+1-i\le l<n;\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}
\sum_{i,l=1}^{n}\hat{\varphi}_{i,l}^2\hat{A}_{i,l}&=\sum_{i=2}^{n}\sum_{l=n+1-i}^{n-1}\hat{C}_{i,n}^2\hat{A}_{i,l}\\
&=\sum_{i=2}^{n}\left(\hat{C}_{i,n}^2\sum_{l=n+1-i}^{n-1}\hat{A}_{i,l}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat{\varphi}_{i,l}\hat{\varphi}_{i',l}\hat{B}_l\\
&\quad=\sum_{1\le i<i'\le n}\sum_{l=n+1-i}^{n-1}\hat{C}_{i,n}\hat{C}_{i',n}\hat{B}_l\\
&\quad=\sum_{i=2}^{n}\left(\hat{C}_{i,n}\left(\sum_{i'=i+1}^{n}\hat{C}_{i',n}\right)\left(\sum_{l=n+1-i}^{n-1}\hat{B}_l\right)\right).
\end{aligned}$$

It follows that we estimate $\operatorname{mse}\hat{S}$ by

$$\begin{aligned}
&\sum_{i,l=1}^{n}\hat{\varphi}_{i,l}^2\hat{A}_{i,l}+2\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat{\varphi}_{i,l}\hat{\varphi}_{i',l}\hat{B}_l\\
&\quad=\sum_{i=2}^{n}\left(\hat{C}_{i,n}^2\sum_{l=n+1-i}^{n-1}\hat{A}_{i,l}\right)\\
&\qquad+2\sum_{i=2}^{n}\left(\hat{C}_{i,n}\left(\sum_{i'=i+1}^{n}\hat{C}_{i',n}\right)\left(\sum_{l=n+1-i}^{n-1}\hat{B}_l\right)\right).
\end{aligned}$$

Example 2 (Estimate 4)**.** Let $p,q=1,\ldots,n$ satisfy $p+q\ge n+2$. Set $j_i=n+1-i$ for all $i=1,\ldots,n$, and

$$k_i=\begin{cases}n+1-i & \text{if } i\ne p;\\ q & \text{if } i=p.\end{cases}$$

Then

$$S=C_{p,q}-C_{p,n+1-p}$$

and so

$$\begin{aligned}
\operatorname{mse}\hat{S}&=E\bigl[(S-\hat{S})^2\big|\mathcal{D}\bigr]\\
&=E\Bigl[\bigl((C_{p,q}-C_{p,n+1-p})-(\hat{C}_{p,q}-C_{p,n+1-p})\bigr)^2\Big|\mathcal{D}\Bigr]\\
&=E\bigl[(C_{p,q}-\hat{C}_{p,q})^2\big|\mathcal{D}\bigr]\\
&=\operatorname{mse}\hat{C}_{p,q}.
\end{aligned}$$

Since

$$\hat{\varphi}_{i,l}=\begin{cases}\hat{C}_{p,q} & \text{if } i=p \text{ and } n+1-p\le l<q;\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\sum_{i,l=1}^{n}\hat{\varphi}_{i,l}^2\hat{A}_{i,l}=\sum_{l=n+1-p}^{q-1}\hat{C}_{p,q}^2\hat{A}_{p,l}=\hat{C}_{p,q}^2\sum_{l=n+1-p}^{q-1}\hat{A}_{p,l}$$

and

$$\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat\varphi_{i,l}\hat\varphi_{i',l}\hat B_l=0.$$

It follows that we estimate $\operatorname{mse}\hat S=\operatorname{mse}\hat C_{p,q}$ by

$$\sum_{i,l=1}^{n}\hat\varphi_{i,l}^2\hat A_{i,l}+2\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat\varphi_{i,l}\hat\varphi_{i',l}\hat B_l$$
$$=\hat C_{p,q}^2\sum_{l=n+1-p}^{q-1}\hat A_{p,l}.$$

