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ON PROPER-EFFICIENCY FOR NONSMOOTH MULTIOBJECTIVE OPTIMAL CONTROL PROBLEMS

By

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Abstract

This paper deals with the proper-efficiency conditions for nonsmooth multiobjective optimal control problems. For the problem without inequality constraints, two kinds of Kuhn-Tucker-type necessary conditions are derived. One of them is sufficient for the proper-efficiency under suitable assumptions. Furthermore, we give necessary and sufficient conditions to the problem in a convex case. These conclusions are also extended to the problems with functional inequality constraints.

Key Words and Phrases: Multiobjective optimal control problem, Properly efficient solution, Nonsmooth analysis, Kuhn-Tucker-type necessary conditions, Necessary and sufficient conditions.

1. Introduction

In this paper, we focus our attention mainly on the following *nonsmooth* multiobjective optimal control problem.

$$
\begin{aligned}
(MCP): \quad \text{“}Minimize\text{''}_{(x(\cdot),u(\cdot))}: \quad & \left[\int_0^1 G_1(t,x(t),u(t))dt + g_1(x(1)), \cdots, \right.\\
& \left. \int_0^1 G_k(t,x(t),u(t))dt + g_k(x(1))\right] \\
subject\ to: \quad & \dot{x}(t) = \Phi(t,x(t),u(t)) \quad a.e. \\
& x(0) \in D \\
& u(t) \in U(t) \quad a.e.,
\end{aligned}
$$

in which

$$
\begin{cases}
x(\cdot) \in AC([0,1],R^m), \ u(\cdot) \in M([0,1],R^n) \\
G_i : [0,1] \times R^m \times R^n \to R, \ g_i : R^m \to R, \ i \in I := \{1,\cdots,k\} \\
\Phi : [0,1] \times R^m \times R^n \to R^m \\
D \subset R^m, \ U(\cdot) : [0,1] \to 2^{R^n}
\end{cases}
$$

where, $AC([0,1],R^m)$ is the space of absolutely continuous functions on [0,1] with value in R^m, and $M([0,1],R^n)$ denotes the space of Lebesgue measurable functions from [0,1] to R^n.

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For this problem (MCP), a pair $(x,u) \in AC([0,1],R^m) \times M([0,1],R^n)$ is called an admissible process iff $G_i(\cdot,x(\cdot),u(\cdot))$ is integrable in $[0,1]$ for every $i \in I$, and (x,u) satisfies the differential equation $\dot{x}(t) = \Phi(t,x(t),u(t))$ a.e. with the initial condition $x(0) \in D$ and the control constraint $u(t) \in U(t)$ a.e.. The first component of the process (x,u) is called a trajectory and the second is called a control. Nonsmoothness means that functions G_i, g_i, $i = 1,\cdots,k$, Φ are not assumed to be differentiable in any argument.

Now, we set

$$F_i(x,u) := \int_0^1 G_i(t,x(t),u(t))dt + g_i(x(1)) \quad \textit{for any } i \in I,$$

and denote by Ω the set of all admissible processes of (MCP), by B^n the n-dimensional unit closed ball at the origin. Then, we introduce the following definitions.

DEFINITION 1.1. $(x_*,u_*) \in \Omega$ is called

(i) an efficient solution of (MCP) if there exists no $(x,u) \in \Omega$ satisfying the following condition (C1);

(C1): $F_i(x,u) \leq F_i(x_*,u_*)$ for any $i \in I$ and $F_j(x,u) < F_j(x_*,u_*)$ for some $j \in I$.

(ii) a properly efficient solution of (MCP) if (x_*,u_*) is an efficient solution of (MCP) and there exists a scalar $K > 0$, such that for each $i \in I$ and every $(x,u) \in \Omega$ with $F_i(x,u) < F_i(x_*,u_*)$, the following condition (C2) holds.

(C2): There is at least one $j \in I$ with $F_j(x,u) > F_j(x_*,u_*)$ such that

$$(F_i(x_*,u_*) - F_i(x,u)) \,/\, (F_j(x,u) - F_j(x_*,u_*)) \leq K.$$

DEFINITION 1.2. $(x_*,u_*) \in \Omega$ is called

(i) a weakly-local efficient solution (resp. strongly-local) for (MCP) if there exists no $(x,u) \in \Omega$ with $\|x - x_*\|_{L^\infty} \leq \epsilon$ for some $\epsilon > 0$ (resp. with $x(t) \in x_*(t) + \epsilon B^m$ and $u(t) \in u_*(t) + \epsilon B^n$ a.e. for some $\epsilon > 0$) satisfying (C1);

(ii) a weakly-local properly efficient solution (resp. strongly-local) for (MCP) if (x_*,u_*) is a weakly-local efficient solution (resp. strongly-local) for (MCP), and there exists a scalar $K > 0$, such that for each $i \in I$ and any $(x,u) \in \Omega$ with $F_i(x,u) < F_i(x_*,u_*)$ and $\|x - x_*\|_{L^\infty} \leq \epsilon$ for some $\epsilon > 0$ (resp. with $x(t) \in x_*(t) + \epsilon B^m$ and $u(t) \in u_*(t) + \epsilon B^n$ a.e. for some $\epsilon > 0$), the condition (C2) holds.

Multiobjective optimization problems are important mathematical models for investigating the practical problems with several competing objectives, which arise from economics, engineering and policy sciences. There are many papers dealing with the optimality conditions for multiobjective programming problems and multiobjective optimal control problems under certain smoothness and convexity assumptions, for example, see Bhatia (1995), Geoffrion (1968), Kanniappan (1983), Khanh and Nuong (1988), Mishra and Mukherjee (1999), Yang and Jeyakumar (1997), Zemin (1996) and Zhukovskiy and Salukvadze (1996). On nonsmooth problems, Craven (1989), Wang, Dong and Liu (1994a), (1994b), and Ying (1985) derived optimality conditions for multiobjective programming, Breckner (1997) for an abstract optimal control problem, Hu and Salcudean (1998) for linear-quadratic and H^∞ optimal control problems, and Zhu (2000) for a differential inclusion problem. This paper is devoted to optimality conditions for the properly efficient solutions of the above general optimal control problem. We first study the proper-efficiency in the nonsmooth and nonconvex case. Then, we also discuss the convex problems without smoothness assumptions.

The following sections are organized as follows. Section 2 gives single-objective (scalar) optimization problems, which are equivalent to our multiobjective ones. In Section 3, we derive two kinds of Kuhn-Tucker-type necessary conditions for properly efficient solutions of (MCP), and show that one of them is sufficient to the proper-efficiency under a restricted setting. Section 4 will discuss necessary and sufficient conditions of the proper-efficiency for the "convex" multiobjective problems.

2. Preliminaries

As a general rule, to obtain the optimality conditions for (MCP), we shall reformulate this multiobjective optimal control problem as a single-objective (scalar) optimization problem. The following results give the relationship between (MCP) and the scalar optimal control problems.

