RATES OF CONVERGENCE IN DISTRIBUTION OF A LINEAR COMBINATION OF U-STATISTICS FOR NON- DEGENERATE KERNEL

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RATES OF CONVERGENCE IN DISTRIBUTION OF A LINEAR COMBINATION OF U-STATISTICS FOR NON-DEGENERATE KERNEL

By

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Abstract

As an estimator of an estimable parameter, we consider a linear combination of U-statistics introduced by Toda and Yamato (2001). As a special case, this statistic includes the V-statistic and LB-statistic. In case that the kernel is not degenerate, this linear combination of U-statistics converges to normal distribution. We show some rates of convergence different from Berry-Esseen bound.

Key Words and Phrases: Estimable parameter, Rate of convergence, linear combination of U-statistics, V-statistics.

1. Introduction

Let $\theta(F)$ be an estimable parameter of an unknown distribution F and $g(x_1, \dots, x_k)$ be its kernel of degree $k (\geq 2)$. We assume that the kernel g is symmetric and not degenerate. Let $X_1, \dots, X_n$ be a random sample of size n from the distribution F .

As an estimator of $\theta(F)$, Toda and Yamato (2001) introduces a linear combination Y_n of U-statistics as follows: Let $w(r_1, \dots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \dots, r_j$ such that $r_1 + \dots + r_j = k$ for $j = 1, \dots, k$. We assume that at least one of $w(r_1, \dots, r_j; k)$'s is positive. For $j = 1, \dots, k$, let $g_{(j)}(x_1, \dots, x_j)$ be the kernel given by

$$g_{(j)}(x_1, \dots, x_j) = \frac{1}{d(k, j)} \sum_{r_1 + \dots + r_j = k}^+ w(r_1, \dots, r_j; k) g(\underbrace{x_1, \dots, x_{r_1}}_{r_1}, \dots, \underbrace{x_j, \dots, x_{r_j}}_{r_j}), \quad (1.1)$$

where the summation $\sum_{r_1 + \dots + r_j = k}^+$ is taken over all positive integers $r_1, \dots, r_j$ satisfying $r_1 + \dots + r_j = k$ with j and k fixed and $d(k, j) = \sum_{r_1 + \dots + r_j = k}^+ w(r_1, \dots, r_j; k)$ for $j = 1, 2, \dots, k$. Let $U_n^{(j)}$ be the U-statistic associated with kernel $g_{(j)}(x_1, \dots, x_j; k)$ for $j = 1, \dots, k$. The kernel $g_{(j)}(x_1, \dots, x_j; k)$ is symmetric because of the symmetry of $w(r_1, \dots, r_j; k)$. If $d(k, j)$ is equal to zero for some j , then the associated $w(r_1, \dots, r_j; k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero. Note that

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$U_n^{(k)} = U_n$ for $w(1, \dots, 1; k) > 0$, because of $g_{(k)} = g$. The statistics Y_n is given by

$$Y_n = \frac{1}{D(n, k)} \sum_{j=1}^k d(k, j) \binom{n}{j} U_n^{(j)}, \quad (1.2)$$

where $D(n, k) = \sum_{j=1}^k d(k, j) \binom{n}{j}$. Since w 's are nonnegative and at least one of them is positive, $D(n, k)$ is positive. Y_n includes important statistics as shown in the following examples.

EXAMPLE 1. Let w be the function given by $w(1, 1, \dots, 1; k) = 1$ and $w(r_1, \dots, r_j; k) = 0$ for positive integers $r_1, \dots, r_j$ such that $r_1 + \dots + r_j = k$ for $j = 1, \dots, k-1$. Then the corresponding statistic Y_n is equal to U-statistic U_n , which is given by

$$U_n = \binom{n}{k}^{-1} \sum_{1 \leq j_1 < \dots < j_k \leq n} g(X_{j_1}, \dots, X_{j_k}), \quad (1.3)$$

where $\sum_{1 \leq j_1 < \dots < j_k \leq n}$ denotes the summation over all integers $j_1, \dots, j_k$ satisfying $1 \leq j_1 < \dots < j_k \leq n$.

EXAMPLE 2. Let w be the function given by $w(r_1, \dots, r_j; k) = 1$ for positive integers $r_1, \dots, r_j$ such that $r_1 + \dots + r_j = k$ for $j = 1, \dots, k$. Then the corresponding statistic Y_n is equal to the LB-statistic B_n given by

$$B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1 + \dots + r_n = k} g(\underbrace{X_1, \dots, X_1}_{r_1}, \dots, \underbrace{X_n, \dots, X_n}_{r_n}), \quad (1.4)$$

where $\sum_{r_1 + \dots + r_n = k}$ denotes the summation over all non-negative integers $r_1, \dots, r_n$ satisfying $r_1 + \dots + r_n = k$.