Example 3 (Estimate 6)**.** Let $t=1,\dots,n-1$. Set

$$j_i=\begin{cases}n+1-i & \text{for } i=1,\dots,t;\\ n-i+t & \text{for } i=t+1,\dots,n,\end{cases}$$

and

$$k_i=\begin{cases}n+1-i & \text{for } i=1,\dots,t;\\ n+1-i+t & \text{for } i=t+1,\dots,n.\end{cases}$$

Then

$$S=\sum_{i=t+1}^{n}(C_{i,n+1-i+t}-C_{i,n-i+t}).$$

Since

$$\hat\varphi_{i,l}=\begin{cases}\hat C_{i,n+1-i+t}-\hat C_{i,n-i+t}=\hat X_{i,n+1-i+t}\\ \quad\text{if } t+1\le i\le n \text{ and } n+1-i\le l<n-i+t;\\ \hat C_{i,n+1-i+t} \quad \text{if } t+1\le i\le n \text{ and } l=n-i+t;\\ 0 \quad \text{otherwise,}\end{cases}$$

we have

$$\sum_{i,l=1}^{n}\hat\varphi_{i,l}^2\hat A_{i,l}=\sum_{i=t+1}^{n}\sum_{l=n+1-i}^{n-i+t-1}\hat X_{i,n+1-i+t}^2\hat A_{i,l}$$
$$+\sum_{i=t+1}^{n}\hat C_{i,n+1-i+t}^2\hat A_{i,n-i+t}$$

and

$$\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat\varphi_{i,l}\hat\varphi_{i',l}\hat B_l$$
$$=\sum_{i=t+1}^{n-1}\sum_{i'=i+1}^{\min\{i+t-1,n\}}\sum_{l=n+1-i}^{n-i'+t-1}\hat X_{i,n+1-i+t}\hat X_{i',n+1-i'+t}\hat B_l$$
$$+\sum_{i=t+1}^{n-1}\sum_{i'=i+1}^{\min\{i+t-1,n\}}\hat X_{i,n+1-i+t}\hat C_{i',n+1-i'+t}\hat B_{n-i'+t}.$$

It follows that we estimate $\operatorname{mse}\hat S$ by

$$\sum_{i,l=1}^{n}\hat\varphi_{i,l}^2\hat A_{i,l}+2\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat\varphi_{i,l}\hat\varphi_{i',l}\hat B_l$$
$$=\sum_{i=t+1}^{n}\sum_{l=n+1-i}^{n-i+t-1}\hat X_{i,n+1-i+t}^2\hat A_{i,l}$$
$$+\sum_{i=t+1}^{n}\hat C_{i,n+1-i+t}^2\hat A_{i,n-i+t}$$
$$+2\sum_{i=t+1}^{n-1}\sum_{i'=i+1}^{\min\{i+t-1,n\}}\sum_{l=n+1-i}^{n-i'+t-1}\hat X_{i,n+1-i+t}\hat X_{i',n+1-i'+t}\hat B_l$$
$$+2\sum_{i=t+1}^{n-1}\sum_{i'=i+1}^{\min\{i+t-1,n\}}\hat X_{i,n+1-i+t}\hat C_{i',n+1-i'+t}\hat B_{n-i'+t}.$$

6. Justification for the main formula

Suppose that j_i and k_i are given and S is defined as in Estimate 7.

Lemma 1. *The mean squared error* $\operatorname{mse}\hat S$ *decomposes as*

$$\operatorname{mse}\hat S=V(S|\mathcal D)+\big(E[S|\mathcal D]-\hat S\big)^2,$$

the first term being called the process variance *and the second the* estimation error*.*

