RATES OF CONVERGENCE IN DISTRIBUTION OF A LINEAR COMBINATION OF U-STATISTICS FOR NON-DEGENERATE KERNEL

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RATES OF CONVERGENCE IN DISTRIBUTION OF A LINEAR COMBINATION OF U-STATISTICS FOR NON-DEGENERATE KERNEL

By

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Abstract

As an estimator of an estimable parameter, we consider a linear combination of U-statistics introduced by Toda and Yamato (2001). As a special case, this statistic includes the V-statistic and LB-statistic. In case that the kernel is not degenerate, this linear combination of U-statistics converges to normal distribution. We show some rates of convergence different from Berry-Esseen bound.

Key Words and Phrases: Estimable parameter, Rate of convergence, linear combination of U-statistics, V-statistics.

1. Introduction

Let $\theta(F)$ be an estimable parameter of an unknown distribution F and $g(x_1, ..., x_k)$ be its kernel of degree $k(\geq 2)$. We assume that the kernel g is symmetric and not degenerate. Let $X_1, \ldots, X_n$ be a random sample of size n from the distribution F.

As an estimator of $\theta(F)$, Toda and Yamato (2001) introduces a linear combination Y_n of U-statistics as follows: Let $w(r_1, \ldots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \ldots, r_j$ such that $r_1 + \cdots + r_j = k$ for $j = 1, \ldots, k$. We assume that at least one of $w(r_1, \ldots, r_j; k)$'s is positive. For $j = 1, \ldots, k$, let $g_{(j)}(x_1, ..., x_j)$ be the kernel given by

$$g_{(j)}(x_1, \ldots, x_j) = \frac{1}{d(k,j)} \sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1, \ldots, r_j; k) g(\underbrace{x_1, \ldots, x_1}_{r_1}, \ldots, \underbrace{x_j, \ldots, x_j}_{r_j}), \tag{1.1}$$

where the summation $\sum^{+}_{r_1+\cdots+r_j=k}$ is taken over all positive integers $r_1, ..., r_j$ satisfying $r_1 + \cdots + r_j = k$ with j and k fixed and $d(k,j) = \sum^{+}_{r_1+\cdots+r_j=k} w(r_1, \ldots, r_j; k)$ for $j = 1, 2, ..., k$. Let $U_n^{(j)}$ be the U-statistic associated with kernel $g_{(j)}(x_1, \ldots, x_j; k)$ for $j = 1, \ldots, k$. The kernel $g_{(j)}(x_1, \ldots, x_j; k)$ is symmetric because of the symmetry of $w(r_1, \ldots, r_j; k)$. If $d(k,j)$ is equal to zero for some j, then the associated $w(r_1, \ldots, r_j; k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero. Note that

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$U_n^{(k)} = U_n$ for $w(1,\dots,1;k) > 0$, because of $g_{(k)} = g$. The statistics Y_n is given by

$$Y_n = \frac{1}{D(n,k)} \sum_{j=1}^{k} d(k,j) \binom{n}{j} U_n^{(j)}, \tag{1.2}$$

where $D(n,k) = \sum_{j=1}^{k} d(k,j)\binom{n}{j}$. Since w's are nonnegative and at least one of them is positive, $D(n,k)$ is positive. Y_n includes important statistics as shown in the following examples.

EXAMPLE 1. Let w be the function given by $w(1,1,\dots,1;k) = 1$ and $w(r_1,\dots,r_j;k) = 0$ for positive integers $r_1,\dots,r_j$ such that $r_1 + \cdots + r_j = k$ for $j = 1,\dots,k-1$. Then the corresponding statistic Y_n is equal to U-statistic U_n, which is given by

$$U_n = \binom{n}{k}^{-1} \sum_{1 \le j_1 < \cdots < j_k \le n} g(X_{j_1},\dots,X_{j_k}), \tag{1.3}$$

where $\sum_{1\le j_1<\cdots<j_k\le n}$ denotes the summation over all integers $j_1,\dots,j_k$ satisfying $1 \le j_1 < \cdots < j_k \le n$.