LEMMA 2.1. *$(x_*, u_*) \in \Omega$ is a properly efficient (or local properly efficient) solution of (MCP) if and only if there is a scalar $K > 0$, such that (x_*, u_*) is an optimal (or local optimal) solution of the following scalar optimization problem (P_i) for every $i \in I$.*

$$(P_i) \quad \textit{Minimize}: \quad K \max_{j\in I}\{F_j(x,u) - F_j(x_*,u_*)\} + F_i(x,u)$$
$$\textit{subject to}: \quad (x,u) \in \Omega.$$

PROOF. : [Necessity] Let (x_*, u_*) be a properly efficient solution of (MCP). Then, there is $K > 0$ such that for every $i \in I$ and any $(x,u) \in \Omega$ with $F_i(x,u) < F_i(x_*,u_*)$, there exists $j \in I$ with $F_j(x,u) > F_j(x_*,u_*)$ and

$$F_i(x_*,u_*) - F_i(x,u) \le K(F_j(x,u)) - F_j(x_*,u_*)).$$

This implies that

$$K \max_{j\in I}\{F_j(x,u) - F_j(x_*,u_*)\} + F_i(x,u) \ge F_i(x_*,u_*).$$

On the other hand, for any $(x,u) \in \Omega$ with $F_i(x,u) \ge F_i(x_*,u_*)$, it is obvious that

$$K \max_{j\in I}\{F_j(x,u) - F_j(x_*,u_*)\} + F_i(x,u) \ge F_i(x,u) \ge F_i(x_*,u_*).$$

Hence, (x_*,u_*) is an optimal solution for every (P_i).

[Sufficiency] Suppose that (x_*,u_*) is an optimal solution for every (P_i), $i \in I$. It is easy to see that (x_*,u_*) is an efficient solution of (MCP). In fact, if there is $(x,u) \in \Omega$ with $F_i(x,u) < F_i(x_*,u_*)$ for some $i \in I$ and $F_j(x,u) \le F_j(x_*,u_*)$ for all $j \in I/\{i\}$, thus,

$$K \max_{j\in I}\{F_j(x,u) - F_j(x_*,u_*)\} + F_i(x,u) \le F_i(x,u) < F_i(x_*,u_*),$$

which contradicts that (x_*,u_*) is a minimum for (P_i^*).

For each $i \in I$ and any $(x,u) \in \Omega$ satisfying $F_i(x,u) < F_i(x_*,u_*)$, since

$$K \max_{j\in I}\{F_j(x,u) - F_j(x_*,u_*)\} \ge F_i(x_*,u_*) - F_i(x,u) > 0,$$

there exists $j \in I$ with $F_j(x,u) > F_j(x_*,u_*)$ such that

$$(F_i(x_*,u_*) - F_i(x,u))/(F_j(x,u)) - F_j(x_*,u_*)) \le K.$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. *(See [Theorem 2, Geoffrion (1968)]) Assume that Ω is a convex set, G_i and g_i, $i = 1,\cdots,k$ are convex functions. Then, $(x_*,u_*) \in \Omega$ is a properly efficient solution of (MCP) if and only if there exist $\lambda_i > 0$ for each $i \in I$ with $\sum_{i=1}^k \lambda_i = 1$ such that (x_*,u_*) is an optimal solution of the following.*

$$(P_\lambda) \quad Minimize: \quad \sum_{i=1}^{k} \lambda_i F_i(x,u)$$
$$subject\ to: \quad (x,u) \in \Omega.$$

LEMMA 2.3. *(See [Theorem 1, Geoffrion (1968)]) Let $\lambda_i > 0$ $(i = 1,\cdots,k)$ be fixed. If (x_*,u_*) is optimal for (P_λ) given above, then (x_*,u_*) is properly efficient for (MCP).*

By using the above lemmas and the recent results in nonsmooth analysis, we will study the properly efficient conditions for multiobjective optimal control problems in next sections. In Section 3, the notations $\partial(\ )$ will denote the Clarke generalized gradients, $N_{(\ )}$ will indicate the Clarke normal cones (see Clarke (1983)), except in Theorem 3.4. While, in Section 4, $\partial(\ )$ and $N_{(\ )}$ will stand for the subdifferential and the normal cone in the sense of convex analysis, respectively. We notice that for a convex Lipscitz continuous function the Clarke generalized gradients coincides with the subdifferential, and the Clarke normal cone of a convex set is the same of the normal cone from convex analysis.

For simplicity, throughout next sections the variable t will be omitted when it does not cause confusion. We will freeze $(x_*,u_*) \in \Omega$ which will be a (local) properly efficient solution for (MCP) or other problems, and abbreviate the arguments $(t,x_*(t),u_*(t))$ to $[t]$, for instance, we write $G_i[t] := G_i(t,x_*(t),u_*(t))$.

3. Proper-efficiency conditions for (MCP)

We suppose the following hypotheses.

(A1): D is closed, $U(\cdot)$ is a nonempty compact set-valued map and the graph GrU is $\mathcal{L} \times \mathcal{B}$ measurable.

(A2): $g_i(\cdot)$, $i = 0,\cdots,1$ are Lipschitz continuous in a neighbourhood of $x_*(1) \in R^m$.

(A3): For every admissible control $u(\cdot)$, there are real-valued measurable function $\epsilon(t) > 0$ and $h(t) \ge 0$ such that

$$|G_i(t,x,u(t)) - G_i(t,x',u(t))| \le h(t)\,|x - x'| \quad for\ any\ \ i \in I$$

$$|\Phi(t,x,u(t)) - \Phi(t,x',u(t))| \le h(t)\,|x - x'|\,,$$

whenever $|x - x_*(t)| \le \epsilon(t)$, $|x' - x_*(t)| \le \epsilon(t)$, $t \in [0,1]$. For $u(\cdot) = u_*(\cdot)$ these functions can be chosen in such a way that $\epsilon(t) = \epsilon > 0$ and $h_i(t)$ $(i = 0,\cdots,1)$ are integrable.

(A4): For any $u(\cdot) \in \mathcal{U} := \{u(\cdot) \in M([0,1],R^n) : u(t) \in U(t)\ a.e.\}$, $G_i(t,x,u(t))$, $i = 0,\cdots,1$ and $\Phi(t,x,u(t))$ are measurable.

Now, let us state the "Pontryagin Maximum Principle" for the multiobjective optimal control problem (MCP).

THEOREM 3.1. *Suppose that the above assumptions (A1)-(A4) hold. If (x_*, u_*) is a weakly-local properly efficient solution for (MCP), then there exist $\lambda_i > 0$, $i = 1, \cdots, k$, and an absolutely continuous function $p(\cdot) : [0,1] \to R^n$ such that*

$$-\dot{p}(t) \in \partial_x H(t, x_*(t), p(t), u_*(t), \lambda) \quad a.e. \tag{3.1}$$

$$p(0) \in N_D(x_*(0)), -p(1) \in \sum_{i=1}^{k} \lambda_i \partial g_i(x_*(1)) \tag{3.2}$$

$$H(t, x_*(t), p(t), u_*(t), \lambda) = \max_{v \in U(t)} H(t, x_*(t), p(t), v, \lambda) \quad a.e., \tag{3.3}$$

where H is the Hamiltonian function defined as

$$H(t, x, p, u, \lambda) := \langle p, \Phi(t, x, u) \rangle - \sum_{i=1}^{k} \lambda_i G_i(t, x, u), \;\; \lambda := (\lambda_1, \cdots, \lambda_k).$$

PROOF. Notice that the problems (P_i), $i = 1, \cdots, k$ in Lemma 2.1 can be rewritten as follows.

$$\begin{aligned}
(P_i^*) \quad Minimize: \quad & \mathcal{F}_i(y) := K \max_{j \in I} \{ y_j(1) + g_j(x(1)) - F_j(x_*, u_*) \} \\
& \qquad + y_i(1) + g_i(x(1)) \\
subject\ to: \quad & L_0(x, u) := x(t) - x(0) - \int_0^t \Phi(t, x(t), u(t)) dt = 0 \\
& L_j(x, u) := y_j(t) - \int_0^t G_j(t, x(t), u(t)) dt = 0 \quad j \in I \\
& y(\cdot) \in \mathcal{S}, \; u(\cdot) \in \mathcal{U},
\end{aligned}$$

where $\mathcal{S} := \{x \in C([0,1], R^m) : x(0) \in D\} \times C([0,1], R^k)$, and $\mathcal{U}$ is given by (A4).

