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CRISPNESS IN DEDEKIND CATEGORIES

By

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Abstract

This paper studies notions of scalar relations and crispness of relations in terms of Dedekind categories. It is well-known that a category of L-relations in the sense of Goguen is a Dedekind category. To compare with an ordinary notion of crispness of L-relations, we introduce three notions of crispness in Dedekind categories.

Key Words and Phrases: crispness, Dedekind categories, L-relations, points, scalars

1. Introduction

Just after Zadeh's invention of the concept of fuzzy sets, Goguen (1967) generalized the concepts of fuzzy sets and relations to L-(fuzzy) sets and L-(fuzzy) relations. The sets of membership values of L-sets and L-relations are arbitrary lattices instead of the unit interval $[0, 1]$.

On the other hand, the theory of relations, namely relational calculus, has been investigated since the middle of the nineteen century. Almost all modern formalisations of relation algebras are affected by Tarski (1941). Maddux (1991) summerized the history of relation algebras. Mac Lane (1961) and Puppe (1962) exposed a categorical basis for the calculus of additive relations. Freyd and Scedrov (1990) developed and summarized categorical relational calculus, which they called allegories. In relational calculus, one calculates with relations in an element-free style, which makes relational calculus a very useful framework for the study of mathematics, theoretical computer science and also a useful tool for applications. Kawahara (1995) studied relational approach to set theory. Bird and de Moor (1997) described an algebraic approach to programming in the framework of relational calculus. Kawahara (1990), Kawahara and Mizoguchi (1994) developed relational methodology graph grammer. Schmidt and Ströhlein (1993) wrote a text book on relations and graphs with many useful examples in computer science. Some element-free formalisations of fuzzy relations were provided in Furusawa (1996), Kawahara and Furusawa (1999) and Kawahara et al. (1999).

In this paper we consider Dedekind categories named by Olivier and Serrato (1995). The aim of this paper is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness in terms of Dedekind categories was introduced in Kawahara et al. (1999) under the assumption that Dedekind categories have unit objects which are an abstraction of singleton (or one-point) sets. To generalize the notion

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of crispness, we use a notion of scalar relations. The notion of scalar relations in homogeneous relation algebras was introduced in Furusawa (1997). In L-relations we illustrate a few relationships between the generalized crispness which is called s-crispness and lattice structures of scalars.

Winter (2000) showed that it is impossible to characterize ordinary notion of crispness of L-relations in terms of Dedekind categories if L is an arbitrary complete distributive lattice. Also Winter introduced a notion of Goguen categories, namely, Dedekind categories with a kind of cut operators, and characterized crisp L-relations. Now, a question arises;

> Under which assumption of the underlying lattice is it possible to characterize in terms of Dedekind categories without adding cut operators?

One of main results in this paper answers to this question;

> The notion of s-crispness coincides with ordinary crispness of L-relations if the ordering on L is linear in the neighborhood of the least element.

This condition is fulfilled by the unit interval $[0, 1]$, which is the case of fuzzy relations in the sense of Zadeh (1965).

In addition to the notion of s-crispness we provide another notion of crispness by using a concept of points. Then our notions of crispness is compared to one another and also compared with an ordinary notion of crispness of L-relations.

This paper is organized as follows: In Section 2 we first state the definition of Dedekind categories as a categorical structure formed by L-relations with sup-inf composition. Also we define a preoder among objects of Dedekind categories which compares the lattice structures on objects in a sense. In Section 3 we recall the definition of L-relations, due to Goguen (1967). Section 4 studies notions of scalars for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. In Section 5 we study notions of crispness in Dedekind categories and also in L-relations.

2. Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call Dedekind categories following Olivier and Serrato (1995). Dedekind categories are called locally complete division allegories in Freyd and Scedrov (1990).

Throughout this paper, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoondown Y$, and the composite of a morphism $\alpha : X \rightharpoondown Y$ followed by a morphism $\beta : Y \rightharpoondown Z$ will be written as $\alpha\beta : X \rightharpoondown Z$. We denote the identity morphism on an object X by id_X.

DEFINITION 2.1. A *Dedekind category* $\mathcal{D}$ is a category satisfying the following:
D1. [Complete Distributive Lattice] For all pairs of objects X and Y the hom-set $\mathcal{D}(X, Y)$ consisting of all morphisms of X into Y is a complete distributive lattice with

the least morphism 0_{XY} and the greatest morphism ∇_{XY}. Its lattice structure will be denoted by

$$\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \sqsubseteq, \sqcup, \sqcap, 0_{XY}, \nabla_{XY})$$

and satisfies the following conditions:
(a) $\sqsubseteq$ is a partial order on $\mathcal{D}(X,Y)$, (b) $\forall\alpha \in \mathcal{D}(X,Y) : 0_{XY} \sqsubseteq \alpha \sqsubseteq \nabla_{XY}$, (c) $\sqcup_\lambda\beta_\lambda \sqsubseteq \alpha \iff \forall\lambda : \beta_\lambda \sqsubseteq \alpha$, (d) $\alpha \sqsubseteq \sqcap_\lambda\beta_\lambda \iff \forall\lambda : \alpha \sqsubseteq \beta_\lambda$, (e) $\alpha \sqcap (\sqcup_\lambda\beta_\lambda) = \sqcup_\lambda(\alpha \sqcap \beta_\lambda)$.
D2. [Converse] There is given a converse operator $^\sharp : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$ for all pair of objects X and Y. That is, for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$, the following involutive laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residue] For all morphisms $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoondown Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoondown Y$.

EXAMPLE 2.2. Consider a category Rel_0 whose objects are all nonempty sets and in which a hom-set $Rel_0(X,Y)$ between objects X and Y is the set of all (binary) relations on X if $X = Y$, and $\nabla_{XY} = 0_{XY}$ otherwise. That is, a hom-set $Rel_0(X,Y)$ is a singleton set when X and Y are distinct. Then it is easy to verify that the category Rel_0 is a Dedekind category. The conditions (D1) and (D2) are trivial, and (D3) and (D4) also hold as follows: If $X = Y = Z$, then (D3) and (D4) are clear. If $X = Y \neq Z$, then $\beta = 0_{YZ}$, $\gamma = 0_{XZ}$ and $\gamma \div \beta = \nabla_{XX}$. If $X \neq Y$, then $\alpha = 0_{XY}$ and $\gamma \div \beta = 0_{XY}$.

Throughout the rest of this section, all discussions will assume a fixed Dedekind category $\mathcal{D}$. The greatest morphism ∇_{XY} is called the *universal* morphism and the least morphism 0_{XY} the *zero* morphism. A morphism is *nonzero* if it is not equal to the zero morphism. An object X is called *nonzero* if $\nabla_{XX} \neq 0_{XX}$. A morphism $\alpha : X \rightharpoondown Y$ is *complemented* if it has a complement morphism $\alpha^- : X \rightharpoondown Y$ such that $\alpha \sqcup \alpha^- = \nabla_{XY}$ and $\alpha \sqcap \alpha^- = 0_{XY}$.

PROPOSITION 2.3. *Let $\alpha, \alpha' : X \rightharpoondown Y$, $\beta, \beta' : Y \rightharpoondown Z$, $\gamma : Y \rightharpoondown Z$ and $\delta : Z \rightharpoondown X$ be morphisms in $\mathcal{D}$, and W an object of $\mathcal{D}$.*

(a) $(\alpha\nabla_{YZ} \sqcap \gamma)\nabla_{ZW} = \alpha\nabla_{YW} \sqcap \gamma\nabla_{ZW}$ *and* $\nabla_{WZ}(\nabla_{ZX}\alpha \sqcap \delta) = \nabla_{WX}\alpha \sqcap \nabla_{WZ}\delta$,

(b) $\alpha\nabla_{YX}\nabla_{XW} = \alpha\nabla_{YW}$ *and* $\nabla_{WY}\nabla_{YX}\alpha = \nabla_{WX}\alpha$,

(c) $\nabla_{XX}\nabla_{XY} = \nabla_{XY}\nabla_{YY} = \nabla_{XY}$,

(d) *If* $\alpha \sqcup \alpha' = \nabla_{XY}$, $\alpha \sqcap \alpha' = 0_{XY}$ *and* $\nabla_{XX}\alpha = \alpha$, *then* $\nabla_{XX}\alpha' = \alpha'$,

(e) *If* $u \sqsubseteq \mathrm{id}_X$ *and* $u' \sqsubseteq \mathrm{id}_X$, *then* $u^\sharp = uu = u$ *and* $uu' = u \sqcap u'$,

(f) *If* $u \sqsubseteq \mathrm{id}_X$ *and* $v \sqsubseteq \mathrm{id}_Y$, *then* $u\alpha = \alpha \sqcap u\nabla_{XY}$ *and* $\alpha v = \alpha \sqcap \nabla_{XY}v$.

