# ”U” TYPE FUNCTIONS

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https://doi.org/10.5109/13519

出版情報：Bulletin of informatics and cybernetics. 35 (1/2), pp.35-39, 2003-12. Research Association of Statistical Sciences
バージョン：
権利関係：

*Bulletin of Informatics and Cybernetics, Vol. 35, No. 1~2, 2003*

# "U" TYPE FUNCTIONS

By

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**Abstract**

A function from the exponential family is proved to have U shape in certain interval. When one of the coefficients is negative then the functional form will be the exact mirror image of its original function. Volume of such $\bigcup$ shaped vessel is calculated.

*Key Words and Phrases:* Quadratic exponential, Mathematical induction, Volume.

## 1. Introduction

Let

$$f(x) = e^{ux^2+vx} \tag{1}$$

define $v = (-1)^{2k-1} mu^{2k-1}$. Here $k, m \in \mathbf{N}$. If $u = 0$ then $v = 0$ and equation (1) becomes simple. So I condition that u $\neq 0$. This condition became necessary for later part of the work. Take $k = 1$. As $x$ value increases from integer 1, then the function decreases up to a point where it starts increasing and then for certain point of abscissa the function will be exactly equal to that value when it was for $x = 1$. The result is also true when $u$ is an irrational number $\pi$. We state and prove a theorem below for a particular case when $k = 1$.

## 2. $\bigcup$ Functions

**Theorem 1** *Let $f$ is defined as in (1). Then for $v = (-1)^{2k-1} mu^{2k-1}$ and $k = 1$, $f(1) = f\left(\mid \frac{u+v}{u} \mid\right)$ for all $u \in \mathbf{R} - \{0\}, m \in \mathbf{N}$.*

Proof. Given $k = 1$. First let us begin that $u \in \mathbf{R}^+$. For $m = 1$, we get $v = -u$ and $\mid \frac{u+v}{u} \mid= 0$. Therefore, $f(1) = e^0 = f\left(\mid \frac{u+v}{u} \mid\right) = 1$. For $m = 2$, we get $v = -2u$ and $\mid \frac{u+v}{u} \mid= 1$. Therefore, $f(1) = e^{-u} = f\left(\mid \frac{u+v}{u} \mid\right)$.

Now take $m = n \in \mathbf{N}$, then we have

$$\begin{aligned} v &= -nu \\ \left| \frac{u+v}{u} \right| &= n-1 \end{aligned}$$

Therefore,

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$$
\begin{aligned}
f(1) &= e^{u(1-n)} \\
f\left(\mid \frac{u+v}{u} \mid\right) &= e^{u(n-1)^2-nu(n-1)} \\
&= e^{un^2-2un+u-un^2+nu} \\
&= e^{u(1-n)} \qquad (2)
\end{aligned}
$$

Now substitute $n+1$ for $n$ in (2), then $f\left(\mid \frac{u+v}{u} \mid\right) = e^{-un}$. This can be rewritten as

$$
f\left(\mid \frac{u+v}{u} \mid\right) = e^{u.1^2+\{-(n+1)u\}.1} \qquad (3)
$$

Comparing (1) and (3), we see that,

$$
\begin{aligned}
v = (-1)^{2k-1} mu^{2k-1}(\text{for } k=1) &= -(n+1)u \\
-mu &= -(n+1)u
\end{aligned}
$$

Therefore, $m = n+1$. Hence from (3) $f\left(\mid \frac{u+v}{u} \mid\right) = f(1)$. Hence the statement is true for all $m \in \mathbf{N}$.

Similar proof for $u \in \mathbf{R}^-$can be produced.

When $u \in \mathbf{R}^+$the following three remarks holds good.

**Remark 1** *$f$ in theorem 1 attains minimum at the midpoint of the interval* $\left[1, \mid \frac{u+v}{u} \mid\right]$.

**Remark 2** $f\left(\mid \frac{u+v}{u} \mid -\delta\right) = f(1+\delta)$ *for all real $\delta$ in* $\left[1, \mid \frac{u+v}{u} \mid\right]$.