EXAMPLE 2. Let w be the function given by $w(r_1,\dots,r_j;k) = 1$ for positive integers $r_1,\dots,r_j$ such that $r_1 + \cdots + r_j = k$ for $j = 1,\dots,k$. Then the corresponding statistic Y_n is equal to the LB-statistic B_n given by

$$B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1+\cdots+r_n=k} g(\underbrace{X_1,\dots,X_1}_{r_1},\dots,\underbrace{X_n,\dots,X_n}_{r_n}), \tag{1.4}$$

where $\sum_{r_1+\cdots+r_n=k}$ denotes the summation over all non-negative integers $r_1,\dots,r_n$ satisfying $r_1 + \cdots + r_n = k$.

EXAMPLE 3. Let w be the function given by $w(r_1,\dots,r_j;k) = k!/(r_1!\cdots r_j!)$ for positive integers $r_1,\dots,r_j$ such that $r_1 + \cdots + r_j = k$ for $j = 1,\dots,k$. Then the corresponding statistic Y_n is equal to the V-statistic V_n given by

$$V_n = \frac{1}{n^k} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} g(X_{j_1},\dots,X_{j_k}). \tag{1.5}$$

(See Toda and Yamato, 2001).

EXAMPLE 4. Let w be the function given by $w(r_1,\dots,r_j;k) = k!/(r_1\cdots r_j)$ for positive integers $r_1,\dots,r_j$ such that $r_1+\cdots+r_j = k$ for $j = 1,\dots,k$. Then, for example, the corresponding statistic Y_n for the third central moment of the distribution F is given by

$$S_n = \frac{n}{n^2+1} \sum_{i=1}^{n} (X_i - \bar{X})^3,$$

where $\bar{X}$ is the sample mean of $X_1,\dots,X_n$ (see Nomachi et al., 2002).

For the non-degenerate kernel g, U-statistic U_n converges to normal distribution. The purpose of this paper is to show some rates of convergences different from the Berry-Esseen bound, for linear combination of U-statistics Y_n given by (1.3). In Section 2, we quote three rates of convergence different from the Berry-Esseen bound, from Zhao (1983), Zhao and Chen (1983), Koroljuk and Borovskich (1994) and Borovskikh (1996). Furthermore we give a new rate described by using a polynomial. In Section 3, for the statistic Y_n we shall show three rates of convergence to normal distribution, using the propositions of Section 2. Furthermore, we give a rate different from these ones, using a polynomial.

2. Rates of convergence for U-statistics

For kernel $g(x_1, \ldots, x_k)$, we put

$$\psi_1(x_1) = E\big(g(X_1, \ldots, X_k) \mid X_1 = x_1\big),$$

$$\psi_2(x_1, x_2) = E\big(g(X_1, \ldots, X_k) \mid X_1 = x_1, X_2 = x_2\big),$$

$$g^{(1)}(x_1) = \psi_1(x_1) - \theta, \quad \sigma_1^2 = E[g^{(1)}(X_1)^2] > 0,$$

and

$$g^{(2)}(x_1, x_2) = \psi_2(x_1, x_2) - \psi_1(x_1) - \psi_1(x_2) - \theta.$$

Let $\Phi(x)$ be the standard normal distribution function. We shall quote two rates of convergence of the distribution for U-statistic U_n.

LEMMA 2.1. (Koroljuk and Borovskich, 1994, Theorem 6.2.4) *If for some* $0 \le \delta \le 1$ *kernel* g *satisfies the conditions*

$$\sigma_1 > 0, \quad E \mid g^{(1)}(X_1) \mid^{2+\delta} < \infty, \quad E \mid g(X_1, \ldots, X_k) \mid^{\frac{4+\delta}{3}} < \infty,$$

then

$$\sup_{-\infty < x < \infty} \Big| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - \Phi(x) \Big| = O\big(n^{-\frac{\delta}{2}}\big) \tag{2.1}$$

as $n \to \infty$, *and for* $\delta = 0$ *we can replace* $O(1)$ *on the right-hand side by* $o(1)$.