PROOF. (a) It is trivial that $(\alpha\nabla_{YZ} \sqcap \gamma)\nabla_{ZW} \sqsubseteq \alpha\nabla_{YW} \sqcap \gamma\nabla_{ZW}$. Conversely,

$$\begin{aligned} \alpha\nabla_{YW} \sqcap \gamma\nabla_{ZW} &\sqsubseteq (\alpha\nabla_{YW}\nabla_{ZW}^\sharp \sqcap \gamma)\nabla_{ZW} \\ &= (\alpha\nabla_{YZ} \sqcap \gamma)\nabla_{ZW} \ . \end{aligned}$$

(b) It is trivial that $\alpha\nabla_{YX}\nabla_{XW} \sqsubseteq \alpha\nabla_{YW}$. Conversely,

$$\begin{aligned}\alpha\nabla_{YW} &= \alpha\nabla_{YW} \sqcap \nabla_{XW} \\ &\sqsubseteq (\alpha\nabla_{YW}\nabla^{\sharp}_{XW} \sqcap \mathrm{id}_X)\nabla_{XW} \\ &\sqsubseteq \alpha\nabla_{YW}\nabla_{WX}\nabla_{XW} \\ &\sqsubseteq \alpha\nabla_{YX}\nabla_{XW} .\end{aligned}$$

(c) Immediate from $\nabla_{XY} = \mathrm{id}_X\nabla_{XY} \sqsubseteq \nabla_{XX}\nabla_{XY}$.
(d) It is trivial that $\alpha' \sqsubseteq \nabla_{XX}\alpha'$. Note that $\nabla_{XX}\alpha' \sqcap \alpha = 0_{XY}$, because

$$\nabla_{XX}\alpha' \sqcap \alpha \sqsubseteq \nabla_{XX}(\alpha' \sqcap \nabla_{XX}\alpha) = \nabla_{XX}(\alpha' \sqcap \alpha) = 0_{XY} .$$

Then we have

$$\nabla_{XX}\alpha' = \nabla_{XX}\alpha' \sqcap \nabla_{XY} = \nabla_{XX}\alpha' \sqcap (\alpha \sqcup \alpha') = (\nabla_{XX}\alpha' \sqcap \alpha) \sqcup (\nabla_{XX}\alpha' \sqcap \alpha') \sqsubseteq \alpha' .$$

(e) Assume that $u \sqsubseteq \mathrm{id}_X$. Then $u = u \sqcap \mathrm{id}_X \sqsubseteq u(\mathrm{id}_X \sqcap u^\sharp) \sqsubseteq uu^\sharp \sqsubseteq u^\sharp$. Similarly it can be shown that $u^\sharp \sqsubseteq u$ holds. $uu \sqsubseteq u$ is trivial, and $u = u \sqcap \mathrm{id}_X \sqsubseteq u(\mathrm{id}_X \sqcap u^\sharp) = uu$. Assume that $u' \sqsubseteq \mathrm{id}_X$. Then $uu' = uu' \sqcap u' \sqsubseteq u \sqcap u'$ and $u \sqcap u' \sqsubseteq u(id_X \sqcap u^\sharp u') \sqsubseteq uu'$.
(f) follows from $u\alpha = u\alpha \sqcap \nabla_{XY} \sqsubseteq u(\alpha \sqcap u^\sharp\nabla_{XY}) \sqsubseteq \alpha \sqcap u\nabla_{XY} \sqsubseteq u(u^\sharp\alpha \sqcap \nabla_{XY}) = u\alpha$. □

The statement (c) in the last proposition indicates that if $\nabla_{XY} \neq 0_{XY}$, then both of X and Y are nonzero.

REMARK. In general, $\nabla_{XY}\nabla_{YZ} = \nabla_{XZ}$ does not always hold. Consider a category *Rel* whose objects are all sets and in which a hom-set $Rel(X,Y)$ between objects X and Y is the set of all relations from X to Y. Clearly, the category is a Dedekind category. If a set Y is empty one and X, Z are nonempty sets, $\nabla_{XY}\nabla_{YZ} = 0_{XZ}$ but $\nabla_{XZ} \neq 0_{XZ}$.

PROPOSITION 2.4. *Let $\alpha : X \rightharpoonup Y$ be a morphism such that $\nabla_{XX}\alpha = \alpha$. Then the following three conditions are equivalent: (a) $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$, (b) $\nabla_{XX} = \alpha\alpha^\sharp$, (c) $\nabla_{XX} = \alpha\nabla_{YX}$.*

Proof is omitted; it may be found in Kawahara et al. (1999). □

For a morphism $\alpha : X \rightharpoonup Y$ and an object W define a morphism $\phi_W(\alpha) = \nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W : W \rightharpoonup W$.

LEMMA 2.5. *Let $\alpha : X \rightharpoonup Y$ be a morphism and W an object. Then*

(a) $\phi_W(\alpha)\nabla_{WZ} = \nabla_{WX}\alpha\nabla_{YZ}$ *and* $\nabla_{ZW}\phi_W(\alpha) = \nabla_{ZX}\alpha\nabla_{YW}$ *for each object Z,*

(b) $\phi_W(\phi_X(\alpha)) = \phi_W(\phi_Y(\alpha)) = \phi_W(\alpha)$,

(c) $\phi_W(\alpha) = \phi_W(\alpha^\sharp)$,

(d) *If* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $\alpha \sqsubseteq \nabla_{XW}\phi_W(\alpha)\nabla_{WY}$,

(e) *If* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $\phi_W(\alpha) = 0_{WW}$ *is equivalent to* $\alpha = 0_{XY}$.

PROOF. (a) The former follows from

$$\phi_W(\alpha)\nabla_{WZ} = (\nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} = \nabla_{WX}\alpha\nabla_{YZ}$$

using proposition 2.3(a). The latter is similar.
(b) follows from

$$\begin{array}{rcl}\phi_W(\phi_X(\alpha)) & = & \nabla_{WX}\phi_X(\alpha)\nabla_{XW} \sqcap \mathrm{id}_W \\ & = & \nabla_{WX}\nabla_{XX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \phi_W(\alpha)\end{array}$$

and

$$\begin{array}{rcl}\phi_W(\phi_Y(\alpha)) & = & \nabla_{WY}\phi_Y(\alpha)\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \nabla_{WX}\alpha\nabla_{YY}\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \phi_W(\alpha)\end{array}$$

by (a) and proposition 2.3(c).

$$\begin{array}{ccc}\mathcal{D}(X,Y) & \xrightarrow{\phi_X} & \mathcal{D}(X,X) \\ {\scriptstyle\phi_Y}\downarrow & & \downarrow{\scriptstyle\phi_W} \\ \mathcal{D}(Y,Y) & \xrightarrow[\phi_W]{} & \mathcal{D}(W,W)\end{array}$$