PROOF. When $k = 1, m = n$ then $v = -nu$, then for all real $\delta$

$$
\begin{aligned}
f(1+\delta) &= e^{u(1+\delta)^2-nu(1+\delta)} \\
&= e^{\delta^2 u+\delta u(2-n)+u(1-n)} \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
f\left(\mid \frac{u+v}{u} \mid -\delta\right) &= e^{u(n-1-\delta)^2-nu(n-1-\delta)} \\
&= e^{\delta^2 u-2\delta u(n-1)+u(n-1)^2+n\delta u-un(n-1)} \\
&= e^{\delta^2 u-\delta un+2\delta u-nu+u} \\
&= e^{\delta^2 u+\delta u(2-n)+u(1-n)} \qquad (5)
\end{aligned}
$$

Hence the proof.

**Remark 3** *$f$ in theorem 1 resembles alphabet $\bigcup$ in the interval* $\left[1, \mid \frac{u+v}{u} \mid\right]$.

PROOF. See Remarks 1 & 2.

**Definition 1** *$f^+$and $f^-$ are functions with the same property as in theorem 1 except that $f^+$is applicable when $u \in \mathbf{R}^+$ and $f^-$is applicable when $u \in \mathbf{R}^-$.*

**Remark 4** *$f^-$ resembles alphabet $\bigcap$ in the interval $\left[1, \mid \frac{u+v}{u} \mid\right]$. Interestingly the curve of $f^-$ is exactly the mirror image of $f^+$*

**Remark 5** *For $f^+$ and $f^-$ in definition 1 the following type of curves holds good.*

| *Operation* | | *Type* |
|---|---|---|
| $f^+ + f^-$ | $\sim$ | $\bigcap$ |
| $\mid f^+ - f^- \mid$ | $\sim$ | $\bigcap$ |
| $f^+ . f^-$ | $\sim$ | $\bigcap$ |
| $\frac{f^+}{f^-}$ | $\sim$ | $\bigcup$ |
| $\frac{f^-}{f^+}$ | $\sim$ | $\bigcap$ |

*We generalise the function for all $m$ instead of particular $m = 1$ and define as below.*

**Definition 2** *$g$ be function such that $g(x) = e^{ux^2+vx}$, where $v = (-1)^{2k-1} mu^{2k-1}$ for all $u, m, k \in$* $\mathbf{N}$ .

**Theorem 2** *$g(1) = g\left(\mid \frac{u+v}{u} \mid\right)$ for all $u, m, k \in \mathbf{N}$.*

PROOF. Let us consider the situation where $m = k$ in the definition 2. Then for $m = 1$ it is already proved above. Take $m = 2$, then $v = -2u^3$ and $\mid \frac{u+v}{u} \mid = 2u^2 - 1$. Also we have,

$$\begin{aligned} g(1) &= e^{u\left(1-2u^2\right)} \\ g\left(\mid \frac{u+v}{u} \mid\right) &= e^{u\left(2u^2-1\right)^2-2u^3\left(2u^2-1\right)} \\ &= e^{u\left(1-2u^2\right)} \end{aligned}$$

take $m = n$ then $v = (-1)^{2n-1} nu^{2n-1} = -nu^{2n-1}$ and $\mid \frac{u+v}{u} \mid = \mid \frac{u-nu^{2n-1}}{u} \mid = nu^{2n-2} - 1$.

$$\begin{aligned} g(1) &= e^{u\left(1-nu^{2n-2}\right)} \\ g\left(\mid \frac{u+v}{u} \mid\right) &= e^{u\left(nu^{2n-2}-1\right)^2-nu^{2n-1}\left(nu^{2n-2}-1\right)} \\ &= e^{n^2u^{4n-3}-2u^{2n-1}+u-n^2u^{4n-3}+nu^{2n-1}} \\ &= e^{u\left(1-nu^{2n-2}\right)} \end{aligned} \tag{6}$$

Therefore $g(1) = g\left(\mid \frac{u+v}{u} \mid\right)$ for $m = n$. Now substitute $n = n+1$ in (6) then

$$g\left(\left|\frac{u+v}{u}\right|\right) = e^{u\left\{1-(n+1)u^{2(n+1)-2}\right\}}$$
$$= e^{u\left\{1-(n+1)u^{2n}\right\}} \qquad (7)$$

when $m = n+1$, then $v = -(n+1)\,u^{2n+1}$. Therefore,

$$g(1) = e^{u-(n+1)u^{2n+1}} = e^{u\left\{1-(n+1)u^{2n}\right\}} \qquad (8)$$

Comparing (7) and (8), we see that $g(1) = g\left(\left|\frac{u+v}{u}\right|\right)$ for all $u, m \in \mathbf{N}$. Hence the proof. The case when $m \neq k$ can be similarly proved.