In this problem, the pair of trajectory and control is:

$$(y(\cdot); u(\cdot)) := (x(\cdot), y_1(\cdot), \cdots, y_k(\cdot); u(\cdot)) \in C([0,1], R^{m+k}) \times M([0,1], R^n).$$

Let $y_{j_*}(t) := \int_0^t G_j[t] dt$ for each $j \in I$. By Lemma 2.1, we see that $y_* = (x_*, y_{1_*}, \cdots, y_{k_*})$ corresponding to u_* minimizes $\mathcal{F}_i(y)$ over all admissible processes $(y; u)$ of (P_i^*) with $x(\cdot)$ being sufficiently close to $x_*(\cdot)$ (in the norm L^∞), for every $i \in I$.

For each (P_i^*), as in Section 5 of Ioffe (1984), the assumptions in [Theorem 2, Ioffe (1984)] can be verified. Thus, by this theorem, there exist Lagrange multipliers $\delta \geq 0$, $x^* \in (C([0,1], R^m))^*$ and $y_j^* \in (C([0,1], R))^*$, $j = 1, \cdots, k$ not all zero (C^* is the dual space of C) such that

$$0 \in \partial_y \mathcal{L}_i(y_*, y^*, u_*, \delta) + N_S(y_*), \tag{3.4}$$

$$\mathcal{L}_i(y_*, y^*, u_*, \delta) = \min_{u \in \mathcal{U}} \mathcal{L}_i(y_*, y^*, u, \delta), \tag{3.5}$$

where $y^* := (x^*, y_1^*, \cdots, y_k^*)$, $\mathcal{L}_i$ is the Lagrangian for (P_i^*) defined as follows:

$$\mathcal{L}_i(y(\cdot), y^*, u(\cdot), \delta) := \delta \mathcal{F}_i(y(\cdot)) + \langle x^*, L_0(x(\cdot), u(\cdot)) \rangle + \sum_{j \in I} \left\langle y_j^*, L_i(x(\cdot), u(\cdot)) \right\rangle .$$

(3.4) implies that

$$0 \in \delta\partial_y \mathcal{F}_i(y_*) + \partial_y \left(\langle x^*, L_0(x_*, u_*) \rangle + \sum_{j\in I} \langle y_j^*, L_i(x_*, u_*) \rangle \right) + N_S(y_*)$$

According to the formulas of the Clarke gradients (see Clarke (1983)), we obtain that
(i) for every $\xi \in \delta\partial_y \mathcal{F}_i(y_*)$, there are $\bar{\lambda}_j \geq 0$ for $j \in I$ with $\sum_{j\in I} \bar{\lambda}_j = 1$, $\nu_j \in \partial g(x_*(1))$ for $j \in I/\{i\}$ and $\nu_{i_1}, \nu_{i_2} \in \partial g(x_*(1))$ such that for any $y \in C([0,1], R^{n+k})$.

$$\langle \xi, y \rangle = \delta \left(\sum_{j\in I} K\bar{\lambda}_j y_j(1) + \sum_{j\in I/\{i\}} K\bar{\lambda}_j \langle \nu_j, x(1) \rangle + K\bar{\lambda}_i \langle \nu_{i_1}, x(1) \rangle + y_i(1) + \langle \nu_{i_2}, x(1) \rangle \right).$$

Put $\tilde{\lambda}_i := \delta K\bar{\lambda}_i + \delta$ and $\tilde{\lambda}_j := \delta K\bar{\lambda}_j$ for $j \in I/\{i\}$. Since $\partial g(x_*(1))$ is convex, we have

$$\nu_i := K\bar{\lambda}_i \nu_{i_1} / \left(K\bar{\lambda}_i + 1\right) + \nu_{i_2} / \left(K\bar{\lambda}_i + 1\right) \in \partial g(x_*(1)).$$

Thus, we get

$$\langle \xi, y \rangle = \sum_{j\in I} \tilde{\lambda}_j y_j(1) + \sum_{j\in I} \tilde{\lambda}_j \langle \nu_j, x(1) \rangle .$$

Discussing as in the proof of [Theorem 3, Ioffe (1984)], we have the following.
(ii) Corresponding to x^* and y_j^*, $j = 1, \cdots, k$, there exist pairs of the nonnegative Radon measure and Radon-integrable functions (μ_j, ξ_j), $j = 0, \cdots, k$, with

$$\langle x^*, L_0(x, u) \rangle = \int_0^1 \left\langle x(t) - x(0) - \int_0^t \Phi(t, x(t), u(t))dt, \xi_0 \right\rangle d\mu_0,$$

$$\langle y_j^*, L_j(x, u) \rangle = \int_0^1 \left\langle y_j(t) - \int_0^t G_j(t, x(t), u(t))dt, \xi_j \right\rangle d\mu_j.$$

For every $\xi \in \partial_y(\langle x^*, L_0(x_*, u_*) \rangle + \sum_{j\in I} \langle y_j^*, L_i(x_*, u_*) \rangle)$, there is a Lebesgue measurable function $\eta(\cdot)$ with

$$\eta(t) \in \partial_x \left(\left\langle \int_t^1 \xi_0 d\mu_0, \Phi(t, x_*(t), u_*(t)) \right\rangle + \left\langle \int_t^1 \xi_j d\mu_j, G_j(t, x_*(t), u_*(t)) \right\rangle \right) \ a.e.$$

such that for any $y \in C([0,1], R^{m+k})$,

$$\langle \xi, y \rangle = \int_0^1 \langle x(t) - x(0), \xi_0 \rangle \, d\mu_0 + \sum_{j\in I} \int_0^1 \langle y_j, \xi_j \rangle \, d\mu_j - \int_0^1 \langle \eta, x \rangle \, dt.$$

(iii) For every $\xi \in N_S(y_*)$, there is $\alpha \in N_D(x_*(0))$ such that

$$\langle \xi, y \rangle = \langle \alpha, x(0) \rangle \quad \textit{for any} \ \ y \in C([0,1], R^{n+k}).$$

In summary, we see that there are $\tilde{\lambda}_j$, ν_j, $j = 1, \cdots, k$, (μ_j, ξ_j), $j = 0, \cdots, k$, $\eta(\cdot)$ and α stated above such that

$$\begin{aligned}\langle \xi, y\rangle = & \sum_{j\in I}\tilde{\lambda}_j y_j(1) + \sum_{j\in I}\tilde{\lambda}_j \langle \nu_j, x(1)\rangle + \int_0^1 \langle x(t) - x(0), \xi_0\rangle \, d\mu_0 \\ & + \sum_{j\in I}\int_0^1 \langle y_j, \xi_j\rangle \, d\mu_j - \int_0^1 \langle \eta, x\rangle \, dt + \langle \alpha, x(0)\rangle = 0.\end{aligned} \tag{3.6}$$

for any $x \in C([0,1], R^n)$ and $y_j \in C([0,1], R)$, $j = 1, \cdots, k$.