EXAMPLE 3. Let w be the function given by $w(r_1, \dots, r_j; k) = k!/(r_1! \dots r_j!)$ for positive integers $r_1, \dots, r_j$ such that $r_1 + \dots + r_j = k$ for $j = 1, \dots, k$. Then the corresponding statistic Y_n is equal to the V-statistic V_n given by

$$V_n = \frac{1}{n^k} \sum_{j_1=1}^n \dots \sum_{j_k=1}^n g(X_{j_1}, \dots, X_{j_k}). \quad (1.5)$$

(See Toda and Yamato, 2001).

EXAMPLE 4. Let w be the function given by $w(r_1, \dots, r_j; k) = k!/(r_1! \dots r_j!)$ for positive integers $r_1, \dots, r_j$ such that $r_1 + \dots + r_j = k$ for $j = 1, \dots, k$. Then, for example, the corresponding statistic Y_n for the third central moment of the distribution F is given by

$$S_n = \frac{n}{n^2 + 1} \sum_{i=1}^n (X_i - \bar{X})^3,$$

where $\bar{X}$ is the sample mean of $X_1, \dots, X_n$ (see Nomachi et al., 2002).

For the non-degenerate kernel g , U-statistic U_n converges to normal distribution. The purpose of this paper is to show some rates of convergences different from the Berry-Esseen bound, for linear combination of U-statistics Y_n given by (1.3). In Section 2, we quote three rates of convergence different from the Berry-Esseen bound, from Zhao (1983), Zhao and Chen (1983), Koroljuk and Borovskich (1994) and Borovskikh (1996). Furthermore we give a new rate described by using a polynomial. In Section 3, for the statistic Y_n we shall show three rates of convergence to normal distribution, using the propositions of Section 2. Furthermore, we give a rate different from these ones, using a polynomial.

2. Rates of convergence for U-statistics

For kernel $g(x_1, \dots, x_k)$, we put

$$\psi_1(x_1) = E(g(X_1, \dots, X_k) | X_1 = x_1),$$

$$\psi_2(x_1, x_2) = E(g(X_1, \dots, X_k) | X_1 = x_1, X_2 = x_2),$$

$$g^{(1)}(x_1) = \psi_1(x_1) - \theta, \quad \sigma_1^2 = E[g^{(1)}(X_1)^2] > 0,$$

and

$$g^{(2)}(x_1, x_2) = \psi_2(x_1, x_2) - \psi_1(x_1) - \psi_1(x_2) - \theta.$$

Let $\Phi(x)$ be the standard normal distribution function. We shall quote two rates of convergence of the distribution for U-statistic U_n .

LEMMA 2.1. (Koroljuk and Borovskich, 1994, Theorem 6.2.4) *If for some $0 \leq \delta \leq 1$ kernel g satisfies the conditions*

$$\sigma_1 > 0, \quad E |g^{(1)}(X_1)|^{2+\delta} < \infty, \quad E |g(X_1, \dots, X_k)|^{\frac{4+\delta}{3}} < \infty,$$

then

$$\sup_{-\infty < x < \infty} |P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - \Phi(x)| = O(n^{-\frac{\delta}{2}}) \quad (2.1)$$

as $n \rightarrow \infty$, and for $\delta = 0$ we can replace $O(1)$ on the right-hand side by $o(1)$.

LEMMA 2.2. (Koroljuk and Borovskich, 1994, Theorem 6.2.5, Zhao, 1983) *Let $\sigma_1 > 0$ and $E |g(X_1, \dots, X_k)|^3 < \infty$. Then the inequality*

$$|P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - \Phi(x)| \leq \frac{C}{\sqrt{n}(1+x^2)} \quad (2.2)$$

holds for all $x \in R$, where C depends on kernel g only via σ_1 and $E |g|^3$ and does not depend on x and n .

Hereafter we use $C, C_1, C_2, C_3, \dots$ as generic constants which do not depend on x and n . We shall show the similar result to (2.2) for Y-statistic Y_n . For this purpose we quote the following.