(c) First, for each $\alpha' : X \rightharpoonup Y$, it holds that $\nabla_{WX}\alpha'\nabla_{YW} = (\nabla_{WX}\alpha'\nabla_{YW})^\sharp$ since

$$\begin{array}{rcl}\nabla_{WX}\alpha'\nabla_{YW} & = & \nabla_{WX}\alpha'\nabla_{YW} \sqcap \nabla_{WW} \\ & \sqsubseteq & \nabla_{WX}\alpha'(\nabla_{YW} \sqcap \alpha'^\sharp\nabla_{WX}^\sharp\nabla_{WW}) \\ & \sqsubseteq & \nabla_{WY}\alpha'^\sharp\nabla_{XW} \\ & = & (\nabla_{WX}\alpha'\nabla_{YW})^\sharp\end{array}$$

and

$$\begin{array}{rcl}(\nabla_{WX}\alpha'\nabla_{YW})^\sharp & = & \nabla_{WY}\alpha'^\sharp\nabla_{XW} \sqcap \nabla_{WW} \\ & \sqsubseteq & \nabla_{WY}\alpha'^\sharp(\nabla_{XW} \sqcap \alpha'\nabla_{YW}\nabla_{WW}) \\ & \sqsubseteq & \nabla_{WX}\alpha'\nabla_{YW}\ .\end{array}$$

Then it follows from

$$\phi_W(\alpha^\sharp) = \nabla_{WY}\alpha^\sharp\nabla_{XW}\sqcap\mathrm{id}_W = (\nabla_{WY}\alpha^\sharp\nabla_{XW})^\sharp\sqcap\mathrm{id}_W = \nabla_{WX}\alpha\nabla_{YW}\sqcap\mathrm{id}_W = \phi_W(\alpha)\ .$$

(d) If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then

$$\begin{array}{rcl}\alpha & = & \alpha \sqcap \nabla_{XY} \\ & = & \alpha \sqcap \nabla_{XW}\nabla_{WY} \\ & \sqsubseteq & \nabla_{XW}(\nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WY} \\ & = & \nabla_{XW}\phi_W(\alpha)\nabla_{WY}\ .\end{array}$$

(e) is immediate from (d). □

A binary relation $\prec$ among objects of $\mathcal{D}$ is defined as follows: For two objects X and Y, a relation $X \prec Y$ holds if and only if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$. (Note that the three conditions $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$, $\mathrm{id}_X \sqsubseteq \nabla_{XY}\nabla_{YX}$ and $\phi_X(\mathrm{id}_Y) = \mathrm{id}_X$ are mutually equivalent.) It is easy to see that $\prec$ is a preorder, that is, reflexive and transitive. For $\nabla_{XX} = \nabla_{XX}\nabla_{XX}$, and if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ and $\nabla_{YY} = \nabla_{YZ}\nabla_{ZY}$, then $\nabla_{XX} = \nabla_{XY}\nabla_{YY}\nabla_{YX} = \nabla_{XY}\nabla_{YZ}\nabla_{ZY}\nabla_{YX} \sqsubseteq \nabla_{XZ}\nabla_{ZX}$. Hence its symmetric kernel with $X \sim Y$ if and only if $X \prec Y$ and $Y \prec X$, is an equivalence relation. Remark that in the category Rel_0 of 2.2, two distinct objects are never equivalent.

PROPOSITION 2.6. *Assume that* $X \prec Y$. *If* $u \sqsubseteq \mathrm{id}_X$, $u' \sqsubseteq \mathrm{id}_X$ *and* $u\nabla_{XY} \sqsubseteq u'\nabla_{XY}$, *then* $u \sqsubseteq u'$.

PROOF. It follows from $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ that $u = \mathrm{id}_X \sqcap u\nabla_{XX} = \mathrm{id}_X \sqcap u\nabla_{XY}\nabla_{YX}$. □

DEFINITION 2.7. A Dedekind category $\mathcal{D}$ is *uniform* if all pairs of objects of $\mathcal{D}$ are equivalent, that is, if $X \sim Y$ for all objects X and Y of $\mathcal{D}$.

A morphism $f : X \rightharpoondown Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) and $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function* and may be introduced as $f : X \to Y$.

PROPOSITION 2.8. (a) *If there exists a total morphism* $\alpha : X \rightharpoondown Y$, *then* $X \prec Y$.

(b) *If there exists a function* $f : X \to Y$, *then* $X \prec Y$.

(c) *If* $X \prec W$ *or* $Y \prec W$, *then* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$.

(d) *If* $X \prec Y$ *and* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $X \prec W$.

(e) *If* $X \prec Y$ *and* $\nabla_{XY} = p^\sharp q$ *for some functions* $p : W \to X$ *and* $q : W \to Y$, *then* $X \sim W$.

PROOF. (a) $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp \sqsubseteq \nabla_{XY}\nabla_{YX}$.
(b) is a just corollary of (a).
(c) If $\nabla_{XX} = \nabla_{XW}\nabla_{WX}$, then $\nabla_{XY} = \nabla_{XX}\nabla_{XY} = \nabla_{XW}\nabla_{WX}\nabla_{XY} \sqsubseteq \nabla_{XW}\nabla_{WY}$.
(d) $\nabla_{XX} = \nabla_{XY}\nabla_{YX} = \nabla_{XW}\nabla_{WY}\nabla_{YX} \sqsubseteq \nabla_{XW}\nabla_{WX}$.
(e) First note that $W \prec X$ by (a). Since $\nabla_{XY} = p^\sharp q \sqsubseteq \nabla_{XW}\nabla_{WY}$, it follows from (d) that $X \prec W$. □

3. L-Relations

Let L be a complete distributive lattice (or, a complete Heyting algebra) with least element 0 and greatest element 1. The supremum (least upper bound) and the infimum (greatest lower bound) of a family $\{k_\lambda\}$ of elements in L will be denoted by $\vee_\lambda k_\lambda$ and $\wedge_\lambda k_\lambda$, respectively. For two elements $a, b \in L$ the relative pseudo-complement of a relative to b will be written as $a \Rightarrow b$. Now recall some fundamentals on L-relations.

Let X and Y be sets. An L-relation R from X into Y, written $R : X \rightharpoondown Y$, is a function $R : X \times Y \to L$. L-relations on finite sets may be expressed by matrices on a lattice L of membership values. For instance, let $L = \{0, a, b, 1\}$, $X = \{1, 2\}$, $Y = \{3, 4\}$. Then a matrix

$$\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$$

expresses an L-relation $R : X \rightharpoondown Y$ given by $R(1,3) = 0, R(1,4) = b, R(2,3) = 1, R(2,4) = a$. The set of all L-relations from X into Y will be denoted by $L\text{-}Rel(X, Y)$. An L-relation R is contained in an L-relation S, written $R \subseteq S$, if $R(x,y) \leq S(x,y)$ for all $(x,y) \in X \times Y$. The zero relation 0_{XY} and the universal relation ∇_{XY} are L-relations with $0_{XY}(x,y) = 0$ and $\nabla_{XY}(x,y) = 1$ for all $(x,y) \in X \times Y$, respectively. It is trivial that $\subseteq$ is a partial order, and $0_{XY} \subseteq R \subseteq \nabla_{XY}$ for all L-relations R. For a family $\{R_\lambda\}_\lambda$ of L-relations we define L-relations $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ as follows:

$$(\cup_\lambda R_\lambda)(x,y) = \vee_\lambda R_\lambda(x,y) \quad \text{and} \quad (\cap_\lambda R_\lambda)(x,y) = \wedge_\lambda R_\lambda(x,y)$$

for all $(x,y) \in X \times Y$. It is obvious that $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ are the least upper bound and the greatest lower bound of a family $\{R_\lambda\}_\lambda$, respectively, with respect to the order $\subseteq$. The composite $RS(= R; S) : X \rightharpoondown Z$ of an L-relation $R : X \rightharpoondown Y$ followed by an L-relation $S : Y \rightharpoondown Z$ is defined by

$$(RS)(x,z) = \vee_{y \in Y}[R(x,y) \wedge S(y,z)]$$

for all $(x,z) \in X \times Z$. This composition of L-relations is called sup-inf composition. The composition is associative, i.e. the equation $(RS)T = R(ST)$ holds for all L-relations R, S and T. The identity relation id_X of a set X is an L-relation such that $\mathrm{id}_X(x,x') = 1$ if $x = x'$ and $\mathrm{id}_X(x,x') = 0$ otherwise. The unit laws $\mathrm{id}_X R = R$ and $R\mathrm{id}_Y = R$ hold for all $R : X \rightharpoondown Y$. The converse (or transpose) $R^\sharp : Y \rightharpoondown X$ of an L-relation $R : X \rightharpoondown Y$ is defined by

$$R^\sharp(y,x) = R(x,y)$$

for all $(y,x) \in Y \times X$. For L-relations $S : Y \rightharpoondown Z$ and $T : X \rightharpoondown Z$, the residue $T \div S : X \rightharpoondown Y$ is defined by

$$(T \div S)(x,y) = \wedge_{z \in Z}[S(y,z) \Rightarrow T(x,z)]$$

for all $(x,y) \in X \times Y$. The readers can easily see that L-relations and their operations defined above satisfy all axioms of Dedekind categories; only D3 (Dedekind formula) and D4 (Residues) are not so obvious, and will be proved in the following:

PROPOSITION 3.1. *Let $R : X \rightharpoondown Y, S : Y \rightharpoondown Z$ and $T : X \rightharpoondown Z$ be L-relations. Then*

(a) $RS \cap T \subseteq R(S \cap R^\sharp T)$ *(Dedekind formula),*

(b) $RS \subseteq T$ *if and only if* $R \subseteq T \div S$.

PROOF. (a) Since $R^\sharp(y,x) \wedge T(x,z) \leq (R^\sharp T)(y,z)$, for all $(x,z) \in X \times Z$ we obtain that

$$\begin{aligned}
(RS \cap T)(x,z) &= \vee_{y \in Y}[R(x,y) \wedge S(y,z)] \wedge T(x,z) \\
&= \vee_{y \in Y}[R(x,y) \wedge S(y,z) \wedge T(x,z)] \\
&= \vee_{y \in Y}[R(x,y) \wedge S(y,z) \wedge R^\sharp(y,x) \wedge T(x,z)] \\
&\leq \vee_{y \in Y}[R(x,y) \wedge S(y,z) \wedge (R^\sharp T)(y,z)] \\
&= \vee_{y \in Y}[R(x,y) \wedge (S \cap R^\sharp T)(y,z)] \\
&= [R(S \cap R^\sharp T)](x,z) \ .
\end{aligned}$$

(b) follows from the following equivalence:

$$\begin{aligned} RS \subseteq T &\iff \forall x \forall z : (RS)(x,z) \leq T(x,z) \\ &\iff \forall x \forall z : \vee_{y \in Y}[R(x,y) \wedge S(y,z)] \leq T(x,z) \\ &\iff \forall x \forall z \forall y : R(x,y) \wedge S(y,z) \leq T(x,z) \\ &\iff \forall x \forall z \forall y : R(x,y) \leq S(y,z) \Rightarrow T(x,z) \\ &\iff \forall x \forall y : R(x,y) \leq \wedge_{z \in Z}[S(y,z) \Rightarrow T(x,z)] \\ &\iff \forall x \forall y : R(x,y) \leq (T \div S)(x,y) \\ &\iff R \subseteq T \div S \ . \end{aligned}$$

□

An L-relation $k : X \rightharpoondown X$ is a scalar (represented as a scalar matrix) if and only if

$$\forall x, x' \in X : k(x,x) = k(x',x') \text{ and } x \neq x' \Rightarrow k(x,x') = 0 \ .$$

Scalar L-relations can be characterized algebraically:

PROPOSITION 3.2. *R is a scalar relation on a set X if and only if $R \subseteq \mathrm{id}_X$ and $R\nabla_{XX} = \nabla_{XX}R$.*

PROOF. Remark that

$$\begin{aligned} R\nabla_{XX}(x,y) &= \vee_{z \in X}(R(x,z) \wedge \nabla_{XX}(z,y)) \\ &= R(x,x) \wedge \nabla_{XX}(x,y) \\ &= R(x,x) \end{aligned}$$

for all $x, y \in X$ if $R \subseteq \mathrm{id}_X$. (Similarly $\nabla_{XX}R(x,y) = R(y,y)$.) Now assume that R is a scalar L-relation. Then it is trivial that $R \subseteq \mathrm{id}_X$. Thus

$$R\nabla_{XX}(x,y) = R(x,x) = R(y,y) = \nabla_{XX}R(x,y) \ .$$

Next assume that $R \subseteq \mathrm{id}_X$ and $R\nabla_{XX} = \nabla_{XX}R$. By $R \subseteq \mathrm{id}_X$ we obtain $R(x,y) = 0$ if $x \neq y$. Also $R(x,x) = R\nabla_{XX}(x,y) = \nabla_{XX}R(x,y) = R(y,y)$ for all $x, y \in X$. Therefore R is a scalar L-relation. □

4. Scalars

We now introduce a notion of scalars in Dedekind categories.

DEFINITION 4.1. Let X be an object of a Dedekind category $\mathcal{D}$. A *scalar* k on X is a morphism $k : X \rightharpoondown X$ of $\mathcal{D}$ such that $k \sqsubseteq \mathrm{id}_X$ and $k\nabla_{XX} = \nabla_{XX}k$.

A scalar k on X commutes with all endomorphisms $\alpha : X \rightharpoondown X$, that is, $k\alpha = \alpha k$, because

$$k\alpha = \alpha \sqcap k\nabla_{XX} = \alpha \sqcap \nabla_{XX}k = \alpha k \ .$$

It is trivial that the zero morphism $0_{XX} : X \rightharpoondown X$ and the identity morphism $\mathrm{id}_X : X \rightharpoondown X$ are scalars on X. The set of all scalars on X is denoted by $\mathcal{F}(X)$. It is clear that $\mathcal{F}(X)$ is a complete distributive lattice for all objects X. A morphism $\xi : X \rightharpoondown Y$ is called an *ideal* if $\nabla_{XX}\xi\nabla_{YY} = \xi$. The notion of ideals in relation algebras was initially introduced by Jónsson and Tarski (1952). The following lemma shows that scalars bijectively correspond to ideals in a sense.

LEMMA 4.2. (a) *If* $\xi : X \rightharpoondown Y$ *is an ideal, then* $k = \xi\nabla_{YX} \sqcap \mathrm{id}_X$ *is a scalar on* X *such that* $\xi = k\nabla_{XY}$.

(b) *If* $X \prec Y$ *and* k *is a scalar on* X, *then* $\xi = k\nabla_{XY}$ *is an ideal such that* $k = \xi\nabla_{YX} \sqcap \mathrm{id}_X$.

PROOF. (a) We have

$$(\xi\nabla_{YX} \sqcap \mathrm{id}_X)\nabla_{XX} = \xi\nabla_{YX} \sqcap \nabla_{XX} = \xi\nabla_{YX}$$

and

$$\nabla_{XX}(\xi\nabla_{YX} \sqcap \mathrm{id}_X) = \nabla_{XX}(\nabla_{XX}\xi\nabla_{YY}\nabla_{YX} \sqcap \mathrm{id}_X) = \nabla_{XX}\xi\nabla_{YY}\nabla_{YX} = \xi\nabla_{YX}$$

by proposition 2.3(a), which means that $\xi\nabla_{YX} \sqcap \mathrm{id}_X$ is a scalar on X. Also it holds that $(\xi\nabla_{YX} \sqcap \mathrm{id}_X)\nabla_{XY} = \xi\nabla_{YY} = \xi$.
(b) By $\nabla_{XY}\nabla_{YX} = \nabla_{XX}$ we have

$$\nabla_{XX}(k\nabla_{XY})\nabla_{YY} = k\nabla_{XX}\nabla_{XY}\nabla_{YY} = k\nabla_{XY}$$

and

$$(k\nabla_{XY})\nabla_{YX} \sqcap \mathrm{id}_X = k\nabla_{XX} \sqcap \mathrm{id}_X = k\mathrm{id}_X = k\ .\Box$$

Remark that $\phi_X(k) = k$ for all scalars k on X.