Put $\tilde{p}(t) := \int_t^1 \xi_0 d\mu_0$. As in Ioffe (1984), form (3.6), it follows that

$$\int_t^1 \xi_j d\mu_j = -\tilde{\lambda}_j \quad \textit{for any } j \in I. \tag{3.7}$$

$$\dot{\tilde{p}}(t) = -\eta(t) \text{ a.e.}, \ \tilde{p}(0) = \alpha, \ \tilde{p}(1) = -\sum_{j\in I}\tilde{\lambda}_j \nu_j. \tag{3.8}$$

Notice that the above $\tilde{\lambda}_j$, $j = 1, \cdots, k$ and $\tilde{p}(t)$ are Lagrange multipliers of P_i^*. Then, we set $\tilde{\lambda} := \lambda_j^i$, $j = 1, \cdots, k$ and $\tilde{p}(t) := p_i(t)$. Thus, (3.7), (3.8), (i), (ii) and (iii) yield that

$$-\dot{p}_i(t) \in \partial_x H_i(t, x_*(t), p_i(t), u_*(t), \lambda^i) \quad a.e. \tag{3.9}$$

$$p_i(0) \in N_D(x_*(0)), -p_i(1) \in \sum_{j\in I} \lambda_j^i \partial g_j(x_*(1)) \tag{3.10}$$

where

$$H_i(t, x, p_i, u, \lambda^i) := \langle p_i, \Phi(t, x, u)\rangle - \sum_{j\in I}\lambda_j^i G_j(t, x, u), \ \lambda^i = (\lambda_1^i, \cdots, \lambda_k^i).$$

For the above Lagrange multipliers δ and y^* (depend on i), if $\delta = 0$, then $y_j^* = -\lambda_j^i = 0$ for any $j \in I$. From (3.9) and (3.10), we derive that $p_i(\cdot) = 0$. Thus, $x^* = 0$ which contradict that δ and y^* are not all zero. Hence, $\lambda_i^i = \delta K \bar{\lambda}_i + \delta > 0$.

On the other hand, by (3.5), (3.7) and (3.8), we have

$$\int_0^1 (\langle p_i, \Phi[t]\rangle - \sum_{j\in I}\lambda_j^i G_j[t])dt = \max_{u\in\mathcal{U}} \int_0^1 (\langle p_i, \Phi(t, x_*, u)\rangle - \sum_{j\in I}\lambda_j^i G_j(t, x_*, u))dt.$$

Analyze as in Ioffe (1984), we can get

$$H_i(t, x_*(t), p_i(t), u_*(t), \lambda^i) = \max_{v\in U(t)} H_i(t, x_*(t), p_i(t), v, \lambda^i) \quad a.e.. \tag{3.11}$$

Now we see that there are $p_i(t) \in AC$, $\lambda_j^i \geq 0$, $j = 1, \cdots, k$, $i = 1, \cdots, k$ with $\lambda_i^i > 0$ such that (3.9)-(3.11) hold. Set

$$p(t) := \sum_{j\in I} p_j(t), \ \lambda_i := \sum_{j\in I} \lambda_i^j.$$

Because for every $i \in I$, $H_i(t,x,p_i,u,\lambda^i)$ is linear in p_i and λ^I, we have

$$\sum_{i\in I}\partial_x\left(\langle p_i,\Phi[t]\rangle-\sum_{j\in I}\lambda_j^iG_j[t]\right)=\partial_x\left(\left\langle\sum_{i\in I}p_i,\Phi[t]\right\rangle-\sum_{i\in I}\sum_{j\in I}\lambda_i^jG_i[t]\right)$$
$$=\partial_x\left(\langle p,\Phi[t]\rangle-\sum_{i\in I}\lambda_iG_i[t]\right).$$

Notice that $\partial g_i(x_*(1))$ is convex set and $N_D(x_*(0))$ is convex cone, (3.1)-(3.3) are derived directly form (3.9)-(3.11).

Thus we have proved the theorem. □

Theorem 3.1 gives a kind of Kuhn-Tucker type necessary proper-efficiency conditions for (MCP). Now, let us add the following condition to the constraints in (MCP),

$$\bar{F}_j(x,u):=\int_0^1\bar{G}_j(t,x(t),u(t))dt+\bar{g}_j(x(1))\le 0\quad j\in\bar{I}:=\{1,\cdots l\},$$

where $\bar{G}_j$ and $\bar{g}_j$ are defined as G_i and g_i in Section 1, and we denote this problem by $(\overline{MCP})$. We say (x,u) is admissible for $(\overline{MCP})$ iff $(x,u)\in\Omega$ and $\bar{F}_j(x,u)\le 0$ for any $j\in\bar{I}$, and we define properly efficient solution (resp. local properly efficient solution) for $(\overline{MCP})$ as Definition 1.1 (resp. Def. 1.2) for (MCP). It is clear that Lemma 2.1 and Lemma 2.2 also hold for $(\overline{MCP})$. For this multiobjective problem, in the same manner as above, we can get the following Fritz-John type necessary conditions.

COROLLARY 3.2. *Let (A1)-(A4) be satisfied. Furthermore, we assume that (A2)-(A4) also hold with G_i and g_i $(i\in I)$ replaced by $\bar{G}_j$ and $\bar{g}_j$ $(j\in\bar{I})$, respectively. If (x_*,u_*) is a weakly-local properly efficient solution for $(\overline{MCP})$, then there exist $\lambda_i\ge 0$, $i=1,\cdots,k+l$ not all zero and $p(\cdot)\in AC$ such that*

$$-\dot{p}(t)\in\partial_x\bar{H}(t,x_*(t),p(t),u_*(t),\lambda)\quad a.e.$$

$$p(0)\in N_D(x_*(0)),\ -p(1)\in\sum_{i\in I}\lambda_i\partial g_i(x_*(1))+\partial\sum_{j\in\bar{I}}\lambda_{k+j}\bar{g}_j(x_*(1))$$

$$\bar{H}(t,x_*(t),p(t),u_*(t),\lambda)=\max_{v\in U(t)}\bar{H}(t,x_*(t),p(t),v,\lambda)\quad a.e.$$

$$\lambda_{k+j}\left(\int_0^1\bar{G}_j[t]dt+\bar{g}_j(x_*(1))\right)=0\quad for\ j\in\bar{I}$$

where,

$$\bar{H}(t,x,p,u,\lambda):=\langle p,\Phi(t,x,u)\rangle-\sum_{j\in I}\lambda_iG_i(t,x,u)-\sum_{j\in\bar{I}}\lambda_{k+j}\bar{G}_j(t,x,u)$$
$$\lambda:=(\lambda_1,\cdots,\lambda_{k+l}).$$

For scalar nonsmooth optimal control problems, Pinho and Vinter (1995) pointed out that the Pontryagin Maximum Principle may fail to be sufficient conditions for optimality even in the "convex" case. Similarly, it is not difficult to show that the conditions (3.1)-(3.3) in Theorem 3.1 may be not sufficient for the proper- efficiency to (MCP) in the "convex" case.

Next, we extend the results of Pinho and Vinter (1995) to our problem (MCP), to establish another kind of necessary conditions for the local proper-efficiency, the 'weak maximum principle'. We will show that this kind of conditions is sufficient for the proper-efficiency if appropriate convexity assumptions are imposed on the Harmiltonian function and constraint sets. Furthermore, in the next section, we will prove that this kind of conditions is necessary and sufficient for the proper-efficiency to the "convex" problems.

For $(\overline{MCP})$, it is not difficult to get a Fritz-John type 'weak maximum principle'. In this paper, we will only discuss the 'weak maximum principle' for $(\overline{MCP})$ in the "convex" case.