LEMMA 2.3. (Zhao, 1983, Lemma 7) Suppose that $W_n = W_{n1} + W_{n2}$, $n = 1, 2, \dots$ be a sequence of random variables. Denote the distribution functions of W_n and W_{n1} by F_n and F_{n1} , respectively. If

$$|F_{n1} - \Phi(x)| \leq \frac{C_1}{\sqrt{n}(1+x^2)}$$

for all $x \in R$ and for $|x| \geq 1$

$$P(|W_{n2}| \geq \frac{C_2}{\sqrt{n}} |x|) \leq \frac{C_3}{\sqrt{n}(1+x^2)},$$

then for all $x \in R$

$$|F_n - \Phi(x)| \leq \frac{C_4}{\sqrt{n}(1+x^2)}.$$

In the following lemma, we consider kernel g of degree $k = 2$.

LEMMA 2.4. (Zhao and Chen, 1983) Let $\sigma_1 > 0$ and $E|g(X_1, X_2)|^3 < \infty$. Then the inequality

$$|P\left(\frac{\sqrt{n}}{2\sigma_1}(U_n - \theta) \leq x\right) - \Phi(x)| \leq \frac{C}{\sqrt{n}(1+|x|)^3} \quad (2.3)$$

holds for all $n(\geq 2)$ and all $x \in R$.

For this lemma, see also, Koroljuk and Borovskikh (1994), Theorem 6.2.6 and Borovskikh (1996), Theorem 6.4.1. We shall show the similar result to (2.3) for the Y-statistic Y_n . For this purpose we quote the following.

LEMMA 2.5. (Zhao and Chen, 1983, Lemma 3) Suppose that $W_n = W_{n1} + W_{n2}$, $n = 1, 2, \dots$ be a sequence of random variables. Denote the distribution functions of W_n and W_{n1} by F_n and F_{n1} , respectively. If

$$|F_{n1} - \Phi(x)| \leq \frac{C_1}{\sqrt{n}(1+|x|)^3}$$

for all $x \in R$ and for $|x| \geq 1$

$$P(|W_{n2}| \geq \frac{C_2}{\sqrt{n}} |x|) \leq \frac{C_3}{\sqrt{n}(1+|x|)^3},$$

then for all $x \in R$

$$|F_n - \Phi(x)| \leq \frac{C_4}{\sqrt{n}(1+|x|)^3}.$$

Again we consider the kernel of degree $k \geq 2$. Let us consider a bound related with a polynomial including $1+x^2$ of (2.2) and $(1+x)^3$ of (2.3). If we allow n to depend on x , then we have the following.

THEOREM 2.6. *Let $\sigma_1 > 0$ and $E |g(X_1, \dots, X_k)|^3 < \infty$. In addition, we suppose that $\lim_{|t| \rightarrow \infty} |\eta(t)| < 1$. Let p be a polynomial which is positive and increasing over $[0, \infty)$. Then inequality*

$$|P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - \Phi(x)| \leq \frac{C}{\sqrt{np}(|x|)}, \quad x \in R \quad (2.4)$$

holds for a sufficiently large n which depends on x .

Before its proof we note the Berry-Esseen bound and the Edgeworth expansion. Let ϕ be the density of the standard normal distribution and

$$\kappa_3 = \sigma_1^{-3} [E[(g^{(1)}(X))^3] + 3(k-1)E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_2)]]$$

Under the condition of this theorem we have the Berry-Esseen bound

$$\sup_{-\infty < x < \infty} |P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - \Phi(x)| \leq \frac{C_1}{\sqrt{n}} \quad (2.5)$$

and the Edgeworth expansion

$$\sup_{-\infty < x < \infty} |P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - Q_n(x)| \leq \frac{\epsilon_n}{\sqrt{n}}, \quad (2.6)$$

where

$$Q_n(x) = \Phi(x) - \frac{1}{6\sqrt{n}}(x^2 - 1)\kappa_3\phi(x)$$

and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ (see, for example, Maesono and Yamato, 1994).

Proof of Theorem 2.6. Let M be a positive constant such that

$$|x^2 - 1| p(|x|)\phi(x) \leq 1 \quad \text{for } |x| \geq M. \quad (2.7)$$

By the definition of Q_n we have

$$\begin{aligned} I_n &= |P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - \Phi(x)| \\ &\leq \sup |P\left(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \leq x\right) - Q_n(x)| + \frac{1}{6\sqrt{n}} |x^2 - 1|\kappa_3 |\phi(x)|. \end{aligned}$$

For a given x , we can choose a sufficiently large n such that $\epsilon_n < 1/p(|x|)$. Using (2.6), for $|x| \geq M$, we have for a sufficiently large n

$$I_n \leq \frac{1}{\sqrt{np}(|x|)} + \frac{\kappa_3}{6\sqrt{np}(|x|)} = \frac{C_1}{\sqrt{np}(|x|)}.$$