Proof. We have

$$\begin{aligned}\operatorname{mse}\hat S&=E\big[(S-\hat S)^2\big|\mathcal D\big]\\&=V(S-\hat S|\mathcal D)+E[S-\hat S|\mathcal D]^2\\&=V(S|\mathcal D)+\big(E[S|\mathcal D]-\hat S\big)^2\end{aligned}$$

because $\hat S$ is $\mathcal D$-measurable. □

Lemma 2. *We have*

$$\sum_{i,l=1}^{n}\frac{\hat\varphi_{i,l}^2\hat v_l}{\hat C_{i,l}^{2-\alpha}\hat f_l^2}+\sum_{i,i',l=1}^{n}\frac{\hat\varphi_{i,l}\hat\varphi_{i',l}\hat v_l}{\hat f_l^2\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}$$
$$=\sum_{i,l=1}^{n}\hat\varphi_{i,l}^2\hat A_{i,l}+2\sum_{1\le i<i'\le n}\sum_{l=1}^{n}\hat\varphi_{i,l}\hat\varphi_{i',l}\hat B_l.$$

Proof. Straightforward. □

By Lemmas 1 and 2, it suffices to justify estimating the process variance $V(S|\mathcal D)$ by

$$\sum_{i,l=1}^{n}\frac{\hat\varphi_{i,l}^2\hat v_l}{\hat C_{i,l}^{2-\alpha}\hat f_l^2}$$

and the estimation error $\big(E[S|\mathcal D]-\hat S\big)^2$ by

$$\sum_{i,i',l=1}^{n}\frac{\hat\varphi_{i,l}\hat\varphi_{i',l}\hat v_l}{\hat f_l^2\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}.$$

6.1. Process variance

We shall hereafter adopt the standard convention that the product $a_p \cdots a_q$, where we normally have $p \leq q$, is interpreted as 1 if $p > q$.

Lemma 3. *If $i + j \geq n + 1$, then*

$$V(C_{i,j}|\mathcal{D}) = \sum_{l=n+1-i}^{j-1} E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2 \cdots f_{j-1}^2.$$

Proof. For $l = n + 1 - i, \ldots, j - 1$, we have

$$\begin{aligned}
V(C_{i,l+1}|\mathcal{D}) &= V(C_{i,l+1}|\mathcal{G}_{i,n+1-i}) \\
&= V\big(E[C_{i,l+1}|\mathcal{G}_{i,l}]\big|\mathcal{G}_{i,n+1-i}\big) \\
&\quad + E\big[V(C_{i,l+1}|\mathcal{G}_{i,l})\big|\mathcal{G}_{i,n+1-i}\big] \\
&= V(C_{i,l}f_l|\mathcal{G}_{i,n+1-i}) + E[C_{i,l}^{\alpha}v_l|\mathcal{G}_{i,n+1-i}] \\
&= V(C_{i,l}|\mathcal{G}_{i,n+1-i})f_l^2 + E[C_{i,l}^{\alpha}|\mathcal{G}_{i,n+1-i}]v_l \\
&= V(C_{i,l}|\mathcal{D})f_l^2 + E[C_{i,l}^{\alpha}|\mathcal{D}]v_l.
\end{aligned}$$

Multiplying $f_{l+1}^2 \cdots f_{j-1}^2$ gives

$$\begin{aligned}
V(C_{i,l+1}|\mathcal{D})f_{l+1}^2 \cdots f_{j-1}^2 &= V(C_{i,l}|\mathcal{D})f_l^2 \cdots f_{j-1}^2 \\
&\quad + E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2 \cdots f_{j-1}^2.
\end{aligned}$$

Taking the sum over $l = n+1-i, \ldots, j-1$ and noting that $V(C_{i,n+1-i}|\mathcal{D}) = 0$ because $C_{i,n+1-i}$ is $\mathcal{D}$-measurable, we get the desired result. □

Definition 4. Define $\varphi_{i,l}$ by

$$\varphi_{i,l} = \begin{cases} E[C_{i,k_i} - C_{i,j_i}|\mathcal{D}] & \text{if } n+1-i \leq l < j_i; \\ E[C_{i,k_i}|\mathcal{D}] & \text{if } j_i \leq l < k_i; \\ 0 & \text{otherwise} \end{cases}$$

for $i, l = 1, \ldots, n$.

Remark 7. Note that $\hat{\varphi}_{i,l}$ is an estimator of $\varphi_{i,l}$ for each $i, l = 1, \ldots, n$.