In the next, the following assumption is required.

(A5): $G_i(\cdot,x,u)$, $i=1,\cdots,k$, $\Phi(\cdot,x,u)$ are Lebesgue measurable, and there exist $\epsilon>0$ and $h(t)\in L^1([0,1],R)$, such that

$$|G_i(t,x,u)-G_i(t,x',u')|\le h(t)\left(|x-x'|+|u-u'|\right)\quad \textit{for any}\ \ i\in I$$

$$|\Phi(t,x,u)-\Phi(t,x',u')|\le h(t)\left(|x-x'|+|u-u'|\right),$$

whenever $x,x'\in x_*(t)+\epsilon B^n$, $u,u'\in u_*(t)+\epsilon B^m$ *a.e.*.

THEOREM 3.3. *Let (A1), (A2) and (A5) be satisfied. If (x_*,u_*) is a strongly-local properly efficient solution for (MCP), then there exist $\lambda_i>0$, $i=1,\cdots,k$, an absolutely continuous function $p(\cdot):[0,1]\to R^n$ and an integrable function $\zeta(\cdot):[0,1]\to R^m$ such that*

$$(-\dot p(t),\zeta(t))\in\partial_{(x,u)}H(t,x_*(t),p,u_*(t),\lambda)\quad a.e. \tag{3.12}$$

$$p(0)\in N_D(x_*(0)),-p(1)\in\sum_{i=1}^k\lambda_i\partial g_i(x_*(1)) \tag{3.13}$$

$$\zeta(t)\in N_{U(t)}(u_*(t))\quad a.e., \tag{3.14}$$

where, $H(t,x,p,u,\lambda)$ is the Harmiltonian function given in Theorem 3.1.

PROOF. Reformulate the problems (P_i) in Lemma 2.1 as follows.

$$\begin{array}{lll}(P_i^\dagger) & Minimize: & \Psi_i(y(1)):=\ K\max\limits_{j\in I}\{y_j(1)+g_j(x(1))-F_j(x_*,u_*)\}\\ & & \qquad\qquad +y_i(1)+g_i(x(1))\\ & subject\ to: & \dot x(t)=\Phi(t,x(t),u(t))\quad a.e.\\ & & \dot y_j(t)=G_j(t,x(t),u(t))\quad a.e.\quad j\in I\\ & & x(0)\in D,\ y_j(0)=0\quad j\in I\\ & & u(t)\in U(t)\quad a.e..\end{array}$$

where $(y(\cdot),u(\cdot)):=(x(\cdot),y_1(\cdot),\cdots,y_k(\cdot),u(\cdot))\in AC([0,1],R^{m+k})\times M([0,1],R^n)$ is the pair of trajectory y and control u.

Define $y_*(\cdot)$ as in the proof of Theorem 3.1. By Lemma 2.1, we know that y_* corresponding to u_* minimizes $\Psi_i(y(1))$ over all admissible processes $(y;u)$ of $(P_i^\dagger)$ with $x(t)\in x_*(t)+\epsilon B^m$ and $u(t)\in u_*(t)+\epsilon B^n$ a.e. $t\in[0,1]$ for every $i\in I$.

For each $(P_i^\dagger)$, by [Proposition 6.1, Pinho and Vinter (1995)], there exist an absolutely continuous function $\bar p_i=(p_i,p_1^i,\cdots,p_k^i)$ and an integrable function ζ_i such that

$$(-\bar p_i(t),\dot y(t),\zeta_i(t))\in\partial\tilde H_i(t,y_*(t),\bar p_i(t),u_*(t))\quad a.e. \tag{3.15}$$

$$\bar{p}_i(0) \in N_{D\times\underbrace{\{0\}\times\cdots\times\{0\}}_{k}}(y_*(0)) \tag{3.16}$$

$$-\bar{p}_i(1) \in \partial\Psi_i(y_*(1)) \tag{3.17}$$

$$\zeta_i(t) \in N_{U(t)}(u_*(t)) \quad a.e., \tag{3.18}$$

where

$$\bar{H}_i(t,y,\bar{p}_i,u) := \langle p_i, \Phi(t,x,u)\rangle + \sum_{j\in I}\langle p_j^i, G_i(t,x,u)\rangle .$$

First, we observe (3.17). By the formulas of Clarke gradients, there exist $\tilde{\lambda}_j^i \geq 0$ for every $j \in I$ with $\sum_{j\in I}\tilde{\lambda}_j^i = 1$, $\gamma_j \in \partial_x g_j(x_*(1))$ for every $j \in I/\{i\}$ and $\gamma_{i_1}, \gamma_{i_2} \in \partial_x g_i(x_*(1))$ such that

$$-p_i(1) = \sum_{j\in I/\{i\}} K\tilde{\lambda}_j^i\gamma_j + K\tilde{\lambda}_i^i\gamma_{i_1} + \gamma_{i_2} \tag{3.19}$$

$$-p_j^i(1) = \tilde{\lambda}_j^i K \quad \textit{for any} \quad j \in I/\{i\},\ -p_i^i(1) = \tilde{\lambda}_i^i K + 1. \tag{3.20}$$

Putting $\lambda_i^i := \tilde{\lambda}_i^i K + 1$, $\lambda_j^i := \tilde{\lambda}_j^i K$, (3.19) and (3.16) yield that

$$-p_i(1) \in \sum_{j\in I}\lambda_j^i\partial g_j(x_*(1)), \quad p_i(0) \in N_D(x_*(0)). \tag{3.21}$$

On the other hand, since $\bar{H}_i$ does not contain the arguments y_j, $j = 1,\cdots,k$, (3.15) implies that $\dot{p}_j^i(\cdot) = 0$. Combining (3.15) with (3.20), we have $p_j^i(\cdot) = -\lambda_j^i$ and

$$(-\dot{p_i}\,(t), \dot{y}(t), \zeta_i(t)) \in \partial_{(x,\bar{p}_i,u)}(\langle p_i(t), \Phi[t]\rangle - \sum_{j\in I}\lambda_j^i G_j[t]) \quad a.e..$$

From this inclusion, we deduce that

$$(-\dot{p}_i(t), \zeta_i(t)) \in \partial_{(x,u)}(\langle p_i(t), \Phi[t]\rangle - \sum_{j\in I}\lambda_j^i G_j[t]) \quad a.e.. \tag{3.22}$$

Now, let us put

$$p(t) := \sum_{j\in I} p_i(t),\ \zeta := \sum_{i\in I}\zeta_i,\ \lambda_i := \sum_{j\in I}\lambda_i^j.$$

Thus, $\lambda_i > 0$ for every $i \in I$. Notice that $\langle p_i(t), \Phi[t]\rangle - \sum_{j\in I}\lambda_j^i G_j[t]$ is linear in $p_i(t)$ and λ_j^i, $j = 1,\cdots,k$ for each $i \in I$. As in the proof of Theorem 3.1, it follows immediately (3.12)-(3.14) from (3.22), (3.21) and (3.18). This completes the proof. □

Now, we show that necessary conditions (3.12)-(3.14) are sufficient for the proper-efficiency under suitable convexity assumptions on H, g_i, D and U.