PROPOSITION 4.3. *Let* $\alpha : X \rightharpoondown Y$ *be a morphism. Then*

(a) $\phi_W(\alpha)$ *is a scalar on* W,

(b) *If* $X \prec Y$, *then* $\phi_X(\phi_Y(k)) = k$ *for all scalars* $k \in \mathcal{F}(X)$,

(c) *If* $X \sim Y$, *then* $\mathcal{F}(X)$ *and* $\mathcal{F}(Y)$ *are isomorphic as lattices,*

(d) $\phi_X(k)\alpha = \alpha\phi_Y(k)$ *for all scalars* k *on* W,

(e) *If* $\alpha \neq 0_{XY}$, *then there is a nonzero scalar* $k \in \mathcal{F}(X)$ *such that* $\nabla_{XX}\alpha\nabla_{YY} = k\nabla_{XY}$.

PROOF. (a) Set $W = Z$ in Lemma 2.5(a). Then $\phi_W(\alpha)\nabla_{WW} = \nabla_{WX}\alpha\nabla_{YW} = \nabla_{WW}\phi_W(\alpha)$.
(b) First note that $\phi_Y(k)\nabla_{YX} = \nabla_{YX}k\nabla_{XX}$ by Lemma 2.5(a) and so

$$\nabla_{XY}\phi_Y(k)\nabla_{YX} = \nabla_{XY}\nabla_{YX}k\nabla_{XX} = \nabla_{XX}k\nabla_{XX} = k\nabla_{XX}\ .$$

Hence we have $\phi_X(\phi_Y(k)) = \nabla_{XY}\phi_Y(k)\nabla_{YX} \sqcap \mathrm{id}_X = k\nabla_{XX} \sqcap \mathrm{id}_X = k$ by proposition 2.3(f).
(c) is obvious from (b).
(d) By Lemma 2.5(a) we have $\phi_X(k)\nabla_{XY} = \nabla_{XW}k\nabla_{WY} = \nabla_{XY}\phi_Y(k)$. Consequently it holds that $\phi_X(k)\alpha = \alpha \sqcap \phi_X(k)\nabla_{XY} = \alpha \sqcap \nabla_{XY}\phi_Y(k) = \alpha\phi_Y(k)$.
(e) Set $k = \phi_X(\alpha)$. Then it is clear that k is a scalar on X by (a) and $\nabla_{XX}\alpha\nabla_{YY} = k\nabla_{XY}$ by Lemma 2.5(a). And k is nonzero by Lemma 2.5(e), since α is nonzero. (Cf. Kawahara et al. (1999), Theorem 5.4) $\Box$

From the above Lemma 4.2(a) we have ϕ_W as a mapping $\phi_W : \mathcal{D}(X,Y) \to \mathcal{F}(W)$.

Fact 1

$$\begin{aligned}\phi_W(\phi_X(\alpha)) &= \nabla_{WX}\phi_X(\alpha)\nabla_{XW} \sqcap \mathrm{id}_W \\ &= \nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W \\ &= \nabla_{WX}\alpha\nabla_{YX}\nabla_{XW} \sqcap \mathrm{id}_W\end{aligned}$$

and

$$\begin{aligned}\phi_W(\phi_Y(\alpha)) &= \nabla_{WY}\phi_Y(\alpha)\nabla_{YW} \sqcap \mathrm{id}_W \\ &= \nabla_{WX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W \\ &= \nabla_{WY}\nabla_{YX}\alpha\nabla_{YW} \sqcap \mathrm{id}_W\end{aligned}$$

by Lemma 2.5(a). In particular, the following holds for $\alpha = \nabla_{XY}$:

$$\begin{aligned}\nabla_{WX}\nabla_{XY}\nabla_{YW} \sqcap \mathrm{id}_W &= \nabla_{WX}\nabla_{XY}\nabla_{YX}\nabla_{XW} \sqcap \mathrm{id}_W \\ &= \nabla_{WY}\nabla_{YX}\nabla_{XY}\nabla_{YW} \sqcap \mathrm{id}_W\ .\end{aligned}$$

PROPOSITION 4.4. *If all nonzero morphisms $\alpha : X \rightharpoonup X$ satisfy $\nabla_{XX}\alpha\nabla_{XX} = \nabla_{XX}$ (Tarski rule), then there is no scalar on X except for the zero morphism 0_{XX} and the identity id_X.*

PROOF. Let k be a nonzero scalar on X. Then, by the Tarski rule, we have $k\nabla_{XX} = k\nabla_{XX}\nabla_{XX} = \nabla_{XX}k\nabla_{XX} = \nabla_{XX}$, and so $k = \mathrm{id}_X \sqcap k\nabla_{XX} = \mathrm{id}_X$ since $k \sqsubseteq \mathrm{id}_X$. □

5. Crispness

In this section we study notions of crispness in Dedekind categories. First of all recall the definition of (0-1) crispness of L-relations.

An L-relation $R : X \rightharpoonup Y$ is called *0-1 crisp* if $R(x,y) = 0$ or $R(x,y) = 1$ for all $(x,y) \in X \times Y$. Of course O_{XY}, ∇_{XY} and id_X are 0-1 crisp. For a 0-1 crisp L-relation $R : X \rightharpoonup Y$ define an L-relation $\overline{R} : X \rightharpoonup Y$ by $\overline{R}(x,y) = 0$ if $R(x,y) = 1$ and $\overline{R}(x,y) = 1$ otherwise. Then $R \cup \overline{R} = \nabla_{XY}$ and $R \cap \overline{R} = O_{XY}$. This fact means that all 0-1 crisp L-relations are complemented. Note that all identity L-relations id_X are always complemented and a singleton set I is a unique set (up to isomorphisms) such that $0_{II} \neq \mathrm{id}_I = \nabla_{II}$.

Now we introduce a notion of crispness in Dedekind categories which is called s-crispness.

DEFINITION 5.1. A morphism $\alpha : X \rightharpoonup Y$ is *s-crisp* (scalar crisp) if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all nonzero scalars $k : X \rightharpoonup X$ and all morphisms $\tau : X \rightharpoonup Y$.

It is trivial from the above definition that every universal morphism ∇_{XY} is s-crisp.

A unit object I of $\mathcal{D}$ is an object of $\mathcal{D}$ such that $0_{II} \neq \mathrm{id}_I = \nabla_{II}$. Using a notion of a unit object we define a notion of I-crispness in Kawahara et al. (1999). The notion of I-crispness is defined only in hom-sets $\mathcal{D}(I,\ \_)$ from a unit object I but the notion of s-crispness is defined in any hom-sets. And an I-crisp relation is a s-crisp relation from a unit object. So s-crispness is a generalized notion from I-crispness.