Lemma 4. *We have*

$$V(S|\mathcal{D}) = \sum_{i,l=1}^{n} \frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}.$$

Proof. Since

$$V(S|\mathcal{D}) = \sum_{i=1}^{n} V(C_{i,k_i} - C_{i,j_i}|\mathcal{D}),$$

we only need to prove that

$$V(C_{i,k_i} - C_{i,j_i}|\mathcal{D}) = \sum_{l=n+1-i}^{k_i-1} \frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}$$

for $i = 1, \ldots, n$, noting that $\varphi_{i,l} \neq 0$ only if $n+1-i \leq l \leq k_i - 1$.

Fix i and write $j = j_i$ and $k = k_i$ for simplicity. For $l = j, \ldots, k-1$, we have

$$\begin{aligned}
&V(C_{i,l+1}f_{l+1} \cdots f_{k-1} - C_{i,j}|\mathcal{D}) \\
&\quad = V(C_{i,l+1}f_{l+1} \cdots f_{k-1} - C_{i,j}|\mathcal{G}_{i,n+1-i}) \\
&\quad = V\big(E[C_{i,l+1}f_{l+1} \cdots f_{k-1} - C_{i,j}|\mathcal{G}_{i,l}]\big|\mathcal{G}_{i,n+1-i}\big) \\
&\qquad + E\big[V(C_{i,l+1}f_{l+1} \cdots f_{k-1} - C_{i,j}|\mathcal{G}_{i,l})\big|\mathcal{G}_{i,n+1-i}\big] \\
&\quad = V(C_{i,l}f_l \cdots f_{k-1} - C_{i,j}|\mathcal{G}_{i,n+1-i}) \\
&\qquad + E[C_{i,l}^{\alpha}v_l f_{l+1}^2 \cdots f_{k-1}^2|\mathcal{G}_{i,n+1-i}] \\
&\quad = V(C_{i,l}f_l \cdots f_{k-1} - C_{i,j}|\mathcal{D}) + E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2 \cdots f_{k-1}^2.
\end{aligned}$$

Taking the sum over $l = j, \ldots, k-1$ gives

$$\begin{aligned}
V(C_{i,k} - C_{i,j}|\mathcal{D}) &= V(C_{i,j}f_j \cdots f_{k-1} - C_{i,j}|\mathcal{D}) \\
&\quad + \sum_{l=j}^{k-1} E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2 \cdots f_{k-1}^2.
\end{aligned}$$

Since Lemma 3 shows that

$$\begin{aligned}
&V(C_{i,j}f_j \cdots f_{k-1} - C_{i,j}|\mathcal{D}) \\
&\quad = V(C_{i,j}|\mathcal{D})(f_j \cdots f_{k-1} - 1)^2 \\
&\quad = \sum_{l=n+1-i}^{j-1} E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2 \cdots f_{j-1}^2(f_j \cdots f_{k-1} - 1)^2
\end{aligned}$$

and since Proposition 4 gives

$$\begin{aligned}
&f_{l+1}^2 \cdots f_{j-1}^2(f_j \cdots f_{k-1} - 1)^2 \\
&\quad = (f_{l+1} \cdots f_{k-1} - f_{l+1} \cdots f_{j-1})^2 \\
&\quad = \frac{(C_{i,n+1-i}f_{n+1-i} \cdots f_{k-1} - C_{i,n+1-i}f_{n+1-i} \cdots f_{j-1})^2}{(C_{i,n+1-i}f_{n+1-i} \cdots f_{l-1})^2 f_l^2} \\
&\quad = \frac{E[C_{i,k} - C_{i,j}|\mathcal{D}]^2}{E[C_{i,l}|\mathcal{D}]^2 f_l^2} = \frac{\varphi_{i,l}^2}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}
\end{aligned}$$

for $l = n+1-i, \ldots, j-1$, we have

$$V(C_{i,j}f_j \cdots f_{k-1} - C_{i,j}|\mathcal{D}) = \sum_{l=n+1-i}^{j-1} \frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}.$$