THEOREM 3.4. *Assume $U(t)$ for $t \in [0,1]$ and D are convex sets, $g_i(\cdot)$ $(i = 1,\cdots,k)$ are convex functions. Let (x_*, u_*) be an admissible process for (MCP). If there exist $\lambda_i > 0$ $(i = 1,\cdots,k)$, $p(\cdot) \in AC$ and $\zeta(\cdot) \in L^1$ such that (3.12)-(3.14) be satisfied and $H(t,x,p(t),u,\lambda)$ is concave in x and u for t a.e., then (x_*, u_*) is properly efficient for (MCP). Here, $\partial(\ )$ and $N_{(\ )}$ indicate the subdifferential and the normal cone in the sense of convex analysis, respectively.*

PROOF. . Let (x,u) be an arbitrary admissible process for (MCP). Recall the definitions of the subdifferential and the normal cone in the sense of convex analysis, from (3.12)-(3.14), it is easy to derive that

$$\begin{aligned}
&\sum_{i\in I}\lambda_I F_i(x,u) - \sum_{i\in I}\lambda_i F_i(x_*,u_*)) \\
&= \int_0^1 (H(t,x_*,p,u_*,\lambda) - H(t,x,p,u,\lambda))dt - \int_0^1 \langle p,\dot{x}_* - \dot{x}\rangle\, dt + \sum_{i\in I}\lambda_i(g_i(x(1)) - g_i(x_*(1))) \\
&\geq \int_0^1 (\langle -\dot{p}, x_* - x\rangle + \langle \zeta, u_* - u\rangle)dt - \int_0^1 \langle p, \dot{x}_* - \dot{x}\rangle\, dt + \langle -p(1), x(1) - x_*(1)\rangle \\
&= \langle p(0), x_*(0) - x(0)\rangle + \int_0^1 \langle \zeta, u_* - u\rangle\, dt \\
&\geq 0.
\end{aligned}$$

Then, (x_*,u_*) is a minimizer for (P_λ). According to Lemma 2.3, (x_*,u_*) is a properly efficient solution for (MCP). □

4. Necessary and sufficient conditions for convex problems

In this section, we consider the following multiobjective optimal control problem.

$$\begin{aligned}
(MCP^*): \quad & Minimize: && \left[\int_0^1 G_1(t,x(t),u(t))dt + g_1(x(1)), \cdots, \right. \\
& && \left. \int_0^1 G_k(t,x(t),u(t))dt + g_k(x(1))\right] \\
& subject\ to: && \dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \quad a.e. \\
& && x(0) \in D \\
& && u(t) \in U(t) \quad a.e.,
\end{aligned}$$

where $x(\cdot) \in AC([0,1],R^m)$, $u(\cdot) \in L^1([0,1],R^n)$, $A(\cdot):[0,1] \to R^{n\times n}$, $B(\cdot):[0,1] \to R^{n\times m}$, $b(\cdot):[0,1] \to R^n$, and the other data are given as in (MCP). For this problem the following hypotheses are in force.

(H1): $A(\cdot)$ is integrable, $B(\cdot)$ is measurable essentially bounded, $b(\cdot)$ is measurable.

(H2): For every $i \in I$, $G_i(\cdot,x(\cdot),u(\cdot))$ is integrable for any $(x,u) \in AC \times L^1$, $G_i(t,\cdot,\cdot)$ is convex lower semicontinuous, and there are $v(t) \in L^\infty([0,1],R^{m+n})$ and $w(t) \in L^1([0,1],R)$, such that $G_i(t,x,u) \geq \langle v(t),(x,u)\rangle + w(t)$ for any $x \in R^m$, $u \in R^n$, a.e..

(H3): The functions $g_i(\cdot)$, $i = 1,\cdots,k$ is proper convex and lower semicontinuous.

(H4): The set D is convex, and $U(t)$ is convex for a.e. t, and there is $\rho(t) \in L^1$ such that $|u| \leq \rho(t)$ for any $u \in U(t)$ a.e..

THEOREM 4.1. *Assume that (H1)-(H4) and (A1) are satisfied. An admissible process (x_*,u_*) is a properly efficient solution for (MCP^*) if and only if there exist $\lambda_i > 0$, $i = 1,\cdots,k$, $p(\cdot) \in AC([0,1],R^m)$, and $\zeta(\cdot) \in L^\infty([0,1],R^n)$ such that*

$$(\dot{p}(t) + p(t)A(t), p(t)B(t) - \zeta(t)) \in \sum_{i=1}^k \lambda_i \partial G_i[t] \quad a.e. t \in [0,1] \tag{4.1}$$

$$p(0) \in N_D(x_*(0)), -p(1) \in \sum_{i=1}^{k} \lambda_i \partial g_i(x_*(1)) \tag{4.2}$$

$$\zeta(t) \in N_{U(t)}(u_*(t)) \quad a.e.. \tag{4.3}$$

REMARK. In this theorem, (4.1) is equivalent to (3.12) in Theorem 3.3.

PROOF. [Necessity] According to Lemma 2.2, there are $\lambda_i > 0$, $i = 1, \cdots, k$, such that (x_*, u_*) is an optimal solution of the following.

$$\begin{aligned}
(P_\lambda^*): \quad & Minimize: && \sum_{j \in I} \lambda_i \left(\int_0^1 G_i(t, x, u)dt + g_i(x(1)) \right) \\
& subject\ to: && \dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \quad a.e. \\
& && x \in \{x \in AC([0,1], R^m): \ x(0) \in D\} \\
& && u \in \mathcal{C} := \{u \in L^1([0,1], R^n): \ u(t) \in U(t) \ a.e.\}.
\end{aligned}$$

Thus, $(x_*, u_*, x_*(0), x_*(1))$ is a minimizer for the following optimization problem,

$$\begin{aligned}
Minimize: \quad & \Lambda(z, u, \alpha, \beta) := \sum_{j \in I} \lambda_i (\int_0^1 G_i(t, z, u)dt + g_i(\beta)) \\
subject\ to: \quad & \Gamma_1(z, u, \alpha, \beta) := z - \alpha - \int_0^t (Az + Bu + b)d\tau = 0 \quad a.e. \\
& \Gamma_2(z, u, \alpha, \beta) := \beta - \alpha - \int_0^1 (Az + Bu + b)d\tau = 0 \\
& (z, u, \alpha, \beta) \in \mathcal{M} := L^1([0,1], R^m) \times \mathcal{C} \times D \times R^m.
\end{aligned}$$

Put

$$\theta := (z, u, \alpha, \beta) \in L^1([0,1], R^m) \times L^1([0,1], R^m) \times R^m \times R^m.$$

It is obvious that $\Gamma_1(\theta), \Gamma_2(\theta)$ are affine mappings, $\Lambda(\theta)$ is convex function, $\mathcal{M}$ is convex set. By [Theorem 5, p.74, Ioffe and Tihomirov (1979)], there exist $\delta \geq 0$, $q(\cdot) \in (L^1)^*$ and $\sigma \in R^m$ not all zero such that

$$\delta\Lambda(\theta_*) + \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle \, dt + \langle \sigma, \Gamma_2(\theta_*) \rangle = \min_{\mathcal{M}} \left(\delta\Lambda(\theta) + \int_0^1 \langle q, \Gamma_1(\theta) \rangle \, dt + \langle \sigma, \Gamma_2(\theta) \rangle \right), \tag{4.4}$$

where $\theta_* := (x_*, u_*, x_*(0), x_*(1))$.