LEMMA 2.2. (Koroljuk and Borovskich, 1994, Theorem 6.2.5, Zhao, 1983) *Let* $\sigma_1 > 0$ *and* $E \mid g(X_1, \ldots, X_k) \mid^3 < \infty$. *Then the inequality*

$$\Big| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - \Phi(x) \Big| \le \frac{C}{\sqrt{n}(1 + x^2)} \tag{2.2}$$

holds for all $x \in R$, *where* C *depends on kernel* g *only via* σ_1 *and* $E \mid g \mid^3$ *and does not depend on* x *and* n.

Hereafter we use $C, C_1, C_2, C_3, \ldots$ as generic constants which do not depend on x and n. We shall show the similar result to (2.2) for Y-statistic Y_n. For this purpose we quote the following.

LEMMA 2.3. (Zhao, 1983, Lemma 7) *Suppose that* $W_n = W_{n1} + W_{n2}$, $n = 1, 2, \ldots$ *be a sequence of random variables. Denote the distribution functions of* W_n *and* W_{n1} *by* F_n *and* F_{n1}, *respectively. If*

$$| F_{n1} - \Phi(x) | \leq \frac{C_1}{\sqrt{n}(1+x^2)}$$

for all $x \in R$ *and for* $| x | \geq 1$

$$P\big(| W_{n2} | \geq \frac{C_2}{\sqrt{n}} | x | \big) \leq \frac{C_3}{\sqrt{n}(1+x^2)},$$

then for all $x \in R$

$$| F_n - \Phi(x) | \leq \frac{C_4}{\sqrt{n}(1+x^2)}.$$

In the following lemma, we consider kernel g of degree $k = 2$.

LEMMA 2.4. (Zhao and Chen, 1983) *Let* $\sigma_1 > 0$ *and* $E \mid g(X_1, X_2) \mid^3 < \infty$. *Then the inequality*

$$| P\Big(\frac{\sqrt{n}}{2\sigma_1}(U_n - \theta) \leq x\Big) - \Phi(x) | \leq \frac{C}{\sqrt{n}(1+ | x |)^3} \tag{2.3}$$

holds for all $n (\geq 2)$ *and all* $x \in R$.

For this lemma, see also, Koroljuk and Borovskich (1994), Theorem 6.2.6 and Borovskikh (1996), Theorem 6.4.1. We shall show the similar result to (2.3) for the Y-statistic Y_n. For this purpose we quote the following.

LEMMA 2.5. (Zhao and Chen, 1983, Lemma 3) *Suppose that* $W_n = W_{n1} + W_{n2}$, $n = 1, 2, \ldots$ *be a sequence of random variables. Denote the distribution functions of* W_n *and* W_{n1} *by* F_n *and* F_{n1}, *respectively. If*

$$| F_{n1} - \Phi(x) | \leq \frac{C_1}{\sqrt{n}(1+ | x |)^3}$$

for all $x \in R$ *and for* $| x | \geq 1$

$$P\big(| W_{n2} | \geq \frac{C_2}{\sqrt{n}} | x | \big) \leq \frac{C_3}{\sqrt{n}(1+ | x |)^3},$$

then for all $x \in R$

$$| F_n - \Phi(x) | \leq \frac{C_4}{\sqrt{n}(1+ | x |)^3}.$$

Again we consider the kernel of degree $k \geq 2$. Let us consider a bound related with a polynomial including $1 + x^2$ of (2.2) and $(1 + x)^3$ of (2.3). If we allow n to depend on x, then we have the following.

THEOREM 2.6. *Let $\sigma_1 > 0$ and $E \mid g(X_1, \dots, X_k) \mid^3 < \infty$. In addition, we suppose that $\lim_{|t|\to\infty} \mid \eta(t) \mid < 1$. Let p be a polynomial which is positive and increasing over $[0, \infty)$. Then inequality*

$$\left| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - \Phi(x) \right| \le \frac{C}{\sqrt{n}p(\mid x \mid)}, \quad x \in R \tag{2.4}$$

holds for a sufficiently large n which depends on x.