PROPOSITION 5.2. (a) *If $X \prec Y$ and a morphism $\alpha : X \rightharpoondown Y$ is s-crisp, then $\alpha^\sharp : Y \rightharpoondown X$ is s-crisp.*

(b) *The infimum of s-crisp morphisms is s-crisp.*

(c) *If $f : X \to Y$ is a function and a morphism $\beta : Y \rightharpoondown Z$ is s-crisp, then the composite $f\beta : X \rightharpoondown Z$ is s-crisp.*

(d) *If the identity id_Y is s-crisp, then all functions $f : X \to Y$ are s-crisp.*

(e) *A morphism $\alpha : X \rightharpoondown Y$ is s-crisp if and only if its relative pseudo-complement $\alpha' \Rightarrow \alpha$ is s-crisp for every morphism $\alpha' : X \rightharpoondown Y$.*

(f) *If $x \sqcap \rho \neq 0_{IX}$ and I is a unit object for a function $x : I \to X$ and an s-crisp relation $\rho : I \rightharpoondown X$, then $x \sqsubseteq \rho$.*

PROOF. (a) Assume that $\alpha : X \rightharpoondown Y$ is s-crisp and $k\tau \sqsubseteq \alpha^\sharp$ for a nonzero scalar k on Y and a morphism $\tau : Y \rightharpoondown X$. Then $\phi_X(k)\tau^\sharp = \tau^\sharp k = (k\tau)^\sharp \sqsubseteq (\alpha^\sharp)^\sharp = \alpha$ and so $\tau^\sharp \sqsubseteq \alpha$, since $\phi_X(k)$ is a nonzero scalar on X by Lemma 2.5(e). Hence $\tau \sqsubseteq \alpha^\sharp$.
(b) Assume that $\alpha_i : X \rightharpoondown Y$ is s-crisp for $i \in I$ and $k\tau \sqsubseteq \sqcap_{i\in I}\alpha_i$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoondown Y$. Then we have $k\tau \sqsubseteq \alpha_i$ for all $i \in I$, and so $\tau \sqsubseteq \alpha_i$ by s-crispness. Hence $\tau \sqsubseteq \sqcap_{i\in I}\alpha_i$.
(c) Assume that $k\tau \sqsubseteq f\beta$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoondown Z$. First note that $\phi_Y(k)$ is a nonzero scalar by Lemma 2.5(e) and proposition 2.8(b), and $\phi_Y(k)f^\sharp = f^\sharp k$ by proposition 4.3(d). Then we have $\phi_Y(k)f^\sharp\tau = f^\sharp k\tau \sqsubseteq f^\sharp f\beta \sqsubseteq \beta$ and so $f^\sharp\tau \sqsubseteq \beta$ by the s-crispness of β. Therefore $\tau \sqsubseteq ff^\sharp\tau \sqsubseteq f\beta$, which completes the proof.
(d) is a special case of (b).
(e) First assume that $\alpha : X \rightharpoondown Y$ is s-crisp and $k\tau \sqsubseteq \alpha' \Rightarrow \alpha$ for a nonzero scalar k and morphisms $\tau, \alpha' : X \rightharpoondown Y$. Then we have $k(\tau \sqcap \alpha') = k\tau \sqcap \alpha' \sqsubseteq \alpha$ and so $\tau \sqcap \alpha' \sqsubseteq \alpha$, since $\alpha : X \rightharpoondown Y$ is s-crisp. Therefore $\tau \sqsubseteq \alpha' \Rightarrow \alpha$. Conversely, if $\alpha' \Rightarrow \alpha$ is s-crisp for all morphisms $\alpha' : X \rightharpoondown Y$, then $\alpha = \nabla_{XY} \Rightarrow \alpha$ is s-crisp.
(f) Note that $(x \sqcap \rho)(x \sqcap \rho)^\sharp x \sqsubseteq \rho x^\sharp x \sqsubseteq \rho$ and $(x \sqcap \rho)(x \sqcap \rho)^\sharp$ is a nonzero scalar by $x \sqcap \rho \neq 0_{IX}$. It follows from the s-crispness of ρ that $x \sqsubseteq \rho$. (Consequently ρ is total since it contains a total relation x.) This completes the proof. □

It immediately follows from the last proposition 5.2(c) that every composite of s-crisp functions is also an s-crisp function.

THEOREM 5.3. *The following four statements are equivalent:*

(a) *If $k \neq 0_{XX}$ and $k \sqcap k' = 0_{XX}$ for scalars $k, k' \in \mathcal{F}(X)$, then $k' = 0_{XX}$,*

(b) *The zero morphism 0_{XY} is s-crisp for every object Y (that is, if $k\tau = 0_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoondown Y$, then $\tau = 0_{XY}$),*

(c) *For every morphism $\alpha : X \rightharpoondown Y$, its pseudo-complement $\neg\alpha : X \rightharpoondown Y$ is s-crisp,*

(d) *Every complemented morphism $\alpha : X \rightharpoondown Y$ is s-crisp.*

PROOF. (a)⇒(b) Assume that $k\tau = 0_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. Recall that $\phi_X(\tau)$ is a scalar on X. Hence we have $k \sqcap \phi_X(\tau) = k\phi_X(\tau) = k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \sqsubseteq k\nabla_{XX}\tau\nabla_{YX} = \nabla_{XX}k\tau\nabla_{YX} = 0_{XX}$. It follows from (a) that $\phi_X(\tau) = 0_{XX}$ and so $\tau = 0_{XY}$ by Lemma 2.5(e). Hence 0_{XY} is s-crisp. (b)⇒(a) is trivial. (b)⇔(c)⇔(d) is a corollary of the last proposition 5.2. □

Next we investigate a relationship between the notions of s-crispness and 0-1 crispness of L-relations.

PROPOSITION 5.4. *All s-crisp L-relations are 0-1 crisp.*

PROOF. Let an L-relation $R : X \rightharpoonup Y$ be s-crisp. Assume that $a = R(x_0, y_0)$ is not equal to $0 \in L$ for some point $(x_0, y_0) \in X \times Y$. Consider a scalar k on X such that $k(x, x') = a$ if $x = x'$ and $k(x, x') = 0$ otherwise, and an L-relation $T : X \rightharpoonup Y$ such that $T(x, y) = a \Rightarrow R(x, y)$ for all $(x, y) \in X \times Y$. Then we have $kT \subseteq R$, since $(kT)(x, y) = a \wedge (a \Rightarrow R(x, y)) \leq R(x, y)$ for all $(x, y) \in X \times Y$. Hence $T \subseteq R$ follows from the fact that $R : X \rightharpoonup Y$ is s-crisp. Finally we have $1 = (a \Rightarrow a) = T(x_0, y_0) \leq R(x_0, y_0)$, which shows R is 0-1 crisp. □

The converse of the last proposition does not hold in general. Its necessary and sufficient condition is given by the following:

PROPOSITION 5.5. *For L-relations the following statements are equivalent:*

C0. $\forall a, b \in L : a \wedge b = 0 \Rightarrow a = 0$ *or* $b = 0$,

K0. All 0-1 crisp L-relations are s-crisp.

PROOF. First assume that C0 and $kT \subseteq R$ for a scalar k on X, an L-relation $T : X \rightharpoonup Y$, and a 0-1 crisp L-relation $R : X \rightharpoonup Y$. To prove that R is s-crisp we have to show that $T(x, y) \leq R(x, y)$ for all $(x, y) \in X \times Y$. Since $R(x, y) = 0$ or 1 by the 0-1 crispness of R it is enough to show that if $R(x, y) = 0$ then $T(x, y) = 0$. But $(kT)(x, y) = k(x, x) \wedge T(x, y) \leq R(x, y)$. Hence when $R(x, y) = 0$, we have $T(x, y) = 0$ from C0 and $k(x, x) \neq 0$. Conversely assume that K0 and $a \wedge b = 0$ for $a, b \in L$. Define a scalar k on a singleton set $I = \{*\}$ and an L-relation $R : I \rightarrow I$ by $k(*, *) = a$ and $T(*, *) = b$, respectively. Then $kT = 0_{II}$ and so $k = 0_{II}$ or $T = 0_{II}$ since 0_{II} is s-crisp by the assumption K0. □

PROPOSITION 5.6. *For L-relations the following statements are equivalent:*

C1. $\forall a, b \in L : a \wedge b = 0$ *and* $a \vee b = 1 \Rightarrow a = 0$ *or* $b = 0$,

K1. All complemented L-relations are 0-1 crisp,

K2. All L-relations which are functions are 0-1 crisp.

PROOF. Trivial. □

The following proposition means that the s-crispness for L-relations is too strong. This alerts that the notion of s-crispness does not always work as desired.