Similarly, since

$$\begin{aligned}
f_{l+1}^2 \cdots f_{k-1}^2 &= \frac{(C_{i,n+1-i}f_{n+1-i} \cdots f_{k-1})^2}{(C_{i,n+1-i}f_{n+1-i} \cdots f_{l-1})^2 f_l^2} \\
&= \frac{E[C_{i,k}|\mathcal{D}]^2}{E[C_{i,l}|\mathcal{D}]^2 f_l^2} = \frac{\varphi_{i,l}^2}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}
\end{aligned}$$

for $l = j, \ldots, k-1$, we have

$$\sum_{l=j}^{k-1} E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2 \cdots f_{k-1}^2 = \sum_{l=j}^{k-1} \frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}.$$

It follows that

$$\begin{aligned}V(C_{i,k}-C_{i,j}|\mathcal{D}) &= V(C_{i,j}f_j\cdots f_{k-1}-C_{i,j}|\mathcal{D})\\ &\quad+\sum_{l=j}^{k-1}E[C_{i,l}^{\alpha}|\mathcal{D}]v_l f_{l+1}^2\cdots f_{k-1}^2\\ &=\sum_{l=n+1-i}^{j-1}\frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}\\ &\quad+\sum_{l=j}^{k-1}\frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}\\ &=\sum_{l=n+1-i}^{k-1}\frac{E[C_{i,l}^{\alpha}|\mathcal{D}]\varphi_{i,l}^2 v_l}{E[C_{i,l}|\mathcal{D}]^2 f_l^2}.\end{aligned}$$ □

This lemma leads to the following estimate:

Estimate 8. *We estimate the process variance $V(S|\mathcal{D})$ by*

$$\sum_{i,l=1}^{n}\frac{\hat{\varphi}_{i,l}^2\hat{v}_l}{\hat{C}_{i,l}^{2-\alpha}\hat{f}_l^2}.$$

6.2. Estimation error

Definition 5. For each $i=1,\dots,n$, we define

$$\Phi_i=E[C_{i,k_i}-C_{i,j_i}|\mathcal{D}]-(\hat{C}_{i,k_i}-\hat{C}_{i,j_i}).$$

Definition 6. Define $\psi_{i,l}$ by

$$\psi_{i,l}=\begin{cases}C_{i,n+1-i}\hat{f}_{n+1-i}\cdots\hat{f}_{l-1}(f_l-\hat{f}_l)f_{l+1}\cdots f_{j_i-1}\\ \qquad\times(f_{j_i}\cdots f_{k_i-1}-1)\quad \text{if } n+1-i\le l<j_i;\\ C_{i,n+1-i}\hat{f}_{n+1-i}\cdots\hat{f}_{l-1}(f_l-\hat{f}_l)f_{l+1}\cdots f_{k_i-1}\\ \qquad\qquad\qquad\qquad\quad \text{if } j_i\le l<k_i;\\ 0 \qquad\qquad\qquad\qquad\quad \text{otherwise}\end{cases}$$

for $i,l=1,\dots,n$.

Lemma 5. *We have*

$$\Phi_i=\sum_{l=1}^{n}\psi_{i,l}$$

for $i=1,\dots,n$.

Proof. Fix i and write $j=j_i$ and $k=k_i$ for simplicity. Then we have

$$\begin{aligned}\frac{\Phi_i}{C_{i,n+1-i}} &= \frac{E[C_{i,k}-C_{i,j}|\mathcal{D}]-(\hat{C}_{i,k}-\hat{C}_{i,j})}{C_{i,n+1-i}}\\ &=(f_{n+1-i}\cdots f_{k-1}-f_{n+1-i}\cdots f_{j-1})\\ &\quad-(\hat{f}_{n+1-i}\cdots\hat{f}_{k-1}-\hat{f}_{n+1-i}\cdots\hat{f}_{j-1})\\ &=(f_{n+1-i}\cdots f_{j-1}-\hat{f}_{n+1-i}\cdots\hat{f}_{j-1})\\ &\qquad\times(f_j\cdots f_{k-1}-1)\\ &\quad+\hat{f}_{n+1-i}\cdots\hat{f}_{j-1}(f_j\cdots f_{k-1}-\hat{f}_j\cdots\hat{f}_{k-1})\\ &=\sum_{l=n+1-i}^{j-1}\hat{f}_{n+1-i}\cdots\hat{f}_{l-1}(f_l-\hat{f}_l)f_{l+1}\cdots f_{j-1}\\ &\qquad\times(f_j\cdots f_{k-1}-1)\\ &\quad+\sum_{l=j}^{k-1}\hat{f}_{n+1-i}\cdots\hat{f}_{l-1}(f_l-\hat{f}_l)f_{l+1}\cdots f_{k-1}\\ &=\frac{1}{C_{i,n+1-i}}\sum_{l=1}^{n}\psi_{i,l},\end{aligned}$$