**Remark 6** *Theorem 2 is not true for real u. This is the another difference between theorem 1 and theorem 2 apart from generalisation of the result for all $m > 1$.*

**Remark 7** *The functional values of g resembles alphabet $\bigcup$ in the interval $\left[1, \left|\frac{u+v}{u}\right|\right]$. If u is a negative integer in above theorem 2 then g resembles alphabet $\bigcap$ in the interval $\left[1, \left|\frac{u+v}{u}\right|\right]$. Various operations on these two can also be obtained as shown for the theorem 1. Even for general integer n the curve of $g^-$ is exactly the mirror image of $g^+$.*

## 3. Area and Volume

In this section, area covered by the function $g$ ($G$, say) i.e. area of the shape $\bigcup$ and volume of the same shaped vessel ($V$, say) are calculated. Area is calculated using integration of the function between the limits 1 and $\left|\frac{u+v}{u}\right|$. Then $V = G\phi$, where $\phi$ is the multiplier. Imagine a cricle whose diameter ($d$, say) of length $\left|\frac{u+v}{u}\right| - 1$. $\phi$ is calculated as ratio of the area of the circle of diameter $d$ and diameter $d$.

Let $x \in \left[1, \left|\frac{u+v}{u}\right|\right]$ and $m \neq n$, then area of $\bigcup$ for some $k = n \in \mathbf{N}$ is

$$\int_1^{\left|\frac{u+v}{u}\right|} g(x)dx = \int_1^{mu^{2n-2}-1} e^{ux^2 - mu^{2n-1}x}dx \qquad (9)$$

substitute $x = mu^{2n-2}$ in (9) and change the limits accordingly. When

$$x \longrightarrow 1 \;\Rightarrow\; u \longrightarrow m^{2-2n}$$
$$x \longrightarrow mu^{2n-2} \;\Rightarrow\; u \longrightarrow (n-m)^{2-2n}$$

Hence (9) will become

$$m\left[(n-m)^{-(2-2n)^2} - m^{-(2-2n)^2}\right]$$

Therefore

$$G = g(1)\frac{u}{v} - m\left(\frac{m}{n-m}\right)^{(2-2n)^2}$$

Diameter **d** is $mu^{2n-2} - 2$. Now, the multiplier is calculated as the ratio of circle of diameter **d** to the diamater **d**. Hence by simplification, we get $\phi$ is $\frac{\pi \mathbf{d}}{4}$. Now the volume of $\bigcup$ shaped vessel is given by

$$V = \frac{\pi m\left(mu^{2n-2}-2\right)}{4}\left(\frac{m}{n-m}\right)^{(2-2n)^2} \qquad (10)$$

## 4. Conclusions

We are here dealing with properties of the function in the interval of finite length and where functions have bounds, hence applying traditional analysis methods on functions defined here does not give any interesting results. Though there were useful literature on quadratic exponential earlier (McCullagh, P (1994)), it was not explored by defining the coefficients in a special way (definitions 1 & 2). One can also obtain the maxima and minima values for these functions in the above intervals through traditional differentiation method. The volume calculated is practically applicable. The procedure to obtain volume in section 3 is also worked for the volume of the cone ( Rade, L and Westergren, B (1990)).

**Acknowledgement**

This work was carried out when the author was a visitor at the Mathematical Institute, Oxford University. I thank P. Ramu for reading this draft and for comments. This research was funded by Department of Science & technology, New Delhi (SR/FTP/2001/MS-12). Part of the work was funded by The London Mathematical Society.

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*Received May 16, 2003*
*Revised March 16, 2004*