Before its proof we note the Berry-Esseen bound and the Edgeworth expansion. Let ϕ be the density of the standard normal distribution and

$$\kappa_3 = \sigma_1^{-3}\big[E[(g^{(1)}(X))^3] + 3(k-1)E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_2)]\big].$$

Under the condition of this theorem we have the Berry-Esseen bound

$$\sup_{-\infty<x<\infty} \left| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - \Phi(x) \right| \le \frac{C_1}{\sqrt{n}} \tag{2.5}$$

and the Edgeworth expansion

$$\sup_{-\infty<x<\infty} \left| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - Q_n(x) \right| \le \frac{\epsilon_n}{\sqrt{n}}, \tag{2.6}$$

where

$$Q_n(x) = \Phi(x) - \frac{1}{6\sqrt{n}}(x^2 - 1)\kappa_3\phi(x)$$

and $\epsilon_n \to 0$ as $n \to \infty$ (see, for example, Maesono and Yamato, 1994).

Proof of Theorem 2.6. Let M be a positive constant such that

$$\mid x^2 - 1 \mid p(\mid x \mid)\phi(x) \le 1 \quad \text{for} \quad \mid x \mid \ge M. \tag{2.7}$$

By the definition of Q_n we have

$$I_n = \left| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - \Phi(x) \right|$$

$$\le \sup \left| P\Big(\frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) \le x\Big) - Q_n(x) \right| + \frac{1}{6\sqrt{n}} \mid (x^2 - 1)\kappa_3 \mid \phi(x).$$

For a given x, we can choose a sufficiently large n such that $\epsilon_n < 1/p(\mid x \mid)$. Using (2.6), for $\mid x \mid \ge M$, we have for a sufficiently large n

$$I_n \le \frac{1}{\sqrt{n}p(\mid x \mid)} + \frac{\kappa_3}{6\sqrt{n}p(\mid x \mid)} = \frac{C_1}{\sqrt{n}p(\mid x \mid)}.$$

If $\mid x \mid \le M$, then $p(\mid x \mid)$ is bounded and $1/p(M) \le 1/p(\mid x \mid) \le 1/p(0)$. Therefore by (2.5) we have

$$I_n \le \frac{C_2}{\sqrt{n}p(\mid x \mid)}.$$

Thus we get (2.4). □

3. Rates of convergence for Y-statistics

If $d(k,k) = w(1,\ldots,1;k) > 0$, then there exists a constant $\beta(\geq 0)$ such that

$$\frac{d(k,k)}{D(n,k)}\binom{n}{k} = 1 - \frac{\beta}{n} + O(\frac{1}{n^2}) \tag{3.1}$$

and

$$\sum_{j=1}^{k-1}\frac{d(k,j)}{D(n,k)}\binom{n}{j} = \frac{\beta}{n} + O(\frac{1}{n^2}). \tag{3.2}$$

For U-statistic U_n, $\beta = 0$. In the following we assume that

$$\beta > 0,$$

because the corresponding results for U-statistic are given in Section 2. For V-statistic V_n and S-statistic S_n, $\beta = k(k-1)/2$. For the LB-statistic B_n, $\beta = k(k-1)$.