verifying the lemma. □

Lemma 6. *The estimation error can be written as*

$$\left(E[S|\mathcal{D}]-\hat{S}\right)^2=\sum_{i,i',l,l'=1}^{n}\psi_{i,l}\psi_{i',l'}.$$

Proof. By the definition of Φ_i and Lemma 5, we have

$$\begin{aligned}\left(E[S|\mathcal{D}]-\hat{S}\right)^2 &= \left(\sum_{i=1}^{n}\Phi_i\right)^2=\left(\sum_{i,l=1}^{n}\psi_{i,l}\right)^2\\ &=\sum_{i,i',l,l'=1}^{n}\psi_{i,l}\psi_{i',l'}.\end{aligned}$$ □

Lemma 7. *We have $E[\psi_{i,l}|\mathcal{B}_l]=0$ for $i,l=1,\dots,n$, and $\psi_{i,l}$ is $\mathcal{B}_{l+1}$-measurable for $i=1,\dots,n$ and $l=1,\dots,n-1$.*

Proof. Immediate from Propositions 1 and 2. □

Lemma 8. *We have*

$$E\left[(f_l-\hat{f}_l)^2\middle|\mathcal{B}_l\right]=\frac{v_l}{\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}$$

for $l=1,\dots,n$.

Proof. By Proposition 1 and the proof of Proposition 2, we have

$$\begin{aligned}E\left[(f_l-\hat{f}_l)^2\middle|\mathcal{B}_l\right] &= V(f_l-\hat{f}_l|\mathcal{B}_l)+E[f_l-\hat{f}_l|\mathcal{B}_l]^2\\ &=V(\hat{f}_l|\mathcal{B}_l)=\frac{v_l}{\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}.\end{aligned}$$ □

Lemma 9. *We have*

$$\psi_{i,l}=\frac{\hat{C}_{i,l}\varphi_{i,l}}{E[C_{i,l}|\mathcal{D}]f_l}(f_l-\hat{f}_l)$$

for $i,l=1,\dots,n$.

Proof. Fix i and write $j = j_i$ and $k = k_i$ for simplicity. If $n+1-i \le l < j$, then

$$\begin{aligned}
&\frac{\hat{C}_{i,l}\varphi_{i,l}}{E[C_{i,l}|\mathcal{D}]f_l}(f_l - \hat{f}_l) \\
&\quad = C_{i,n+1-i}\hat{f}_{n+1-i}\cdots\hat{f}_{l-1} \times E[C_{i,k} - C_{i,j}|\mathcal{D}] \\
&\qquad \times (C_{i,n+1-i}f_{n+1-i}\cdots f_{l-1}f_l)^{-1} \times (f_l - \hat{f}_l) \\
&\quad = \frac{\hat{f}_{n+1-i}\cdots\hat{f}_{l-1}(f_l - \hat{f}_l)}{f_{n+1-i}\cdots f_{l-1}f_l} \\
&\qquad \times C_{i,n+1-i}(f_{n+1-i}\cdots f_{k-1} - f_{n+1-i}\cdots f_{j-1}) \\
&\quad = C_{i,n+1-i}\hat{f}_{n+1-i}\cdots\hat{f}_{l-1}(f_l - \hat{f}_l)f_{l+1}\cdots f_{j-1} \\
&\qquad \times (f_j\cdots f_{k-1} - 1) \\
&\quad = \psi_{i,l}.
\end{aligned}$$