PROPOSITION 5.7. *Let X be an object of a Dedekind category $\mathcal{D}$. If there exist two nonzero scalars k and k' on X such that $k \sqcap k' = 0_{XY}$, then there is no s-crisp relation $\alpha : X \rightharpoondown Y$ except for the universal relation ∇_{XY} for all objects Y of $\mathcal{D}$.*

PROOF. We have to show that if a relation $\alpha : X \rightharpoondown Y$ is s-crisp then $\alpha = \nabla_{XY}$. Now let α be s-crisp. Then $k(k'\nabla_{XY}) = (k \sqcap k')\nabla_{XY} = 0_{XX}\nabla_{XY} = 0_{XY} \sqsubseteq \alpha$. Hence $k'\nabla_{XY} \sqsubseteq \alpha$ and $\nabla_{XY} \sqsubseteq \alpha$ by the s-crispness of α. □

COROLLARY 5.8. *Assume that there exist two nonzero elements $a, b \in L$ such that $a \wedge b = 0$. Then there is no s-crisp L-relations except for the universal L-relations.*

COROLLARY 5.9. *Let (X, Y) be an object of the product Dedekind category $\mathcal{D} \times \mathcal{D}$. Then the lattice of scalars on (X, Y) is the product of two lattices of scalars on X and Y, that is, $\mathcal{F}(X, Y) = \mathcal{F}(X) \times \mathcal{F}(Y)$, and two nonzero scalars $(\mathrm{id}_X, 0_{YY})$ and $(0_{XX}, \mathrm{id}_Y)$ are mutually complements. Hence all s-crisp relations in $\mathcal{D} \times \mathcal{D}$ are just universal relations.*

Although, as we have seen in proposition 5.5, s-crispness and 0-1 crispness are equivalent in L-relations when L satisfies the condition C0. For example, in fuzzy relations that may be the most applicable case, s-crispness exactly represents 0-1 crispness.

Next we will state a kind of crispness which was originally suggested by Wolfram Kahl.

DEFINITION 5.10. A scalar k on X is called *linear* if and only if for every scalar k' on X an equation $k \sqcap k' = 0_{XX}$ implies $k' = 0_{XX}$.

Let $\mathcal{W}(X)$ denote the set of all linear scalars on X. Every identity id_X is obviously linear. Note that a scalar k on X is linear if and only if its pseudo-complement $\neg k$ ($= \mathrm{id}_X \sqcap (k \Rightarrow 0_{XX})$) in $\mathcal{F}(X)$ is equal to 0_{XX}.

LEMMA 5.11. *If X is a nonzero object, then $\mathcal{W}(X)$ is a filter of $\mathcal{F}(X)$.*

PROOF. 0) It is trivial that 0_{XX} is not a linear scalar, whenever X is nonzero. i) If $k_0, k_1 \in \mathcal{W}(X)$, then $k_0 \sqcap k_1 \in \mathcal{W}(X)$: Assume $(k_0 \sqcap k_1) \sqcap k' = 0_{XX}$. Then $k_0 \sqcap (k_1 \sqcap k') = 0_{XX}$ and so $k_1 \sqcap k' = 0_{XX}$, which shows $k' = 0_{XX}$. ii) If $k_0 \in \mathcal{W}(X)$ and $k_1 \in \mathcal{F}(X)$ with $k_0 \sqsubseteq k_1$, then $k_1 \in \mathcal{W}(X)$: Assume $k_1 \sqcap k' = 0_{XX}$. Then $k_0 \sqcap k' = 0_{XX}$ and so $k' = 0_{XX}$. □

So the set of linear scalars on X is a sublattice of the lattice $\mathcal{F}(X)$ of all scalars on X, and as such it is distributive.

DEFINITION 5.12. A morphism $\alpha : X \rightharpoondown Y$ is *l-crisp* (linear crisp) if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all linear scalars $k : X \rightharpoondown X$ and all morphisms $\tau : X \rightharpoondown Y$.

PROPOSITION 5.13. *Every zero morphism 0_{XY} is l-crisp.*

PROOF. Assume that $k\tau = 0_{XY}$ for a linear scalar k on X and a morphism $\tau : X \rightharpoondown Y$. Then

$$\begin{aligned} k \sqcap \phi_X(\tau) &= k\phi_X(\tau) \\ &= k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \\ &\sqsubseteq k\nabla_{XX}\tau\nabla_{YX} \\ &\sqsubseteq \nabla_{XX}k\tau\nabla_{YX} \\ &= 0_{XY} \end{aligned}$$

and so $\phi_X(\tau) = 0_{XX}$. Hence $\tau = 0_{XY}$ by Lemma 2.5(e). □

DEFINITION 5.14. An element a of a lattice L is called *linear* if $a \wedge b = 0$ implies $b = 0$ for $b \in L$.

Let $k : X \rightharpoondown X$ be an L-relation on a nonempty set X. If k is a linear scalar, then $k(x,x)$ is linear in L for all $x \in X$.

Assume that $k(x,x) \wedge a = 0$ for $a \in L$. Now consider a scalar $k' : X \rightharpoondown X$ such that $k'(x,x') = a$ if $x = x'$, and $k'(x,x') = 0$ otherwise. Then $k \cap k' = O_{XX}$ and so $k' = O_{XX}$ by the linearity of k. Hence $a = 0$, which proves that $k(x,x)$ is linear.

PROPOSITION 5.15. *All 0-1 crisp L-relations are l-crisp.*

PROOF. Let an L-relation $R : X \rightharpoondown Y$ be 0-1 crisp and assume that $kT \sqsubseteq R$ for a linear scalar k on X and an L-relation $T : X \rightharpoondown Y$. We have to show that $T(x,y) \leq R(x,y)$ for all $(x,y) \in X \times Y$. Now $k(x,x) \wedge T(x,y) \leq (kT)(x,y) \leq R(x,y)$, and since $k(x,x)$ is linear, it follows that $R(x,y) = 0$ implies $T(x,y) = 0$, which is sufficient since $R(x,y)$ can only be 0 or 1 by 0-1 crispness. □

The converse of the above proposition does not hold: Consider a Boolean lattice L having a nontrivial element s such that $s \neq 0$ and $s \neq 1$, and define an L-relation $R : X \rightharpoondown X$ by $R(x,x') = s$ if $x = x'$ and $R(x,x') = 0$ otherwise. Then it is clear that R is l-crisp, but not 0-1 crisp. Generally for a Boolean lattice L every L-relation is l-crisp since the identity id_X is a unique linear scalar on X.

At the end of this section, we will state a kind of crispness which is called p-crispness.

A strict unit I of $\mathcal{D}$ is an object of $\mathcal{D}$ such that $0_{II} \neq \mathrm{id}_I = \nabla_{II}$ and $\nabla_{XI}\nabla_{IX} = \nabla_{XX}$ for all objects X. Let X be an object X of $\mathcal{D}$. An I-point x of X is a relation (or a morphism) $x : I \rightharpoondown X$ such that $x^\sharp x \sqsubseteq \mathrm{id}_X$ (univalent) and $\mathrm{id}_I \sqsubseteq xx^\sharp$ (total), that is, an I-point x of X is a function from a strict unit I into X. A notation $x \in X$ denotes x is an I-point of an object X.

Note that I-points in Kawahara et al. (1999) were required to be I-crisp as well as to be functions. A notion of point relations (points) in homogeneous relation algebras was introduced by Schmidt and Ströhlein (1985).

DEFINITION 5.16. A relation $\alpha : X \rightharpoondown Y$ is *p-crisp* (point-wise crisp) if $x\alpha\nabla_{YI} = \mathrm{id}_I$ or $x\alpha = 0_{IY}$ for all I-points $x : I \to X$. (Note that $x\alpha = 0_{IY}$ iff $x\alpha\nabla_{YI} = 0_{II}$.)