As stated in Toda and Yamato (2001), we can write

$$Y_n = U_n + R_n \tag{3.3}$$

and R_n satisfies the following: For $r(\geq 1)$ and integers $j_1,\ldots,j_k$ $(1 \leq j_1 \leq \cdots \leq j_k \leq k)$, we assume $E \mid g(X_{j_1},\ldots,X_{j_k}) \mid^r < \infty$. Then we have

$$E \mid R_n - ER_n \mid^r \leq C_1 n^{-\frac{3r}{2}}, \quad r \geq 2 \tag{3.4}$$

and

$$E \mid R_n - ER_n \mid^r \leq C_2 n^{-(2r-1)}, \quad 1 \leq r < 2, \tag{3.5}$$

(we note here these inequalities hold even if r is not integer by the reason of the proof of Proposition 3.6 of Toda and Yamato, 2001). From (3.1), we have

$$Y_n - EY_n = U_n - \theta + (R_n - ER_n).$$

THEOREM 3.1. *If for some $0 \leq \delta \leq 1$ the kernel g satisfy the conditions*

$$\sigma_1 > 0, \quad E \mid g^{(1)}(X_1) \mid^{2+\delta} < \infty, \quad E \mid g(X_1,\ldots,X_k) \mid^{\frac{4+\delta}{3}} < \infty,$$

and

$$E \mid g(X_{j_1},\ldots,X_{j_k}) \mid^{\frac{8+\delta}{6}} < \infty, \quad 1 \leq j_1 \leq \cdots \leq j_k \leq k,$$

then

$$\sup_{-\infty<x<\infty} \mid P\Big(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\Big) - \Phi(x) \mid = O\big(n^{-\frac{\delta}{2}}\big)$$

as $n \to \infty$.

Proof. Let G_n and Φ_n be the distribution functions of $(\sqrt{n}/(k\sigma_1))[Y_n - EY_n]$ and $(\sqrt{n}/(k\sigma_1))[U_n - \theta]$, respectively. Then for any $\varepsilon > 0$

$$\sup \mid G_n(x) - \Phi(x) \mid \leq \sup \mid \Phi_n(x) - \Phi(x) \mid + P\Big(\frac{\sqrt{n} \mid R_n - ER_n \mid}{k\sigma_1} \geq \varepsilon\Big) + \frac{\varepsilon}{\sqrt{2\pi}}, \tag{3.6}$$

(see, for example, Lee, 1990, p.187). By taking $\varepsilon = n^{-\delta/2}$ and using Markov's inequality and (3.3),

$$P\Big(\frac{\sqrt{n} \mid R_n - ER_n \mid}{k\sigma_1} \geq \varepsilon\Big) \leq \frac{1}{\varepsilon^{\frac{8+\delta}{6}}} E\Big[\frac{\sqrt{n} \mid R_n - ER_n \mid}{k\sigma_1}\Big]^{\frac{8+\delta}{6}} \leq Cn^{-\frac{\delta}{2}+\frac{1}{12}(\delta+12)(\delta-1)}.$$

Since $0 \leq \delta \leq 1$,

$$P\Big(\frac{\sqrt{n} \mid R_n - ER_n \mid}{k\sigma_1} \geq \varepsilon\Big) = O(n^{-\frac{\delta}{2}}).$$

Thus applying this relation and Lemma 2.1 to (3.4) with $\varepsilon = n^{-\delta/2}$, we get sup $\mid G_n(x) - \Phi(x) \mid = O(n^{-\frac{\delta}{2}})$. □

THEOREM 3.2. *Suppose that* $\sigma_1 > 0$, $E \mid g(X_1, \ldots, X_k) \mid^3 < \infty$ *and*

$$E \mid g(X_{j_1}, \ldots, X_{j_k}) \mid^2 < \infty, \quad 1 \leq j_1 \leq \cdots \leq j_k \leq k.$$

Then, inequality

$$\mid P\Big(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\Big) - \Phi(x) \mid \leq \frac{C}{\sqrt{n}(1+x^2)}$$

holds for all $x \in R$.