The other cases can be dealt with in a similar fashion. □

We follow Mack in looking at $E[\psi_{i,l}\psi_{i',l'}|\mathcal{B}_{\max\{l,l'\}}]$ for the estimator of $\psi_{i,l}\psi_{i',l'}$:

Lemma 10. *For $i, i', l, l' = 1, \dots, n$, we have*

$$\begin{aligned}
&E[\psi_{i,l}\psi_{i',l'}|\mathcal{B}_{\max\{l,l'\}}] \\
&\quad = \begin{cases} \frac{\hat{C}_{i,l}\hat{C}_{i',l}\varphi_{i,l}\varphi_{i',l}v_l}{E[C_{i,l}|\mathcal{D}]E[C_{i',l}|\mathcal{D}]f_l^2\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}} & \text{if } l = l'; \\ 0 & \text{if } l \neq l'. \end{cases}
\end{aligned}$$

Proof. We first consider the case $l \neq l'$. We may assume that $l < l'$. Then since $\psi_{i,l}$ is $\mathcal{B}_{l'}$-measurable and $E[\psi_{i',l'}|\mathcal{B}_{l'}] = 0$ by Lemma 7, the assertion follows.

Now suppose that $l = l'$. Observe that $\hat{C}_{i,l}$, $\varphi_{i,l}$, and $E[C_{i,l}|\mathcal{D}]$ are all $\mathcal{B}_l$-measurable, no matter whether $l \le n+1-i$ or $l \ge n+2-i$. The same is true if i is replaced with i'. Therefore, by Lemmas 9 and 8, we have

$$\begin{aligned}
&E[\psi_{i,l}\psi_{i',l}|\mathcal{B}_l] \\
&\quad = E\left[\frac{\hat{C}_{i,l}\varphi_{i,l}}{E[C_{i,l}|\mathcal{D}]f_l}(f_l - \hat{f}_l) \times \frac{\hat{C}_{i',l}\varphi_{i',l}}{E[C_{i',l}|\mathcal{D}]f_l}(f_l - \hat{f}_l)\middle|\mathcal{B}_l\right] \\
&\quad = \frac{\hat{C}_{i,l}\hat{C}_{i',l}\varphi_{i,l}\varphi_{i',l}}{E[C_{i,l}|\mathcal{D}]E[C_{i',l}|\mathcal{D}]f_l^2}E\left[(f_l - \hat{f}_l)^2\middle|\mathcal{B}_l\right] \\
&\quad = \frac{\hat{C}_{i,l}\hat{C}_{i',l}\varphi_{i,l}\varphi_{i',l}v_l}{E[C_{i,l}|\mathcal{D}]E[C_{i',l}|\mathcal{D}]f_l^2\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}.
\end{aligned}$$
□

This lemma leads to the following estimate:

Estimate 9. *For $i, i', l, l' = 1, \dots, n$, we estimate $\psi_{i,l}\psi_{i',l'}$ by*

$$\frac{\hat{\varphi}_{i,l}\hat{\varphi}_{i',l}\hat{v}_l}{\hat{f}_l^2\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}$$

if $l = l'$, and by 0 if $l \neq l'$.

This estimate and Lemma 6 give the following:

Estimate 10. *We estimate $\bigl(E[S|\mathcal{D}] - \hat{S}\bigr)^2$ by*

$$\sum_{i,i',l=1}^{n}\frac{\hat{\varphi}_{i,l}\hat{\varphi}_{i',l}\hat{v}_l}{\hat{f}_l^2\sum_{m=1}^{n-l}C_{m,l}^{2-\alpha}}.$$

Acknowledgements

The author obtained these results whilst working as a member of the joint research group between Nisshin Fire & Marine Insurance Co., Ltd. and the Faculty of Mathematics, Kyushu University. He wishes to express his sincere gratitude to the group members, including Professor Setsuo Taniguchi and Dr Tatsushi Tanaka of Kyushu University, for many useful discussions.

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