If $|x| \leq M$, then $p(|x|)$ is bounded and $1/p(M) \leq 1/p(|x|) \leq 1/p(0)$. Therefore by (2.5) we have

$$I_n \leq \frac{C_2}{\sqrt{np}(|x|)}.$$

Thus we get (2.4). □

3. Rates of convergence for Y-statistics

If $d(k, k) = w(1, \dots, 1; k) > 0$, then there exists a constant $\beta (\geq 0)$ such that

$$\frac{d(k, k)}{D(n, k)} \binom{n}{k} = 1 - \frac{\beta}{n} + O\left(\frac{1}{n^2}\right) \quad (3.1)$$

and

$$\sum_{j=1}^{k-1} \frac{d(k, j)}{D(n, k)} \binom{n}{j} = \frac{\beta}{n} + O\left(\frac{1}{n^2}\right). \quad (3.2)$$

For U-statistic U_n , $\beta = 0$. In the following we assume that

$$\beta > 0,$$

because the corresponding results for U-statistic are given in Section 2. For V-statistic V_n and S-statistic S_n , $\beta = k(k-1)/2$. For the LB-statistic B_n , $\beta = k(k-1)$.

As stated in Toda and Yamato (2001), we can write

$$Y_n = U_n + R_n \quad (3.3)$$

and R_n satisfies the following: For $r (\geq 1)$ and integers $j_1, \dots, j_k$ ($1 \leq j_1 \leq \dots \leq j_k \leq k$), we assume $E |g(X_{j_1}, \dots, X_{j_k})|^r < \infty$. Then we have

$$E |R_n - ER_n|^r \leq C_1 n^{-\frac{3r}{2}}, \quad r \geq 2 \quad (3.4)$$

and

$$E |R_n - ER_n|^r \leq C_2 n^{-(2r-1)}, \quad 1 \leq r < 2, \quad (3.5)$$

(we note here these inequalities hold even if r is not integer by the reason of the proof of Proposition 3.6 of Toda and Yamato, 2001). From (3.1), we have

$$Y_n - EY_n = U_n - \theta + (R_n - ER_n).$$

THEOREM 3.1. *If for some $0 \leq \delta \leq 1$ the kernel g satisfy the conditions*

$$\sigma_1 > 0, \quad E |g^{(1)}(X_1)|^{2+\delta} < \infty, \quad E |g(X_1, \dots, X_k)|^{\frac{4+\delta}{3}} < \infty,$$

and

$$E |g(X_{j_1}, \dots, X_{j_k})|^{\frac{8+\delta}{3}} < \infty, \quad 1 \leq j_1 \leq \dots \leq j_k \leq k,$$

then

$$\sup_{-\infty < x < \infty} \left| P\left(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\right) - \Phi(x) \right| = O(n^{-\frac{\delta}{2}})$$

as $n \rightarrow \infty$.

Proof. Let G_n and Φ_n be the distribution functions of $(\sqrt{n}/(k\sigma_1))[Y_n - EY_n]$ and $(\sqrt{n}/(k\sigma_1))[U_n - \theta]$, respectively. Then for any $\varepsilon > 0$

$$\sup |G_n(x) - \Phi(x)| \leq \sup |\Phi_n(x) - \Phi(x)| + P\left(\frac{\sqrt{n} |R_n - ER_n|}{k\sigma_1} \geq \varepsilon\right) + \frac{\varepsilon}{\sqrt{2\pi}}, \quad (3.6)$$

(see, for example, Lee, 1990, p.187). By taking $\varepsilon = n^{-\delta/2}$ and using Markov's inequality and (3.3),

$$P\left(\frac{\sqrt{n} |R_n - ER_n|}{k\sigma_1} \geq \varepsilon\right) \leq \frac{1}{\varepsilon^{\frac{s+\delta}{\delta}}} E\left[\frac{\sqrt{n} |R_n - ER_n|}{k\sigma_1}\right]^{\frac{s+\delta}{\delta}} \leq Cn^{-\frac{\delta}{2} + \frac{1}{12}(\delta+12)(\delta-1)}.$$

Since $0 \leq \delta \leq 1$,

$$P\left(\frac{\sqrt{n} |R_n - ER_n|}{k\sigma_1} \geq \varepsilon\right) = O(n^{-\frac{\delta}{2}}).$$

Thus applying this relation and Lemma 2.1 to (3.4) with $\varepsilon = n^{-\delta/2}$, we get $\sup |G_n(x) - \Phi(x)| = O(n^{-\frac{\delta}{2}})$. $\square$

THEOREM 3.2. Suppose that $\sigma_1 > 0$, $E |g(X_1, \dots, X_k)|^3 < \infty$ and

$$E |g(X_{j_1}, \dots, X_{j_k})|^2 < \infty, \quad 1 \leq j_1 \leq \dots \leq j_k \leq k.$$

Then, inequality

$$|P\left(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\right) - \Phi(x)| \leq \frac{C}{\sqrt{n}(1+x^2)}$$

holds for all $x \in R$.