Let $I_{\mathcal{M}}(\theta)$ denote the indicator function of $\mathcal{M}$. We see that $I_{\mathcal{M}}(\theta)$, $\Lambda(\theta)$ are proper convex and lower semicontinuous (refer to Section 1 of Chapter 1 in Barbu (1994)). According to [Theorem 1.1, Barbu (1994)], (4.4) implies that

$$0 \in \partial\delta\Lambda(\theta_*) + \partial \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle \, dt + \partial \langle \sigma, \Gamma_2(\theta_*) \rangle + N_{\mathcal{M}}(\theta_*). \tag{4.5}$$

Now, let us calculate (4.5). By the formulas of subdifferentials (see Barbu (1994), Ioffe and Tihomirov (1979)), we have the following.

For every $\xi \in \partial\delta\Lambda(\theta_*)$, there are $(\mu_i, \eta_i) \in L^\infty([0,1], R^{m+n})$ with $(\mu_i(t), \eta_i(t)) \in \partial G_i[t]$ and $\nu_i \in \partial g_i(x_*(1))$ for each $i \in I$ such that for any $\theta \in L^1 \times L^1 \times R^m \times R^m$

$$\langle \xi, \theta \rangle = \delta \sum_{i\in I} \lambda_i \left(\int_0^1 (\langle \mu_i, z \rangle + \langle \eta_i, u \rangle)\, dt + \langle \nu_i, \beta \rangle \right).$$

Corresponding to any $\xi \in N_{\mathcal{M}}(\theta_*)$, there are $\gamma \in N_D(x_*(0))$, and $\zeta(\cdot) \in N_C(u_*(\cdot))$ such that for any $\theta \in L^1 \times L^1 \times R^m \times R^m$, one has

$$\langle \xi, \theta \rangle = \langle \gamma, \alpha \rangle + \int_0^1 \langle \zeta, u \rangle\, dt.$$

$\partial \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle\, dt$ is a singleton $\{\xi\}$, where

$$\langle \xi, \theta \rangle = \int_0^1 \left\langle q, z - \alpha - \int_0^t (Az + Bu) d\tau \right\rangle dt$$

for any $\theta \in L^1 \times L^1 \times R^m \times R^m$.

Similarly, $\partial \langle \sigma, \Gamma_2(\theta_*) \rangle = \{\xi\}$ with

$$\langle \xi, \theta \rangle = \left\langle \sigma, \beta - \alpha - \int_0^1 (Az + Bu) dt \right\rangle$$

for any $\theta \in L^1 \times L^1 \times R^m \times R^m$.

Summarize the above interpretation of (4.5), we obtain that there are (μ_i, η_i), ν_i, $i = 1, \cdots, k$, γ and ζ stated above such that

$$\begin{aligned} &\delta \sum_{i\in I} \lambda_i \int_0^1 (\langle \mu_i, z \rangle + \langle \eta_i, u \rangle)\, dt + \delta \sum_{i\in I} \lambda_i \langle \nu_i, \beta \rangle + \int_0^1 \left\langle q, z - \int_0^t (Az + Bu) d\tau \right\rangle dt \\ &- \left\langle \int_0^1 q dt, \alpha \right\rangle + \left\langle \sigma, \beta - \alpha - \int_0^1 (Az + Bu) dt \right\rangle + \langle \gamma, \alpha \rangle + \int_0^1 \langle \zeta, u \rangle\, dt = 0 \end{aligned} \tag{4.6}$$

for any $(z, u, \alpha, \beta) \in L^1 \times L^1 \times R^m \times R^m$.

Let $p(t) := \int_t^1 q(\tau) d\tau + \sigma$, then,

$$q(t) = -\dot{p}(t) \ a.e., \ \int_0^1 q(\tau) d\tau + \sigma = p(0), \ \sigma = p(1).$$

Thus, (4.6) yields that

$$\begin{aligned} &\int_0^1 \left\langle \delta \sum_{i\in I} \lambda_i \mu_i, z \right\rangle dt - \int_0^1 \langle \dot{p} + pA, z \rangle\, dt + \int_0^1 \left\langle \delta \sum_{i\in I} \lambda_i \eta_i, u \right\rangle dt - \int_0^1 \langle pB - \zeta, u \rangle\, dt \\ &+ \left\langle \delta \sum_{i\in I} \lambda_i \nu_i, \beta \right\rangle + \langle \sigma, \beta \rangle - \left\langle \int_0^1 q dt, \alpha \right\rangle - \langle \sigma, \alpha \rangle + \langle \gamma, \alpha \rangle = 0 \end{aligned}$$

for any $(z, u, \alpha, \beta) \in L^1([0,1], R^m) \times L^1([0,1], R^n) \times R^m \times R^n$.

From this equation, we conclude that

$$\dot{p} + pA = \delta \sum_{i \in I} \lambda_i \mu_i, \; pB - \zeta = \delta \sum_{i \in I} \lambda_i \eta_i \tag{4.7}$$

$$p(1) = \sigma = -\delta \sum_{j \in I} \lambda_j \nu_j, \; p(0) = \int_0^1 q(\tau) d\tau + \sigma = \gamma. \tag{4.8}$$

Here, if the above multiplier $\delta = 0$, then (4.7) and (4.8) imply that $p(\cdot) = 0$ and $\sigma = 0$ which contradicts that δ, $p(\cdot)$ and σ are not all zero. Hence, we have $\delta > 0$, and we can set $\delta = 1$.

Thus, it follows (4.1) and (4.2) from (4.7) and (4.8), respectively.

Finally, from $\zeta(\cdot) \in N_{\mathcal{C}}(u_*(\cdot))$, it is easy to see that $\zeta(t)(u(t) - u_*(t)) \leq 0$ *a.e.* for any $u(\cdot) \in \mathcal{C}$. Then, from the theory of measurable selection we get (4.3).

[Sufficiency] Let (x, u) be an arbitrary admissible process for (MCP). Using (4.1)-(4.3) we deduce that

$$\begin{aligned}
&\sum_{j \in I} \lambda_i \left(\int_0^1 G_i(t,x,u) dt + g_i(x(1)) \right) - \sum_{j \in I} \lambda_i \left(\int_0^1 G_i[t] dt + g_i(x_*(1)) \right) \\
&= \sum_{j \in I} \lambda_i \left(\int_0^1 G_i(t,x,u) dt + g_i(x(1)) - \int_0^1 G_i[t] dt - g_i(x_*(1)) \right) \\
&\quad + \int_0^1 \langle p, \dot{x} - Ax - Bu - b \rangle \, dt - \int_0^1 \langle p, \dot{x}_* - Ax_* - Bu_* - b \rangle \, dt \\
&= \int_0^1 \sum_{i \in I} \lambda_i \left(G_i(t,x,u) dt - G_i[t] \right) dt + \sum_{i \in I} \lambda_i \left(g_i(x(1)) - g_i(x_*(1)) \right) \\
&\quad - \int_0^1 \left(\langle \dot{p} + pA, x - x_* \rangle + \langle pB - \zeta, u - u_* \rangle \right) dt + \langle p(1), x(1) - x_*(1) \rangle \\
&\quad - \langle p(0), x(0) - x_*(0) \rangle - \int_0^1 \langle \zeta, u - u_* \rangle \, dt \\
&\geq 0,
\end{aligned}$$

which means that (x_*, u_*) is a minimizer for (P^*_λ). By Lemma 2.2, (x_*, u_*) is a properly efficient solution for (MCP).

The proof of Theorem 3.1 is complete. □

REMARK. From the proof of [Sufficiency] above, it is easy to see that the conditions (4.1)-(4.3) are also sufficient for the proper-efficiency for (MCP^*) under the following weaker assumptions: $A(t)$, $B(t)$ are integrable, $b(t)$ is measurable, G_i, $i = 1, \cdots, k$ are convex in (x, u), measurable in t, g_i, $i = 1, \cdots, k$ are convex functions, C is a convex set and $U(t)$ is convex a.e..