It is trivial from the above definition that all zero relations 0_{XY} and all total relations are p-crisp. So all functions $f : X \to Y$ are p-crisp. Remark that every universal relations ∇_{XY} is not always p-crisp (total). For example, take a universal relation $\nabla_{(I,I)(I,\emptyset)} = (\mathrm{id}_I, 0_{I\emptyset})$ in the product category *Rel* × *Rel* of the category *Rel* of sets and relations, where $I = \{*\}$. Then $\mathrm{id}_{(I,I)}\nabla_{(I,I)(I,\emptyset)}\nabla_{(I,\emptyset)(I,I)} = (\mathrm{id}_I, 0_{I\emptyset})(\mathrm{id}_I, 0_{\emptyset I}) = (\mathrm{id}_I, 0_{II})$ ($\neq 0_{(I,I)(I,I)}$ and $\neq \mathrm{id}_{(I,I)}$). (Note that $\mathrm{id}_{(I,I)}$ is a unique I-point of (I,I).) Thus $\nabla_{(I,I)(I,\emptyset)}$ is not p-crisp (of course not total).

Let $L = \{0,1\}^2 = \{0,a,b,1\}$. Then an L-relation $(a,b) : \{*\} \rightharpoonup \{1,2\}$ is total and so p-crisp, since

$$(a,b)(a,b)^\sharp = (a,b)\begin{pmatrix} a \\ b \end{pmatrix} = (1) = \mathrm{id}_{\{*\}} \ .$$

But (a,b) is not s-crisp because it is not 0-1 crisp. (Cf. proposition 5.4.) Hence this example shows that p-crisp relations are not always s-crisp.

PROPOSITION 5.17. (a) *The supremum of p-crisp relations are p-crisp.*

(b) *If x is an I-point of X and $f : X \to Y$ is a p-crisp partial function, then $xf = 0_{IY}$ or xf is an I-point of Y.*

(c) *If x is an I-point of X and $u : X \rightharpoonup X$ is a p-crisp relation such that $u \sqsubseteq \mathrm{id}_X$, then $xu = 0_{IX}$ or $xu = x$.*

(d) *The composite of a p-crisp partial function followed by a p-crisp relation is p-crisp.*

PROOF. (a) Let $\alpha_j : X \rightharpoonup Y$ $(j \in J)$ be p-crisp relations and $x : I \to X$ an I-point. If $x(\sqcup_{j\in J}\alpha_j) \neq 0_{IY}$, then $x\alpha_j\nabla_{YI} = \mathrm{id}_I$ for some $j \in J$ and so $x(\sqcup_{j\in J}\alpha_j)\nabla_{YI} = \mathrm{id}_I$.
(b) Let $f : X \rightharpoonup Y$ be a p-crisp partial function and $x : I \to X$ an I-point. As x and f are univalent, the composite xf is also univalent, that is $(xf)^\sharp(xf) \sqsubseteq \mathrm{id}_Y$. Now assume that $xf \neq 0_{IY}$. Then $xf\nabla_{YI} = \mathrm{id}_I$ and so $xf\nabla_{YY} = \nabla_{IY}$ from $\nabla_{YI}\nabla_{IY} = \nabla_{YY}$. Hence xf is an I-point of Y.
(c) It is clear from (b) that $xu = 0_{IX}$ or xu is an I-point of X. When xu is an I-point, we have $xu = x$ from $xu \sqsubseteq x$.
(d) Let $f : X \rightharpoonup Y$ be a p-crisp partial function, $\beta : Y \rightharpoonup Z$ a p-crisp relation and $x : I \to X$ an I-point. Assume that $xf\beta \neq 0_{IZ}$. Then $xf \neq 0_{IY}$ and xf is an I-point of Y by (b). Hence $xf\beta\nabla_{ZI} = \mathrm{id}_I$ by the p-crispness of β. This means that $f\beta$ is p-crisp. □

PROPOSITION 5.18. *Assume that there are nonzero relations $k, k' : I \rightharpoonup I$ such that $k \sqcup k' = \mathrm{id}_I$ and $k \sqcap k' = 0_{II}$. If $x\alpha\nabla_{YI} = 0_{II}$ for an I-point $x : I \to X$ and a p-crisp relation $\alpha : X \rightharpoonup Y$, then $y\alpha\nabla_{YI} = 0_{II}$ for all I-points $y : I \to X$.*

PROOF. Let $x, y : I \to X$ be an I-point of X and set $z = kx \sqcup k'y$. Then z is also an I-point of X, since $z^\sharp z = (x^\sharp k \sqcup y^\sharp k')(kx \sqcup k'y) = x^\sharp kx \sqcup y^\sharp k'y \sqsubseteq x^\sharp x \sqcup y^\sharp y \sqsubseteq \mathrm{id}_X$ and $zz^\sharp = (kx \sqcup k'y)(x^\sharp k \sqcup y^\sharp k') \sqsupseteq kxx^\sharp k \sqcup k'yy^\sharp k' \sqsupseteq kk \sqcup k'k' = k \sqcup k' = \mathrm{id}_I$. Now assume that $y\alpha\nabla_{YI} \neq 0_{II}$. Then $y\alpha\nabla_{YI} = \mathrm{id}_I$ by the p-crispness of α and so $z\alpha\nabla_{YI} = (kx \sqcup k'y)\alpha\nabla_{YI} = kx\alpha\nabla_{YI} \sqcup k'y\alpha\nabla_{YI} = k'$. Again from the p-crispness of α it follows that $z\alpha\nabla_{YI} = k'$ is equal to 0_{II} or id_I, which is a contradiction. □

PROPOSITION 5.19. *Assume that $\sqcup_{y\in Y} y = \nabla_{IY}$. Then all s-crisp relations $\alpha : X \rightharpoonup Y$ are p-crisp.*

PROOF. Let $\alpha : X \rightharpoondown Y$ be s-crisp. We have to see that for all I-points $x : I \to X$ the composite $x\alpha$ is total unless $x\alpha = 0_{II}$. Assume that $x\alpha \neq 0_{II}$. Then by the assumption $\sqcup_{y \in Y} y = \nabla_{IY}$ there is some I-point $y : I \to Y$ such that $x\alpha \sqcap y \neq 0_{IY}$. Recall that $x\alpha$ is s-crisp by proposition 5.2(c) and so $y \sqsubseteq x\alpha$ by proposition 5.2(f), which claims that $x\alpha$ is total. □

COROLLARY 5.20. *All s-crisp L-relations are p-crisp.*

Remark that every 0-1 crisp L-relation is not always p-crisp. Let $L = \{0,1\}^2 = \{0, a, b, 1\}$ and take a 0-1 crisp L-relation $\alpha : \{1,2\} \rightharpoondown \{1,2\}$ given by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

Then for an I-point $(a,b) : \{*\} \to \{1,2\}$ we have $(a,b)\alpha\nabla_{\{1,2\}\{*\}} = (a)$, which means that α is not p-crisp.

$$(a,b)\alpha\nabla_{\{1,2\}\{*\}} = (a,b)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (a) .$$

PROPOSITION 5.21. *Assume that the class of s-crisp relations is closed under composition. Then all s-crisp relations are p-crisp.*

PROOF. Let $\alpha : X \rightharpoondown Y$ be s-crisp and $x : I \to X$ an I-point. Recall that ∇_{YI} is s-crisp and so $\alpha\nabla_{YI}$ is s-crisp by the assumption. By proposition 5.2(c) $x\alpha\nabla_{YI}$ is also s-crisp. Since a nonzero s-crisp relation k $(= x\alpha\nabla_{YI}) : I \rightharpoondown I$ is equal to id_I (for $\mathrm{id}_I \sqsubseteq k$ from $k\mathrm{id}_I \sqsubseteq k$), $x\alpha\nabla_{YI} = 0_{II}$ or id_I. □

6. Conclusion

In this paper, we introduce new notions of s-crispness, l-crispness and p-crispness in Dedekind categories to specify crisp L-relations in terms of Dedekind categories and compared with each other. As Winter (2000) reported, Dedekind categories do not have enough operators to characterize crisp L-relations in the case that L is an arbitrary lattice. So, all of our new notions of crispness may not work to capture crispness of L-relations in general. Although, we made clear that the notion of s-crispness coincides with ordinary crispness of L-relations if the ordering on L is linear in the neighborhood of the least element. In fact the condition is fulfilled by the unit interval $[0,1]$, which is the case of fuzzy relations in the sense of Zadeh (1965).

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