Proof. For the first term of the left-hand side of the inequality

$$\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) = \frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) + \frac{\sqrt{n}}{k\sigma_1}(R_n - ER_n), \quad (3.7)$$

By Markov's inequality and (3.2) we have for $x \neq 0$

$$P\left(\frac{\sqrt{n}}{k\sigma_1} |R_n - ER_n| \geq \frac{C_1}{\sqrt{n}} |x| \right) \leq \frac{C_2}{n|x|^2}.$$

For $|x| \geq 1$, we have $1 + |x|^2 \leq 2|x|^2$ and so

$$P\left(\frac{\sqrt{n}}{k\sigma_1} |R_n - ER_n| \geq \frac{C_1}{\sqrt{n}} |x| \right) \leq \frac{C_3}{n(1+|x|^2)}. \quad (3.8)$$

Applying Lemma 2.2, (3.5) and (3.6) to Lemma 2.3, we get the theorem. $\square$

THEOREM 3.3. Suppose that $\sigma_1 > 0$, $E |g(X_1, X_2)|^3 < \infty$, and $E |g(X_1, X_1)|^3 < \infty$. Then, the inequality

$$|P\left(\frac{\sqrt{n}}{2\sigma_1}(Y_n - EY_n) \leq x\right) - \Phi(x)| \leq \frac{C}{\sqrt{n}(1+|x|)^3} \quad (3.9)$$

holds for $n \geq 8$ and all $x \in R$.

Proof. By Markov's inequality and (3.2) we have for $x \neq 0$,

$$P\left(\frac{\sqrt{n}}{2\sigma_1} | R_n - ER_n | \geq \frac{C_1}{\sqrt{n}} |x|\right) \leq \frac{C_2}{n^{3/2} |x|^3}.$$

For $|x| \geq 1$, we have $(1 + 1/|x|)^3 \leq 2^3 \leq n$ and so

$$P\left(\frac{\sqrt{n}}{k\sigma_1} | R_n - ER_n | \geq \frac{C_1}{\sqrt{n}} |x|\right) \leq \frac{C_3}{\sqrt{n}(1 + |x|)^3}. \quad (3.10)$$

Applying Proposition 2.4, (3.5) and (3.10) to Lemma 2.5, we get (3.9) $\square$

Let us consider a bound related with a polynomial. If we allow n to depend on x , then we have the following.

THEOREM 3.4. *Let $\sigma_1 > 0$, $E |g(X_1, \dots, X_k)|^3 < \infty$ and $E |g(X_{j_1}, \dots, X_{j_k})|^2 < \infty$ ($1 \leq j_1 \leq \dots \leq j_k \leq k$). In addition, we suppose that $\lim_{|t| \rightarrow \infty} |\eta(t)| < 1$. Let p be a polynomial which is positive and increasing over $[0, \infty)$. Then inequality*

$$|P\left(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\right) - \Phi(x)| \leq \frac{C}{\sqrt{np}(|x|)}, \quad x \in R \quad (3.11)$$

holds for a sufficiently large n which depends on x .

We can prove this theorem by the similar method to Theorem 2.6, using the Berry-Esseen bound of $(\sqrt{n}/(k\sigma_1))[Y_n - EY_n]$ (Toda and Yamato, 2001) and its Edgeworth expansion (Yamato et al., 2002). We note that $Y_n - \theta$ has a bias but $Y_n - EY_n$ has no bias. Under the condition of this proposition we have the Berry-Esseen bound

$$\sup_{-\infty < x < \infty} |P\left(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\right) - \Phi(x)| \leq \frac{C_1}{\sqrt{n}} \quad (3.12)$$

and the Edgeworth expansion

$$\sup_{-\infty < x < \infty} |P\left(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\right) - Q_n(x)| \leq \frac{\epsilon_n}{\sqrt{n}}, \quad (3.13)$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. We can prove Theorem 3.4 by using these results.

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