Now, let $(\overline{MCP}^*)$ indicate the problem $(\overline{MCP})$ in the case where the differential state equation is linear as in (MCP^*). For this problem, we have the following.

COROLLARY 4.2. *In addition to the assumptions in Theorem 4.1, suppose that* $\bar{G}_j$, $\bar{g}_j$, $j \in \bar{I}$ *also satisfy (H2) and (H3), respectively. Let admissible process* (x_*, u_*) *be* *a properly efficient solution for* $(\overline{MCP}^*)$. *Then, there exist* $\lambda_i \geq 0$, $i = 1, \cdots, k + l$ *not all zero,* $p(\cdot) \in AC$ *and* $\zeta(\cdot) \in L^\infty$ *such that*

$$(\dot{p}(t) + p(t)A(t), p(t)B(t) - \zeta(t)) \in \sum_{i \in I} \lambda_i \partial G_i[t] + \sum_{j \in I} \lambda_{k+j} \partial \bar{G}_j[t] \quad a.e. \tag{4.9}$$

$$p(0) \in N_D(x_*(0)), \; -p(1) \in \sum_{i \in I} \lambda_i \partial g_i(x_*(1)) + \sum_{j \in I} \lambda_{k+j} \partial \bar{g}_j(x_*(1)) \tag{4.10}$$

$$\zeta(t) \in N_{U(t)}(u_*(t)) \quad a.e. \tag{4.11}$$

$$\lambda_{k+j} \left(\int_0^1 \bar{G}_j[t]dt + \bar{g}_j(x_*(1)) \right) = 0 \quad for \; j \in \bar{I}. \tag{4.12}$$

In particular, assume that the following Slater constraint qualifications hold: there is an admissible process (x_0, u_0) *of* $(\overline{MCP}^*)$ *with* $\bar{F}_j(x_0, u_0) < 0$ *for any* $j \in \{j \in \bar{I} : \bar{F}_j(x_*, u_*) = 0\}$. *Then, we have* $\lambda_i > 0$ *for all* $i \in I$. *In this case, the above conditions (4.9)-(4.12) are also sufficient for* (x_*, u_*) *to be a properly efficient solution of* $(\overline{MCP}^*)$.

PROOF. Let (x_*, u_*) be a properly efficient solution for $(\overline{MCP}^*)$. Then, as in the preceding proof, there are $\lambda_i^* > 0$, $i = 1, \cdots, k$, such that $\theta_* := (x_*, u_*, x_*(0), x_*(1))$ is a minimizer for the following problem,

$$\begin{aligned}
Minimize: \quad & \Lambda_0(z, u, \alpha, \beta) := \sum_{j \in I} \lambda_i^* \Big(\int_0^1 G_i(t, z, u)dt + g_i(\beta) \Big) \\
subject\ to: \quad & \Gamma_1(z, u, \alpha, \beta) := z(t) - \alpha - \int_0^t (Az + Bu + b)d\tau = 0 \quad a.e. \\
& \Gamma_2(z, u, \alpha, \beta) := \beta - \alpha - \int_0^1 (Az + Bu + b)d\tau = 0 \\
& \Lambda_j(z, u, \alpha, \beta) := \int_0^1 \bar{G}_j(t, z, u)dt + \bar{g}_j(\beta) \leq 0 \quad j \in \bar{I} \\
& (z, u, \alpha, \beta) \in \mathcal{M} := L^1([0,1], R^m) \times C \times D \times R^m.
\end{aligned}$$

Using [Theorem 5, Ioffe and Tihomirov (1979)] again, there exist $\delta_0, \delta_1, \cdots, \delta_l \geq 0$, $q(\cdot) \in (L^1)^*$ and $\sigma \in R^m$ not all zero, such that

$$\begin{aligned}
& \sum_{j=0}^{l} \delta_j \Lambda_j(\theta_*) + \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle \, dt + \langle \sigma, \Gamma_2(\theta_*) \rangle \\
&= \min_{\mathcal{M}} \left(\sum_{j=0}^{l} \delta_j \Lambda_j(z, u, \alpha, \beta) + \int_0^1 \langle q, \Gamma_1(z, u, \alpha, \beta) \rangle \, dt + \langle \sigma, \Gamma_2(z, u, \alpha, \beta) \rangle \right),
\end{aligned}$$

$$\delta_j \Lambda_j(\theta_*) = 0 \quad j \in \bar{I}.$$

Setting $p(t) := \int_t^1 q(\tau)d\tau + \sigma$, $\lambda_i := \delta_0 \lambda_i^*$ for $i \in I$ and $\lambda_{k+j} := \delta_j$ for $j \in \bar{I}$. Following the proof of Theorem 4.1 above, we can show that $\lambda_i, \cdots, \lambda_{k+l}$ are not all zero, and there is $\zeta(\cdot) \in L^\infty$ such that $\lambda_i, \cdots, \lambda_{k+l}$, ζ and p satisfy (4.9)-(4.12).

Now, let us show the second part of this corollary. Assume that the Slater constraint qualifications are satisfied. If the above multiplier $\delta_0 = 0$, then $\lambda_i = 0$ for all $i \in I$, (4.9) and (4.10) implies that

$$(\dot{p}(t) + p(t)A(t), p(t)B(t) - \zeta(t)) \in \partial_{(x,u)} \sum_{j \in \bar{I}} \lambda_{k+j} \bar{G}[t]) \quad a.e. \tag{4.13}$$

$$p(0) \in N_D(x_*(0)),\ -p(1) \in \sum_{j \in \bar{I}} \lambda_{k+j} \partial \bar{g}_j(x_*(1)) \tag{4.14}$$

Notice that $\sum_{j=1}^{l} \lambda_{k+j} > 0$, by (4.12) and the Slater constraint qualifications, we have

$$\begin{aligned} 0 > \ & \sum_{j \in \bar{I}} \lambda_{k+j} \bar{F}_j(x_0, u_0) - \sum_{j \in \bar{I}} \lambda_{k+j} \bar{F}_j(x_*, u_*) \\ = \ & \sum_{j \in \bar{I}} \lambda_{k+j} \left(\int_0^1 \bar{G}_j(t, x_0, u_0) dt + \bar{g}_j(x_0(1)) - \int_0^1 \bar{G}_j[t] dt - \bar{g}_j(x_*(1)) \right) + \\ & \int_0^1 \langle p, \dot{x}_0 - Ax_0 - Bu_0 - b \rangle \, dt - \int_0^1 \langle p, \dot{x}_* - Ax_* - Bu_* - b \rangle \, dt \Big) \cdots\cdots (*) \\ =: \ & \Delta \end{aligned}$$

Calculate $(*)$ as in the [Sufficiency] of the Proof of Theorem 4.1. By using (4.13), (4.14) and (4.11), we can obtain that $\Delta \geq 0$, which is a contradiction. Thus $\delta_0 \neq 0$, so $\lambda_i > 0$ for all $I \in I$.

Let (x, u) be an arbitrary admissible process of $(\overline{MCP}^*)$, comparing $\sum_{i \in I} \lambda_i F_i(x, u)$ with $\sum_{i \in I} \lambda_i F_i(x_*, u_*)$ as in the [Sufficiency] of Proof of Theorem 4.1, we can see that (x_*, u_*) is a properly-solution for $(\overline{MCP}^*)$. Thus, the Corollary is proved. □

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