Proof. For the first term of the left-hand side of the inequality

$$\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) = \frac{\sqrt{n}}{k\sigma_1}(U_n - \theta) + \frac{\sqrt{n}}{k\sigma_1}(R_n - ER_n), \tag{3.7}$$

By Markov's inequality and (3.2) we have for $x \neq 0$

$$P\Big(\frac{\sqrt{n}}{k\sigma_1} \mid R_n - ER_n \mid \geq \frac{C_1}{\sqrt{n}} \mid x \mid\Big) \leq \frac{C_2}{n \mid x \mid^2}.$$

For $\mid x \mid \geq 1$, we have $1 + \mid x \mid^2 \leq 2 \mid x \mid^2$ and so

$$P\Big(\frac{\sqrt{n}}{k\sigma_1} \mid R_n - ER_n \mid \geq \frac{C_1}{\sqrt{n}} \mid x \mid\Big) \leq \frac{C_3}{n(1 + \mid x \mid^2)}. \tag{3.8}$$

Applying Lemma 2.2, (3.5) and (3.6) to Lemma 2.3, we get the theorem. □

THEOREM 3.3. *Suppose that* $\sigma_1 > 0$, $E \mid g(X_1, X_2) \mid^3 < \infty$, *and* $E \mid g(X_1, X_1) \mid^3 < \infty$. *Then, the inequality*

$$\mid P\Big(\frac{\sqrt{n}}{2\sigma_1}(Y_n - EY_n) \leq x\Big) - \Phi(x) \mid \leq \frac{C}{\sqrt{n}(1 + \mid x \mid)^3} \tag{3.9}$$

holds for $n \geq 8$ *and all* $x \in R$.

Proof. By Markov's inequality and (3.2) we have for $x \neq 0$,

$$P\Big(\frac{\sqrt{n}}{2\sigma_1} \mid R_n - ER_n \mid \geq \frac{C_1}{\sqrt{n}} \mid x \mid\Big) \leq \frac{C_2}{n^{3/2} \mid x \mid^3}.$$

For $\mid x \mid \geq 1$, we have $(1 + 1/\mid x \mid)^3 \leq 2^3 \leq n$ and so

$$P\Big(\frac{\sqrt{n}}{k\sigma_1} \mid R_n - ER_n \mid \geq \frac{C_1}{\sqrt{n}} \mid x \mid\Big) \leq \frac{C_3}{\sqrt{n}(1 + \mid x \mid)^3}. \tag{3.10}$$

Applying Proposition 2.4, (3.5) and (3.10) to Lemma 2.5, we get (3.9) □

Let us consider a bound related with a polynomial. If we allow n to depend on x, then we have the following.

THEOREM 3.4. *Let $\sigma_1 > 0$, $E \mid g(X_1, \ldots, X_k) \mid^3 < \infty$ and $E \mid g(X_{j1}, \ldots, X_{j_k}) \mid^2 < \infty$ $(1 \leq j_1 \leq \cdots \leq j_k \leq k)$. In addition, we suppose that $\lim_{|t| \to \infty} \mid \eta(t) \mid < 1$. Let p be a polynomial which is positive and increasing over $[0, \infty)$. Then inequality*

$$\mid P\Big(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\Big) - \Phi(x) \mid \leq \frac{C}{\sqrt{n}p(\mid x \mid)}, \quad x \in R \tag{3.11}$$

holds for a sufficiently large n which depends on x.

We can prove this theorem by the similar method to Theorem 2.6, using the Berry-Esseen bound of $(\sqrt{n}/(k\sigma_1))[Y_n - EY_n]$ (Toda and Yamato, 2001) and its Edgeworth expansion (Yamato et al., 2002). We note that $Y_n - \theta$ has a bias but $Y_n - EY_n$ has no bias. Under the condition of this proposition we have the Berry-Esseen bound

$$\sup_{-\infty < x < \infty} \mid P\Big(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\Big) - \Phi(x) \mid \leq \frac{C_1}{\sqrt{n}} \tag{3.12}$$

and the Edgeworth expansion

$$\sup_{-\infty < x < \infty} \mid P\Big(\frac{\sqrt{n}}{k\sigma_1}(Y_n - EY_n) \leq x\Big) - Q_n(x) \mid \leq \frac{\epsilon_n}{\sqrt{n}}, \tag{3.13}$$

where $\epsilon_n \to 0$ as $n \to \infty$. We can prove Theorem 3.4 by using these results.

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